Harmonic and holomorphic Prym differentials on a compact Riemann surface

Victor V. Chueshev

Fakultät für Mathematik
Universität Bielefeld
D-33501 Bielefeld, Deutschland

and

Department of Mathematics
Kemerovo State University
Kemerovo 650043, Russia

e-mail: chueshev@kemgu.kemerovo.su

Abstract

In 1980, R. C. Gunning studied the holomorphic (multiplicative) Prym differentials on a compact Riemann surface of genus $g \geq 2$ and their periods for the so-called essential characters. These period classes are an important set of transcendental invariants attached to the surface. Gunning introduced the cohomology bundle consisting of such period classes and the Prym bundle consisting of the holomorphic Prym differentials, and gave their explicit description for the case $g = 2$. In 1990, the author gave the explicit description of the Gunning bundle for surfaces of genus $g \geq 2$.

In the present article we introduce and study also the harmonic Prym bundle consisting of the harmonic Prym differentials for the case when $g \geq 2$. It is shown that the harmonic Prym bundle is the orthogonal sum of the holomorphic vector bundles of holomorphic Prym differentials and antiholomorphic Prym differentials. Some properties for the period classes of such differentials are established.

Introduction

The harmonic and holomorphic (multiplicative) Prym differentials on compact Riemann surfaces and their periods were investigated by F. Prym, and G. Rost [1] at the end of the last century; and by R. Koenig [2], O. Haupt [3], and H. Petersson [4] at the first half of our century. At present they are considered by many authors from different points of view: in connection with the study of Poincaré's series by C. Earle [5], and I. Kra [6]; in connection with application to number theory by G. Kempf [7], J. Fay [8], and J. Jorgenson [9]; and in variation theory by E. Jablow [10]. However, all these authors studied holomorphic Prym differentials as a rule with respect to two special kinds of complex one-dimensional representations (characters) of the fundamental group of a compact Riemann surface: when either the characters take their values only on the unit circle, or when they are equal to unit on half the generators.

In 1980, R. C. Gunning [11] studied the holomorphic Prym differentials and their periods for the so-called essential characters. The period classes of the holomorphic

Prym differentials on compact Riemann surfaces of genus $g \geq 2$ are an important set of transcendental invariants attached to the surfaces. Gunning introduced the cohomology bundle consisting of such period classes and the Prym bundle consisting of the holomorphic Prym differentials, and gave their explicit description for the case $g = 2$. The explicit description of the Gunning bundle for surfaces of genus $g \geq 2$ has been given in [12].

In the present article we introduce and study also the harmonic Prym bundle consisting of the harmonic Prym differentials for the case when $g \geq 2$. It will be shown that the harmonic Prym bundle is the orthogonal sum of the holomorphic vector bundles of holomorphic Prym differentials and of antiholomorphic Prym differentials. Some properties of the period classes for such differentials will be established.

I would like to thank Prof. A. D. Mednykh for the constructive conversations on this subject. This work was written while the author was staying at the University of Bielefeld. I would like to express my sincere gratitude to Prof. H. Helling and Prof. J. Mennicke for their kind hospitality.

1. Preliminaries

Let $\left[F, \{\alpha_k, \beta_k\}_{k=1}^{g}\right]$ be a marked compact Riemann surface of genus $g \geq 2$, and Γ be a marked Fuchsian group of the first kind acting on the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$, so that U is the universal covering for F, and Γ is the group of covering (deck) transformations for the natural projection $\pi : U \to U/\Gamma = F$. The choice of generators for the fundamental group

$$\pi_1(F; O) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g : \prod_{j=1}^{g} [\alpha_j, \beta_j] = 1 \rangle$$

specifies the choice of generators for the group

$$\Gamma = \langle A_1, \ldots, A_g, B_1, \ldots, B_g : \prod_{j=1}^{g} C_j = 1 \rangle;$$

here O is a fixed base point on F; and $C_j = [A_j, B_j] = A_j B_j A_j^{-1} B_j^{-1}$, $j = 1, \ldots, g$.

Denote by $Hom(\Gamma, \mathbf{C}^*)$ the group of all one-dimensional representations ρ of Γ in $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ with the natural product operation; i.e., if $\rho_1, \rho_2, \rho \in Hom(\Gamma, \mathbf{C}^*)$, then $\rho_1 \cdot \rho_2 \in Hom(\Gamma, \mathbf{C}^*)$, where $(\rho_1 \cdot \rho_2)(T) = \rho_1(T)\rho_2(T)$ for all $T \in \Gamma$; and also $\rho^{-1} \in Hom(\Gamma, \mathbf{C}^*)$, where $(\rho^{-1})(T) = \rho(T)^{-1}$ for all $T \in \Gamma$.

Note that $Hom(\Gamma, \mathbf{C}^*)$ is a $2g$-dimensional complex analytic Lie group. Indeed, the map

$$\rho \to (\rho(A_1), \ldots, \rho(A_g), \rho(B_1), \ldots, \rho(B_g)) \in [\mathbf{C}^*]^{2g}$$

defines a global complex analytic structure on $Hom(\Gamma, \mathbf{C}^*)$. Therefore, $Hom(\Gamma, \mathbf{C}^*)$ is biholomorphically isomorphic to the group $[\mathbf{C}^*]^{2g}$ with the coordinate-wise product operation.

For $\rho \in Hom(\Gamma, \mathbf{C}^*)$, denote by $Z^1(\Gamma, \rho)$ the set of all mappings $\phi : \Gamma \to \mathbf{C}$ such that

$$\phi(ST) = \phi(S) + \rho(S)\phi(T) \tag{1}$$

for all S, $T \in \Gamma$. Each element $\phi \in Z^1(\Gamma, \rho)$ is uniquely defined by an ordered system of complex numbers $\phi(A_1), \ldots, \phi(A_g), \phi(B_1), \ldots, \phi(B_g)$ satisfying the equation

$$\sum_{j=1}^{g} \{\sigma(B_j)\phi(A_j) - \sigma(A_j)\phi(B_j)\} = 0 \tag{2}$$

which is obtained from the relation $\prod_{j=1}^{g} C_j = 1$ in Γ, where $\sigma(T) = 1 - \rho(T)$ for all $T \in \Gamma$. Therefore, $Z^1(\Gamma, \rho)$ is a $(2g-1)$-dimensional complex vector space if $\rho \neq 1$ (that is $\rho(S) \neq 1$ for some $S \in \Gamma$), but $2g$-dimensional space if $\rho = 1$.

Let $B^1(\Gamma, \rho)$ be the one-dimensional subspace of $Z^1(\Gamma, \rho)$ generated by the element σ. Then $H^1(\Gamma, \rho) = Z^1(\Gamma, \rho)/B^1(\Gamma, \rho)$ is a $(2g-2)$-dimensional complex vector space for $\rho \neq 1$. We shall call the set $G = \cup_{\rho \neq 1} H^1(\Gamma, \rho)$ the *Gunning cohomology bundle* for the surface F.

