INFINITE PROJECTION PROPERTIES

CHRISTIAN DELHOMMÉ

Abstract. A structure $\mathfrak{A}$ is κ-projective if the projections are the only retraction of $\mathfrak{A}^\kappa$ on its diagonal. We investigate projectivity with respect to infinite exponents.

1. Introduction

The definition of κ-projectivity (which makes sense in any category with products) extends a notion introduced by E. Corominas [Cor]. Given a structure $\mathfrak{A}$ and a cardinal κ, denoting by $\delta : \mathfrak{A} \to \mathfrak{A}^\kappa$ the diagonal embedding, say that $\mathfrak{A}$ is *κ-projective* if the projections are the only morphisms $f : \mathfrak{A}^\kappa \to \mathfrak{A}$ satisfying $f \circ \delta = \mathrm{id}_{\mathfrak{A}}$. Reflexive κ-projective structures enjoy the following irreducibility property (see [Cor]) : if such a structure is a retract of a product $\prod_{\alpha<\kappa} \mathfrak{A}_\alpha$ then it is a retract of at least one $\mathfrak{A}_\alpha$. (A structure $\mathfrak{A}$ on a set A is -weakly- reflexive if the constant selfmappings of A are all endomorphisms of $\mathfrak{A}$.)

A κ-projective structure is obviously κ'-projective for every $\kappa' < \kappa$. As for the opposite direction, it was shown in [P,R,S] and [D,McK,N,P] that, provided it is not "2-elementary-group-like", a reflexive 2-projective structure is *finitely-projective*, ie. it is n-projective for each integer n. The question of the existence of countable, ω-projective and reflexive multirelations was raised in [P,R,S], following the observation that a first order structure $\mathfrak{A}$ on a finite set A (with $|A| \geq 2$) can be κ-projective for no infinite κ (consider $\lim_{\mathcal{U}} : A^\kappa \to A$ for some non-principal ultrafilter $\mathcal{U}$ on κ) ; note that this argument also holds for any *topologically compact* structure $\mathfrak{A} = (A; \dots)$ (ie. such that there exists a Hausdorff-compact topology on A, for which the relations and functions of $\mathfrak{A}$ are respectively closed and countinuous with respect to the correponding product topologies). More generally, atomic compact structures can be κ-projective for no infinite κ either, as they are precisely the structures which are retracts of all their ultrapowers -with respect to the diagonal embedding- (a first order structure $\mathfrak{A}$ on a set A is atomic compact, if each of its atomic systems with parameters in A is solvable, provided it is finitely solvable). Conversely, given a finitely-projective structure $\mathfrak{A}$ and a cardinal κ, for each retraction f of $\mathfrak{A}^\kappa$ on its diagonal, there exists a ultrafilter $\mathcal{U}_f$ on κ, such that f and $\lim_{\mathcal{U}_f}$ coincide on the κ-tuples which take only finitely many distinct values [Weh]. Observe that such an f yields a retraction of the ultrapower $\mathfrak{A}^\kappa/\mathcal{U}_f$ on its diagonal precisely when it is quotientable by the ultrafilter $\mathcal{U}_f$, ie. when any two couples coinciding on an element of $\mathcal{U}_f$ get the same image by f ; in this case, either $\mathcal{U}_f$ is principal and f is a projection, or $\mathfrak{A}^\kappa/\mathcal{U}_f$ is ω^+-saturated and $\mathfrak{A}$ is compact with respect to countable atomic systems with parameters. But in general, f need not be quotientable by $\mathcal{U}_f$. However, for certain classes of structures, such retractions are always prequotientable by their associated ultrafilter, ie. f and $\lim_{\mathcal{U}_f}$ coincide on each κ-tuple which is constant on some element of $\mathcal{U}_f$. Under this assumption, we shall relate some deficits of atomic compactness to restrictions on the ultrafilters $\mathcal{U}_f$ (lemma 2), eventually getting general criteria of infinite projectivity, that we illustrate with examples mainly taken from binary relational structures.

2. Preliminaries

2.1. **Definitions and conventions.** Given a set A, a cardinal κ, a family $\mathcal{K} = (K_\xi : \xi < \lambda)$ of pairwise disjoint subsets of κ and $\mathbf{a} \in A^{\kappa \setminus \cup\{K_\xi : \xi<\lambda\}}$, for every λ-tuple $(a_\xi : \xi < \lambda) \in A^\lambda$, denote by $\mathbf{a} \otimes \bigotimes_{\xi<\lambda} a_\xi^{K_\xi}$ the the κ-tuple which takes the value a_ξ on each K_ξ and coincides with $\mathbf{a}$ elsewhere ; to each $f : A^\kappa \to A$, associate the selfmapping of A, $f^{\mathbf{a}}_{\mathcal{K}} : (a_\xi : \xi < \lambda) \mapsto f(\mathbf{a} \otimes \bigotimes_{\xi<\lambda} a_\xi^{K_\xi})$ (we shall freely elaborate these notations).

