ASYMPTOTIC DISTRIBUTION OF QUADRATIC FORMS II

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Abstract. We consider the quadratic forms

$$Q_n = \sum_{1 \le j,k \le n} a_{jk} \big(X_j X_k - \mathbf{E}\, X_j X_k \big),$$

where X_j are i.i.d. random variables with finite sixth moment. For a large class of matrices a_{jk} the distribution of Q_n can be approximated by the distribution of a second order polynomial in Gaussian random variables. We provide optimal bounds for the Kolmogorov distance of these distributions, extending previous results for matrices with zero diagonals to the general case.

1. Introduction and results

Let $A = A^{(n)} = \{a_{jk}^{(n)}\}_{j,k=1}^{n}$ denote a sequence of symmetric matrices such that

$$\sum_{j,k=1}^{n} \big(a_{jk}^{(n)}\big)^2 = 1. \tag{1.1}$$

In order to simplify the notation we will drop in the following, the index n.

$X, X_1, X_2, \ldots$ denote independent identically distributed (i.i.d.) random variables (r.v.) such that $\mathbf{E}\, X = 0$. Consider the quadratic form

$$Q = \sum_{j,k=1}^{n} a_{jk} \big(X_j X_k - \mathbf{E}\, X_j X_k \big).$$

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The aim of this paper is to bound the error of approximation of the distribution of the quadratic form Q by that of a suitable polynomial of second order in Gaussian random variables, for sufficiently general classes of matrices.

Introduce $\mu_k = \mathbf{E}\, X^k$ and assume that

$$\mu_6 < \infty, \tag{1.2}$$

and $\mu_2 > 0$. Let $Y, Y_1, Y_2, \dots$ denote a sequence of independent Gaussian random variables. Consider the polynomial

$$G = \mu_2 \sum_{1\le j\ne k\le n} a_{jk} Y_j Y_k + \mu_3 \mu_2^{-1/2} \sum_{j=1}^{n} a_{jj} Y_j + \sqrt{\mu_4 - \mu_2^2 - \mu_3^2\mu_2^{-1}}\, V Y,$$

where $V^2 = \sum_{j=1}^{n} a_{jj}^2$. The inequality $|\mathbf{E}\,(X^2 - \mathbf{E}\, X^2) X|^2 \le \mathbf{E}\,(X^2 - \mathbf{E}\, X^2)^2 \mathbf{E}\, X^2$ implies that $\mu_4 - \mu_2^2 - \mu_3^2\mu_2^{-1} \ge 0$. Without loss of generality we may assume that $\mu_2 = 1$. Let us denote

$$\mathcal{L}_j^2 = \sum_{\substack{k\ne j\\ k=1,\dots,n}} a_{jk}^2, \qquad \mathcal{L}^2 = \max_{1\le j\le n} \mathcal{L}_j^2, \qquad \mathcal{M} = \max_{1\le j\le n} |a_{jj}|,$$

and

$$\delta(A,F) := \sup_x \Big| \mathbf{P}\,\big\{ Q \le x \big\} - \mathbf{P}\,\big\{ G \le x \big\} \Big|.$$

We will write $\Phi(x)$ for the standard Gaussian distribution function and denote

$$\Delta(A,F) := \sup_x \Big| \mathbf{P}\,\big\{ Q \le x \big\} - \Phi(x) \Big|,$$

where F is the distribution function (d.f.) of X, i.e. $F(x) = \mathbf{P}\,\{X \le x\}$. Throughout this paper C (with index or without it) denote absolute constants, whereas $C(\cdot,\cdot)$ is used for constants depending on the arguments in parentheses. Introduce the matrix A_0 with elements $A_0(j,k) = a_{jk}$ for $1 \le j \ne k \le n$, and $A_0(j,j) = 0$. By λ_k, $\quad k = 1, 2, \dots, n$, we denote the eigenvalues of the matrix A_0 ordered so that $|\lambda_1| \ge \dots \ge |\lambda_n|$. Note that, by (1.1), $\sum_{k=1}^{n} \lambda_k^2 = 1$.

Define $\log^+ x = \max\{1/2, |\log x|\}$, $x > 0$. The main results of this paper are contained in the following theorems.

Theorem 1. *Assume that the conditions* (1.1)–(1.2) *hold. Then there exists an absolute constant C such that*

$$\max\big\{\, \delta(A,F), \Delta(A,F) \,\big\} \le C\,\mu_6 \Big(\big|\lambda_1\big| + \mathcal{M} \Big).$$

Theorem 2. *Assume that the conditions* (1.1)–(1.2) *hold and*

$$\left|\lambda_q\right| \geq b_0 \tag{1.3}$$

for some positive constant b_0 and $q \geq 2$. Then there exist a constant $C(b_0)$ such that

$$\delta(A,F) \leq \begin{cases} C(b_0)\,\mu_6\,\big(\mathcal{L}_n^{4/(16-q)} + \mathcal{M}_n\big), & \text{if } 2 \leq q \leq 11, \\ C(b_0)\,\mu_6\,(\mathcal{L}+\mathcal{M}\big)\log^+\mathcal{L}, & \text{if } q = 12, \\ C(b_0)\,\mu_6\,(\mathcal{L}+\mathcal{M}\big), & \text{if } q \geq 13. \end{cases}$$

In the literature only special cases of quadratic forms with non-zero diagonal part were considered so far: Rotar' (1973) proved that the distribution of Q is close to that of G assuming the condition (1.3) and the additional assumption $V_n \to 0$ as $n \to \infty$; a similar result was proved by Wet and Venter (1973), but under a more restrictive assumption on the matrix of the quadratic form. Guttorp and Lockhard (1988) obtained some modification of Rotar's result, but in the zero-diagonal case only. Rotar' (1978, 1985) gave an estimate of the rate of convergence of $\delta_n(A,F)$. Optimal bounds for the case $V=0$ were obtained in Götze and Tikhomirov (1996), called GT for shot in the following. Here we extend this result for quadratic forms with non-zero diagonal part.

The proof is based on the results of paper by GT. We use a basic symmetrization inequality due to Götze (1979) and the differential equation method of Tikhomirov (1991, 1993) for small frequencies of the characteristic function (ch.f.) of Q_n.

We denote by $\mathcal{X}$, $\mathcal{Y}$ the random vectors in $\mathbb{R}^n$ with coordinates $X_1,\dots,X_n$; $Y_1,\dots,Y_n$ respectively. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the Euclidean scalar product and the Euclidean norm in $\mathbb{R}^n$ respectively. Introduce the vector $\mathcal{D}$ with coordinates $d_1,\dots,d_n$, where $d_i = a_{ii}$ for all i, and the diagonal matrix D with diagonal elements $D(i,i) = d_i$. Hence we may split A as $A = A_0 + D$.

In this notation we have

$$Q = \langle A\mathcal{X},\mathcal{X}\rangle, \quad G = \langle A_0\mathcal{Y},\mathcal{Y}\rangle + \mu_3\langle\mathcal{D},\mathcal{Y}\rangle + \sqrt{\mu_4 - 1 - \mu_3^2}\,VY.$$

In the sequel we will write $\mathrm{e}\{x\} := \exp\{itx\}$. Thus, the ch.f. of Q and G can be written as

$$f(t) = \mathbf{E}\exp\{itQ\} = \mathbf{E}\,\mathrm{e}\{Q\}, \qquad g(t) = \mathbf{E}\exp\{itG\} = \mathbf{E}\,\mathrm{e}\{G\}.$$

2. The method of symmetrization

In this section we consider the bounds of $|f(t)|$ and $|g(t)$, and for some auxiliary functions. Let ξ_j denote the vector with zero coordinates except for the j^{th} coordinate, where $\xi_j^{(j)} = X_j$ and let ζ_j denote a similar vector with $\zeta_j^{(j)} = Y_j$. Introduce the notation

$$\begin{aligned} &\mathcal{X}_j = \mathcal{X} - \xi_j, \quad Q_j = \langle A\mathcal{X}_j,\mathcal{X}_j\rangle, \quad \mathcal{Y}_j = \mathcal{Y} - \zeta_j, \\ &G_j = \langle\mathcal{Y}_j, A_0\mathcal{Y}_j\rangle + \mu_3\langle\mathcal{D},\mathcal{Y}_j\rangle + \sqrt{\mu_4 - \mu_3^2 - \mu_2^2}\,\langle\mathcal{D}, Z\rangle, \\ &\Delta_j = Q - Q_j = 2\langle\xi_j, A\mathcal{X}_j\rangle + \langle\xi_j, A\xi_j\rangle, \\ &\widetilde{\Delta}_j = G - G_j = 2\langle\zeta_j, A_0\mathcal{Y}_j\rangle + \mu_3\langle\mathcal{D}, A_0\zeta_j\rangle. \end{aligned}$$

For $j = 1, \dots, n$ introduce the functions

$$\mathcal{H}_j(p, q, \eta, \zeta) = \Big|\mathbf{E}\,\langle \eta, R_t A_0 \mathcal{X}_j\rangle^p \langle \eta, R_t A_0 \mathcal{X}_j\rangle^q\, \mathrm{e}\big\{\,Q_j + \langle \zeta, A_0 \mathcal{X}_j\rangle\,\big\}\Big|,$$

and for the Gaussian r.v.'s

$$\widetilde{\mathcal{H}}_j(p, q, \eta, \zeta) = \Big|\mathbf{E}\,\langle \eta, R_t A_0 \mathcal{Y}_j\rangle^p \langle \eta, R_t A_0 \mathcal{Y}_j\rangle^q\, \mathrm{e}\big\{\,G_j + \langle \zeta, A_0 \mathcal{Y}_j\rangle\,\big\}\Big|.$$

In this section we called some results of GT.

Theorem 2.1. *Let $\varepsilon_1, \cdots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then for all $t \ge 0$, $K > 0$, and $H > 0$

$$\begin{aligned}\big|f(t)\big|^4 &\le \mathbf{E}\,\exp\Big\{\,\frac{1}{4}\,i(t\wedge T_0)\sigma(H)\sum_{1\le j,k\le n} a_{jk}\varepsilon_k(1-\varepsilon_j)Y_jY_k\Big\} + \mathbf{I}\{|t| > \gamma\mathcal{L}^{-1}\}\\ &+ \mathcal{L}^{KH^{-1}-2-p\varkappa} + p^{K/H}\mathcal{L}^{-2}(\log^+\mathcal{L})^{-2K/H},\end{aligned}$$

where $T_0 = \gamma\mathcal{L}^{-1}(\log^+\mathcal{L})^{-1/2}$, $t\wedge T_0 = \min\{t, T_0\}$ *and* $\gamma = \dfrac{\sigma(H)}{2\sqrt{K\mu_4(H)}}$.