In [12], it was shown that for each $g \geq 2$, G is a holomorphic vector bundle of rank $(2g-2)$ over $Hom(\Gamma, \mathbf{C}^*) \setminus 1$. In order to describe G for $\rho \neq 1$, we use the isomorphism (established in [11]) between the complex vector space $H^1(\Gamma, \rho)$ and the vector space $Hom_\rho([\Gamma, \Gamma], \mathbf{C})$ that consists of homomorphisms $\phi_0 : [\Gamma, \Gamma] \to (\mathbf{C}, +)$ such that

$$\phi_0(STS^{-1}) = \rho(S)\phi_0(T)$$

for all $T \in [\Gamma, \Gamma]$ and $S \in \Gamma$; here $[\Gamma, \Gamma]$ is the commutator subgroup of Γ.

Thus, the bundle G is isomorphic to the vector bundle $\cup_{\rho \neq 1} Hom_\rho([\Gamma, \Gamma], \mathbf{C})$. Moreover, the holomorphic transition matrices of this bundle can be completely described by $2g$ coordinate neighbourhoods $U_j = \{\rho : \rho(A_j) \neq 1\}$ and $U_{g+j} = \{\rho : \rho(B_j) \neq 1\}$, $j = 1, \ldots, g$, which cover the base of $Hom(\Gamma, \mathbf{C}^*) \setminus 1$.

It follows from [13, p.14] that each representation $\rho \in Hom(\Gamma, \mathbf{C}^*)$ is a flat factor of automorphy of rank one for the action of Γ on U which can be described by a flat holomorphic line bundle over F. Indeed, we have the relation $\rho(ST) = \rho(S)\rho(T)$ for all S, $T \in \Gamma$, and the factor of automorphy $\rho : \Gamma \times U \to \mathbf{C}^*$ is independent of $z \in U$. Consider the trivial holomorphic line bundle $\tilde{\xi} = U \times \mathbf{C}$ and the group $\tilde{\Gamma}$ of holomorphic automorphisms for $\tilde{\xi}$ such that $\tilde{T}(z, t) = (Tz, \rho(T) \cdot t)$, where $z \in U$, $T \in \Gamma$. Then the factor bundle $\tilde{\xi}/\tilde{\Gamma} = \xi$ is a flat holomorphic line bundle over $U/\Gamma = F$, and its factor of automorphy equals ρ.

Let us denote by L_g the set of those points $\rho \in Hom(\Gamma, \mathbf{C}^*)$ each of which corresponds to a flat holomorphic line bundle ξ over F that is analytically equivalent to the trivial holomorphic line bundle over F. In [13], it was shown that L_g is a g-dimensional complex analytic connected Lie subgroup of the Lie group $Hom(\Gamma, \mathbf{C}^*)$. We need the following properties of L_g.

The quotient group $Hom(\Gamma, \mathbf{C}^*)/L_g$ can be identified with the marked Jacobi variety $J(F) = \mathbf{C}^g/\{I, \Omega\}$, where I is the identity matrix of order g; and $\Omega = (\Omega_{jk})$ is the matrix of B-periods for the canonical basis $\zeta_1, \ldots, \zeta_g$ of the holomorphic abelian differentials of the first kind on F dual with $\{\alpha_k, \beta_k\}_{k=1}^g$ (that is, $\int_{\alpha_k} \zeta_j = \delta_{jk}$, $\int_{\beta_k} \zeta_j = \Omega_{jk};\ j,\, k = 1, \ldots, g$ [13, 14]).

There are other descriptions of the groups $Hom(\Gamma, \mathbf{C}^*)$ and L_g [13, 14]. For each point $(x, y) = (x_1, \ldots, x_g; y_1, \ldots, y_g) \in \mathbf{C}^g \times \mathbf{C}^g$, there exists a representation $\rho_{x,y} \in Hom(\Gamma, \mathbf{C}^*)$ such that

$$\rho_{x,y}(A_j) = \exp 2\pi i x_j, \quad \rho_{x,y}(B_j) = \exp 2\pi i y_j, \quad j = 1, \ldots, g. \tag{3}$$

The map $\rho : (x, y) \mapsto \rho_{x,y}$ is surjective with kernel $\mathbf{Z}^{2g} < \mathbf{C}^{2g}$. Hence $Hom(\Gamma, \mathbf{C}^*) \cong \mathbf{C}^{2g}/\mathbf{Z}^{2g}$; and it is not a compact set. We have $L_g = \rho(L_0)$, where $L_0 = \{(x, y) \in \mathbf{C}^g \times \mathbf{C}^g : y = \Omega x\}$ is a g-dimensional vector subspace of $\mathbf{C}^{2g}$.

Note that L_g coinsides with the set of the so-called *inessential characters* $\rho_{x,y}$ for Γ which can be represented in form (3) for $x_j = c_j,\ y_j = \sum_{k=1}^g \Omega_{jk} c_k,\ j = 1, \ldots, g$, where $(c_1, \ldots, c_g)$ is an arbitrary point of $\mathbf{C}^g$. If $\rho_{x,y} \in Hom(\Gamma, \mathbf{C}^*) \setminus L_g$, then $\rho_{x,y}$ can be presented in form (3) for $x_j = c_j,\ y_j = \sum_{k=1}^g \Omega_{jk} c_k + g_j$, where $g_j \notin \mathbf{Z}$ for some j, $j = 1, \ldots, g$. These characters are called the *essential characters* for Γ [14].

Notice that the mapping ρ is a holomorphic bijection of L_0 onto L_g. Indeed, if $\rho_{x,y} = \rho_{\widetilde{x},\widetilde{y}}$, then $\widetilde{x} = x + k$, $k \in \mathbf{Z}^g$, and $\widetilde{y} = \Omega\widetilde{x} = \Omega(x + k)$. Hence, $\Omega(x + k) = \widetilde{y} = \Omega x + n_1$, $n_1 \in \mathbf{Z}^g$, which is equivalent to the following two identities $(Re\Omega)k = n_1$, $(Im\Omega)k = 0$. Therefore, since $Im\Omega$ is a non-singular matrix, we have $k = 0$. Thus, $x = \widetilde{x}$, and $y = \widetilde{y}$.

2. Holomorphic Prym differential

Definition 1. A *holomorphic Prym differential* on F associated to a representation $\rho \in Hom(\Gamma, \mathbf{C}^*)$ is a holomorphic (single-valued) differential 1-form $\phi(z)$ on U such that $\phi(Tz) = \rho(T)\phi(z)$ for all $T \in \Gamma$ and $z \in U$.

This 1-form is multiplicative and many-valued on the surface $F = U/\Gamma$, because after analytical continuation of some of its branches along a closed curve one can obtain another branch which differs from given one on a non-zero complex multiplier.