Given a first order structure $\mathfrak{A} = (A; \dots)$ over a first order language $\mathcal{L}$, let $\mathcal{L}_A$ denote the expansion of $\mathcal{L}$ by A as a set of new constant symbols. So, the *retractions* of $\mathfrak{A}^\kappa$ on its diagonal (which we shall also just call the *κ-retractions* of $\mathfrak{A}$) are precisely the $\mathcal{L}_A$-morphisms $f : \mathfrak{A}^\kappa \to \mathfrak{A}$, where each $a \in A$ is interpreted by a^κ in A^κ. Then $\mathfrak{A}$ is *κ-projective* if and only if every such retraction coincides with a *projection* $p_\alpha : (a_\beta : \beta < \kappa) \mapsto a_\alpha$.

Let us say that $\mathfrak{A}$ is *reflexive* if for each atomic $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ (with all free variables displayed) and every $a \in A$, $\mathfrak{A} \models \varphi(a, \dots, a)$. In particular, a relational structure $\mathfrak{A} = (A; \mathcal{R}_i : i \in I)$ is reflexive if and only if $(a, \dots, a) \in \mathcal{R}_i$ for each $i \in I$ and every $a \in A$ (each $\mathcal{R}_i \subset A^{n_i}$ interprets the corresponding n_i-ary relational symbol r_i).

We shall mainly be interested in κ-projectivity for reflexive structures (cf. the related notion of κ-strong rigidity [Rosb]). Some of them, such as the homogeneous structures, may have many endomorphisms (see 6.3).

2.2. **Examples.** Let us first mention that the conclusion of the theorem in [P,R,S] and [D,McK,N,P] applies to reflexive binary relations and metric spaces : 2-projective ones are indeed finitely-projective.

2.2.1. *Graphs.* Our *graphs* will be the binary relations $\mathfrak{G} = (A; \mathcal{R})$ which are reflexive and symmetric. One easily sees that a 2-projective graph must be connected and have no *dangling tree* (the dangling trees of $\mathfrak{G}$ are its induced subgraphs which contain no cycle and are linked to the rest of the graph by exactly one edge). Conversely, if such a graph contains no triangle, then it is 2-projective (cf. [Haz]).

2.2.2. *Metric spaces.* To each metric space $\mathfrak{X} = (A, d)$, one can associate a reflexive binary relational structure $\mathfrak{M}_\mathfrak{X} = (A; \mathcal{D}_r : r \in \mathbb{Q}_+)$ in such a way that the non-expansive mappings between metric spaces are exactly the homomorphisms between the corresponding relational structures : for each $r \in \mathbb{Q}_+$ and every a and b in A, $(a, b) \in \mathcal{D}_r$ if and only if $d(a, b) \leq r$ (note that the distance can be recovered ; indeed $d(a, b) = \sup\{r \in \mathbb{Q}_+ : (a, b) \in \mathcal{D}_r\}$.) Hence, a metric space is κ-projective (with respect to the non-expansive mappings) if and only if its associated relational structure is. In the same way, a connected graph $\mathfrak{G} = (A; \mathcal{R})$ is κ-projective if and only if the metric space $\mathfrak{X}_\mathfrak{G} = (A, d)$ is, with $d : A \times A \to \mathbb{N}$ denoting the "shortest path distance". Furthermore, $\mathfrak{X}_\mathfrak{G}$ embeds in the metric space obtained by "materializing" each edge of $\mathfrak{G}$ by a segment of length 1 ; it can be shown (cf. [Del2]) that this new metric space is κ-projective if and only if $\mathfrak{G}$ is. For example the circle $\mathfrak{C}$ of length 1 equipped with its geodetic distance (ie. $\mathbb{R}/\mathbb{Z}$) is

2-projective. This last fact can also be checked directly : Given any continuous $f : \mathfrak{C}^2 \to \mathfrak{C}$, the mappings $a \mapsto f(a,b)$ (resp. $b \mapsto f(a,b)$) are homotopic with one another, so their degrees do not depend on the fixed value : let us denote them respectivlely by d_0 and d_1. It can easily be checked that the sum $d_0 + d_1$ is equal the degree d_δ of the diagonal mapping $c \mapsto f(c,c)$. Now, when f is moreover non-expansive, d_0 and d_1 belong to $\{0,-1,1\}$ and $d_\delta = 1$; hence $d_0 = 1$ for example, and for each b, $a \mapsto f(a,b)$ is the identity mapping of $\mathfrak{C}$ (being non-expansive with degree equal to 1, it is a rotation ; but it has a fixed point -namely b-).

