ON SMOOTH BEHAVIOUR OF PROBABILITY DISTRIBUTIONS UNDER POLYNOMIAL MAPPINGS

F. Götze[1], Yu. V. Prohorov[1], V.V. Ulyanov[2]

Universität Bielefeld
Steklov Mathematical Institute, Moscow
Moscow State University

Abstract. Let X be a random variable with probability distribution P_X concentrated on $[-1, 1]$ and $Q(x)$ be a polynomial of degree $k \geq 2$. The characteristic function of a random variable $Y = Q(X)$ is of order $O(1/|t|^{1/k})$ as $|t| \to \infty$ if P_X is sufficiently smooth. In comparison for every $\varepsilon : 1/k > \varepsilon > 0$ there exists a singular distribution P_X such that every convolution $P_X^{n\star}$ is also singular while the characteristic function of Y is of order $O(1/|t|^{1/k-\varepsilon})$. While the characteristic function of X is small when "averaged" the characteristic function of the polynomial transformation Y of X is *uniformly* small.

1. Introduction

Let $Q(x)$ denote a polynomial of degree $k \geq 2$:

$$Q(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x. \tag{1-1}$$

Let X be a random variable with a probability distribution P_X concentrated on $[-1, 1]$. The characteristic function $g_Y(t)$ of a random varable $Y = Q(X)$, i.e. $g_Y(t) = E\exp\{itY\}$ is of order $O(1/|t|^{1/k})$ as $|t| \to \infty$ when P_X is sufficiently smooth (see Remark 1.3 below).

In the first part of the paper we extend this result (see Theorem 1.1 and its Corollaries) and obtain in particular that under some conditions on the characteristic function g_X we have

$$|g_Y(t)| = O(1/|t|^{\frac{1}{k}-\varepsilon}) \text{ as } |t| \to \infty, \tag{1-2}$$

where $0 \leq \varepsilon < 1/k$.

The second part is devoted to an example of X with a singular distribution such that any finite number of convolutions with itself remains singular (see Lemma 3.3). Furthermore, since (1-2) holds for characteristic function of $Q(X)$ (see Corollary 4.3), its behaviour is similar to the case where X has a smooth distribution.

[1]Supported by SFB 343 at Bielefeld, FRG.
[2]Supported by SFB 343 at Bielefeld, FRG, and Ministry of Education, Science and Culture, Japan.

In the following we shall introduce some notions concerning the smoothing of distributions. We introduce a smoothing random variable V independent of X with probability density

$$p_V(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2, \quad -\infty < x < \infty,$$

and characteristic function

$$g_V(t) = \begin{cases} 1-|t|, & |t| \le 1, \\ 0, & |t| > 1. \end{cases}$$

Fix any $T \ge 1$ and put

$$W_T = X + VT^{-1}.$$

Then

$$g_{W_T}(t) = \begin{cases} g_X(t)(1-|t|/T), & |t| \le T, \\ 0, & |t| > T. \end{cases}$$

Since W_T converges weakly to X as $T \to \infty$, we have for any real t

$$E\, e^{itQ(W_T)} \to E\, e^{itQ(X)} \quad \text{as} \quad T \to \infty. \tag{1-3}$$

Moreover for any positive number η we get

$$P\{|X + VT^{-1}| > 1+\eta\} \le P\{|V| > \eta T\} \to 0 \quad \text{as} \quad T \to \infty.$$

Hence

$$\int_{|x|>1+\eta} e^{itQ(x)} p_{W_T}(x)dx \to 0 \quad \text{as} \quad T \to \infty, \tag{1-4}$$

where $p_{W_T}(x)$ is a probability density of W_T.
Combining (1-3) and (1-4) we conclude that in order to prove $|g_Y(t)| \le A(t)$ it suffices to show

$$\lim_{\eta\to 0}\lim_{T\to\infty} |J_\eta(t,T)| \le A(t), \tag{1-5}$$

where

$$J_\eta(t,T) = \int_{|x|\le 1+\eta} e^{itQ(x)} p_{W_T}(x)dx.$$

2. General Results and their Proofs

Recall that $g_X(t) = E\, \exp\{itX\}$. Put for $T > 0$

$$\phi_X(T) = \int_{-T}^{T} |g_X(t)|\, dt.$$

Theorem 1.1. *Let X be a random variable such that $|X| \le 1$. Assume that there exist a non-decreasing positive function $\phi(t)$ on $(0,\infty)$ and a constant $\varepsilon : 0 \le \varepsilon < 1/k$, such that for all $t \ge 1$ and $b \ge 1$*

$$\phi_X(bt) \le b^{\varepsilon}\, \phi(t). \tag{2-1}$$

Then for $|t| \ge 1$ we have

$$|g_{Q(X)}(t)| \le C\, \phi(|t|)\, |t|^{-1/k}, \tag{2-2}$$

where the constant C does not depend on t (see (2-5) and (2-10)).

