A New Upper Bound for the Isoperimetric Number of deBruijn Networks

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Abstract

C. Delorme and J.-P. Tillich found an upper bound and a lower bound for the isoperimetric number $i^n(d)$ of deBruijn Networks over the alphabet $\{0,1,\ldots,d-1\}$ using eigenvalue techniques (see [DT 96]). We improve their upper bound for $i^n(d)$ and give constructions for the sets of vertices of the deBruijn Network, which lead to our bound.

1 Introduction

The **d-ary deBruijn Network $\{0,1,\ldots,d-1\}^n_{dB}$** is an undirected graph without loops and without multiple edges. The set of vertices V^n consists of all n-tuples over the alphabet $\{0,1,\ldots,d-1\}$. Let $u,v \in V^n$ with $u \neq v$. Assume that $v = [v_0,\ldots,v_{n-1}]$.

The vertices u and v are connected by an edge, if

$$u \in \{[b,v_0,\ldots,v_{n-2}],[v_1,\ldots,v_{n-1},b] \mid b \in \{0,1,\ldots,d-1\}\}.$$

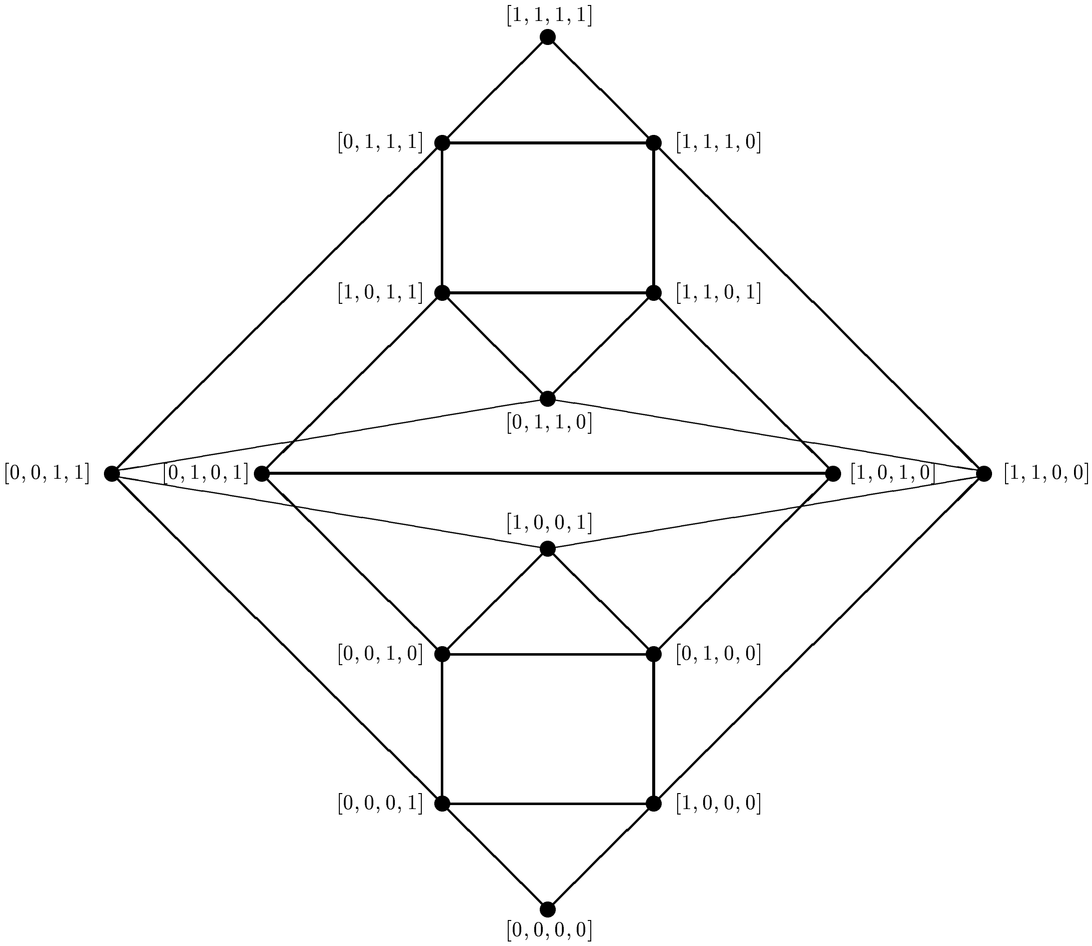

Figure 1: The $\{0,1\}^4_{dB}$

We define the **isoperimetric number $\boldsymbol{i^n(d)}$ of the deBruijn Network $\boldsymbol{\{0,1,\ldots,d-1\}^n_{dB}}$** by

$$i^n(d) := \min_{\substack{S \subset V^n \\ 1 \leq |S| \leq V^n/2}} \frac{|N(S)|}{|S|}.$$

where $N(S)$ is the set of vertices of $V^n \setminus S$ which are connected to some vertex in S.

2 A Construction of Sets with Small Neighbourhoods

It is convenient to compress the elements in V^n in the following way: A substring with a constant symbol is represented by the symbol and by the length of this substring.