3. Atomic compactness

3.1. **Definitions.** Given an $\mathcal{L}$-structure $\mathfrak{A}$ on A, let X be a -possibly uncountably infinite- set of variables. An *atomic system* of $\mathfrak{A}$ with *unknowns* set X is any set $\Phi(X)$ of $\mathcal{L}_A$-atomic formulas with free variables in X. Given cardinals λ and μ, $\Phi(X)$ is a λ-*system*, resp. a $[\mu]$-*system*, resp. a $[\mu^-]$-*system*, if $|\Phi(X)| \leq \lambda$, resp. $|X| \leq \mu$, resp. $|X| < \mu$. A *solution* of $\Phi(X)$ is any family $(a_x : x \in X) \in A^X$ such that $\mathfrak{A} \models \Phi[a_x : x \in X]$. The system $\Phi(X)$ is *solvable* if it has a solution, it is *finitely solvable* if all its finite subsets have a solution. The structure $\mathfrak{A}$ is *atomic compact* if all its finitely solvable atomic systems are solvable, it is λ- (resp. $[\mu]$-, resp. $[\mu^-]$-) *atomic compact* if all its finitely solvable λ- (resp. $[\mu]$-, resp. $[\mu^-]$-) atomic systems are solvable. One shows [Myc,Ry-Na] that only "small" systems are needed to check atomic compactness : if $\mathfrak{A}$ is λ-atomic compact for some infinite λ greater or equal to the cardinality of $\mathcal{L}_A$, then it is atomic compact. Hence a countable structure over a finite or countable language is atomic compact if and only if it is ω-atomic compact.

Atomic compact structures can be characterized as those structures which are retract (with respect to the diagonal embedding) of all their ultrapowers [Weg], hence they are κ-projective for no infinite κ (provided $|A| \geq 2$). Recall that topologically compact structures are always atomic compact [Myc]. For example, the binary relational structure $\mathfrak{M}_{\mathfrak{X}}$ associated to a compact metric space $\mathfrak{X} = (A,d)$ is topologically compact, hence, such a metric space is never ω-projective (see 2.2.2).

3.2. **Examples.** — It can be checked that a graph with only finitely many vertices of infinite degree is always $[\omega^-]$-atomic compact ; it is even topologically compact when it has at most two such vertices [Del3]. We shall give later an example of an ω-projective graph with exactly three vertices of infinite degree (section 6.3).

— Given a graph $\mathfrak{G} = (A;\mathcal{R})$, if some set of vertices $B \subset A$ has no neighbour while each of its finite subset has one, then $\mathfrak{G}$ is not [1]-atomic compact. Indeed the hypothesis means exactly that the [1]-atomic system $\Phi(x) = \{r(x,a) : a \in B\}$ is finitely solvable but not solvable. For example, given an infinite cardinal μ, consider the graph $\mathfrak{H}_\mu = (M,\mathcal{H})$ on $M = (\mu \times \{0,1\}) \backslash \{(0,1)\}$, characterized by the fact that any two distinct vertices (α,i) and (β,j) with $\alpha \leq \beta$ are linked if and only if $i = 0$ and $j = 1$. Note that this graph is connected, has no dangling tree and contains no triangle, so according to 2.2.1, it is finitely-projective.

— Given a non-complete metric space $\mathfrak{X} = (A,d)$, the relational structure $\mathfrak{M}_{\mathfrak{X}}$ is not [1]-atomic compact : denoting by $\overline{\mathfrak{X}} = (\overline{A},\overline{d})$,the completion of $\mathfrak{X}$, consider, for some $c \in \overline{A} \backslash A$, the [1]-atomic system $\Phi(x) = \{\delta_r(x,a) : a \in A, \overline{d}(c,a) \leq r\}$ (each $\mathcal{D}_r$ interpreting the symbol δ_r).

4. Prequotientability

4.1. **Notations.** Given infinite cardinal μ and κ, let us recall the following characterization of the *μ-complete* ultrafilters on κ : A set $\mathcal{F}$ of subsets of κ is a μ-complete ultrafilter if and only if for each decomposition $\kappa = \cup\{K_\xi : \xi < \lambda\}$ with $\lambda < \mu$ and the K_ξ's pairwise disjoint, precisely one K_ξ is an element of $\mathcal{F}$.

Given a set A and a ultrafilter $\mathcal{U}$ on a cardinal κ, denote by $A^{\kappa,\mathcal{U}}$ the set of κ-tuples $(a_\alpha : \alpha < \kappa) \in A^\kappa$ such that, for some $a \in A$, $\{\alpha : a_\alpha = a\} \in \mathcal{U}$ (for such an $\mathbf{a}$, this -unique- a is denoted by $\lim_{\mathcal{U}} \mathbf{a}$).