Proof. By the remark in (1-5) it is enough to prove

$$\lim_{\eta\to 0} \lim_{T\to\infty} |J_\eta(t,T)| \le C\, \phi(|t|)\, |t|^{-1/k}. \tag{2-3}$$

We write

$$\begin{aligned} J_\eta = J_\eta(t,T) &= \int_{|x|\le 1+\eta} e^{itQ(x)} \left(\frac{1}{2\pi} \int_{-T}^{T} e^{-i\tau x} g_X(\tau)(1-|\tau|/T)\, d\tau \right) dx \\ &= \frac{1}{2\pi} \int_{-T}^{T} g_X(\tau)(1-|\tau|/T) \left(\int_{-(1+\eta)}^{1+\eta} e^{itQ(x)-i\tau x} dx \right) d\tau. \end{aligned} \tag{2-4}$$

The inner integral in the right-hand side of (2-4) can be estimated by the following

Lemma 1.2. *(Vinogradov (1985), sec. II, ch. 2, Lemma 4, p. 199). Let $R(x)$ be a polynomial of degree $k \ge 2$ with real coefficients*

$$R(x) = r_k x^k + r_{k-1} x^{k-1} + \cdots + r_1 x$$

and

$$r = \max_{1\le j\le k} |r_j|.$$

Then

$$\left| \int_0^1 e^{2\pi i R(x)}\, dx \right| \le \min(1, 32\, r^{-1/k}).$$

We put

$$I_\eta = \int_{-(1+\eta)}^{(1+\eta)} e^{itQ(x)-i\tau x} dx$$

and

$$\rho_\eta = \max_{2\le j\le k} |(1+\eta)^j \alpha_j|, \tag{2-5}$$

where α_j are the coefficients of the polynomial $Q(x)$ (see (1-1)).

According to Lemma 1.2 we have

$$
\begin{aligned}
|I_\eta| &\le \left|\int_0^{1+\eta}\right| + \left|\int_{-(1+\eta)}^0\right| \\
&= (1+\eta)\left|\int_0^1 \exp\{itQ((1+\eta)y) - i\tau(1+\eta)y\}dy\right| \\
&\quad + (1+\eta)\left|\int_0^1 \exp\{itQ(-(1+\eta)y) + i\tau(1+\eta)y\}dy\right| \\
&\le (1+\eta)\cdot D\cdot \min\left(\frac{1}{\rho_\eta^{1/k}},\ \frac{|t|^{1/k}}{|\tau-\alpha_1 t|^{1/k}}\right), \qquad (2\text{-}6)
\end{aligned}
$$

where

$$D = 64\cdot\frac{(2\pi)^{1/k}}{|t|^{1/k}}.$$

We proceed now to bound J_η (see (2-4)).
Put

$$d = \rho_\eta + |\alpha_1|.$$

Let $J_{\eta,0}$ be the part of the integral J_η for values of τ satisfying

$$|\tau| \le 2\,d\,|t|,$$

and $J_{\eta,m}$, $m \ge 1$, be the part of J_η for values of τ satisfying

$$2^m d\,|t| < |\tau| \le 2^{m+1} d\,|t|.$$

Then by (2-6) we get

$$
\begin{aligned}
2\pi\cdot|J_{\eta,0}| &\le \left(\int_{|\tau|\le 2d|t|} |g_X(\tau)|\,d\tau\right)\cdot(1+\eta)\cdot D\cdot\rho_\eta^{-1/k} \\
&= (1+\eta)\cdot D\cdot\rho_\eta^{-1/k}\cdot\phi_X(2d|t|) \qquad (2\text{-}7)
\end{aligned}
$$

and for $m = 1, 2, \ldots$ we have

$$
\begin{aligned}
2\pi\cdot|J_{\eta,m}| &\le \left(\int_{2^m d|t|<|\tau|\le 2^{m+1}d|t|} |g_X(\tau)|\,d\tau\right)\frac{(1+\eta)\,D}{(2^m-1)^{1/k}(\rho_\eta+|\alpha_1|)^{1/k}} \\
&\le \frac{(1+\eta)\,D}{(2^m-1)^{1/k}\,(\rho_\eta+|\alpha_1|)^{1/k}}\left(\phi_X(2^{m+1}d|t|) - \phi_X(2^m d|t|)\right). \qquad (2\text{-}8)
\end{aligned}
$$