The Prym differential $\phi(z)$ defined on the simply connected domain U can be written in the form $\phi(z) = df(z)$ for a suitable function $f(z)$ holomorphic on U. We will call such an $f(z)$ a *Prym integral* for differential $\phi(z)$, it is uniquely defined up to an additive complex constant. The following equalities hold:

$$f(Tz) = \rho(T)f(z) + \phi(T), \tag{4}$$

and

$$\phi(ST) = \phi(S) + \rho(S)\phi(T), \tag{5}$$

where $\phi(T) = f(Tz_0) - \rho(T)f(z_0)$, S, $T \in \Gamma$ and $\pi(z_0) = O$ (recall that π has been defined in Section 1). Indeed, if we integrate the relation $df(Tz) = \rho(T)df(z)$ from the fixed point z_0 to an arbitrary $z \in U$, then we obtain $f(Tz) - f(Tz_0) = \rho(T)(f(z) - f(z_0))$, whence $f(Tz) = \rho(T)f(z) + f(Tz_0) - \rho(T)f(z_0)$. Formula (5) easily follows from formula (4). Thus, the map $\phi : T \to \phi(T)$, or the period mapping $\phi : \Gamma \to \mathbf{C}$ with respect to the Prym integral $f(z)$, is an element of $Z^1(\Gamma, \rho)$.

If $f_1(z)$ is another Prym integral for $\phi(z)$, then $df_1(z) = \phi(z) = df(z)$ and $f_1(z) = f(z) + c$. Whence $\phi_1(T) = f_1(Tz_0) - \rho(T)f_1(z_0) = f(Tz_0) - \rho(T)f(z_0) + c(1 - \rho(T)) = \phi(T) + c\sigma(T)$ for all $T \in \Gamma$. Therefore, the period mappings with respect to different Prym integrals for the same Prym differential differ by an element of $B^1(\Gamma, \rho)$. Hence, the $\mathbf{C}$-linear mapping $p : \phi(z) \to [\phi] \in H^1(\Gamma, \rho)$, which maps any Prym differential $\phi(z)$ onto its period class $[\phi]$, is well defined.

Let $\phi(z) = df(z) = f'(z)dz = \varphi(z)dz$, then we have $\varphi(Tz)T'(z) = \rho(T)\varphi(z)$ for all $T \in \Gamma$ and $z \in U$. Therefore, the holomorphic Prym differential $\phi(z) = \varphi(z)dz$ for ρ on $F = U/\Gamma$ can be considered as a holomorphic multiplicative cusp-form for (Γ, ρ) of weight (-2) [14].

This is equivalent also to the condition

$$\varphi(Tz) = (1/T'(z))\rho(T)\varphi(z) = k(T, z)\rho(T)\varphi(z)$$

for all $T \in \Gamma$ and $z \in U$, where $k(T, z)$ is a canonical factor of automorphy for U/Γ which depends only on the complex analytic structure on U/Γ. Hence $\varphi(z)$ is defined as a holomorphic section of holomorphic line bundle $k \otimes \rho$ over U/Γ, or $\varphi(z)$ is a relatively-automorphic function for the factor of automorphy $k \otimes \rho$, where $\otimes$ denotes the tensor product of line bundles over U/Γ [13].

For $\rho = 1$, the holomorphic Prym differential is an abelian differential of the first kind on U/Γ, since we have $\phi(Tz) = \phi(z)$ for all $T \in \Gamma$ and $z \in U$. So it can be lowered to a single-valued differential on F.

For every ρ, the equality $\phi(Sz) = \phi(z)$ holds for all $S \in [\Gamma, \Gamma]$. Hence $\phi(z)$ is $[\Gamma, \Gamma]$-invariant on U.

Next, the equalities

$$\int_{z_0}^{[A,B]z_0} \phi(z) = f([A, B]z_0) - f(z_0) =$$

$$\rho([A, B])f(z_0) + \phi([A, B]) - f(z_0) = \phi([A, B]) \tag{6}$$

imply that the classical commutator period for $\phi(z)$ depends only on the commutator $[A, B]$, but not on the choice of z_0 or the choice of Prym integral for differential $\phi(z)$. Indeed, if $z_1 \neq z_0$, then

$$\int_{z_1}^{[A,B]z_1} \phi(z) = \int_{z_0}^{[A,B]z_0} \phi(z) + \int_{z_1}^{z_0} \phi(z) + \int_{[A,B]z_0}^{[A,B]z_1} \phi(z) =$$

$$\phi([A, B]) + \int_{z_0}^{z_1} d(f([A, B]z) - f(z)) = \phi([A, B]);$$

and if $f_1(z)$ be another Prym integral for $\phi(z)$, then

$$\phi([A, B]) = \int_{z_0}^{[A,B]z_0} df(z) = \int_{z_0}^{[A,B]z_0} df_1(z) = f_1([A, B]z_0) - f_1(z_0) = \phi_1([A, B]).$$

Note that we have used the commutator periods of the form $\phi([A,B])$ to obtain the chart of structure of holomorphic vector bundle on G over $Hom(\Gamma, \mathbf{C}^*) \setminus 1$ for every $g \geq 2$ [11, 12].

For the other classical periods we have

$$\int_{z_0}^{Tz_0} \phi(z) = f(Tz_0) - f(z_0) = \rho(T)f(z_0) + \phi(T) - f(z_0) = \phi(T) - f(z_0)\sigma(T),$$

for all $T \in \Gamma$. Thus, the classical period mapping $T \to \int_{z_0}^{Tz_0} df(z)$ determines the same period class as the map $T \to \phi(T)$. For example, for the Prym integral $f(z)$ with $f(z_0) = 0$ we have

$$\int_{z_0}^{Tz_0} \phi(z) = f(Tz_0) = \phi(T), \ T \in \Gamma. \tag{7}$$

Therefore, only the commutator periods $\phi([A,B])$ of a holomorphic Prym differential $\phi(z)$ associated to $\rho \neq 1$ are equal to the classical commutator periods $\int_{z_0}^{[A,B]z_0} \phi(z)$ for every choice of Prym integrals for this differential, for every choice of z_0, and for all $A, B \in \Gamma$.

The set of all holomorphic Prym differentials $\phi(z)$ associated to $\rho \in Hom(\Gamma, \mathbf{C}^*)$ forms a $(g-1)$-dimensional complex vector space $\Gamma(F, O^{1,0}(\rho))$ if $\rho \notin L_g$, but g-dimensional space if $\rho \in L_g$ [11, 14]. Gunning [11] has shown that in a sufficiently small open neighbourhood of any point $\rho_0 \notin L_g$ it is possible to choose a basis $\{\phi_i(\rho : z)\}_{i=1}^{g-1}$ for $\Gamma(F, O^{1,0}(\rho))$ depending holomorphically on ρ. Thus, on the set $P_{1,0} = \cup_{\rho \notin L_g} \Gamma(F, O^{1,0}(\rho))$ we have the structure of a holomorphic vector bundle of rank $(g-1)$ over $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$. This is the so-called *Prym bundle* for F.