Theorem 2.2. *Let J_0, J_1 denote non-intersecting subsets of $\{1, 2, \dots, n\}$ such that $J_0 \cup J_1 = \{1, 2, \dots, n\}$ and let $J_{i1}, \dots, J_{i5}$ denote non-intersecting subsets in J_i, $i = 1, 2$, such that $\cup_{j=1}^5 J_{ij} = J_i$. Then*

$$\begin{aligned}\mathcal{H}_j^2(p, q, \eta, \zeta) &\le C\max_{i=0,1}\ \max_{1\le l\le 5}\Big(\mathbf{E}\,\big|\langle \eta, R_t A Z_{i,l}\rangle\big|^p\big|\langle \eta, A Z_{i,l}\rangle\big|^q\Big)^2\\ &\times\Big(\max_{1\le l\le 5}\mathbf{E}\,\exp\big\{\,2it\big\langle\,\widetilde{Z}_0, A\widetilde{Z}_{1,l}\,\big\rangle\big\} + \max_{1\le l\le 5}\ \max_{1\le m\le 5}\mathbf{E}\,\exp\big\{\,2it\big\langle\,\widetilde{Z}_{1l}, A\widetilde{Z}_{0m}\,\big\rangle\big\}\Big),\end{aligned}$$

where $Z_l := \sum_{k\in J_l, k\ne j}\xi_k$, $Z_{0l} := \sum_{k\in J_{0l}, k\ne j}\xi_k$, $l = 1, \dots, 5$.

Replacing the r.v.'s X_k, $k = 1, \dots, n$ by corresponding Gaussian ones, we obtain the following result.

Theorem 2.3. *Let the conditions of Theorem 2.2 be satisfied. Let r, $q \ge 0$ denote integers such that $r + q \le 3$. Then*

$$\begin{aligned}\widetilde{\mathcal{H}}_j^2(r, q, \eta, \zeta) &\le C\max_{i=0,1}\ \max_{1\le l\le 5}\Big(\mathbf{E}\,\big|\langle \eta, A Z_{i,l}\rangle\big|^r\big|\langle \eta, A Z_{i,l}\rangle\big|^q\Big)^2\\ &\times\Big(\max_{1\le l\le 5}\mathbf{E}\,\exp\big\{\,2it\big\langle\widetilde{Z}_0, A\widetilde{Z}_{1,l}\big\rangle\big\} + \max_{1\le m\le 5}\ \max_{1\le l\le 5}\mathbf{E}\,\exp\big\{\,2it\big\langle\widetilde{Z}_{1m}, A\widetilde{Z}_{0l}\big\rangle\big\}\Big),\end{aligned}$$

where $Z_l := \sum_{k\in J_l, k\ne j}\zeta_k$, $Z_{0l} := \sum_{j\in J_{0l}, k\ne j}\zeta_k$, $l = 1, \dots, 5$.

For $j = 1, \dots, n$ introduce the functions

$$\mathcal{H}_j(p, q, \xi_j) := \mathcal{H}_j(p, q, \xi_j, \tau\xi_j), \text{ and } \widetilde{\mathcal{H}}_j(p, q, \zeta_j) := \mathcal{H}_j(p, q, \zeta_j, \tau\zeta_j).$$

Theorem 2.4. *Assume that the conditions of Theorem* 2.1 *are fulfilled. Then*

$$\max\{\widetilde{\mathcal{H}}_j^2(p,q,\zeta_j),\mathcal{H}_j^2(p,q,\xi_j)\}\le\mathcal{L}_j^{2(p+q)}\beta_3^{\frac{2(p+q)}{3}}\varkappa_1(t),\tag{2.1}$$

where

$$\varkappa_1(t)=\big(\exp\{-\delta' t^2\}+\mathcal{L}^6\exp\{-\delta' T_0^2\}\big)+\mathbf{I}\{|t|\ge\gamma|\lambda_1|\log^+|\lambda_1|\}.$$

If we additionally assume (1.3), *then*

$$\max\big\{\widetilde{\mathcal{H}}_j(p,q,\zeta_j)\,,\mathcal{H}_j(p,q,\xi_j)\big\}\le C\beta_3^{p+q/3}\mathcal{L}_j^{(p+q)}\varkappa_2(t)\tag{2.2}$$

with

$$\varkappa_2(t)=\Big(\frac{1}{1+|t|^{q/4}}+\mathcal{L}^6\Big)+\mathbf{I}\big\{|t|\ge\mathcal{L}_n|\ln\mathcal{L}_n|\big\}.$$

Proof of Theorem 2.1. Note that the inequality

$$|f(t)|\le\mathbf{E}\,\exp\{it\sum_{j,k=1}^n a_{jk}(\varepsilon_j-\varepsilon_k)^2\widetilde{X}_j\widetilde{X}_k\}\tag{2.3}$$

still holds if $a_{jj}\ne0$. Moreover, the right hand side of (2.3) is independent of diagonal part of the matrix A. The conclusion of Theorem 2.1 follows now from Lemmas 3.2, 5.1 and 5.2 in GT. □

Proof of Theorem 2.2. We have

$$\begin{aligned}\big\langle R_tA\eta,\mathcal{X}_j\big\rangle^p\big\langle A\eta,\mathcal{X}_j\big\rangle^q=&\sum_{\substack{\alpha_0+\alpha_{1,1}+\alpha_{1,2}+\ldots+\alpha_{1,5}=p\\ \alpha_0'+\alpha_{1,1}'+\alpha_{1,2}'+\ldots+\alpha_{1,5}'=q}}\big\langle R_tA\eta,Z_0\big\rangle^{\alpha_0}\big\langle A\eta_k,Z_0\big\rangle^{\alpha_0'}\\&\times\prod_{i=1}^5\big\langle R_tA\eta,Z_{1,i}\big\rangle^{\alpha_{1,i}}\big\langle A\eta_k,Z_{1,i}\big\rangle^{\alpha_{1,i}'}.\end{aligned}\tag{2.4}$$

The equality (2.4) implies

$$\begin{aligned}&\mathbf{E}\,\big\langle R_tA\eta,\mathcal{X}_j\big\rangle^p\big\langle A\eta,\mathcal{X}_j\big\rangle^q\exp\big\{iz\big\langle A\zeta,\mathcal{X}_j\big\rangle+itQ_{n,k}'\big\}\\&=\sum_{\substack{\alpha_0+\alpha_{1,1}+\alpha_{1,2}+\ldots+\alpha_{1,5}=p\\ \alpha_0'+\alpha_{1,1}'+\alpha_{1,2}'+\ldots+\alpha_{1,5}'=q}}H(\alpha_0,\alpha_{1,1},\ldots,\alpha_{1,5},\alpha_0',\alpha_{1,1}',\ldots,\alpha_{1,5}'),\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}H(\alpha_0,&\alpha_{1,1},\ldots,\alpha_{1,5},\alpha_0',\alpha_{1,1}',\ldots,\alpha_{1,5}')=\mathbf{E}\,\big\langle A\eta,Z_0\big\rangle^{\alpha_0}\big\langle R_tA\eta,Z_0\big\rangle^{\alpha_0'}\\&\times\prod_{i=1}^5\big\langle A\eta,Z_{1,i}\big\rangle^{\alpha_{1,i}'}\big\langle R_tA\eta,Z_{1,i}\big\rangle^{\alpha_{1,i}'}\exp\big\{iz\big\langle A\zeta,\mathcal{X}_j\big\rangle+itQ_{n,k}'\big\}.\end{aligned}$$

We have

$$\alpha_0 + \alpha_0' + \alpha_{1,1} + \alpha_{1,1}' + \ldots + \alpha_{1,5} + \alpha_{1,5}' = p + q \le 3.$$

From this point on the rest of the proof is similar to the proof of Lemmas 3.3 and 3.4 in GT. □

Proof of Theorem 2.4. We first note that

$$\max_{1\le l\le 5} \mathbf{E}\,\big|\langle \xi_j, R_t A Z_{i,l}\rangle\big|^r \big|\langle \xi_j, A Z_{i,l}\rangle\big|^q \le |\xi_j|^{r+q} \beta_3^{r+q/3} \Big(\sum_{l\ne J} |R_t A(l,j)|^2\Big)^{r/2} \Big(\sum_{l\ne J} |A(l,j)|^2\Big)^{q/2}.$$

Since the spectrum of the operator R_t is contained in the unit circle, we have

$$\sum_{l\ne J} |R_t A(l,j)|^2 \le \sum_{l\ne J} |A(l,j)|^2.$$

From here it follows that

$$\max_{1\le l\le 5} \mathbf{E}\,\big|\langle \xi_j, R_t A Z_{i,l}\rangle\big|^r \big|\langle \xi_j, A Z_{i,l}\rangle\big|^q \le |\xi_j|^{r+q} \mathcal{L}_j^{\frac{r+q}{2}}.$$

Now we can repeat the proof of Theorem 1 in GT. The inequality (2.1) follows from Lemmas 3.4, and inequalities (6.8)–(6.13), in this paper. If (1.3) is fulfilled, the inequalities (6.16)–(6.19) in GT imply (2.2). □

3. An error bound for the differential equation of $f(t)$

In this Section we shall assume that the conditions Theorem 1 are fulfilled. Let R_t be the operator $R_t = \big(I - 2itA\big)^{-1}$, where I is identity operator in $\mathbb{R}^n$, and $\mathcal{D}^{(t)}$ denotes the vector with coordinates $\mathcal{D}^{(t)}(j) = R_t A(j,j)$, for $j = 1, \ldots, n$. In the following θ denotes different complex number with $|\theta| \le 1$. Set $\varkappa(t) = \min\{\varkappa_1(t), \varkappa_2(t)\}$, and

$$\mathcal{N}(t) = \theta C t^2 \Big(\mathcal{L} + \mathcal{M}\Big)\varkappa.$$

The main result of the Section is the following Theorem.