For given T and t we define $M = M(T,t)$ in the following way: put $M = 0$, if $T \le 2d|t|$, and let $M = M_0 \ge 1$ otherwise, where M_0 is chosen such that

$$2^{M_0} d\,|t| < T \le 2^{M_0+1} d\,|t|.$$

Then

$$|J_\eta| \le \sum_{m=0}^{M} |J_{\eta,m}|.$$

Therefore, adding the inequalities (2-7) and (2-8) and using condition (2-1) for a non-decreasing function $\phi(t)$ we obtain for any $T \ge 1$

$$\begin{aligned} 2\pi \cdot |J_\eta| &\le (1+\eta)\ D\ \phi(|t|) \left(\frac{(2\,(1+d))^\varepsilon}{\rho_\eta^{1/k}} + \sum_{j=1}^{\infty} \frac{2^{2\varepsilon}\, 2^{(j-1)\varepsilon}\,(1+d)^\varepsilon}{(2^j-1)^{1/k}\,(\rho_\eta+|\alpha_1|)^{1/k}} \right) \\ &\le (1+\eta)\, \frac{64\,(2\pi)^{1/k}\, 2^\varepsilon\,(1+\rho_\eta+|\alpha_1|)^\varepsilon}{|t|^{1/k}}\ \phi(|t|) \\ &\quad \times \left(\frac{1}{\rho_\eta^{1/k}} + \frac{2^\varepsilon}{(\rho_\eta+|\alpha_1|)^{1/k}} \cdot \frac{1}{1-q} \right), \end{aligned} \tag{2-9}$$

where

$$q = 2^{-1/k+\varepsilon} < 1.$$

As $\eta \downarrow 0$ and we obtain in view of (2-3), (2-5) and (2-9) the inequality (2-2) with

$$\begin{aligned} C &= 64\,(2\pi)^{1/k-1}\,(1+\rho_0+|\alpha_1|)^\varepsilon \cdot 2^\varepsilon \\ &\quad \times \left(\frac{1}{\rho_0^{1/k}} + \frac{1}{(\rho_0+|\alpha_1|)^{1/k}} \cdot \frac{1}{2^{-\varepsilon}-2^{-1/k}} \right). \end{aligned} \tag{2-10}$$

Remark 1.3. If the characteristic function $g_X(t)$ satisfies

$$\int_{-\infty}^{\infty} |g_X(t)|\, dt = A < \infty, \tag{2-11}$$

we may choose in Theorem 1.1

$$\varepsilon = 0 \quad \text{and} \quad \phi(t) = A.$$

Thus assuming condition (2-11) we get

$$|g_{Q(X)}(t)| \le C\, A\, |t|^{-1/k}. \tag{2-12}$$

Moreover, if (2-11) holds we drop the assumption $|X| \le 1$ (see Lemma 7 and estimates (22), (23) in Götze et al.(1996), where the multidimensional case is discussed as well; see also Sadikova (1966)).

Remark 1.4. Now we consider the case when there exists a constant B such that (2-1) holds with

$$\phi(t) \le B\, \phi_X(t). \tag{2-13}$$

This happens, for example, for a distribution with probability density

$$p_X(x) = \frac{e^{-x}}{(1-\lambda)}, \quad 0 \le x \le 1,$$

and characteristic function

$$g_X(t) = \frac{1-\lambda e^{it}}{(1-it)(1-\lambda)},$$

where $\lambda = e^{-1}$.
Here, (2-1) is fulfilled for any positive ε if we choose $\phi(t)$ as $\phi(t) = B\phi_X(t)$ with

$$B = B(\varepsilon) = \Big(1 + \varepsilon^{-1} \ln^{-1}(1+\sqrt{2})\Big) \cdot \frac{1+\lambda}{1-\lambda}.$$