For instance $[0,0,\ldots,0,0] \in V^n$ becomes $[0^n]$, and $[1,0,\ldots,0,1] \in V^n$ becomes $[1,0^{n-2},1]$. By "$*$" we denote any element of $\{0,1,\ldots,d-1\}$, and by "$+$" we denote any element of $\{1,\ldots,d-1\}$. For instance if $d=3$ and $n=3$ then $[+,0,*] = \{[1,0,0],[1,0,1],[1,0,2],[2,0,0],[2,0,1],[2,0,2]\}$.

With the help of this notation we formulate the following two theorems.

Theorem 2.1 *Consider the deBruijn Network $\{0,\ldots,d-1\}^n_{dB}$ for $n = 4m+1$ and $m \geq 2$. Let S be a subset of V^n with*

$$S = \bigcup_{k=0}^{m-1} [*^{m+k},+,0^{2m},*^{m-k}] \;\cup\; [*^m,0^{2m+1},*^m] \;\cup\; \bigcup_{k=0}^{m-1} [*^{m-k},0^{2m},+,*^{m+k}].$$

Then $|S| = d^{2m} + 2m(d-1)d^{2m}$ and $|N(S)| = 2(d-1)d^{2m} - (d-1)$.

Proof: Without loss of generality it suffices to check only three cases:

Case 1:

$$N([*^m,0^{2m+1},*^m]) = \underbrace{[*^{m-1},0^{2m+1},*^{m+1}]}_{\subset S} \cup \underbrace{[*^{m+1},0^{2m+1},*^{m-1}]}_{\subset S}.$$

Case 2: Let $0 \leq k \leq m-2$. Then

$$N([*^{m+k},+,0^{2m},*^{m-k}]) = \underbrace{[*^{m+k-1},+,0^{2m},*^{m-k+1}]}_{\subset S} \cup \underbrace{[*^{m+k+1},+,0^{2m},*^{m-k-1}]}_{\subset S}.$$

Case 3:

$$N([*^{2m-1},+,0^{2m},*]) = \underbrace{[*^{2m-2},+,0^{2m},*^2]}_{\subset S} \cup \underbrace{[*^{2m},+,0^{2m}]}_{\not\subset S}.$$

According to the symmetry of S, we have $N(S) = [*^{2m},+,0^{2m}] \cup [0^{2m},+,*^{2m}]$. Since $[0^{2m},+,*^{2m}] \cap [*^{2m},+,0^{2m}] = [0^{2m},+,0^{2m}]$ we obtain $|N(S)| = (d-1)d^{2m} + (d-1)d^{2m} - (d-1) = 2(d-1)d^{2m} - (d-1)$. $\square$