Theorem 3.1. *The characteristic function $f(t)$ satisfies the following differential equation*

$$f'(t) = i\mathcal{J}(t) f(t) + \delta(t)$$

with

$$\mathcal{J}(t) = \mathrm{Tr}(R_t A) + it(\mu_4 - 1)\,\mathrm{Tr}(R_t A D) - \mathrm{Tr}(A) - t^2\mu_3^2\big\langle \mathcal{D}, R_t^2 A \mathcal{D}\big\rangle - 2t^2\mu_3^2\big\langle \mathcal{D}, R_t A \mathcal{D}^{(t)}\big\rangle$$

and

$$\delta(t) = \theta_1\, C|t|^2\Big(\mathcal{L} + \mathcal{M}\Big)\varkappa(t).$$

The proof of Theorem 3.1 is based on Lemmas 3.2–3.5. To study the behavior of $f(t)$, we need to investigate auxiliary functions

$$h(t) = \mathbf{E}\,\big\langle R_t A \mathcal{X}, \mathcal{X}\big\rangle\, \mathrm{e}\big\{\big\langle A\mathcal{X}, \mathcal{X}\big\rangle\big\}, \tag{3.1}$$

and

$$\varphi(\eta) = \mathbf{E}\,\big\langle \eta, A\mathcal{X}\big\rangle\, \mathrm{e}\big\{\big\langle A\mathcal{X}, \mathcal{X}\big\rangle\big\}, \quad \eta \in \mathbb{R}^n. \tag{3.2}$$

Lemma 3.2. *We have*
$$\varphi(\eta) = it\mu_3\big\langle\mathcal{D}, R_tA\eta\big\rangle f(t) + \mathcal{N}(t)$$
for any $\eta \in \mathbb{R}^n$.

Represent the function $h(t)$ in the form
$$h(t) = h^{(1)}(t) + h^{(2)}(t),$$
where
$$h^{(1)}(t) = \sum_{j=1}^{n} \mathbf{E}\,\big\langle \xi_j, R_tA\mathcal{X}_j\big\rangle\,\mathrm{e}\big\{\,Q\,\big\}, \quad h^{(2)}(t) = \sum_{j=1}^{n} \mathbf{E}\,\big\langle \xi_j, R_tA\xi_j\big\rangle\,\mathrm{e}\big\{\,Q\,\big\}.$$

Lemma 3.3. *The function* $h^{(1)}(t)$ *satisfies the following relation*
$$h^{(1)}(t) = 2it\mathbf{E}\,\big\langle\mathcal{X}, AR_tA\mathcal{X}\big\rangle\,\mathrm{e}\{Q\} + \mathcal{J}_1(t)f(t) + \mathcal{N}(t)$$
with
$$\mathcal{J}_1(t) = -t^2\mu_3^2\big\langle\mathcal{D}, R_t^2A\mathcal{D}\big\rangle - 2it\,\mathrm{Tr}\{R_tA^2\}.$$

Denote by $\mathbf{I}\,\{B\}$ the indicator function of an event B.

Lemma 3.4. *The function* $h^{(2)}(t)$ *satisfies*
$$h^{(2)}(t) = \mathcal{J}_2(t)f(t) + \mathcal{N}(t),$$
with
$$\mathcal{J}_2(t) = \mathrm{Tr}(R_tA) + it\ \mathrm{Tr}\big\{\,R_tAD\,\big\} + it\mu_4\ \mathrm{Tr}\{R_tAD\} \\ - 2it\ \mathrm{Tr}(R_tAD)\mu_4 - 2t^2\mu_3^2\big\langle\mathcal{D}, R_tA\mathcal{D}^{(t)}\big\rangle.$$

Introduce the function $r(t) = 2it\mathbf{E}\,\big\langle\mathcal{X}, DR_tA\mathcal{X}\big\rangle\,\mathrm{e}\{Q\}$.

Lemma 3.5. *The function* $r(t)$ *satisfies the following relation*
$$r(t) = \mathcal{J}_3(t)f(t) + \mathcal{N}(t)$$
with $\mathcal{J}_3(t) = 2it\,\mathrm{Tr}\{DR_tA\}$.

Below we will use the notation
$$\mathcal{E}_j = \big\langle\eta, R_tA\xi_j\big\rangle, \quad \mathcal{T}_j = \big\langle\xi_j, A\xi_j\big\rangle, \quad \mathcal{F}_j = \big\langle\xi_j, A\mathcal{X}_j\big\rangle, \quad \mathcal{S}_j = \big\langle\xi_j, R_tA\mathcal{X}_j\big\rangle.$$

Proof of Lemma 3.2. Consider the function
$$q(\eta) = \mathbf{E}\,\big\langle\eta, R_tA\mathcal{X}\big\rangle\,\mathrm{e}\big\{\,\big\langle A\mathcal{X}, \mathcal{X}\big\rangle\,\big\}.$$

Since ξ_j and Q_j are independent, we have $\sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j\,\mathrm{e}\big\{\,Q_j\,\big\} = 0$, and we can represent $q(\eta)$ in the form

$$q(\eta) = q_1(\eta) + q_2(\eta) + q_3(\eta), \tag{3.3}$$

where

$$q_1(\eta) = \sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j(\mathrm{e}\left\{\,2\mathcal{F}_j\,\right\} - 1)\,\mathrm{e}\big\{\,Q_j\,\big\}, \quad q_2(\eta) = \sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j(\mathrm{e}\left\{\,\mathcal{T}_j\,\right\} - 1)\,\mathrm{e}\big\{\,Q_j\,\big\},$$

$$q_3(\eta) = \sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j\Big(\,\mathrm{e}\left\{\,2\mathcal{F}_j\,\right\} - 1\Big)\Big(\,\mathrm{e}\big\{\,\mathcal{T}_j\,\big\} - 1\,\Big)\,\mathrm{e}\big\{\,Q_j\,\big\}.$$

Note that

$$\mathcal{F}_j = X_j \sum_{l \neq j} a_{lj} X_l, \quad \mathcal{T}_j = d_j X_j^2. \tag{3.4}$$

Since $\mathbf{E}\,\big|\sum_{l\neq j} a_{lj} X_l\,\big| \leq \Big(\,\sum_{l\neq j} a_{lj}^2\,\Big)^{1/2} = \mathcal{L}_j$, we obtain the inequality

$$\big|\,q_3(\eta)\,\big| \leq C\,t^2\,\|\eta\|\,\|\mathcal{D}\| \max_{1\leq j\leq n} \mathbf{E}\,|X_j|^3 \mathcal{H}_j(0,1,\xi_j).$$

Theorem 2.4 and the last inequality together imply that

$$\big|\,q_3(\eta)\,\big| \leq C\,t^2\,\|\eta\|\,\|\mathcal{D}\|\,\mathcal{L}\,\varkappa(t) \leq \mathcal{N}(t) \tag{3.5}$$

Furthermore we can represent the function $q_1(\eta)$ in the form

$$q_1(\eta) = q_4(\eta) + q_5(\eta), \tag{3.6}$$

where

$$q_4(\eta) = 2it \sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j \mathcal{F}_j\,\mathrm{e}\big\{\,Q_j\,\big\}, \quad q_5(\eta) = -4t^2 \sum_{j=1}^{n} \mathbf{E}\,(1-\tau)^2 \mathcal{E}_j \mathcal{F}_j^2\,\mathrm{e}\big\{\,Q_j + 2\tau\mathcal{F}_j\,\big\}.$$

For the function $q_5(\eta)$ we get the bound

$$\big|\,q_5(\eta)\,\big| \leq C\,t^2\,\|\eta\|\,\mathcal{L} \sum_{j=1}^{n} \mathbf{E}\,X_j^2 \mathcal{H}_j(0,2,\xi_j). \tag{3.7}$$

From Theorem 2.4 and (3.7) it follows that

$$\big|\,q_5(\eta)\,\big| \leq C\mu_4 |t|^2 \mathcal{L}\varkappa(t) \leq \mathcal{N}(t). \tag{3.8}$$

In order to investigate the function $q_4(\eta)$, we decompose it as follows

$$q_4(\eta) = q_6(\eta) + q_7(\eta) + q_8(\eta),$$

where

$$q_6(\eta) = 2it\sum_{j=1}^{n}\mathbf{E}\,\langle\eta, R_tA\widehat{\xi}_j\rangle\langle\widehat{\xi}_j, A\mathcal{X}\rangle\,\mathrm{e}\{\,Q\,\},$$
$$q_7(\eta) = -2it\sum_{j=1}^{n}\mathbf{E}\,\langle\eta, R_tA\widehat{\xi}_j\rangle\langle\widehat{\xi}_j, A\mathcal{X}\rangle\big(\mathrm{e}\{\Delta_j\}-1\big)\,\mathrm{e}\{\,Q_j\,\},$$
$$q_8(\eta) = -2it\sum_{j=1}^{n}\mathbf{E}\,\langle\eta, R_tA\widehat{\xi}_j\rangle\langle\widehat{\xi}_j, A\xi_j\rangle\,\mathrm{e}\{\,Q_j\,\},$$

and $\widehat{\xi}_j$ is an independent copy of ξ_j. It is easy to see that

$$q_6(\eta) = 2it\mathbf{E}\,\langle\eta, R_tA^2\mathcal{X}\rangle\,\mathrm{e}\{\,Q\,\}. \tag{3.9}$$

Similarly as in (3.5), (3.8), we prove that

$$\big|\,q_7(\eta)\,\big| \le C\|\eta\||t|^2\mathcal{L}\sum_{j=1}^{n}\Big(|a_{jj}|\mathbf{E}\,X_j^2\mathcal{H}_j(0,1,\xi_j) + \mathbf{E}\,\mathcal{H}_j(0,2,\xi_j)\Big). \tag{3.10}$$

It is evident that

$$q_8(\eta) = 0. \tag{3.11}$$

From (3.7)–(3.11) it follows that

$$q_1(\eta) = 2it\mathbf{E}\,\langle\eta, R_tA^2\mathcal{X}\rangle\,\mathrm{e}\{\,Q\,\} + S_3. \tag{3.12}$$

Representing the function $q_2(x)$ in the form

$$q_2(\eta) = q_9(\eta) + q_{10}(\eta),$$

where

$$q_9(\eta) = it\sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\,\mathrm{e}\{\,Q_j\,\},\quad q_{10}(\eta) = -t^2\sum_{j=1}^{n}\mathbf{E}\,(1-\tau)^2\mathcal{E}_j\mathcal{T}_j^2\,\mathrm{e}\{\,Q_j+\tau\mathcal{T}_j\,\},$$

and proceeding similarly as in (3.5), (3.8) and (3.10), we obtain that

$$|q_{10}(\eta)| \le C\|\eta\||t|^2\mathcal{L}\sum_{j=1}^{n}a_{jj}^2\mathbf{E}\,|X_j|^3\mathcal{H}_j(0,0,\xi_j).$$