Then we have

$$\phi(t) \le 2\, B \,\frac{1+\lambda}{1-\lambda}\, \ln(t + \sqrt{1+t^2}).$$

Furthermore, using the known inequalities for concentration functions we can establish a connection between the concentration functions of $Q(X)$ and X.
Let $G_X(z)$ denote the concentration function of X, i.e.

$$G_X(z) = \sup_{-\infty<x<\infty} P\{-z/2 + x \le X \le z/2 + x\}, \quad z \ge 0.$$

Lemma 1.5. *(Hengartner and Theodorescu (1973), Lemma 2.2.1). There are two positive constants A_1 and A_2 such that for any positive z*

$$A_1\, z\, (1+bz)^{-1} \int_{|t|\le b/2} |g_X(t)|^2\, dt \le G_X(z) \le A_2\, a^{-1} \int_{|t|\le a} |g_X(t)|\, dt,$$

where a, b are arbitrary positive numbers such that $0 < az \le 1$.

Lemma 1.6. *Assume that the characteristic function $g_X(t)$ of X is nonnegative. Then there exists a constant A_3 such that for any positive z*

$$A_3\, z \int_{|t|\le z^{-1}} g_X(t)\, dt \le G_X(z).$$

The Lemma follows from Lemmas 2.5 and 2.6 in Zaitsev (1988a), see also Zaitsev (1988b), Lemma 6.1 in Esseen (1968).

Applying Lemmas 1.5 and 1.6 we get from Theorem 1.1 the following

Corollary 1.7. *Assume that the characteristic function $g_X(t)$ is nonnegative and there exist constants $\varepsilon : 0 \le \varepsilon < 1/k$, and $B > 0$ such that for all $t \ge 1$ and $b \ge 1$*

$$\phi_X(bt) \le b^{\varepsilon} B\, \phi_X(t). \tag{2-14}$$

Then for any $0 < z \le 1$ we have

$$G_{Q(X)}(z) \le C_1\, G_X(z)\, z^{1/k-1}, \tag{2-15}$$

where C_1 does not depend on z (see (2-17)).

Proof. By Theorem 1.1, Lemmas 1.5 and 1.6 and (2-14) we have for any $0 < z \le 1$

$$\begin{aligned} G_{Q(X)}(z) &\le A_2\, z \int_{|t|\le z^{-1}} |g_{Q(X)}(t)|\, dt \\ &\le A_2\, z \left(2 + 2\, C\, B \int_1^{z^{-1}} \phi_X(t)\, t^{-1/k}\, dt\right) \\ &\le 2\, A_2\, z\, \phi_X(z^{-1}) \left(1/\phi_X(1) + C\, B\, (1 - 1/k)^{-1}\, (z^{1/k-1} - 1)\right) \\ &\le C_1\, G_X(z)\, z^{1/k-1}, \end{aligned} \tag{2-16}$$

where

$$C_1 = 2\, A_3\, A_2\, (1/\phi_X(1) + C\, B\, (1 - 1/k)^{-1}). \tag{2-17}$$

Remark 1.8. The power of z in the right-hand side of (2-15) can not be larger. This is shown by the following simple example.

Let X has a density function $p(x) = 1 - |x|$ when $|x| \le 1$. Then for all $z : 0 < z < 2^{-k+1}$, we have (cf.(2-15))

$$G_{Q(X)}(z) \ge G_X(z)\, z^{1/k-1}.$$

For a distribution P_X with atoms the bound (2-15) is trivial.

If we do not assume that $g_X(t)$ is nonnegative we can prove

Corollary 1.9. *Let conditions of Theorem 1.1 are fulfilled. Then for any $0 < z \le 1$*

$$G_{Q(X)}(z) \le C_2\, \phi(z^{-1})\, z^{1/k},$$

where

$$C_2 = 2\, A_2\, (1/\phi(1) + C\, (1 - 1/k)^{-1}).$$

Proof. It follows from Lemma 1.5, Theorem 1.1 and the first three inequalities in (2-16).

The right-hand side inequality in Lemma 1.5 and Corollary 1.9 together immediately imply

Corollary 1.10. *Let X be a random variable such that $|X| \le 1$. Assume that for some $\varepsilon : 0 \le \varepsilon < 1/k$*

$$\phi_X(t) = O(t^\varepsilon) \quad as \quad t \to \infty.$$

Then the distribution function F_X of X satisfies a Lipschitz condition of order $1-\varepsilon$ and $F_{Q(X)} \in Lip(1/k - \varepsilon)$.