The period mapping $p : \Gamma(F, O^{1,0}(\rho)) \to H^1(\Gamma, \rho)$ such that $\phi(z) \to p(\phi(z)) = [\phi] = \{\phi + c\sigma : c \in \mathbf{C}\} = \phi + B^1(\Gamma, \rho)$ is fibrewise a $\mathbf{C}$-linear mapping from $P_{1,0}$ into G over $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$. The mapping p is injective on every fibre $\Gamma(F, O^{1,0}(\rho))$. Indeed, if $\phi(z) \neq 0$, but $[\phi] = 0$ in $H^1(\Gamma, \rho)$, then there exists a non-constant holomorphic Prym integral $f(z)$ for differential $\phi(z)$ such that $\phi(z) = df(z)$, and $f(Tz) = \rho(T)f(z) + c\sigma(T)$ for all $T \in \Gamma$, $z \in U$. If we replace $f(z)$ with $\widehat{f}(z) = f(z) - c$, then the last equality can be rewritten in the form $\widehat{f}(Tz) = \rho(T)\widehat{f}(z)$; and $\widehat{\phi}(T) = 0$ for all $T \in \Gamma$. But this is possible if and only if $\rho \in L_g$ [13, 14]. Therefore, $\phi(z) = 0$. Hence the period mapping $p : P_{1,0} \to G$ is an injective mapping for holomorphic vector bundles over $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$.

The main result of the present paper is the following

Theorem 1. *The sequence of holomorphic vector bundles and mappings over* $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$

$$0 \to P_{1,0} \xrightarrow{p} G \to G/P_{1,0} \to 0$$

is exact for every $g \geq 2$.

Proof. It is known that both $P_{1,0}$ and G have the structure of holomorphic vector bundles over $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$ [11, 12]. Moreover, the period mapping p is fibrewise a $\mathbf{C}$-linear injection for every $\rho \in Hom(\Gamma, \mathbf{C}^*) \setminus L_g$.

First of all, we will prove that p is a holomorphic mapping with corresponding structures on $P_{1,0}$ and G. Let $U(\rho_0)$ be a sufficiently small simply connected neighbourhood of some point ρ_0 of $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$. Then $U(\rho_0)$ is contained in the domains $U_1, \ldots, U_{2g}$ forming the cover of $Hom(\Gamma, \mathbf{C}^*) \setminus 1$ (see Chapter 1). Suppose that $U(\rho_0) \subset U_1 = \{\rho : \rho(A_1) \neq 1\}$. Then there is a basis that consists of holomorphic Prym differentials $\phi_1(\rho; z), \ldots, \phi_{g-1}(\rho; z)$ and depends holomorphically on $\rho \in U(\rho_0)$, and every element $\phi(\rho; z) \in \Gamma(F, O^{1,0}(\rho))$ has the following expansion $\phi(z) = \phi(\rho; z) = \sum_{j=1}^{g-1} \lambda_j(\rho)\phi_j(\rho; z)$. In the chart $\varphi(\rho_0)$ (over $U(\rho_0)$), this element has the coordinates $(\rho; \lambda_1(\rho), \ldots, \lambda_{g-1}(\rho))$. So, in the chart $\varphi(U_1)$ over U_1, the element $[\phi] = p(\phi(z)) = \sum_{j=1}^{g-1} \lambda_j(\rho)[\phi_j] \in H^1(\Gamma, \rho)$ has the following fibre coordinates

$$\xi_j^1 = \widetilde{\phi^1}([A_{j+1}, A_1]) = \int_{z_0}^{[A_{j+1}, A_1]z_0} \phi(\rho; z),$$

$$\eta_j^1 = \widetilde{\phi^1}([B_{j+1}, A_1]) = \int_{z_0}^{[B_{j+1}, A_1]z_0} \phi(\rho; z),$$

$j = 1, \ldots, g-1$, where $\widetilde{\phi^1} \in Z^1(\Gamma, \rho)$ is any representative of the period class $[\phi]$ for $\rho \in U_1$. Thus, the column vector $(\xi_1^1, \ldots, \xi_{g-1}^1; \eta_1^1, \ldots, \eta_{g-1}^1)$ gives us the action of some matrix $A(\rho)$ on the column vector $(\lambda_1(\rho), \ldots, \lambda_{g-1}(\rho))$; here $A(\rho)$ is the matrix of order $(g-1) \times (2g-2)$ for which the jth row is

$$\left(\widetilde{\phi_1^1}([A_{j+1}, A_1]), \ldots, \widetilde{\phi_{g-1}^1}([A_{j+1}, A_1])\right),$$

and the $(g-1+j)$th row is

$$\left(\widetilde{\phi_1^1}([B_{j+1}, A_1]), \ldots, \widetilde{\phi_{g-1}^1}([B_{j+1}, A_1])\right),$$

$j = 1, \ldots, g-1$; where

$$\widetilde{\phi_k^1}([A_{j+1}, A_1]) = \int_{z_0}^{[A_{j+1}, A_1]z_0} \phi_k(\rho; z), \text{ and } \widetilde{\phi_k^1}([B_{j+1}, A_1]) = \int_{z_0}^{[B_{j+1}, A_1]z_0} \phi_k(\rho; z),$$

$k = 1, \ldots, g-1$. The elements of $A(\rho)$ depend holomorphically on $\rho \in U(\rho_0)$.

Similarly, we have matrices depending holomorphically on $\rho \in U(\rho_0)$ for p in the other cases, when $U(\rho_0) \subset U_l$, $l = 2, 3, \ldots, 2g$. To prove this it is sufficient to consider the following commutators

$$[A_1, A_l], \ldots, \widehat{\ }, \ldots, [A_g, A_l],$$

and

$$[B_1, A_l], \ldots, \widehat{\ }, \ldots, [B_g, A_l].$$

where, in both rows, the symbol $\widehat{\ }$ means that the lth element is omitted; $l = 2, 3, \ldots, g$. For $l = g+1, g+2, \ldots, 2g$, it is necessary to replace A_l by B_l in the second place of the commutators in these rows. Therefore, p is a holomorphic injection from $P_{1,0}$ into G over $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$.