The last inequality and Theorem 2.4 together imply that

$$|q_{10}(\eta)| \le \mu_4\|\eta\|\mathcal{N}(t). \tag{3.13}$$

Finally, we shall determine an asymptotic approximation of $q_9(\eta)$. As since ξ_j and $\mathcal{X}_j$ are independent, we can write the following identity

$$q_9(\eta) = it\Big(\sum_{j=1}^{n} \mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\Big)\mathbf{E}\ \mathrm{e}\big\{\,Q\,\big\} + q_{11}(\eta). \tag{3.14}$$

Here $q_{11}(\eta) = -t^2\sum_{j=1}^n \mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\mathbf{E}\,\Delta_j\,\mathrm{e}\big\{\,Q_j + \tau\Delta_j\,\big\}$. Similarly as in (3.5), (3.8), (3.10) we obtain

$$|q_{11}(\eta)| \le C\|\eta\||t|^2\mathcal{P}_1, \tag{3.15}$$

where

$$\mathcal{P}_1 = \max_{1\le j\le n}\Big(|a_{jj}|\,\mathbf{E}\ X_j^2\mathcal{H}_j(0,0,\xi_j) + \mathbf{E}\,\mathcal{H}_j(0,1,\xi_j)\Big).$$

Using Theorem 2.4 and inequality (3.15) we show that

$$|q_{11}(\eta)| \le \|\eta\|\mathcal{N}(t). \tag{3.16}$$

From (3.14)–(3.16) it follows that

$$q_2(\eta) = it\mu_3\big\langle\mathcal{D}, R_tA\eta\big\rangle\mathbf{E}\ \mathrm{e}\big\{\,Q\,\big\} + \|\eta\|\mathcal{N}(t). \tag{3.17}$$

Relations (3.3), (3.12) and (3.17) together imply

$$\begin{aligned} q(\eta) = \mathbf{E}\,\big\langle\eta, R_tA\mathcal{X}\big\rangle\,\mathrm{e}\big\{\,\big\langle A\mathcal{X},\mathcal{X}\big\rangle\,\big\} &= 2it\mathbf{E}\,\big\langle\eta, R_tA^2\mathcal{X}\big\rangle\,\mathrm{e}\big\{\,Q\,\big\} \\ &+ it\mu_3\big\langle\mathcal{D}, R_tA\eta\big\rangle\mathbf{E}\ \mathrm{e}\big\{\,Q\,\big\} + \|\eta\|\mathcal{N}(t). \end{aligned} \tag{3.18}$$

Since $A^2 = A_0A + DA$ and $R_t(I - 2itA_0) = I$, we derive from (3.18) that

$$\varphi(\eta) = it\mu_3\big\langle\mathcal{D}, R_tA\eta\big\rangle\mathbf{E}\ \mathrm{e}\big\{\,Q\,\big\} + q_{12}(\eta) + \|\eta\|\mathcal{N}(t),$$

where

$$q_{12}(\eta) = 2it\mathbf{E}\,\big\langle\eta, R_tDA\mathcal{X}\big\rangle\,\mathrm{e}\big\{\,Q\,\big\}.$$

Since $\sum_{j=1}^n \mathbf{E}\,\big\langle\eta, R_tDA\xi_j\big\rangle\,\mathrm{e}\big\{\,Q_j\,\big\} = 0$, we have

$$|q_{12}(\eta)| \le C\|\eta\||t|^2\mathcal{L}\mathcal{P}_2^{1/2}, \tag{3.19}$$

where $\mathcal{P}_2 = \sum_{j=1}^n \Big(|a_{jj}|^2\mathbf{E}^{\,2}\,|X_j|^3\mathcal{H}_j(0,0,\xi_j) + \mathbf{E}^{\,2}\,|X_j|\mathcal{H}_j(0,1,\xi_j)\Big)$. Similarly as in (3.5), it follows

$$|q_{12}(\eta)| \le \|\eta\|\mathcal{N}(t). \tag{3.20}$$

The result of Lemma 3.2 now follows from (3.19) and (3.20). □

Proof of Lemma 3.3. Consider the function $h^{(1)}(t)$. Since ξ_j and $\mathcal{X}_j$, Q_j are independent, we get $\sum_{j=1}^n \mathbf{E}\,\mathcal{S}_j\,\mathrm{e}\big\{\,Q_j\,\big\} = 0$. Decomposing $h^{(1)}(t)$ we have

$$h^{(1)}(t) = u_1(t) + u_2(t) + u_3(t),$$

where

$$u_1(t) = \sum_{j=1}^{n} \mathbf{E}\, \mathcal{S}_j \Big(\,\mathrm{e}\{2\mathcal{T}_j\} - 1\Big)\, \mathrm{e}\{Q_j\},$$

$$u_2(t) = \sum_{j=1}^{n} \mathbf{E}\, \mathcal{S}_j \big(\mathrm{e}\{\mathcal{T}_j\} - 1\big)\, \mathrm{e}\big\{\, Q_j \,\big\},$$

$$u_3(t) = -2t^2 \sum_{j=1}^{n} \mathbf{E}\, \mathcal{T}_j \mathcal{S}_j \mathcal{F}_j \, \mathrm{e}\big\{\, Q_j + \tau_1 \mathcal{T}_j + \tau_2 \mathcal{F}_j \,\big\}.$$

Proceeding similarly as in (3.5), we obtain the estimate

$$\big|\, u_3(t) \,\big| \le C|t|^2 \mu_4 \sum_{j=1}^{n} |a_{jj}| \mathbf{E}\, X_j^2 \mathcal{H}_j(1, 1, \xi_j). \tag{3.21}$$

The function $u_1(t)$ can be written as

$$u_1(t) = u_4(t) + u_5(t), \tag{3.22}$$

where

$$u_4(t) = 2it \sum_{j=1}^{n} \mathbf{E}\, \mathcal{S}_j \mathcal{F}_j \, \mathrm{e}\big\{\, Q_j \,\big\}, \quad u_5(t) = -4t^2 \sum_{j=1}^{n} \mathbf{E}\, (1-\tau)^2 \mathcal{S}_j \mathcal{F}_j \, \mathrm{e}\big\{\, Q_j + \tau \mathcal{F}_j \,\big\}.$$

It is easy to see that

$$u_4(t) = 2it \sum_{j=1}^{n} \mathbf{E}\, \big\langle B_j R_t A \mathcal{X}_j, A \mathcal{X}_j \big\rangle \, \mathrm{e}\big\{\, Q_j \,\big\},$$

where B_j is the covariance operator of ξ_j, $j = 1, \dots, n$. Represent $u_4(t)$ in the form

$$u_4(t) = u_6(t) + u_7(t) + u_8(t),$$

where

$$u_6(t) = 2it \sum_{j=1}^{n} \mathbf{E}\, \big\langle B_j R_t A \mathcal{X}, A \mathcal{X} \big\rangle \, \mathrm{e}\big\{\, Q \,\big\},$$

$$u_7(t) = -2it \sum_{j=1}^{n} \mathbf{E}\, \big\langle B_j R_t A \mathcal{X}, A \mathcal{X} \big\rangle \big(\mathrm{e}\big\{\, \Delta_j \,\big\} - 1\big)\, \mathrm{e}\big\{\, Q_j \,\big\},$$

$$u_8(t) = -2it \sum_{j=1}^{n} \mathbf{E}\, \big\langle B_j R_t A \xi_j, A \xi_j \big\rangle \, \mathrm{e}\big\{\, Q_j \,\big\}.$$

Since $\sum_{j=1}^{n} B_j = I$, we get

$$u_6(t) = 2it\mathbf{E}\,\big\langle \mathcal{X}, AR_tA\mathcal{X}\big\rangle\,\mathrm{e}\{Q\} \tag{3.26}$$

Similarly as in (3.5), we derive

$$u_7(t) \le C|t|^2\mathcal{P}_2, \tag{3.24}$$

where

$$\mathcal{P}_2 = \sum_{j=1}^{n}\big(\,|a_{jj}|\mathbf{E}\,X_j^2\mathcal{H}_j(1,1,\xi_j) + \mathbf{E}\,\mathcal{H}_j(1,2,\xi_j)\big).$$

An easy computation shows that

$$u_7(t) \le C|t|^2\Big(\mathcal{L}+\mathcal{M}\Big)\varkappa(t).$$

Using the independence of ξ_j and Q_j, and replacing Q_j by Q, we can obtain

$$\begin{aligned} u_8(t) &= -2it\sum_{j=1}^{n}\mathbf{E}\,\langle B_jR_tA\xi_j, A\xi_j\rangle\,\mathrm{e}\big\{\,Q\,\big\} + \mathcal{N}(t)\\ &= -2it\,\mathrm{Tr}\{R_tAD\}f(t) + \mathcal{N}(t) \end{aligned}\tag{3.28}$$

The function $u_5(t)$ can be bounded as follows

$$|u_5(t)| \le C|t|^2\sum_{j=1}^{n}\mathbf{E}\,\mathcal{H}_j(1,2,\xi_j). \tag{3.26}$$

Combined with the results of Theorem 2.4 the inequality (3.26) implies

$$|u_5(t)| \le C|t|^2\mathcal{L}\varkappa(t). \tag{3.27}$$

The relations (3.23)–(3.27) together imply

$$u_1(t) = -2it\,\mathrm{Tr}\{R_tAD\}f(t) + 2it\mathbf{E}\,\big\langle \mathcal{X}, AR_tA\mathcal{X}\big\rangle\,\mathrm{e}\{Q\} + \mathcal{N}(t). \tag{3.31}$$

Consider now the function $u_2(t)$. Write

$$u_2(t) = u_9(t) + u_{10}(t),$$

where

$$u_9(t) = it\sum_{j=1}^{n}\mathbf{E}\,\mathcal{S}_j\mathcal{T}_j\,\mathrm{e}\big\{\,Q_j\,\big\},\quad u_{10}(t) = \sum_{j=1}^{n}\mathbf{E}\,(1-\tau)^2\mathcal{S}_j\mathcal{T}_j^2\,\mathrm{e}\big\{\,Q_j+\tau\mathcal{T}_j\,\big\}.$$

It is easy to see that

$$|u_{10}(t)| \le C|t|^2\|\mathcal{D}\|^2\max_{1\le j\le n}\mathbf{E}\,\mathcal{H}_j(1,0,\xi_j). \tag{3.32}$$