Cf. Theorems 1 and 2 in Makabe (1960).

Remark 1.11. It follows from Corollary 1.10 that a distribution function of X which satisfies the conditions of Theorem 1.1 is necessarily a Lipschitz function of order $1 - \varepsilon$.
Condition (2-1) is satisfied by many absolutely continuous distributions (cf. Zygmund (1968), ch. VI, §3), and by some singular distributions (cf. Wiener, Wintner (1938), Salem (1942), Schaeffer (1939), also Zygmund (1968), ch. XII, §10, 11). The general problem of providing upper and lower bounds for the Fourier-Stieltjes transforms of singular measures has been intensively studied (see Hewitt, Kakutani (1960), (1964), Hewitt, Zuckerman (1966)).

3. Singularity and Cramer's condition

We show that the well-known Cantor distribution with characteristic function

$$L(t) = \prod_{j=1}^{\infty} \cos(2\pi \cdot 3^{-j} t) \tag{3-1}$$

satisfies the Cramer condition, i. e.

$$\overline{\lim}_{|t|\to\infty} |L(t)| < 1. \tag{3-2}$$

In fact, (3-2) was proved by Wiener and Wintner (1938) in the case when t assumes integer values. Our arguments are a refinement of those used by Wiener and Wintner. We obtain sharper bound than (3-2), namely we prove

Theorem 3.1. *If*

$$|t| \ge 8.5 \tag{3-3}$$

then

$$|L(t)| \le e^{-0.027} \tag{3-4}$$

We shall give a proof of Theorem 3.1 at the end of this section. First recall some properties of the Cantor distributions. It is concentrated on $[-\pi, \pi]$ and straightforward calculations yield

Lemma 3.2. *For every integer $n \geq 1$*

$$|L(3^n)| = |L(1)| = 0.37\ldots. \tag{3-5}$$

Not only is the Cantor distribution singular but its convolutions of any order are singular as well, i. e. we have

Lemma 3.3. *A probability distribution with characteristic function $L^m(t)$ is singular for any integer $m \geq 1$.*

Remark. We mention one interesting consequence of (3-4) and Lemma 3.3. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i. i. d. random variables with characteristic function $L(t)$. Put $\sigma^2 = DX_1$ and $S_n = (X_1 + \cdots + X_n)/(\sigma\sqrt{n})$. By the Central Limit Theorem the probability distribution of S_n converges weakly to a standart normal law Φ and the distribution function $P\{S_n < x\}$ admits asymptotic expansion of any order.

At the same time the "distance in variation" between Φ and the distribution of S_n is equal to 1 for any $n \geq 1$, i. e.

$$\sup_{a\in\mathcal{B}} |P\{S_n \in A\} - \Phi(A)| = 1, \tag{3-5a}$$

where $\mathcal{B}$ denotes the class of all Borel sets in $\mathbb{R}$.

However, it is easy to see, that

$$\sup_t |Ee^{itS_n} - e^{-t^2/2}| \to 0 \text{ as } n \to \infty. \tag{3-6}$$

This provides another example demonstrating that the convergence in the sense (3-6) is weaker than the convergence in variation (3-5a).

Proof of Lemma 3.3 A probability distribution with characteristic function $L^m(t)$, $m \geq 1$, is an infinite convolution of discrete distributions. Therefore (see Esseen (1945), pt.I, ch.1, p.28, Lemma 4), it has to be either purely discrete or purely singular or absolutely continuous. By Lemma 3.2 it cannot be absolutely continuous. Furthermore, it cannot be purely discrete since the maximal jump of a distribution function corresponding to the characteristic function

$$\left(\prod_{j=1}^{N} \cos(2\pi\, t/3^j)\right)^m$$

is bounded from above by 2^{-N} which tends to zero as $N \to \infty$ (cf. Esseen (1945), Lemma 5). This proves Lemma 3.3.

We introduce the following notation. For any positive t we write

$$t = j_t + \Theta_t,$$

where j_t is an integer and Θ_t is such that $-0.5 \le \Theta_t < 0.5$.
For j_t we use its ternary representation

$$j_t = \delta_{k_t} \cdot 3^{k_t} + \delta_{k_t - 1} \cdot 3^{k_t - 1} + \cdots + \delta_0 \cdot 1, \tag{3-6a}$$

where $\delta_j = 0$ or 1 or 2 and for $j_t > 0$ a factor $\delta_{k_t} \ne 0$. We say that a sequence

$$\delta_{k_t}, \ldots, \delta_1 \tag{3-6b}$$

has a change on the j-th place, $2 \le j \le k_t$, if $\delta_j \ne \delta_{j-1}$.