Further, we prove that $p(P_{1,0})$ is a holomorphic vector subbundle of rank $(g-1)$ in the holomorphic vector bundle G. We will show that on the quotient bundle $G/P_{1,0} = G/p(P_{1,0})$ one can define a structure of holomorphic vector bundle so that the natural mapping $h : G \to G/P_{1,0}$ is holomorphic. Fix a basis $\phi_1(z) = \phi_1(\rho; z), \ldots, \phi_{g-1}(z) = \phi_{g-1}(\rho; z)$ over sufficiently small neighbourhood $U(\rho_0)$ which depends holomorphically on $\rho \in U(\rho_0)$. In the chart $\varphi(\rho_0)$ it has the fibre coordinates $e_1 = (1, 0, \ldots, 0), \ldots, e_{g-1} = (0, \ldots, 0, 1) \in \mathbf{C}^{g-1}$, respectively. The holomorphic injection p translates this basis into linearly independent (over $\mathbf{C}$) systems $\{p(\phi_j(z)\}_{j=1}^{g-1}$ of sections of the bundle G over $U(\rho_0)$, depending holomorphically on $\rho \in U(\rho_0)$. As above, it is sufficient to consider only the case $U(\rho_0) \subset U_1$. In chart $\varphi(U_1)$, these sections have the following coordinates:

$$p(\phi_1(z))-\text{coordinates}: \quad (\xi_1, \ldots, \xi_{g-1}; \eta_1, \ldots, \eta_{g-1}) = (1, 0, \ldots, 0)A^{'}(\rho) =$$

$$\big(\widetilde{\phi_1^1}([A_2, A_1]), \ldots, \widetilde{\phi_1^1}([A_g, A_1]),\ \widetilde{\phi_1^1}([B_2, A_1]), \ldots, \widetilde{\phi_1^1}([B_g, A_1])\big),$$

$\ldots,$

$$p(\phi_{g-1}(z))-\text{coordinates}: \quad \big(\widetilde{\phi_{g-1}^1}([A_2, A_1]), \ldots, \widetilde{\phi_{g-1}^1}([B_g, A_1])\big),$$

where $A^{'}(\rho)$ denotes the transpose of $A(\rho)$. This rows form a matrix of order $(g-1) \times (2g-2)$. It has exactly $(g-1)$ linearly independent (over $\mathbf{C}$) rows, as well as columns. Consider this matrix for fixed ρ_0. There is a biholomorphic automorphism α on $\mathbf{C}^{2g-2}$ which acts as a permutation of the coordinates so that after its natural action on matrix columns, the matrix can be represented in the form $(C_1(\rho_0), C_2(\rho_0))$, where $\det C_1(\rho_0) \neq 0$. There is a sufficiently small neighbourhood in $U(\rho_0)$ (denote it again by $U(\rho_0)$) such that $\det C_1(\rho) \neq 0$, when $\rho \in U(\rho_0)$. Then the rows of the matrix form a system of linearly independent sections in the trivial bundle $U(\rho_0) \times \mathbf{C}^{2g-2}$ that depends holomorphically on $\rho \in U(\rho_0)$. For every fixed $\rho \in U(\rho_0)$ these sections form a $(g-1)$-dimensional subspace in $\mathbf{C}^{2g-2}$. Extend this system by basis vectors (constant section) $e_g, \ldots, e_{2g-2} \in \mathbf{C}^{2g-2}$ to basis sections in $U(\rho_0) \times \mathbf{C}^{2g-2}$. Then we have quadratic matrix of order $(2g-2)$ of the form $\begin{pmatrix} C_1 & C_2 \\ O & I \end{pmatrix}$, where I is the identity matrix of order $(g-1)$. Biholomorphic automorphism β for this product which for fixed ρ has the non-singular transformation matrix of the form $\begin{pmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ O & I \end{pmatrix}$ translates given basis sections into the standart basis $e_1, e_2, \ldots, e_{2g-2}$ for $U(\rho_0) \times \mathbf{C}^{2g-2}$. Therefore, in the new chart $\psi_1 = \beta \circ \alpha \circ \varphi(U_1)$ (which belongs to the same structure of the holomorphic vector bundle G over $U(\rho_0)$), the system of the holomorphic sections $p(\phi_1(z)), \ldots, p(\phi_{g-1}(z))$ has the coordinates $(\rho; e_1), \ldots, (\rho; e_{g-1})$, respectively. Thus, we have the fibre isomorphism

$$\psi_1 : p(P_{1,0}) \mid U(\rho_0) \cap U_1 \to (U(\rho_0) \cap U_1) \times (\mathbf{C}^{g-1}; 0)(\subset U(\rho_0) \times \mathbf{C}^{2g-2}).$$

Hence, $p(P_{1,0})$ is a holomorphic vector subbundle of rank $(g-1)$ in G [15].

For each pair k and l, $k \neq l$, $k, l = 1, \ldots, 2g$, in the charts ψ_k and ψ_l over U_k and U_l, respectively, transformation matrix T_{kl} for G over $U(\rho_0)$ has the form $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{O} & \mathbf{D} \end{pmatrix}$, where $\mathbf{A} = I$, because the operator T_{kl} translates each of the vectors $e_1, \ldots, e_{g-1}$ onto itself, and $\mathbf{A} = \mathbf{A}(\rho) = I$, $\mathbf{B} = \mathbf{B}(\rho)$, $\mathbf{D} = \mathbf{D}(\rho)$ depend holomorphically on $\rho \in U(\rho_0) \cap U_k \cap U_l$. Matrices $\mathbf{A}(\rho)$ and $\mathbf{D}(\rho)$ are holomorphic matrices of transformation for bundles $p(P_{1,0})$ and $G/p(P_{1,0})$, respectively. Hence $G/p(P_{1,0})$ is a holomorphic vector bundle of rank $(g-1)$ with fibre $H^1(\Gamma, \rho)/(p(P_{1,0}) \mid_\rho)$ over ρ [15]. The mapping $h : [\phi] \to [\phi] + p(P_{1,0}) \mid_\rho$ in such charts ψ have been represented as $(\rho; \xi_1, \ldots, \xi_{g-1}; \eta_1, \ldots, \eta_{g-1}) \to (\rho; 0, \ldots, 0; \eta_1, \ldots, \eta_{g-1})$. Thus h is a holomorphic mapping with respect to the given structures of the holomorphic vector bundles on G and $G/p(P_{1,0})$. Theorem 1 is proved.

Corollary 1. *For an essential character a holomorphic Prym differential on a compact Riemann surface of genus $g \geq 2$ is uniquely defined by any linearly independent (over $\mathbf{C}$) system of $(g-1)$ commutator basic periods from its $(2g-2)$ commutator basic periods.*

Remark. In the partial case when $\rho \in U_1 \setminus L_g$, the holomorphic Prym differential $\phi(z)$ for ρ is uniquely determined by every linearly independent system of $(g-1)$ commutator basic periods from its $(2g-2)$ commutator basic periods

$$\left\{ \phi([A_j, A_1]) = \int_{z_0}^{[A_j, A_1] z_0} \phi(z), \quad \phi([B_j, A_1]) = \int_{z_0}^{[B_j, A_1] z_0} \phi(z) \right\}_{j=2}^{g}.$$

3. Harmonic Prym differential

Definition 2. A *harmonic Prym differential* on F associated to a representation $\rho \in Hom(\Gamma, \mathbf{C}^*)$ is a harmonic (single-valued) differential 1-form $\phi(z)$ on U such that $\phi(Tz) = \rho(T)\phi(z)$ for all $T \in \Gamma$ and $z \in U$.