Using that ξ_j and Q_j are independent, we get

$$u_9(t) = it\sum_{j=1}^{n} \mathbf{E}\,\langle \xi_j, D\xi_j\rangle \varphi(R_t\xi_j) + \theta C|t|^2\mathbf{E}\,|X|^3\mathcal{P}_{11}, \tag{3.30}$$

where

$$\mathcal{P}_{11} = \sum_{j=1}^{n}\Big(|a_{jj}|^2\mathbf{E}\,X_j^4\mathcal{H}_j(1,0,\xi_j) + |a_{jj}|\mathbf{E}\,X_j^2\mathcal{H}_j(1,1,\xi_j)\Big).$$

Applying Lemma 3.2 and results of Theorem 2.4 to the right hand side of (3.30), we see that

$$\begin{aligned} u_9(t) &= \mu_3(it)^2\sum_{j=1}^{n}\mathbf{E}\,\langle \xi_j, D\xi_j\rangle\langle \mathcal{D}, R_tAR_t\xi_j\rangle f(t) + \mathcal{N}(t) \\ &= -t^2\mu_3^2\langle \mathcal{D}, R_t^2A\mathcal{D}\rangle f(t) + \mathcal{N}(t). \end{aligned} \tag{3.31}$$

By (3.29)–(3.31) we obtain that

$$u_2(t) = -t^2\mu_3^2\langle \mathcal{D}, R_t^2A\mathcal{D}\rangle f(t) + \mathcal{N}(t). \tag{3.32}$$

Combining (3.21), (3.28) and (3.32), we get

$$\begin{aligned} h^{(1)}(t) = &-t^2\mu_3^2\langle \mathcal{D}, R_t^2A\mathcal{D}\rangle f(t) - 2it\,\mathrm{Tr}\{R_tAD\}f(t) \\ &+ 2it\mathbf{E}\,\langle \mathcal{X}, AR_tA\mathcal{X}\rangle\,\mathrm{e}\{Q\} + \mathcal{N}(t). \end{aligned} \tag{3.33}$$

□

Proof of Lemma 3.4. Represent the function $h^{(2)}(t)$ in the form

$$h^{(2)}(t) = v_1(t) + v_2(t),$$

where

$$v_1(t) = \sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\,\mathrm{e}\{\,Q_j\,\}, \quad v_2(t) = \sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j(\mathrm{e}\{\Delta_j\}-1)\,\mathrm{e}\{\,Q_j\,\}.$$

We shall study these functions separately. Consider first $v_1(t)$. We have

$$v_1(t) = v_3(t) + v_4(t) + v_5(t), \tag{3.34}$$

where

$$v_3(t) = \mathrm{Tr}(R_tA)f(t), \quad v_4(t) = -it\sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\mathbf{E}\,\mathcal{T}_j\mathbf{E}\;\mathrm{e}\{\,Q_j\,\},$$

$$v_5(t) = \sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\mathbf{E}\,(1-\tau)^2\Delta_j^2\,\mathrm{e}\{\,Q_j + \tau\Delta_j\,\}.$$

Here we used that $\mathbf{E}\,\mathcal{F}_j\,\mathrm{e}\big\{\,Q_j\,\big\} = 0$. It is easy to show that

$$|v_5(t)| \le C|t|^2\mathcal{L}\mathcal{P}_3, \tag{3.35}$$

where

$$\mathcal{P}_3 = \sum_{j=1}^{n}\Big(|a_{jj}|^2\mathbf{E}\,X_j^4\mathcal{H}_j(0,0,\xi_j) + 2|a_{jj}|\mathbf{E}\,X_j^2\mathcal{H}_j(0,1,\xi_j) + \mathbf{E}\,\mathcal{H}_j(0,2,\xi_j)\Big).$$

This gives

$$|v_5(t)| \le \mathcal{N}(t) \tag{3.36}$$

Using the independence of ξ_j and $\mathcal{X}_j$, approximating Q_j by Q, and using (3.39), we conclude

$$v_4(t) = -it\ \mathrm{Tr}\big\{\,R_tAD\,\big\}f(t) + \theta C|t|^2\mu_4\mathcal{P}_5. \tag{3.37}$$

The relations (3.34) and (3.37) imply that

$$v_1(t) = \mathrm{Tr}(R_tA)f(t) - it\ \mathrm{Tr}\big\{\,R_tAD\,\big\}f(t) + \mathcal{N}(t). \tag{3.38}$$

Now write the function $v_2(t)$ in the form

$$v_2(t) = v_6(t) + v_7(t) + v_8(t),$$

where

$$v_6(t) = \sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j(\mathrm{e}\{2\mathcal{F}_j\} - 1)\,\mathrm{e}\big\{\,Q_j\,\big\}, \quad v_7(t) = \sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j(\mathrm{e}\{\mathcal{T}_j\} - 1)\,\mathrm{e}\big\{\,Q_j\,\big\},$$

$$v_8(t) = -2t^2\sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\mathcal{F}_j\,\mathrm{e}\big\{\,Q_j + \tau_1\mathcal{T}_j + \tau_2\mathcal{F}_j\,\big\}.$$

It is easy to see that

$$|v_8(t)| \le C|t|^2\mu_6\max_{1\le j\le n}\mathbf{E}\,\mathcal{H}_j(0,1,\xi_j), \tag{3.39}$$

and by Theorem 2.4

$$|v_8(t)| \le C|t|^2\mu_6\mathcal{N}(t). \tag{3.40}$$

The functions $v_6(t)$ and $v_7(t)$ may be written in the form

$$v_6(t) = v_9(t) + v_{10}(t),$$

where

$$v_9(t) = 2it\sum_{j=1}^{n}\mathbf{E}\,\mathcal{E}_j\mathcal{F}_j\,\mathrm{e}\big\{\,Q_j\,\big\}, \quad v_{10}(t) = 2it\sum_{j=1}^{n}\mathbf{E}\,(1-\tau)^2\mathcal{E}_j\mathcal{F}_j\,\mathrm{e}\big\{\,Q_j + \tau\mathcal{F}_j\,\big\},$$

and

$$v_7(t) = v_{11}(t) + v_{12}(t), \tag{3.41}$$

where

$$v_{11}(t) = it\sum_{j=1}^n \mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\,\mathrm{e}\big\{\,Q_j\,\big\}, \quad v_{12}(t) = it\sum_{j=1}^n \mathbf{E}\,(1-\tau)^2\mathcal{E}_j\mathcal{T}_j^2\,\mathrm{e}\big\{\,Q_j+\tau\mathcal{T}_j\,\big\}.$$

The asymptotic approximation of these functions can be determined in the same way as for the function u_{11} above. Replacing ξ_j, $j = 1\ldots,n$, by their independent copies, and approximating Q_j and $\mathcal{X}_j$ by Q and $\mathcal{X}$ respectively, we get

$$v_9(t) = v_{13}(t) + v_{14}(t),$$

where

$$\begin{aligned} v_{13}(t) &= 2it\sum_{j=1}^n \mathbf{E}\,\mathcal{E}_j\varphi(\xi_j), \\ v_{14}(t) &= -2t^2\sum_{j=1}^n \mathbf{E}\,\big\langle\widehat{\xi}_j, R_tA\widehat{\xi}_j\big\rangle\big\langle\widehat{\xi}_j, A\mathcal{X}_n\big\rangle\Delta_j\,\mathrm{e}\big\{\,Q_j+\tau\Delta_j\big\rangle\big\}, \end{aligned} \tag{3.42}$$

and

$$v_{11}(t) = v_{15}(t) + v_{16}(t),$$

where

$$v_{15}(t) = it\mu_4\,\mathrm{Tr}\{R_tAD\}f(t), \quad v_{16}(t) = -t^2\sum_{j=1}^n \mathbf{E}\,\mathcal{E}_j\mathcal{T}_j\mathbf{E}\,\Delta_j\,\mathrm{e}\{Q_j+\tau\Delta_j\}.$$

Using Lemma 3.2 and the results of Theorem 2.4, we get

$$v_9(t) = -2t^2\mu_3^2\big\langle\mathcal{D}, R_tA\mathcal{D}^{(t)}\big\rangle f(t) + \mathcal{N}(t). \tag{3.43}$$

Form (3.39)–(3.43) it follows that

$$v_2(t) = -2t^2\mu_3^2\big\langle\mathcal{D}, R_tA\mathcal{D}^{(t)}\big\rangle f(t) + it\mu_4\,\mathrm{Tr}\{R_tAD\}f(t) + \mathcal{N}(t). \tag{3.44}$$

Next, by (3.38) and by (3.44), we have

$$\begin{aligned} h^{(2)}(t) = {}& \mathrm{Tr}(R_tA)f(t) + it\,\mathrm{Tr}\big\{\,R_tAD\,\big\}f(t) + it\mu_4\,\mathrm{Tr}\{R_tAD\}f(t) \\ &- 2t^2\mu_3^2\big\langle\mathcal{D}, R_tA\mathcal{D}^{(t)}\big\rangle f(t) + \mathcal{N}(t). \end{aligned} \tag{3.45}$$

□

Proof of Lemma 3.5. Represent the function $r(t)$ in the form

$$r(t) = r_1(t) + r_2(t),$$

where

$$r_1(t) = 2it \sum_{j=1}^{n} \mathbf{E} \left\langle \xi_j, DR_t A\mathcal{X}_j \right\rangle \mathrm{e}\{Q\}, \quad r_2(t) = 2it \sum_{j=1}^{n} \mathbf{E} \left\langle \xi_j, DR_t A\xi_j \right\rangle \mathrm{e}\{Q\}.$$

Since

$$\mathbf{E} \left\langle \xi_j, DR_t A\mathcal{X}_j \right\rangle \mathrm{e}\{Q_j\} = 0, \tag{3.48}$$

we have

$$r_1(t) = -2t^2 \sum_{j=1}^{n} \mathbf{E} \left\langle \xi_j, DR_t A\mathcal{X}_j \right\rangle \Delta_j \, \mathrm{e}\{Q_j + \tau\Delta_j\}.$$

An easy computation shows that

$$|r_1(t)| \leq C|t|^2 \mathcal{P}_7, \tag{3.46}$$

where

$$\mathcal{P}_4 = \sum_{j=1}^{n} |a_{jj}| \mathbf{E}\, X_j^2 \mathcal{H}(1, 1, \xi_j) + \max_{1 \leq j \leq n} \mathbf{E}\, X_j^2 \mathcal{H}_j(1, 0, \xi_j).$$