Theorem 2.2 *Consider the deBruijn Network $\{0,\ldots,d-1\}^n_{dB}$ for $n = 4m+1+f$, for $f \in \{1,2,3\}$ and $m \geq 2$. Let S be a subset of V^n with*

$$S = \bigcup_{k=0}^{m-1} [*^{m+k}, +, 0^{2m+f}, *^{m-k}] \;\cup\; [*^m, 0^{2m+f+1}, *^m] \;\cup\; \bigcup_{k=0}^{m-1} [*^{m-k}, 0^{2m+f}, +, *^{m+k}].$$

Then $|S| = d^{2m} + 2m(d-1)d^{2m}$ and $|N(S)| = 2(d-1)d^{2m}$.

Proof: With the same calculation as in the proof of Theorem 2.1 we obtain $N(S) = [*^{2m}, +, 0^{2m+f}] \cup [0^{2m+f}, +, *^{2m}]$. Since $[0^{2m+f}, +, *^{2m}] \cap [*^{2m}, +, 0^{2m+f}] = \emptyset$ we get $|N(S)| = (d-1)d^{2m} + (d-1)d^{2m} = 2(d-1)d^{2m}$.

□

3 A New Upper Bound for the Isoperimetric Number of deBruijn Networks

C. Delorme and J.-P. Tillich used the spectrum of deBruijn Networks to calculate upper and lower bounds for the isoperimetric number[1] (see [DT 96]). They have shown that the following inequalities hold:

$$\frac{1}{n} \leq i^n(d) \leq \frac{2\sqrt{d}\pi}{(n+1)\sqrt{1-\frac{2d\pi^2}{(n+1)^2}}}.$$

Their proof is a proof of existence, but not constructive. With the constructions of section 2 we are able to improve their upper bound.
We have $|S| = d^{2m} + 2m(d-1)d^{2m} \leq \frac{d^n}{2}$. If $n = 4m+1$ we get $|N(S)| = 2(d-1)d^{2m} - (d-1)$. Using this equality we get the following quotient:

$$\begin{aligned} \frac{|N(S)|}{|S|} &= \frac{2(d-1)d^{2m} - (d-1)}{d^{2m} + 2m(d-1)d^{2m}} \\ &\leq \frac{2(d-1)d^{2m}}{d^{2m} + 2m(d-1)d^{2m}} \end{aligned}$$

[1] *They call the isoperimetric number magnifying coefficient*

$$
\begin{aligned}
&= \frac{2(d-1)}{1+2m(d-1)} \\
&\leq \frac{2(d-1)}{2m(d-1)} \\
&= \frac{2}{2m} \\
&= \frac{1}{m}.
\end{aligned}
$$

Substitution of m by $\frac{n-1}{4}$ yields $\frac{|N(S)|}{|S|} \leq \frac{1}{\frac{n-1}{4}} \leq \frac{4}{n-1}$.
If $n = 4m + 1 + f$ we get $|N(S)| = 2(d-1)d^{2m}$. We have

$$
\begin{aligned}
\frac{|N(S)|}{|S|} &= \frac{2(d-1)d^{2m}}{d^{2m} + 2m(d-1)d^{2m}} \\
&= \frac{2(d-1)}{1+2m(d-1)} \\
&\leq \frac{2(d-1)}{2m(d-1)} \\
&= \frac{2}{2m} \\
&= \frac{1}{m}.
\end{aligned}
$$

Substitution of m by $\frac{n-1-f}{4}$ yields $\frac{|N(S)|}{|S|} \leq \frac{1}{\frac{n-1-f}{4}} \leq \frac{4}{n-4}$.
Summarizing the calculations above yields our main result.

Theorem 3.1 *Consider the deBruijn Network* $\{0, 1, \ldots, d-1\}$ *for* $n \geq 9$. *Then*

$$
i^n(d) \leq \frac{4}{n-4}.
$$

In contrast to C. Delorme and J.-P. Tillich we have have explicitly constructed the sets which lead to our upper bound.

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