Harmonic Prym differential $\phi(z)$ on U can be represented in the form $\phi(z) = \phi_1(z) + \phi_2(z)$, where $\phi_1(z) = df_1(z)$ and $\phi_2(z) = d\overline{f_2(z)}$; and $f_j(z)$, $j = 1, 2$, are holomorphic functions on U which are uniquely defined up to additive complex constants. Thus $\phi(z) = df(z)$, where $f(z) = f_1(z) + \overline{f_2(z)}$ is a complex-valued harmonic function on U (a harmonic Prym integral for differential $\phi(z)$). As in Section 2, we can obtain the following relations

$$f(Tz) = \rho(T)f(z) + \phi(T), \ \phi(ST) = \phi(S) + \rho(S)\phi(T), \tag{9}$$

where $\phi(T) = f(Tz_0) - \rho(T)f(z_0) = \phi_1(T) + \phi_2(T)$, $\phi_1(T) = f_1(Tz_0) - \rho(T)f_1(z_0)$, $\phi_2(T) = \overline{f_2(Tz_0)} - \rho(T)\overline{f_2(z_0)}$, and $\pi(z_0) = O$. Here $\phi_1(Tz) = \rho(T)\phi_1(z)$, $\phi_2(Tz) = \rho(T)\phi_2(z)$ for all $T \in \Gamma$ and $z \in U$. Therefore, the period mapping $\phi : T \to \phi(T)$, or $\phi : \Gamma \to \mathbf{C}$ with respect to harmonic Prym integral $f(z)$, is an element of $Z^1(\Gamma, \rho)$. If $\widehat{f_1}(z) = f_1(z) + c_1$ and $\widehat{f_2}(z) = f_2(z) + c_2$ are other Prym integrals for $\phi_1(z)$ and $\phi_2(z)$, respectively, then $\widehat{\phi_1}(T) = \phi_1(T) + c_1\sigma(T)$, $\widehat{\phi_2}(T) = (\overline{f_2(Tz_0) + c_2}) - \rho(T)(\overline{f_2(z_0) + c_2}) = \phi_2(T) + \overline{c_2}\sigma(T)$. Thus, $\widehat{\phi}(T) = \phi(T) + (c_1 + \overline{c_2})(1 - \rho(T))$ for all $T \in \Gamma$; and the period mappings of different harmonic Prym integrals for the same harmonic Prym differential differ by elements of $B^1(\Gamma,\rho)$. Hence, the $\mathbf{C}$-linear mapping $p : \phi(z) \to [\phi] \in H^1(\Gamma,\rho)$, which maps any harmonic Prym differential $\phi(z)$ onto its period class $[\phi]$, is well defined. Indeed, if $\widetilde{\phi}(z) = \alpha\phi(z)+\beta\psi(z)$, where $\alpha,\beta \in \mathbf{C}$ and $\phi(z),\psi(z)$ are harmonic Prym differential for ρ, then $\widetilde{\phi(z)} = \alpha df(z)+\beta dg(z) = d(\alpha f(z)+\beta g(z))$ and $\widetilde{\phi}(T) = (\alpha f+\beta g)(Tz) - \rho(T)(\alpha f+\beta g)(z) = \alpha\phi(T)+\beta\psi(T)$ for all $T \in \Gamma$. If we replace $f(z)$ and $g(z)$ by $f(z)+c_1$ and $g(z)+c_2$, respectively, then we have

$$\widehat{\phi(T)} = (\alpha(f+c_1)+\beta(g+c_2))(Tz) - \rho(T)(\alpha(f+c_1)+\beta(g+c_2))(z) =$$

$$\widetilde{\phi}(T) + (\alpha c_1 + \beta c_2)\sigma(T),\ T \in \Gamma.$$

Therefore, p is really a $\mathbf{C}$-linear mapping:

$$\alpha\phi(z)+\beta\psi(z) = \widetilde{\phi}(z) \to [\widetilde{\phi}] = [\alpha\phi+\beta\psi] = \alpha[\phi]+\beta[\psi].$$

Harmonic Prym differential $\phi(z)$ associated to ρ is the sum $\phi(z) = df_1(z) + d\overline{f_2(z)} = \varphi_1(z)dz + \overline{\varphi_2(z)dz}$, where $\varphi_j(z)$ $(j = 1,2)$ are holomorphic functions on U such that $\varphi_1(Tz)T'(z) = \rho(T)\varphi_1(z)$, and $\varphi_2(Tz)T'(z) = \overline{\rho(T)}\varphi_2(z)$. Hence $\phi(z)$ can be represented by holomorphic sections $\varphi_1(z)dz$ and $\varphi_2(z)dz$ of holomorphic line bundles $k \otimes \rho$ and $k \otimes \overline{\rho}$, respectively, on F.

For $\rho = 1$, a harmonic Prym differential on F is a single-valued harmonic differential on F that is equal to the sum of holomorphic and antiholomorphic differentials on F.

For every ρ, harmonic Prym differential $\phi(z)$ is $[\Gamma,\Gamma]$-invariant on U. For all $A, B \in \Gamma$ the following equality holds:

$$\int_{z_0}^{[A,B]z_0} \phi(z) = \int_{z_0}^{[A,B]z_0} df_1(z) + \int_{z_0}^{[A,B]z_0} d\overline{f_2(z)} =$$

$$\phi_1([A,B]) + \phi_2([A,B]) = \phi([A,B]). \tag{10}$$

This integral does not depend on the choice of z_0 or harmonic Prym integral for differential $\phi(z)$. Moreover, only commutator periods $\phi([A,B])$ for harmonic Prym differential $\phi(z)$ for $\rho \neq 1$ coinside with the classical commutator periods $\int_{z_0}^{[A,B]z_0} \phi(z)$ for any choice of harmonic Prym integrals for given Prym differential, and for any choice of z_0, and for all $A, B \in \Gamma$. As in Section 2, we have $\int_{z_0}^{Tz_0} \phi(z) = f(Tz_0) = \phi(T)$, $T \in \Gamma$, where the harmonic Prym integral $f(z)$ satisfies the condition $f(z_0) = 0$.