Applying Theorem 2.4, we obtain

$$|r_1(t)| \leq C|t|^2 \mathcal{N}(t).$$

We decompose the function $r_2(t)$ as follows

$$r_2(t) = r_3(t) + r_4(t),$$

where

$$r_3(t) = 2it \sum_{j=1}^{n} \mathbf{E} \left\langle \xi_j, DR_t A\xi_j \right\rangle \mathbf{E}\ \mathrm{e}\{Q_j\},$$

$$r_4(t) = 2it \sum_{j=1}^{n} \mathbf{E} \left\langle \xi_j, DR_t A\xi_j \right\rangle \Delta_j \, \mathrm{e}\{Q_j + \tau\Delta_j\}.$$

The function $r_4(t)$ satisfies

$$|r_4(t)| \leq C|t|^2 \max_{1 \leq j \leq n} \mathbf{E}\, X_j^2 \mathcal{H}_j(0, 1, \xi_j). \tag{3.47}$$

Replacing $\mathbf{E}\ \mathrm{e}\{Q_j\}$ by $f(t)$ and estimating the error as above, we find the asymptotic behaviour of $r_3(t)$. We get

$$r_3(t) = 2it \operatorname{Tr}\big(DR_t A\big) f(t) + \mathcal{N}(t) \tag{3.48}$$

The relations (3.47) and (3.48) together complete the proof. $\square$

Proof of Theorem 3.1. From Lemmas 3.2–3.5 it follows that

$$h(t) = 2it\mathbf{E} \left\langle \mathcal{X}, A_0 R_t A\mathcal{X} \right\rangle \mathrm{e}\{Q\} + (\mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t)) f(t) + \delta(t). \tag{3.49}$$

As since $(\mathrm{I} - 2itA_0)R_t = \mathrm{I}$, the equality (3.49) implies

$$f'(t) = i\mathbf{E} \left\langle \mathcal{X}, A\mathcal{X} \right\rangle \mathrm{e}\{Q\} - \operatorname{Tr}(A) f(t) = i\mathcal{J}(t) f(t) + \delta(t). \square$$

4. A differential equation for $g(t)$

Theorem 4.1. *The characteristic function $g(t)$ satisfies the following equality*

$$g'(t) = i\widetilde{\mathcal{J}}(t)g(t) + \mathcal{N}(t),$$

where

$$\widetilde{\mathcal{J}}(t) = iTr(R_tA_0) - it^2\mu_3^2\big\langle\mathcal{D}, R_t^2A_0\mathcal{D}\big\rangle \\ - t\mu_3^2\big\langle\mathcal{D},\mathcal{D}\big\rangle - t(\mu_4-\mu_3^2-\mu_2^2)\big\langle\mathcal{D},\mathcal{D}\big\rangle$$

To investigate $g'(t)$, we consider the functions

$$\varphi(t) = \mathbf{E}\,\big\langle\mathcal{Y}, A_0\mathcal{Y}\big\rangle\,\mathrm{e}\{G\}, \qquad \widetilde{\varphi}(t) = \mathbf{E}\,\big\langle\mathcal{Y}, R_tA_0\mathcal{Y}\big\rangle\,\mathrm{e}\{G\},$$
$$\psi(\eta) = \mathbf{E}\,\big\langle\eta,\mathcal{Y}\big\rangle\,\mathrm{e}\{G\}, \qquad \widetilde{\psi}(\eta) = \mathbf{E}\,\big\langle\eta, R_t\mathcal{Y}\big\rangle\,\mathrm{e}\{G\}.$$

The following lemmas are similar to Lemmas 3.2–3.5.

Lemma 4.2. *The function $\psi(\eta)$ satisfies the relation*

$$\psi(\eta) = it\mu_3\big\langle\eta, R_t\mathcal{D}\big\rangle g(t) + \theta Ct^2\,\|\eta\|\mathcal{N}(t)$$

Lemma 4.3. *The function $\widetilde{\varphi}(t)$ can be represented as follows*

$$\widetilde{\varphi}(t) = it\mathbf{E}\,\big\langle\mathcal{Y}, AR_tA_0\mathcal{Y}\big\rangle\,\mathrm{e}\{G\} + Tr\big(R_tA\big)g(t) - t^2\mu_3^2\big\langle\mathcal{D}, R_t^2\mathcal{D}\big\rangle g(t) + \theta C|t|^2\mathcal{N}(t).$$

Similarly to Section 3 we will use the notation

$$\widetilde{\mathcal{E}}_j = \big\langle\eta, R_t\zeta_j\big\rangle, \quad \widetilde{\mathcal{T}}_j = \big\langle\mathcal{D},\zeta_j\big\rangle, \quad \widetilde{\mathcal{F}}_j = \big\langle\zeta_j, A_0\mathcal{Y}_j\big\rangle, \quad \widetilde{\mathcal{S}}_j = \big\langle\zeta_j, R_tA_0\mathcal{Y}_j\big\rangle.$$

Proof of Lemma 4.2. Represent the function $\widetilde{\psi}(\eta)$ in the form

$$\widetilde{\psi}(\eta) = \widetilde{\psi}_1(\eta) + \widetilde{\psi}_2(\eta) + \widetilde{\psi}_3(\eta), \tag{4.1}$$

where

$$\widetilde{\psi}_1(\eta) = 2it\sum_{j=1}^n \mathbf{E}\,\widetilde{\mathcal{E}}_j\widetilde{\mathcal{F}}_j\,\mathrm{e}\{G_j\},$$
$$\widetilde{\psi}_2(\eta) = it\mu_3\sum_{j=1}^n \mathbf{E}\,\widetilde{\mathcal{E}}_j\widetilde{\mathcal{T}}_j\,\mathrm{e}\{G_j\},$$
$$\widetilde{\psi}_3(\eta) = \sum_{j=1}^n \mathbf{E}\,(1-\tau)^2\widetilde{\mathcal{E}}_j\widetilde{\Delta}_j^2\,\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}.$$

We rewrite the functions $\widetilde{\psi}_i(\eta)$, $i=1,2$, as follows

$$\widetilde{\psi}_1(\eta) = 2it\mathbf{E}\,\big\langle \eta, R_t A_0 \mathcal{Y}\big\rangle\, \mathrm{e}\{G\} + \varepsilon_2(t,\eta) \tag{4.2}$$

and

$$\widetilde{\psi}_2(\eta) = it\mu_3\big\langle \eta, R_t \mathcal{D}\big\rangle g(t) + \varepsilon_3(t,\eta), \tag{4.3}$$

where

$$\varepsilon_2(t,\eta) = -2t^2 \sum_{j=1}^n \mathbf{E}\,\big\langle B_j\eta, R_t A_0 \mathcal{Y}_j\big\rangle \widetilde{\Delta}_j\, \mathrm{e}\{G_j + \tau\widetilde{\Delta}_j\}$$

and

$$\varepsilon_3(t,\eta) = -t^2\mu_3 \sum_{j=1}^n \big\langle B_j\eta, R_t\mathcal{D}\big\rangle \mathbf{E}\,\widetilde{\Delta}_j\, \mathrm{e}\{G_j\}.$$

Here B_j denotes the covariance operator of ζ_j. It is not difficult to prove that

$$|\varepsilon_2(t,\eta)| \le C|t|^2\|\eta\|\mathcal{P}_5^{1/2}, \tag{4.4}$$

where

$$\mathcal{P}_5 = \sum_{j=1}^n \Big(|\mu_3|^2|a_{jj}|^2\mathbf{E}\,Y_j^2\widetilde{\mathcal{H}}_j^2(1,0,\zeta_j) + \mathbf{E}\,\widetilde{\mathcal{H}}_j^2(1,1,\zeta_j)\Big).$$

By Theorem 2.4,

$$|\varepsilon_2(t,\eta)| \le \|\eta\|\mathcal{N}(t).$$

For $\varepsilon_3(t,\eta)$ we have

$$|\varepsilon_3(t,\eta)| \le C|t|^2\|\eta\|\|\mathcal{D}\|\mathcal{P}_6(t),$$

where

$$\mathcal{P}_6 = \max_{1\le j\le n} \Big(|\mu_3||a_{jj}|\mathbf{E}\,Y_j^2\widetilde{\mathcal{H}}_j(0,0,\zeta_j) + \mathbf{E}\,\widetilde{\mathcal{H}}_j(1,0,\zeta_j)\Big).$$

The relations (4.1)–(4.4) imply that

$$\widetilde{\psi}(\eta) = 2it\mathbf{E}\,\big\langle \eta, R_t A_0\mathcal{Y}\big\rangle\,\mathrm{e}\{G\} + it\mu_3\big\langle R_t\eta, R_t\mathcal{D}\big\rangle g(t) + \mathcal{N}(t). \tag{4.5}$$

Proceeding similarly as in Lemma 3.2, we derive from (4.5) that

$$\psi(\eta) = it\mu_3\big\langle AR_t\eta, \mathcal{D}\big\rangle g(t) + \mathcal{N}(t).$$

□

Introduce the abbreviation $\mathcal{Z}_j = \big\langle \zeta_j, R_t A_0 \zeta_j\big\rangle$.