Lemma 3.4. *If t satisfies (3-3) and the sequence (3-6b) has no changes then*

$$|L(t)| \le \exp(-0.465). \tag{3-7}$$

Proof. For brevity we shall write k instead of k_t. Since $L(t)$ is an even function it is enough to prove (3-7) for $t \ge 8.5$. Hence $k \ge 2$.

If $j_t = 3^k + 3^{k-1} + \cdots + 3 + \delta_0$ then

$$3^{k+1}/2 - 2 \le t \le 3^{k+1}/2 + 1$$

and

$$|L(t)| \le \left| \cos\left(\frac{2\pi\, t}{3^{k+2}} \right) \right| \le \cos\left(\frac{\pi}{3} - \frac{4\pi}{3^4} \right) \le e^{-0.465}. \tag{3-8}$$

If $j_t = 2 \cdot 3^k + 2 \cdot 3^{k-1} + \cdots + 2 \cdot 3 + \delta_0$ then

$$3^{k+1} - 3.5 \le t \le 3^{k+1} - 0.5$$

and

$$|L(t)| \le \left| \cos\left(\frac{2\pi t}{3^{k+2}} \right) \right| \le \left| \cos\left(\frac{2\pi}{3} \right) \right| = 0.5. \tag{3-9}$$

Thus, (3-7) follows from (3-8) and (3-9).

Lemma 3.5. *If t satisfies (3-3) and the sequence (3-6b) has a change, i. e. $\delta_j \ne \delta_{j-1}$ for some $j : 2 \le j \le k$, then*

$$\left| \cos\left(\frac{2\pi\, t}{3^{j+1}} \right) \right| \le e^{-\lambda} \tag{3-10}$$

with $\lambda = 0.027$.

Corollary of Lemma 3.5. *If t satisfies (3-3) then*

$$|L(t)| \leq e^{-\lambda\mu(t)}, \tag{3-11}$$

where $\mu(t) = \mu(j_t)$ is a number of changes in the sequence $\delta_k, \delta_{k-1}, \ldots, \delta_1$, which corresponds to j_t.

Proof of Lemma 3.5. We set

$$t/3^{j+1} = A + r,$$

where

$$A = \delta_j/3 + \delta_{j-1}/3^2,\ r = \delta_{j-2}/3^3 + \cdots + \delta_0/3^{j+1} + \theta_t/3^{j+1}.$$

We have

$$r \leq 2 \cdot 3^{-3}(1 + 3^{-1} + \cdots + 3^{2-j}) + 0.5 \cdot 3^{-j-1} < 9^{-1}. \tag{3-12}$$

At the same time we get

$$r > -0.5 \cdot 3^{-j-1} \geq -0.5 \cdot 3^{-3}. \tag{3-13}$$

If (δ_j, δ_{j-1}) equals $(0,1)$ or $(0,2)$ or $(1,0)$ then $A = 1/9$ or $2/9$ or $1/3$ respectively.

Therefore, $1/9 - 0.5/3^3 \leq A + r \leq 4/9$ and $2\pi t/3^{j+1}$ lies in the interval $[2\pi/9 - \pi/3^3;\ 8\pi/9]$.

If (δ_j, δ_{j-1}) equals $(1,2)$ or $(2,0)$ or $(2,1)$ then $A = 5/9$ or $6/9$ or $7/9$ resp. .

Therefore,

$$5/9 - 0.5/3^3 \leq A + r \leq 8/9$$

and $2\pi t/3^{j+1}$ lies in the interval $[10\pi/9 - \pi/3^3,\ 16\pi/9]$.

Thus, in all cases we have

$$|\cos(2\pi\, t/3^{j+1})| \leq |\cos(10\pi/9 - \pi/3^3)| < \exp(-0.027).$$

and Lemma 3.5 is proved.

Proof of Theorem 3.1 immediately follows from Lemmas 3.4 and 3.5.

4. Convolution powers of Cantor distributions and polynomial mappings

We consider the function $L^m(u)$, where m is an integer, $m \geq 1$, and $L(u)$ is defined by (3-1). Put for any $T \geq 1$

$$\varphi_m(T) = 1 + 2 \sum_{1 \leq j \leq T} L_j^m,$$

where $L_j = \max\limits_{-0.5 \leq \Theta \leq 0.5} |L(j + \Theta)|$.