Given a set A and cardinals κ and μ, let us denote by $A^{(\kappa)_\mu}$ the set of κ-tuples $(a_\alpha : \alpha < \kappa) \in A^\kappa$ such that $|\{a_\alpha : \alpha < \kappa\}| < \mu$. So, if $\mathcal{U}$ is a μ-complete ultrafilter on κ, then $A^{(\kappa)_\mu} \subset A^{\kappa,\mathcal{U}}$. The set $A^{(\kappa)_\omega}$ will simply be denoted by $A^{(\kappa)}$.

Lemma 1. *(see* [Weh]*) Consider a structure $\mathfrak{A}$ on a set A and an infinite cardinal μ such that $\mathfrak{A}$ is λ-projective for every cardinal $\lambda < \mu$. Given a cardinal κ, for every κ-retraction f of $\mathfrak{A}$, there exists a μ-complete ultrafilter $\mathcal{U}_f$ on κ, such that $f(\mathbf{a}) = \lim_{\mathcal{U}_f} \mathbf{a}$, for every $\mathbf{a} \in A^{(\kappa)_\mu}$.*

Proof. For every cardinal $\lambda < \mu$, to each family $\mathcal{K} = (K_\xi : \xi < \lambda)$ of pairwise disjoint subsets of κ such that $\cup\{K_\xi : \xi < \lambda\} = \kappa$, associate the λ-retraction $f_{\mathcal{K}} : (a_\xi : \xi < \lambda) \mapsto f(\bigotimes_{\xi<\lambda} a_\xi^{K_\xi})$. Now observe that, (provided $|A| \geq 2$), for every $\xi < \lambda$, $f_{\mathcal{K}}$ is equal to the projection p_ξ if and only if $f_{(K_\xi, \complement K_\xi)} : (a,b) \mapsto f(a^{K_\xi} \otimes b^{\complement K_\xi})$ is the first projection $(a,b) \mapsto a$. Then consider $\mathcal{U}_f$ to be the set of those $K \subset \kappa$ for which $f_{(K,\complement K)}$ is the first projection. $\square$

Corollary 1. *Let $\mathfrak{A}$ be an ω-projective structure on a set A. Given a cardinal κ, if $\mathfrak{A}$ is not κ-projective, then there is an ω^+-complete ultrafilter on κ.* $\square$

4.2. **Example.** We know that a compact metric space is never ω-projective. However, consider $\mathfrak{X} = (A, d)$ a finitely-projective one (such as the circle $\mathfrak{C}$).

Let us check that $\mathfrak{X}$ is infinitely-projective in the following weak sense : For every cardinal κ, the κ-retractions of $\mathfrak{X}$ are exactly the mappings $\lim_{\mathcal{U}}$, with $\mathcal{U}$ any ultrafilter on κ. Indeed, any κ-retraction f of $\mathfrak{X}$ coincides with the mapping $\lim_{\mathcal{U}_f}$ on $A^{(\kappa)}$; now observe that f and $\lim_{\mathcal{U}_f}$ are both continuous (indeed they are both non-expansive) and that $A^{(\kappa)}$ is dense in A^κ ($\mathfrak{X}$ is precompact) -recall that the distance of $\mathfrak{X}^\kappa$ is the so-called "sup-distance"-.

From this, we deduce that any dense strict subspace D of $\mathfrak{X}$ is ω-projective. Indeed any retraction $f : D^\kappa \to D$ is a uniformally continuous mapping from D^κ -which is dense in $\mathfrak{X}^\kappa$- into the complete metric space $\mathfrak{X}$; so it has an expansion $\overline{f} : \mathfrak{X}^\kappa \to \mathfrak{X}$, which is easily seen to be a κ-retraction of $\mathfrak{X}$. Now $\overline{f}$ is equal to $\lim_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on κ ; but if $\mathcal{U}$ was not ω^+-complete, then $f(D^\kappa)$ would have to be equal to A : there would exist a partition $(K_n : n < \omega)$ of κ with no K_n in $\mathcal{U}$, then for each $a \in A$ and any sequence $(a_n : n < \omega)$ of elements of D converging to a, $f(\bigotimes_{n<\omega} a_n^{K_n}) = \lim_{\mathcal{U}} \bigotimes_{n<\omega} a_n^{K_n} = a$.