Proof of Lemma 4.3. We write $\widetilde{\varphi}(t)$ in the form

$$\widetilde{\varphi}(t) = \widetilde{\varphi}^{(1)}(t) + \widetilde{\varphi}^{(2)}(t),$$

where

$$\widetilde{\varphi}^{(1)}(t)=\sum_{j=1}^{n}\mathbf{E}\,\mathcal{Z}_j\,\mathrm{e}\{G\}\qquad \widetilde{\varphi}^{(2)}(t)=\sum_{j=1}^{n}\mathbf{E}\,\widetilde{\mathcal{S}}_j\,\mathrm{e}\{G\}.$$

Furthermore, we have

$$\widetilde{\varphi}^{(1)}(t)=u_{11}(t)+u_{12}(t),$$

where

$$u_{11}(t)=\sum_{j=1}^{n}\mathbf{E}\,\mathcal{Z}_j\,\mathrm{e}\{G_j\},\ u_{12}(t)=\sum_{j=1}^{n}\mathbf{E}\,\mathcal{Z}_j(\mathrm{e}\{\Delta_j\}-1)\,\mathrm{e}\{G_j\}.$$

Since ζ_j and G_j are independent, we obtain

$$u_{11}(t)=\mathrm{Tr}(R_tA_0)f(t)+\sum_{j=1}^{n}\mathbf{E}\,\mathcal{Z}_j\mathbf{E}\,(1-\mathrm{e}\{\Delta_j\})\,\mathrm{e}\{G_j\}. \tag{4.6}$$

Since in the Gaussian case $\mathbf{E}\,\widetilde{\Delta}_j\,\mathrm{e}\{G_j\}=0$, we can write

$$u_{11}(t)=\mathrm{Tr}(R_tA_0)f(t)+\varepsilon_4(t), \tag{4.7}$$

where

$$\varepsilon_4(t)=-t^2\sum_{j=1}^{n}\mathbf{E}\,\mathcal{Z}_j\mathbf{E}\,(1-\tau)^2\widetilde{\Delta}_j^2\,\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}.$$

Note that

$$\big|\varepsilon_4(t)\big|\le Ct^2\mathcal{L}\mathcal{P}_7, \tag{4.8}$$

where

$$\mathcal{P}_7=\sum_{j=1}^{n}\Big(|a_{jj}|^2\,\mathbf{E}\,Y_j^4\widetilde{\mathcal{H}}_j(0,0,\zeta_j)+|\mu_3||a_{jj}|\,\mathbf{E}\,Y_j^2\widetilde{\mathcal{H}}_j(0,1,\zeta_j)+\mathbf{E}\,\widetilde{\mathcal{H}}_j(0,2,\zeta_j)\Big).$$

From here it follows that $\big|\varepsilon_4(t)\big|\le\mathcal{N}(t)$. As since Y_j, $j=1,\dots,n$, are Gaussian, we have

$$\mathbf{E}\,\mathcal{Z}_j\widetilde{\mathcal{F}}_j\,\mathrm{e}\{G_j\}=0\quad\text{and}\quad\mathbf{E}\,\mathcal{Z}_j\widetilde{\mathcal{T}}_j\,\mathrm{e}\{G_j\}=0.$$

Hence,

$$u_{12}(t)=\sum_{j=1}^{n}\mathbf{E}\,(1-\tau)\mathcal{Z}_j\widetilde{\Delta}_j^2\,\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}.$$

Similarly as above we deduce

$$\big|u_{12}(t)\big|\le\mathcal{N}(t). \tag{4.9}$$

The relations (4.5)–(4.9) together imply that

$$\widetilde{\varphi}^{(1)}(t)=\mathrm{Tr}\big(R_tA_0\big)g(t)+\mathcal{N}(t). \tag{4.10}$$

Now we shall study the asymptotic approximations of the function $\widetilde{\varphi}^{(2)}(t)$. We write

$$\widetilde{\varphi}^{(2)}(t) = v_{12}(t) + v_{13}(t),$$

where

$$v_{12}(t) = \sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j\,\mathrm{e}\{G_j\} = 0,$$
$$v_{13}(t) = \sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j(\mathrm{e}\{\widetilde{\Delta}_j\} - 1)\,\mathrm{e}\{G_j\}.$$

Expand the function $v_{13}(t)$ as follows

$$v_{13}(t) = v_{14}(t) + v_{15}(t) + v_{16}(t), \tag{4.11}$$

where

$$v_{14}(t) = 2it\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j\widetilde{\mathcal{F}}_j\,\mathrm{e}\{G_j\}, \quad v_{15}(t) = it\mu_3\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j\widetilde{\mathcal{T}}_j\,\mathrm{e}\{G_j\},$$
$$v_{16}(t) = -2t^2\sum_{j=1}^{n} \mathbf{E}\,(1-\tau)^2\widetilde{\mathcal{S}}_j\widetilde{\Delta}_j^2\,\mathrm{e}\big\{\,G_j + \tau\widetilde{\Delta}_j\,\big\}.$$

It is not difficult to prove that

$$|v_{16}(t)| \le C|t|^2\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{H}}_j(1,2,\zeta_j). \tag{4.12}$$

This implies $|v_{16}(t)| \le \mathcal{N}(t)$. The functions $v_{14}(t)$ and $v_{15}(t)$ may be written in the form

$$v_{14}(t) = 2it\mathbf{E}\,\big\langle \mathcal{Y}, A_0R_t\mathcal{Y}\big\rangle\,\mathrm{e}\{G\} + \varepsilon_5(t),$$
$$v_{15}(t) = it\mu_3\mathbf{E}\,\big\langle R_tA_0\mathcal{D}, \mathcal{Y}\big\rangle\,\mathrm{e}\{G\} + \varepsilon_6(t),$$

where

$$\varepsilon_5(t) = 2it\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j\widetilde{\mathcal{F}}_j\widetilde{\Delta}_j\}\,\mathrm{e}\{G_j + \tau\widetilde{\Delta}_j\},$$
$$\varepsilon_6(t) = it\mu_3\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{S}}_j\widetilde{\mathcal{T}}_j\widetilde{\Delta}_j\,\mathrm{e}\{G_j + \tau\widetilde{\Delta}_j\}.$$

Similarly as in (4.6), we have

$$\big|\,\varepsilon_5(t)\,\big| \le C|t|^2\sum_{j=1}^{n} \mathbf{E}\,\widetilde{\mathcal{H}}_j(1,2,\zeta_j) \tag{4.13}$$

and

$$\big|\varepsilon_6(t)\big| \le C|t|^2\sum_{j=1}^n |a_{jj}|\mathbf{E}\,|Y_j|\widetilde{\mathcal{H}}_j(1,1,\zeta_j). \tag{4.14}$$

By Lemma 8 and by Theorem 2.4

$$v_{15}(t) = it\mu_3\widetilde{\psi}(R_tA_0\mathcal{D}) + \mathcal{N}(t) = -t^2\mu_3^2\big\langle\mathcal{D}, R_t^2A_0\mathcal{D}\big\rangle g(t) + \mathcal{N}(t).$$

Now (4.11)–(4.14) imply

$$\begin{aligned}\widetilde{\varphi}(t) = {}& it\mathbf{E}\,\langle\mathcal{Y}, A_0R_tA_0\mathcal{Y}\rangle\,\mathrm{e}\{G\} + Tr\big(R_tA_0\big)g(t)\\ &- t^2\mu_3^2\big\langle\mathcal{D}, R_t^2A_0\mathcal{D}\big\rangle g(t) + \mathcal{N}(t).\end{aligned} \tag{4.15}$$

□

Proof of Theorem 4.1. Note that

$$g'(t) = i\varphi(t) + i\mu_3\psi(\mathcal{D}) + i\sqrt{\mu_4 - 1 - \mu_3^2}\,V\mathbf{E}\,Y\,\mathrm{e}\{G\}.$$

Since Y is a Gaussian independent of $\big\langle\mathcal{Y}, A_0\mathcal{Y}\big\rangle + \mu_3\big\langle\mathcal{D},\mathcal{Y}\big\rangle$, it follows that

$$i\sqrt{\mu_4 - 1 - \mu_3^2}\,\mathbf{E}\,Y\,\mathrm{e}\{G\} = -t(\mu_4 - 1 - \mu_3^2)\,\langle\mathcal{D},\mathcal{D}\rangle g(t). \tag{4.16}$$

Now Lemmas 4.2, 4.3 and the equality (4.16) together imply

$$\begin{aligned}g'(t) = {}& i\,\mathrm{Tr}(R_tA_0)g(t) - it^2\mu_3^2\big\langle\mathcal{D}, R_t^2A_0\mathcal{D}\big\rangle g(t) - t\mu_3^2\big\langle\mathcal{D},\mathcal{D}\big\rangle g(t)\\ &- t(\mu_4 - \mu_3^2 - 1)\big\langle\mathcal{D},\mathcal{D}\big\rangle g(t) + \mathcal{N}(t).\end{aligned}$$

□

Introduce the function

$$\alpha(t) = \mathcal{J}(t) - \widetilde{\mathcal{J}}(t). \tag{4.17}$$

We have

$$\alpha(t) = \alpha_1(t) + \alpha_2(t) + \alpha_3(t),$$

where

$$\begin{aligned}\alpha_1(t) &= \mathrm{Tr}(R_tA) - \mathrm{Tr}(R_tA_0) - \mathrm{Tr}\,A,\\ \alpha_2(t) &= it(\mu_4 - 1)\,\mathrm{Tr}\{R_tAD\} - it(\mu_4 - 1)\big\langle\mathcal{D},\mathcal{D}\big\rangle\\ \alpha_3(t) &= -t^2\mu_3^2\big\langle\mathcal{D}, R_t^2A\mathcal{D}\big\rangle - 2t^2\mu_3^2\big\langle\mathcal{D}, R_tA\mathcal{D}^{(t)}\big\rangle + t^2\mu_3^2\big\langle\mathcal{D}, R_t^2A_0\mathcal{D}\big\rangle\\ &\qquad - it\mu_3^2\big\langle\mathcal{D}, (R_t - \mathrm{I})\mathcal{D}\big\rangle.\end{aligned}$$

A bound for $|\alpha(t)|$ is given in the following Lemma.