Theorem 4.1. *For all real $T \geq 1$, $1 > \varepsilon > 0$ and integer*

$$m \geq m(\varepsilon) = \lambda^{-1} \ln(2/(3^{\varepsilon} - 1)) \tag{4-1}$$

we have

$$\varphi_m(T) \leq 53 e^{\lambda m} \cdot T^{\varepsilon}, \tag{4-2}$$

where $\lambda = 0.027$.

Corollary 4.2. *Under the conditions of Theorem 4.1 we have for all $T \geq 0$*

$$\int_{-T}^{T} |L^m(t)|\, dt \leq 55\, e^{\lambda m}\, T^{\varepsilon}. \tag{4-3}$$

Proof of Corollary. Inequality (4-3) immediately follows from the obvious estimate

$$\int_{-T}^{T} |L^m(t)|\, dt \leq \varphi_m(T + 0.5) \leq \varphi_m(T) + 2.$$

Corollary 4.3. *Let $Q(x)$ be a polynomial of degree $k \geq 2$ and $\varepsilon : 0 < \varepsilon < 1/k$. Let X denote a random variable with singular distribution and characteristic function $L^m(t)$ for any m satisfying (4-1). Then*

$$|g_{Q(X)}(t)| = O(|t|^{\varepsilon - 1/k}) \quad as \quad |t| \to \infty.$$

Proof of Corollary 4.3 immediately follows from Theorem 1.1 and (4-2).

We shall give a proof of Theorem 4.1 right after Lemma 4.4.

First, we fix an integer $N \geq 1$ and consider a probability space $(\Omega, \mathcal{F}, P)$ with Ω consisting of all sequences $\delta = (\delta_N, \delta_{N-1}, \ldots, \delta_0)$, such that δ_j assumes values $0, 1$ and 2. Furthermore, $\mathcal{F}$ denotes the set of all subsets of Ω and for each sequence δ we put $P\{\delta\} = 3^{-N-1}$.

Lemma 4.4. *The random variable Z on $(\Omega, \mathcal{F}, P)$ defined as the number of changes in a sequence $\delta_N, \delta_{N-1}, \ldots, \delta_0$ has a binomial distribution with parameters N and $2/3$, i. e.*

$$P\{Z = k\} = \binom{N}{k} \cdot 3^{-N} \cdot 2^k, \;\; k = 0, 1, \ldots, N.$$

Proof of the Lemma 4.4. follows by elementary calculations.

Proof of Theorem 4.1. Let us fix any $N \geq 2$. We represent any integer $j : 0 \leq j \leq 3^{N+1} - 1$, in the form (cf.(3-6a))

$$j = \delta_N \cdot 3^N + \delta_{N-1} \cdot 3^{N-1} + \cdots + \delta_0, \tag{4-4}$$

for $\delta_i = 0, 1$ or 2. Let $\mu_1(j)$ denote the number of changes in a sequence $\delta_N, \dots, \delta_1$ in (4-4) and let $\mu(j)$ be defined as before (see Corollary of Lemma 3.5; recall that $\mu(j)$ is defined for $j \geq 9$). It is clear that for $j :\ 9 \leq j \leq 3^{N+1} - 1$

$$\mu(j) \leq \mu_1(j) \leq \mu(j) + 1. \tag{4-5}$$

By (4-5) and Corollary of Lemma 3.5 we derive for $m \geq 1$

$$L_j^m \leq e^{\lambda m(1-\mu_1(j))}. \tag{4-6}$$

Since for $0 \leq j < 9$ we have $\mu_1(j) \leq 1$, inequality (4-6) holds for all integers $j : 0 \leq j \leq 3^{N+1} - 1$, and $m \geq 1$.