Remark (Pouzet) that, if $\mathfrak{X}$ is κ-projective, then the quaternary relation $\mathfrak{Q} = (A; q)$ given by $q(x,y,x',y') \leftrightarrow ([d(x,y) < d(x',y')] \vee [x = y])$ is κ-projective as well. Indeed, every κ-retraction f of $\mathfrak{Q}$ is also a κ-retraction of $\mathfrak{X}$: Assume that $d(\mathbf{a},\mathbf{b}) < d(f(\mathbf{a}), f(\mathbf{b}))$, for some $\mathbf{a}$ and $\mathbf{b}$ in A^κ. Then in particular, $\mathfrak{Q} \models q(\mathbf{a}, \mathbf{b}, [f(\mathbf{a})]^\kappa, [f(\mathbf{b})]^\kappa)$; hence, as f is a retraction, $\mathfrak{Q} \models q(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{a}), f(\mathbf{b}))$, which is contradictory with the fact that, by assmption, $f(\mathbf{a})$ and $f(\mathbf{b})$ cannot be equal.

4.3. **Definitions.** Let us say that a finitely-projective structure $\mathfrak{A} = (A; \dots)$ is *prequotientable* if for every cardinal κ, each κ-retraction f is prequotientable by its associated ultrafilter $\mathcal{U}_f$, ie. $f(\mathbf{a}) = \lim_{\mathcal{U}_f} \mathbf{a}$, for every $\mathbf{a} \in A^{\kappa, \mathcal{U}_f}$.

So a prequotientable finitely-projective structure is κ-projective if and only if for every κ-retraction f, the ultrafilter $\mathcal{U}_f$ is principal.

4.4. **Example.** One easily sees, arguing from the non-connectedness of $\mathfrak{G}^\kappa$ for any infinite κ, that a graph of infinite diameter is never prequotientable. We shall see (corollary 5) that this is the only restriction.

Now given an infinite cardinal κ, consider the graph $\mathfrak{D}_\kappa$ on $\kappa \times \{0, 1\}$ such that any two distinct vertices (α, i) and (β, j) are linked if and only if $i \neq j$ and $\alpha \neq \beta$. (Note that $\mathfrak{D}_\kappa$ may be seen as the comparability graph of $\overrightarrow{\mathfrak{D}_\kappa}$, the set -ordered by inclusion- of singletons and co-singletons of κ.) The graph $\mathfrak{D}_\kappa$ is finitely-projective (see 2.2.1) and as it has a finite diameter, it is prequotientable (corollary 5). Let us check that it is κ-projective, ie. that, given any κ-retraction f, the ultrafilter $\mathcal{U}_f$ is principal : Consider the κ-tuple $\mathbf{a} = \big((\alpha, 1) : \alpha < \kappa\big)$, and for each $\alpha < \kappa$, the κ-tuple $\mathbf{a}_{[\alpha]} = (\alpha, 1)^{\{\alpha\}} \otimes (\alpha, 0)^{\complement\{\alpha\}}$; note that, as each $\mathbf{a}_{[\alpha]}$ is linked to $\mathbf{a}$ (in $\mathfrak{D}_\kappa^\kappa$), $f(\mathbf{a})$ must be linked to every $f(\mathbf{a}_{[\alpha]})$. If $\mathcal{U}_f$ were not principal, then for each α, $f(\mathbf{a}_{[\alpha]})$ would be equal to $(\alpha, 0)$; but no vertex of $\mathfrak{D}_\kappa$ is linked to all $(\alpha, 0)$'s.

Let us remark that the very same arguments establish that the poset $\overrightarrow{\mathfrak{D}_\kappa}$ is κ-projective as well (cf. 6.1.4).

5. Admissibility

5.1. **Definitions.**

5.1.1. *Admissible formulas.* Given an $\mathcal{L}$-structure $\mathfrak{A}$ on a set A, say that an $\mathcal{L}_A$-formula $\varphi(x_1, \dots, x_p, c_1, \dots, c_q)$ -with all parameters (ie. constant symbols from A) and all free variables displayed- is *admissible*, if for every $(a_1, \dots, a_p) \in A^p$, there exists $(b_1, \dots, b_q) \in A^q$ such that $\mathfrak{A} \models \varphi(a_1, \dots, a_p, b_1, \dots, b_q)$.

5.1.2. *Examples.* — For every $\mathcal{L}$-term $t(x_1, \dots, x_p)$ and every $c \in A$, the formula $t(x_1, \dots, x_p) = c$ is admissible.

— When $\mathfrak{A}$ is reflexive, every atomic $\mathcal{L}_A$-formula $\varphi(x, c_1, \dots, c_q)$ is admissible. In particular, if $\mathfrak{A} = (\mathcal{R}_i : i \in I)$ is a reflexive binary relational structure, then the only formulas which may not be admissible are of the form $r_i(x_1, x_2)$ or $x_1 = x_2$, for two distinct variables x_1 and x_2.

5.1.3. *Admissible structures.* For every atomic system Φ of $\mathfrak{A}$, we shall denote by Φ' and Φ'', respectively the sets of admissibles and non-admissible elements of Φ.