The set of all harmonic Prym differentials $\phi(z)$ associated to $\rho \in Hom(\Gamma, \mathbf{C}^*)$ forms a $(2g-2)$-dimensional complex vector space $\Gamma(F, \mathcal{H}^1(\rho))$ if $\rho \notin L_g \cup \overline{L_g}$, because

$$\Gamma(F,\mathcal{H}^1(\rho)) = \Gamma(F, O^{1,0}(\rho)) \oplus \Gamma(F, O^{0,1}(\rho)), \tag{11}$$

where $\overline{L_g}$ is the image of L_g under the map $\rho \to \overline{\rho}$. It is possible to choose a basis $\{\phi_j(\rho;z)\}_{j=1}^{g-1}$ for $\Gamma(F, O^{1,0}(\rho))$, which depends holomorphically on ρ in a sufficiently small open neighbourhood $U(\rho_0) \subset Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$. One can choose simultaneously a basis $\{\phi_j(\overline{\rho}, z)\}_{j=1}^{g-1}$ in $\Gamma(F, O^{1,0}(\overline{\rho}))$, which depends holomorphically on $\overline{\rho}$ of $\overline{U(\rho_0)}$ (the image of $U(\rho_0)$ for the map $\rho \to \overline{\rho}$). This map is an automorphism of $L_g \cup \overline{L_g}$. Therefore, the collection of harmonic Prym differentials

$$\phi_1(\rho; z), \ldots, \phi_{g-1}(\rho; z);\ \overline{\phi_1(\overline{\rho}; z)}, \ldots, \overline{\phi_{g-1}(\overline{\rho}; z)} \tag{12}$$

forms a basis which depends holomorphically on $\rho \in U(\rho_0)$. Thus, there exists a structure of holomorphic vector bundle on the complex vector bundle $HP = \cup_{\rho \notin L_g \cup \overline{L_g}} \Gamma(F, \mathcal{H}^1(\rho)) = P_{1,0} \oplus P_{0,1}$ of rank $(2g-2)$ over $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$. The bundle HP is called a *harmonic Prym bundle* for F.

The inner product on the fibre $\Gamma(F, \mathcal{H}^1(\rho))$ is defined by the formula

$$(\phi_1(z), \phi_2(z)) = \int\int_\Delta \phi_1(z) \wedge *\overline{\phi_2(z)} = i \int\int_\Delta (u_1\overline{u_2} + v_1\overline{v_2}) dz \wedge d\overline{z}, \tag{13}$$

where Δ is a fixed fundamental domain for Γ in U; and $\phi_j(z) = u_j(z)dz + v_j(z)d\overline{z}$, where $j = 1, 2$. For example, the boundary $\partial\Delta$ is the lifting of the commutator path $\prod_{j=1}^{g}[\alpha_j, \beta_j]$ with initial point $z_0 \in U$. The inner product is hermitian, since $(\phi_1(z), \phi_2(z)) = \overline{(\phi_2(z), \phi_1(z))}$. It is easy to see that the $\mathbf{C}$-linear operator $*$ (the Hodge star) is an isometry of the fibre $\Gamma(F, \mathcal{H}^1(\rho))$ onto itself. For the fibre $\Gamma(F, O^{1,0}(\rho))$ this inner product can be simplified as follows: $(\phi_1(z), \phi_2(z)) = i \int\int_\Delta \phi_1(z) \wedge \overline{\phi_2(z)} = 2 \int\int_\Delta u_1\overline{u_2} dx \wedge dy$. The $\mathbf{C}$-linear operator $*$ is also an isometry of the fibre $\Gamma(F, O^{1,0}(\rho))$ onto itself and an isometry of the fibre $\Gamma(F, O^{0,1}(\rho))$ onto itself .

Remark. For any $\phi_1(z)$ and $\phi_2(z) \in \Gamma(F, \mathcal{H}^1(\rho))$ we have

$$\phi_1(Tz) \wedge *\overline{\phi_2(Tz)} = \mid \rho(T) \mid^2 \phi_1(z) \wedge *\overline{\phi_2(z)},$$

for all $T \in \Gamma$. Therefore, on another fundamental domain $T_0(\Delta)$, where $T_0 \in \Gamma$, inner product can be defined by the formula $(\phi_1(\zeta), \phi_2(\zeta)) := \mid \rho(T_0) \mid^{-2} \int\int_{T_0(\Delta)} \phi_1(\zeta) \wedge *\overline{\phi_2(\zeta)} = \int\int_\Delta \phi_1(z) \wedge *\overline{\phi_2(z)} = (\phi_1(z), \phi_2(z))$, where $\zeta \in T_0(\Delta)$, $z \in \Delta$, $\zeta = T_0(z)$.

With respect to this inner product the spaces $\Gamma(F, O^{1,0}(\rho))$ and $\Gamma(F, O^{0,1}(\rho))$ are orthogonal, because if $\phi_1(z) = u(z)dz \in \Gamma(F, O^{1,0}(\rho))$, and $\phi_2(z) = v(z)d\overline{z} \in \Gamma(F, O^{0,1}(\rho))$, we have $(\phi_1(z), \phi_2(z)) = \int\int_\Delta u dz \wedge *\overline{v(z)d\overline{z}} = -i \int\int_\Delta u(z)\overline{v(z)} dz \wedge dz = 0$. The vector bundles $P_{1,0}$, $P_{0,1}$ and HP are hermitian holomorphic vector bundles over $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$. Thus, we have established the following

Theorem 2. *The hermitian holomorphic vector bundle HP (harmonic Prym bundle) of rank $(2g-2)$ is the orthogonal sum of the hermitian holomorphic $*$-invariant vector subbundles $P_{1,0}$ and $P_{0,1}$ of rank $(g-1)$ over $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$ for every $g \geq 2$.*

Proposition 1. *Let $\phi(z)$ be a harmonic Prym differential of type $(0,1)$ for $\rho \notin \overline{L_g}$, and $[\phi] = 0$ in $H^1(\Gamma, \rho)$. Then $\phi(z) = 0$ on $F = U/\Gamma$.*

Proof. Suppose that there exists $\phi_2(z) \in \Gamma(F, O^{0,1}(\rho))$ for $\rho \notin \overline{L_g}$ such that $\phi_2(z) \neq 0$, and $p(\phi_2(z)) = [\phi_2] = 0$ in $H^1(\Gamma, \rho)$. Then $\phi_2(z) = d\overline{f_2(z)}$, where $f_2(z)$ is a non-constant holomorphic function on U. ¿From the condition on the period class we have $\overline{f_2(Tz)} = \rho(T)\overline{f_2(z)} + c(1-\rho(T))$ for all $T \in \Gamma$ and $z \in U$; or $\overline{f_2(Tz)} - \overline{c} = \rho(T)(\overline{f_2(z)} - \overline{c})$. Replace $f_2(z)$ by $\widehat{f_2}(z) = f_2(z) - \overline{c}$. Then there exists a non-constant holomorphic function on U such that $\widehat{f_2}(Tz) = \overline{\rho(T)}\widehat{f_2}(z)$. But it is possible if and only if $\overline{\rho} \in L_g$ or $\rho \in \overline{L_g}$. We have a contradiction. Hence $\phi_2(z) = 0$ on F.