It follows from Lemma 4.4 and (4-6) that for $N \geq 2$ we have

$$\begin{aligned} \varphi_m(3^{N+1} - 1) &\leq 2\, e^{\lambda m} \sum_{j=0}^{3^{N+1}-1} e^{-\lambda m \mu_1(j)} \\ &= 2\, e^{\lambda m}\, 3^{N+1} \left(\sum_{j=0}^{3^{N+1}-1} e^{-\lambda m \mu_1(j)}\, 3^{-N-1} \right). \end{aligned} \tag{4-7}$$

Assigning to each $j :\ 0 \leq j \leq 3^{N+1} - 1$, a probability 3^{-N-1} we may consider a sequence δ_N, $\delta_{N-1}, \dots, \delta_0$ as a sequence of independent random variables to which we can apply Lemma 4.4. Hence $\mu_1(j)$ has a binomial distribution with parameters $(N-1, 2/3)$. Therefore the right-hand side of (4-7) can be rewritten in the following form

$$\begin{aligned} &2\, e^{\lambda m}\, 3^{N+1} \sum_{r=0}^{N-1} e^{-\lambda m r} \cdot \binom{N-1}{r} \cdot 3^{-N+1} \cdot 2^r \\ &= 18\, e^{\lambda m}\, (1 + 2e^{-\lambda m})^{N-1}. \end{aligned} \tag{4-7a}$$

For given $\varepsilon : 1 > \varepsilon > 0$, we define a real number m_ε so that $1 + 2e^{-\lambda m_\varepsilon} = 3^\varepsilon$. Thus, for $m \geq m_\varepsilon = \lambda^{-1} \ln(2/(3^\varepsilon - 1))$ we get from (4-7a) that

$$\varphi_m(3^{N+1} - 1) \leq 18\, e^{\lambda m}\, 3^{(N-1)\varepsilon}.$$

If

$$3^{N+1} - 1 \leq T < 3^{N+2} - 1,$$

then

$$\varphi_m(T) \leq \varphi_m(3^{N+2} - 1) \leq 18\, e^{\lambda m}\, 3^{N\varepsilon} < 18\, e^{\lambda m}\, (T/2)^\varepsilon. \tag{4-8}$$

Recall that (4-8) has been proved for $N \geq 2$, i. e. $T \geq 26$. For $1 \leq T < 26$ we have

$$\varphi_m(T) \leq 1 + 2\, T \leq 53\, T^\varepsilon. \tag{4-9}$$

Combining (4-8) and (4-9) we get (4-2).

Finally we would like to note that this paper was motivated by the classical work of Wiener and Wintner (1938).

References

Esseen, C.-G., *Fourier analysis of distribution functions. A mathematical study of the Laplace-gaussian law*, Acta Math., **77** (1945), 1–125.

Esseen, C.-G., *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheor. verw. Geb., **9** (1968), 290–308.

Götze, F., Prohorov, Yu. V., Ulyanov, V.V., *Bounds for characteristic functions of polynomials in asymptotically normal variables*, Russian Math. Surveys, no.2 (308) **51** (1996), 181–204.

Hengartner, W., and Theodorescu, R., *Concentration functions*, Academic Press N.-Y., London (1973).

Hewitt, E., Kakutani, S., *A class of multiplicative linear functionals on the measure algebra of a locally compact Abelian group*, Illinois J. Math., **4** (1960), 533–574.

Hewitt, E., Kakutani, S., *Some multiplicative linear functionals of $M(G)$*, Ann. of Math., **79** (1964), 485–505.

Hewitt, E., Zuckerman, H.S., *Singular measures with absolutely continuous convolution squares*, Proc. Camb. Phil. Soc. **62** (1966), 399–420.

Makabe, H., *A remark on the smoothness of the distribution function*, Yokohama Math. J. **8** (1960), 59–68.

Sadikova, S.M., *Some inequalities for characteristic functions*, Theory Probab. Appl. **11** (1966), 441–447.

Salem, R., *On singular monotonic functions of the Cantor type*, J. Math. Phys. **21** (1942), 69–82.

Schaeffer, A.C., *The Fourier–Stieltjes coefficients of a function of bounded variation*, Amer. J. Math. **61** (1939), 934–940.

Vinogradov, I.M., *"Selected Works"*, Springer, Berlin (1985).

Wiener, N., Wintner, A., *Fourier–Stieltjes transforms and singular infinite convolutions*, Amer. J. Math. **60** (1938), 513–522.

Zaitsev, A.Yu., *Estimates for the closeness of successive convolutions of multidimensional symmetric distributions*, Probab. Theory Rel. Fields **79** (1988), 175–200.

Zaitsev, A.Yu., *Multidimensional generalized method of triangular functions*, J. Sovit Math. **43** (1988), 2797–2810.

Zygmund, A., *Trigonometric Series*, Cambridge Univ. Press (1968).