Given two infinite cardinals λ and μ, the atomic system Φ is said to be (λ, μ)-*admissible* if $|\Phi'| = \mu$ and for every subset Ψ of Φ'' with $|\Psi| < \lambda$, the system $\Phi' \cup \Psi$ is solvable. The structure $\mathfrak{A}$ is (λ, μ)-*admissible* if all its (λ, μ)-admissible systems are solvable, it is said to be μ-*admissible* when it is (ω, μ)-admissible.

A (λ, μ)-*regular family for* a ultrafilter $\mathcal{U}$ is any family $\{U_\xi : \xi < \mu\} \subset \mathcal{U}$ such that for evey subset M of μ with $|M| = \lambda$, the intersection $\cap\{U_\xi : \xi \in M\}$ is empty. The ultrafilter $\mathcal{U}$ is said to be (λ, μ)-*regular* if it has a (λ, μ)-regular family, it is said to be μ-*regular* when it is (ω, μ)-regular (see [C,N]).

5.1.4. *Example.* If a reflexive binary multirelation $\mathfrak{A} = (A; \mathcal{R}_1, \ldots, \mathcal{R}_m)$ is ω-admissible, then it is $[\omega^-]$-atomic compact as well : each $[\omega^-]$-atomic system of $\mathfrak{A}$ is an ω-system with only finitely many non-admissible elements.

5.2. We now come to the key lemma.

Lemma 2. *Consider a finitely-projective and prequotientable structure $\mathfrak{A}$. Let λ and μ be two infinite cardinals. If $\mathfrak{A}$ is not (λ, μ)-admissible, then for every cardinal κ and each κ-retraction f, the ultrafilter $\mathcal{U}_f$ is not (λ, μ)-regular.*

Proof. Assuming that for some κ-retraction f, the ultrafilter $\mathcal{U}_f$ admits a (λ, μ)-regular family $\{U_\xi : \xi < \mu\}$, let us check that every (λ, μ)-admissible system $\Phi(X)$ is solvable.

Given an indexing $\{\psi_\xi : \xi < \mu\}$ of Φ', for each $\alpha < \kappa$, consider $\mathbf{a}^{(\alpha)} = (a_x^{(\alpha)} : x \in X)$ a solution of $\Phi'' \cup \{\psi_\xi : \xi \in U^{(\alpha)}\}$, with $U^{(\alpha)}$ denoting the set $\xi < \mu : \alpha \in U_\xi\}$ (by assumption, $|U^{(\alpha)}| < \lambda$). For each $x \in X$, denote by $\mathbf{a}_x$ the κ-tuple $(a_x^{(\alpha)} : \alpha < \kappa)$. We claim that the family $\big(f(\mathbf{a}_x) : x \in X\big)$ is a solution of Φ. Indeed, given any formula $\varphi(x_1, \ldots, x_p, c_1, \ldots, c_q)$ in Φ :

— Either this formula lies in Φ'', then obviously $\mathfrak{A}^\kappa \models \varphi(\mathbf{a}_{x_1}, \ldots, \mathbf{a}_{x_p}, c_1^\kappa, \ldots, c_q^\kappa)$, hence $\mathfrak{A} \models \varphi\big(f(\mathbf{a}_{x_1}), \ldots, f(\mathbf{a}_{x_p}), c_1, \ldots, c_q\big)$.

— Or this formula lies in Φ', then it is equal to the admissible formula ψ_ξ for some $\xi < \mu$; in particular, for each $\alpha < \kappa$, there is $(b_1^{(\alpha)}, \ldots, b_q^{(\alpha)}) \in A^q$ such that $\mathfrak{A} \models \varphi(a_{x_1}^{(\alpha)}, \ldots, a_{x_p}^{(\alpha)}, b_1^{(\alpha)}, \ldots, b_q^{(\alpha)})$. So $\mathfrak{A}^\kappa \models \varphi(\mathbf{a}_{x_1}, \ldots, \mathbf{a}_{x_p}, \mathbf{c}_1, \ldots, \mathbf{c}_q)$, with each $\mathbf{c}_k$ equal to $c_k^{U_\xi} \otimes (b_k^{(\alpha)} : \alpha \notin U_\xi)$. Hence $\mathfrak{A} \models \varphi\big(f(\mathbf{a}_{x_1}), \ldots, f(\mathbf{a}_{x_p}), f(\mathbf{c}_1), \ldots, f(\mathbf{c}_q)\big)$. Now, as $\mathfrak{A}$ is assumed to be prequotientable, $f(\mathbf{c}_k) = c_k$ for each k. □

As non-principal ultrafilters on ω are ω-regular -consider the family $\big\{\{p : p \geq n\} : n < \omega\big\}$-, one gets :

Corollary 2. *Let $\mathfrak{A}$ be a finitely-projective and prequotientable structure. If $\mathfrak{A}$ is not ω-admissible, then it is ω-projective. In particular it is ω-projective in the following two cases.*