Corollary 2. *For $\rho \notin \overline{L_g}$, a harmonic Prym differential of type $(0,1)$ on a compact Riemann surface of genus $g \geq 2$ is uniquely defined by any $\mathbf{C}$-linearly independent system of $(g-1)$ commutator basic periods from its $(2g-2)$ commutator basic periods.*

Proof follows from Proposition 1 for $\rho \notin \overline{L_g}$ and from the existence of the isomorphism

$$H^1(\Gamma, \rho) \cong Hom_\rho([\Gamma, \Gamma], \mathbf{C})$$

(see Chapter 1).

Corollary 3. *The sequence of holomorphic vector bundles and mappings over $Hom(\Gamma, \mathbf{C}^*) \setminus \overline{L_g}$*

$$0 \to P_{0,1} \xrightarrow{p} G \to G/P_{0,1} \to 0$$

is exact for every $g \geq 2$.

Proof. The scheme of the proof is similar to that of Theorem 1. It is necessary to use Proposition 1 in a suitable place, and also the fact that the period mapping p is $\mathbf{C}$-linearly holomorphic to the harmonic Prym differential of type $(0,1)$; which can be proved exactly as for holomorphic Prym differential.

Remark. The operator of complex conjugation $A_- : \Gamma(F, \mathcal{H}^1(\rho)) \to \Gamma(F, \mathcal{H}^1(\overline{\rho}))$, which maps a harmonic Prym differential $\phi(\rho; z) = \phi_1(\rho; z) + \phi_2(\rho; z)$ into its complex conjugation $\overline{\phi(\rho; z)} = \overline{\phi_2(\rho; z)} + \overline{\phi_1(\rho; z)}$, is an R-linear isomorphism for these complex vector spaces for $\rho \notin L_g \cup \overline{L_g}$. This operator also defines the R-linear isomorphisms $A_- : \Gamma(F, O^{1,0}(\rho)) \to \Gamma(F, O^{0,1}(\overline{\rho}))$ and $A_- : \Gamma(F, O^{0,1}(\rho)) \to \Gamma(F, O^{1,0}(\overline{\rho}))$ for $\rho \notin L_g \cup \overline{L_g}$. It is clear that $A_-^2 = id$. Thus, the automorphism of $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$ of order two, $A_- : \rho \to \overline{\rho}$, can be lifted canonically (that is independently of the choice of basis) to fibrewise R-linear automorphism A_- of order two of harmonic Prym bundle HP of rank $(2g-2)$ over $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$ for every $g \geq 2$.

In the following proposition we give some additional characterization of our subsets of $Hom(\Gamma, \mathbf{C}^*)$.

Proposition 2. *The Lie subgroups L_g and $\overline{L_g}$ of the Lie group $Hom(\Gamma, \mathbf{C}^*)$ and complex analytic $2g$-dimensional subspaces*

$$Hom(\Gamma, \mathbf{C}^*) \setminus L_g, \quad Hom(\Gamma, \mathbf{C}^*) \setminus 1, \quad Hom(\Gamma, \mathbf{C}^*) \setminus \overline{L_g},$$

and

$$Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$$

of the complex analytic space $Hom(\Gamma, \mathbf{C}^*)$ are invariant under biholomorphic involutory automorphism $\mathcal{B} : \rho \to \rho^{-1}$, where $\rho \in Hom(\Gamma, \mathbf{C}^*)$.

Proof. If $\rho \in L_g$ is an inessential character with parameters $(c_1, \ldots, c_g) \in \mathbf{C}^g$ (that is, $\rho(A_k) = \exp 2\pi i c_k$, $\rho(B_k) = \exp 2\pi i (\sum_{j=1}^{g} \Omega_{kj} c_j)$, $k = 1, \ldots, g$), then ρ^{-1} has parameters $(-c_1, \ldots, -c_g)$ and is also an inessential character; therefore $\rho^{-1} \in L_g$.

If ρ is an essential character with parameters $(c_1, \ldots, c_g;\ s_1, \ldots, s_g)$, where at least one of $s_k \notin \mathbf{Z}$, then ρ^{-1} is also an essential character with parameters $(-c_1, \ldots, -c_g; -s_1, \ldots, -s_g)$, where $(-s_k) \notin \mathbf{Z}$.

Thus, the map $\mathcal{B} : \rho \to \rho^{-1}$ is an invariant on L_g and also on $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$ and on $Hom(\Gamma, \mathbf{C}^*) \setminus 1$, since $\mathcal{B} = \mathcal{B}^{-1}$. It is clear that $\mathcal{B}$ is a holomorphic bijection of L_g, $Hom(\Gamma, \mathbf{C}^*) \setminus L_g$ and $Hom(\Gamma, \mathbf{C}^*) \setminus 1$ onto itself.

Moreover, if ρ_1 and ρ_2 of L_g have parameters $(c_{11}, \ldots, c_{1g})$ and $(c_{21}, \ldots, c_{2g})$, respectively, then $L_g \ni \rho_1\rho_2 \to \frac{1}{\rho_1\rho_2} = (\frac{1}{\rho_1})(\frac{1}{\rho_2}) \in L_g$, where the last element has parameters $(-c_{11} - c_{21}, \ldots, -c_{1g} - c_{2g})$. Whence $\mathcal{B}$ is biholomorphic automorphism of Lie subgroup L_g.

The set $\overline{L_g} = \{\overline{\rho} \in Hom(\Gamma, \mathbf{C}^*) : \rho \in L_g\}$ is the image of the set $\{(x, y) \in \mathbf{C}^g \times \mathbf{C}^g : y = \overline{\Omega} x\}$ with respect to ρ (see Chapter 1); and is a Lie subgroup, because if $\rho \in L_g$ has parameters $(c_1, \ldots, c_g) \in \mathbf{C}^g$, then $\overline{\rho(A_k)} = \exp 2\pi i(-\overline{c_k})$, $\overline{\rho(B_k)} = \exp 2\pi i (\sum_{j=1}^{g} \overline{\Omega_{kj}}(-\overline{c_j}))$, and also $(\overline{\rho_1})(\overline{\rho_2}) = \overline{\rho_1\rho_2}, (\overline{\rho_1})^{-1} = \overline{(\rho_1^{-1})}$, where $\rho_1,\ \rho_2 \in L_g$. Further, $\overline{L_g}$ is $\mathcal{B}$-invariant, because $\overline{L_g} \ni \overline{\rho} \to \frac{1}{\overline{\rho}} = \overline{(\frac{1}{\rho})} \in \overline{L_g}$, and the sets $Hom(\Gamma, \mathbf{C}^*) \setminus \overline{L_g}$ and $Hom(\Gamma, \mathbf{C}^*) \setminus (L_g \cup \overline{L_g})$ are $\mathcal{B}$-invariant, because $\mathcal{B} = \mathcal{B}^{-1}$. Proposition 2 is proved.

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