Lemma 4.4. *There exist an absolute constant C such that*

$$|\alpha(t)| \le C\mu_4 |t|^2 \big(\mathcal{L} + \mathcal{M}\big).$$

Proof of Lemma 4.4. We derive separate bounds for the functions $\alpha_j(t)$, $j = 1, 2, 3,$. Throughout the proof we shall use the identity

$$R_t - I = 2it R_t A_0 \tag{4.18}$$

which implies that

$$\alpha_1 = \mathrm{Tr}(R_t D) - \mathrm{Tr}(A) = \mathrm{Tr}((R_t - \mathrm{I})D) = 2it\, \mathrm{Tr}(R_t A_0 D).$$

Since $\mathrm{Tr}(A_0 D) = 0$, using (4.18) again we obtain that

$$\alpha_1(t) = -4t^2\, \mathrm{Tr}(R_t (A_0)^2 D). \tag{4.19}$$

The spectrum of the operator R_t is contained inside the unit circle, since A_0 is a symmetric operator. Therefore

$$|\alpha_1(t)| \le 4t^2\, \mathrm{Tr}(A_0^2 D') \le 4t^2 \mathcal{L}. \tag{4.20}$$

Here D' denotes the diagonal matrix such that $D'(j,j) = |D(j,j)|$ for all j. Using

$$\big\langle \mathcal{D}, \mathcal{D} \big\rangle = \mathrm{Tr}(D^2), \tag{4.21}$$

we can write

$$\alpha_2(t) = it(\mu_4 - 1)\big(\mathrm{Tr}(R_t A_0 D) + \mathrm{Tr}(R_t D^2) - \mathrm{Tr}(D^2)\big).$$

From (4.18) it follows that

$$\alpha_2(t) = (\mu_4 - 1)\Big(\frac{1}{2}\,\alpha_1(t) - 2t^2\, \mathrm{Tr}(R_t A_0 D^2)\Big). \tag{4.22}$$

Furthermore,

$$|\,\mathrm{Tr}(R_t A_0 D^2)| \le 2|t|\, \mathrm{Tr}({A_0}^2 D^2) \le 2|t| \mathcal{L}^2. \tag{4.23}$$

By (4.18),

$$\alpha_3(t) = -t^2 \mu_3^2 \Big(\big\langle \mathcal{D}, R_t^2 D \mathcal{D} \big\rangle + 2\big\langle \mathcal{D}, R_t^2 D \mathcal{D}^{(t)} \big\rangle + 2\big\langle \mathcal{D}, R_t A_0 \big(\mathcal{D} - \mathcal{D}^{(t)}\big)\big\rangle\Big). \tag{4.24}$$

Note that

$$\big|\big\langle \mathcal{D}, R_t^2 D \mathcal{D} \big\rangle\big| \le \|R_t^2 \mathcal{D}\|\, \|D\mathcal{D}\| \le \|\mathcal{D}\|\, \|D\mathcal{D}\| \le \mathcal{M}. \tag{4.25}$$

Using Cauchy's inequality, we see that

$$\max_{1 \le j \le n} |\mathcal{D}^{(t)}(j)| = \max_{1 \le j \le n} |R_t A(j,j)| \le \mathcal{L}. \tag{4.26}$$

Therefore,

$$|\langle \mathcal{D}, R_t^2 D\mathcal{D}^{(t)}\rangle| \le \|D\mathcal{D}^{(t)}\| \le \mathcal{L}. \tag{4.27}$$

Let $\mathcal{D}^{(1)}$, $\mathcal{D}^{(2)}$ denote the vectors with coordinates $\mathcal{D}^{(1)}(j) = R_t A_0^2(j,j)$ and $\mathcal{D}^{(2)}(j) = R_t A_0 D(j,j)$ for all j. Using (4.18), we obtain with this notation

$$\langle \mathcal{D}, R_t A_0\big(\mathcal{D} - \mathcal{D}^{(t)}\big)\rangle = 2it\langle \mathcal{D}, R_t A_0 \mathcal{D}^{(1)}\rangle + 2it\langle \mathcal{D}, R_t A_0 \mathcal{D}^{(2)}\rangle.$$

It is easy to see that

$$\mathcal{D}^{(2)} = D\mathcal{D}^{(0)},$$

where $\mathcal{D}^{(0)}$ denote the vector with coordinates $\mathcal{D}^{(0)}(j) = R_t A_0(j,j)$, and

$$\|\mathcal{D}^{(2)}\| \le \mathcal{L}$$

Hence

$$\big|\langle \mathcal{D}, R_t A_0 \mathcal{D}^{(2)}\rangle\big| \le \mathcal{L}. \tag{4.28}$$

Let $U = \{u_{ij}\}_{i,j=1}^n$ denote an orthogonal matrix and let D_0 denote a diagonal matrix, with $D_0(i,i) = \lambda_i$ such that

$$A_0 = U\, D_0\, U'$$

where U' denotes the transposed matrix. Thus

$$\mathcal{D}^{(1)}(i) = \sum_{j=1}^n |u_{ij}|^2 (1 - 2it\lambda_j)^{-1}\lambda_j^2$$

Thus

$$\sum_{i=1}^n |\mathcal{D}^{(1)}(i)| \le \sum_{i=1}^n \sum_{j=1}^n |u_{ij}|^2 \big(\lambda_j^{(0)}\big)^2 \le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = 1,$$

and

$$|\langle \mathcal{D}, R_t A_0 \mathcal{D}^{(1)}\rangle| \le \max_{1\le k\le n} \sum_{l=1}^n |\mathcal{D}(l)||R_t A_0(l,k)| \le \mathcal{L} \tag{4.29}$$

The relations (4.22)–(4.29) together imply that

$$|\alpha_3(t)| \le \mu_3^2 |t|^2 \big(\mathcal{L} + \mathcal{M}\big).$$

□

5. Proof of Theorem 1

Put $\beta(t) = f(t) - g(t)$. Theorems 3.1, 4.1 and Lemma 4.4 together imply that $\beta(t)$ satisfies the differential equation

$$\beta'(t) = \widetilde{\mathcal{J}}(t)\beta(t) + \overline{\delta}(t), \tag{5.1}$$

with $\beta(0) = 0$ and

$$\overline{\delta}(t) = \mathcal{N}(t) + \theta\alpha(t)g(t).$$

It is easy to see that $|g(t)| \le \varkappa(t)$. Hence, we can write $\overline{\delta}(t) = \mathcal{N}(t)$. The equation (5.1) has the unique solution

$$\beta(t) = \int_0^t \exp\Big\{\int_u^t \widetilde{\mathcal{J}}(z)dz\Big\}\,\overline{\delta}(u)\,du. \tag{5.2}$$

Let us derive Theorem 1 and 2 from (5.2).

Proof of Theorem 1. If λ_1 is small, we can use the following bound

$$\operatorname{Re}\,(i\widetilde{\mathcal{J}}(t)) \le -t\big((\mu_4 - 1)\rangle\mathcal{D},\mathcal{D}\rangle + 2\operatorname{Tr}\big(A_0^2\big)\big) + \mathbf{I}\,\{\,|t| \ge \gamma|\lambda_1|^{-1}\,\}, \tag{5.3}$$

for some small positive constant γ. The inequality (5.3) implies that for $|u| \le |t| \le \gamma|\lambda_1|^{-1}$

$$\Big|\exp\Big\{\int_u^t \widetilde{\mathcal{J}}(z)dz\Big\}\Big| \le \exp\Big\{-\gamma_1\big(t^2 - u^2\big)\Big\}. \tag{5.4}$$

From Theorem 3.1, 4.1 and Lemma 10 it follows that for $|t| \le \gamma\big(|\lambda_1|\log^+|\lambda_1|\big)^{-1}$

$$\mathcal{N}(t) \le C\big(\mathcal{L} + \mathcal{M}\big)|t|^2\exp\{-\gamma' t^2\}$$

Combining (5.1), (5.3) and (5.4), we obtain that for $|t| \le \gamma_1\big(|\lambda_1|\log^+|\lambda_1|\big)^{-1}$ and some small positive constant γ that

$$|\beta(t)| \le C\big(\mathcal{L} + \mathcal{M}\big)|t|\exp\{-\gamma t^2\}. \tag{5.5}$$

For $\gamma\big(|\lambda_1|\log^+|\lambda_1|\big)^{-1} \le |t| \le \gamma|\lambda_1|^{-1}$ the inequality (6.14) from GT implies that

$$|f(t)| \le \exp\{-\,\frac{1}{8}\,(t\wedge T_0)^2\mathcal{L}^{1/4}\} + \mathbf{I}\,\{t > \,\frac{1}{8}\,\beta_3^{-1}\mathcal{L}^{-1}\} + C\mathcal{L}^{-9/8} + \mathbf{I}\,\{t > |\lambda_1|^{-1}\}, \tag{5.6}$$

with $T_0 = \gamma_1\big(|\lambda_1|\big|\ln|\lambda_1|\big|\big)^{-1}$. An application of Esseen's inequality together with (5.5) and (5.6) completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Assuming the conditions of Theorem 2 we have, for $t \ge 0$,

$$\operatorname{Re}\,(i\widetilde{\mathcal{J}}(t)) \le -q\,\frac{\lambda_q^2 t}{1 + 4\lambda_q^2 t^2}\,. \tag{5.7}$$

The inequality (5.7) implies that

$$|\exp\Big\{\int_u^t \widetilde{\mathcal{J}}(z)dz\Big\}| \le \frac{(1+4u^2\lambda_q^2)^{q/8}}{(1+4t^2\lambda_q^2)^{q/8}},$$

for all u, t such that $|u| \le |t|$. Thus we get

$$|\beta(t)| \le \frac{1}{(1+4t^2\lambda_q^2)^{q/8}} \int_0^{|t|} |\mathcal{N}(u)|\big(1+4u^2\lambda_q^2\big)^{q/8} du. \tag{5.8}$$

By Theorem 2.4 for $|t| \le \gamma\mathcal{L}^{-1}/|\ln\mathcal{L}|$ we have

$$\mathcal{N}(t) \le C\big(\mathcal{L}+\mathcal{M}\big)\Big(\frac{t^2}{1+\alpha|t|^{q/4}} + \mathcal{L}^4\Big). \tag{5.9}$$

Substituting (5.8) in (5.9) and integrating over t, we obtain

$$|\beta(t)| \le C\Big(\mathcal{L}+\mathcal{M}\Big)\Big(\frac{|t|^3}{1+\alpha|t|^{q/4}} + |t|\mathcal{L}^4\Big) \tag{5.10}$$

for $|t| \le \gamma\mathcal{L}^{-1}/|\ln\mathcal{L}|$. Applying Esseen's inequality and the inequality (5.10) with $T = \gamma\mathcal{L}^{-1}/|\ln\mathcal{L}|$, the result of Theorem 2 follows for $q \le 12$. Theorem 2.1 implies that, for $|t| \le \gamma\mathcal{L}^{-1}$,

$$|f(t)|^4 \le \mathbf{E}\exp\big\{i(t\wedge T_0)\delta_1\sum_{l,j=1}^n a_{lj}\varepsilon_l(1-\varepsilon_j)Y_lY_j\big\} + C(\beta)\mathcal{L}^{\beta}. \tag{5.11}$$

This together with inequality (6.19) of GT yields for $\gamma\mathcal{L}^{-1}/\|\ln\mathcal{L}| \le |t| \le \gamma\mathcal{L}^{-1}$ and $q > 8$

$$|f(t)|^{\le} C\mathcal{L}_k^3 t^3\Big(\frac{1}{(1+2\delta t^2)^{q/4}} + \mathcal{L}^6\Big). \tag{5.12}$$

Combining Esseen's inequality and the inequalities (5.11) and (5.12), completes the proof of Theorem 2 for $q \ge 13$. □

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