— *It is a reflexive binary relation which is not $[1]$-atomic compact (for example, it may "be" a non-complete metric space) : evry atomic formula of $\mathfrak{A}$ is admissible.*

— *It is a countable reflexive binary multirelation which is not $[\omega^-]$-atomic compact : see 5.1.4.* □

5.2.1. *Remark.* Given a set M and a set $\mathcal{I}$ of subsets of M, one can define a ultrafilter being $(\mathcal{I}, M)$-regular, a structure being $(\mathcal{I}, M)$-admissible, and then generalize the above lemma in the obvious manner. For instance, one may consider the set $\mathcal{B}_\mu$ of bounded subsets of some infinite cardinal μ. In fact, it is easily seen that $(\mathcal{B}_\mu, \mu)$-regularity is equivalent to $(\mathrm{cf}\,\mu, \mathrm{cf}\,\mu)$-regularity as well as to $\mathrm{cf}\,\mu$-descending incompleteness ($\mathrm{cf}\,\mu$ denotes the cofinality of μ) ; it is also equivalent to the following property of μ-subordination : Given a topological space $(M, \mathcal{T})$ and a ultrafilter $\mathcal{U}$ on some set K, let us say that $\mathcal{U}$ is *subordinated to* $\mathcal{T}$, if for every mapping $f : K \to M$, the ultrafilter image $f(\mathcal{U})$ converges in M with respect ot $\mathcal{T}$. Given a cardinal μ, the ultrafilter $\mathcal{U}$ is *μ-subordinated* if it is subordinated to the interval topology of μ, ie. if each $f : K \to \mu$ is bounded on some element of $\mathcal{U}$.

5.2.2. *Example.* Let μ be an infinite cardinal. Recall that the graph $\mathfrak{H}_\mu = (M, \mathcal{H})$ of example 3.2 is finitely-projective ; moreover it has a finite diameter, hence it is prequotientable (corollary 5). Given $(\alpha_\zeta : \zeta < \operatorname{cf}\mu)$ a cofinal sequence in μ, consider the [1]-atomic system :

$$\Phi(x) = \{r\big((\alpha_\zeta, 0), x\big) : \zeta < \operatorname{cf}\mu\}$$

This system is obviously $(\operatorname{cf}\mu, \operatorname{cf}\mu)$-admissible but not solvable. Hence, according to the above lemma, for every cardinal κ and every κ-retraction f of $\mathfrak{H}_\mu$, the ultrafilter $\mathcal{U}_f$ is μ-subordinated.

Conversely, the sets $\mu \times \{0\}$ and $(\mu\backslash\{0\}) \times \{1\}$ inherit the interval topology of μ ; consider on M their topology sum $\mathcal{T}$. Observe that $\mathcal{H}$ is closed in M^2 and that the μ-subordinated ultrafilters are precisely the ultrafilters which are subordinated to $\mathcal{T}$. Now, for each such ultrafilter $\mathcal{U}$ on κ, the mapping $\lim_\mathcal{U} : M^\kappa \to M$ is indeed a κ-retraction of $\mathfrak{H}_\mu$.

So, the ultrafilters associated to the κ-retraction of $\mathfrak{H}_\mu$ are exactly the μ-subordonated ultrafilters on κ. In particular, $\mathfrak{H}_\mu$ is κ-projective if and only if every μ-subordinated ultrafilter on κ is principal, which occurs if and only if $\operatorname{cf}\mu = \omega$ and there is no ω^+-complete ultrafilter on κ. Note also that the graph $\mathfrak{H}_{\aleph_1}$ is not ω-projective, but the ultrafilters associated to its $\aleph_m$-retractions (m an integer) are precisely the ultrafilters on $\aleph_m$ which contain a countable element, as each $\aleph_1$-subordinated ultrafilter is $\aleph_p$-subordinated for every integer p (see [K,P]).

6. Binary relations

6.1. **Finite projectivity.** We shall consider diameters and connectedness for reflexive binary relations in the weak sense, ie. with respect to non-oriented paths.

Let us recall that every 2-projective reflexive binary relation is finitely-projective.

6.1.1. *Notations.* Let $\mathfrak{R} = (A; \mathcal{R})$ be a reflexive binary relation. For any two endomorphisms of $\mathfrak{R}$, φ and ψ, and any $F \subset A$, consider $h_{\varphi,\psi,F} : A \times A \to A$ given by :

$$h_{\varphi,\psi,F}(a,b) = \begin{cases} \varphi(b) & \text{if } a \in F \\ \psi(b) & \text{if } a \in \complement F \end{cases}$$

The following is straigthforward :

Claim 1. *The mapping $h_{\varphi,\psi,F}$ is a 2-retraction of $\mathfrak{R}$ and is distinct from any projection, when the following four conditions are met :*

— $\big(\varphi(a), \psi(b)\big) \in \mathcal{R}$*, for every* $(a,b) \in \mathcal{R}$ *(in that case we shall write* $\varphi \sim \psi$*).*
— $(F \times \complement F) \cap \mathcal{R}$ *is non-empty.*
— $\varphi_{/F} \neq \mathrm{id}_F$ *and* $\psi_{/\complement F} \neq \mathrm{id}_{\complement F}$
— $\varphi \neq \mathrm{id}_A$. □

Note that if $\mathfrak{R}$ is not connected, then by considering $h_{\rho,\mathrm{id}_A,C}$ for any retraction $\rho : A \to C$ on a connected component of $\mathfrak{R}$, one gets the non-2-projectivity of $\mathfrak{R}$.

6.1.2. *Graphs.* Given a graph $\mathfrak{G} = (A; \mathcal{R})$, for every vertex $a \in A$, let $\mathcal{R}_a$ denote the neighbourhood of a, ie. the set $\{b \in A : (a,b) \in \mathcal{R}\}$.

Corollary 3. *If a graph $\mathfrak{G} = (A; \mathcal{R})$ contains two distinct vertices a and b such that $\mathcal{R}_a \subset \mathcal{R}_b$, then it is not 2-projective.*

Proof. Let $\rho_{a\to b}$ denote the selfmapping of A which maps a to b and fixes every other point. Observe that $\mathcal{R}_a \subset \mathcal{R}_b$ means precisely that $\rho_{a\to b}$ is an endomorphism satisfying $\rho_{a\to b} \sim \mathrm{id}_A$. □

Corollary 4. *If a graph $\mathfrak{G} = (A; \mathcal{R})$ has an endomorphism φ satisfying $\mathrm{id}_A \sim \varphi$, then it is not 2-projective.*

Proof. Observe that if $\mathrm{id}_A \sim \varphi$, then $\mathcal{R}_a \subset \mathcal{R}_{\varphi(a)}$ for every vertex $a \in A$. □

6.1.3. *Posets.* Recall that a *poset* $\mathfrak{P} = (A; \leq)$ is any transitive, symmetric and reflexive binary relation. A *final segment* of $\mathfrak{P}$ is any set $F \subset A$ satisfying : for any vertices a and b, if $a \leq b$ and $a \in F$, then $b \in F$ as well. The poset $\mathfrak{P}$ is *upward directed* if any two of its vertices always have a common upper bound.

Proposition 1. *Consider a poset $\mathfrak{P} = (A; \leq)$ admitting an endomorphism φ, distinct from id_A and satisfying $\varphi \geq \mathrm{id}_A$. If one of the following holds, then $\mathfrak{P}$ is not 2-projective.*
— a/ There exists a non-empty final segment F such that $\varphi_{/F} \neq \mathrm{id}_F$. (Note that this is the case each time $\mathfrak{P}$ has maximal elements, and in particular, when A is finite.)
— b/ $\mathfrak{P}$ is not upward directed.

Of course, the "dual" statement holds as well.

Proof.
— a/ Consider $h_{\varphi,\mathrm{id}_A,F}$.
— b/ We can assume additionally, that case a/ does not hold. Then consider $f : A \times A \to A$ given by :

$$f(a,b) = \begin{cases} \varphi(a) & \text{if } a \text{ and } b \text{ have no common upper bound} \\ a & \text{if } a \text{ and } b \text{ have a common upper bound} \end{cases}$$

It is obviously a 2-retraction of $\mathfrak{P}$. Let us check that it is not a projection : consider a and b with no common upper bound ; as φ does not fix all upper bounds of a, there exists $c \geq a$ such that $\varphi(c) \neq c$. Clearly, b and c have no common upper bound either ; hence $f(c,b) = \varphi(c)$. But $\varphi(c) \neq c$ and $\varphi(c) \neq b$ (as $\varphi(c) \geq c \geq a$ while $b \not\geq a$). □

6.1.4. *Remark.* One easily checks (cf. 2.2.1), that a reflexive binary relation $\mathfrak{R}$ with no dangling -weak- tree and no transitivity (ie. not containing three distinct vertices a, b and c such that $\mathfrak{R} \models r(a,b) \wedge r(b,c) \wedge r(a,c)$) has no endomorphism φ satisfying $\varphi \sim \mathrm{id}_A$ or $\mathrm{id}_A \sim \varphi$. Furthermore, it can be shown that such a relation is 2-projective if and only if it is connected (cf. [Del2]).

6.2. **Infinite-projectivity.** A binary relation with an infinite diameter is κ-projective for no infinite κ (cf. 4.4). Conversely :

Lemma 3. *Consider a reflexive binary relation $\mathfrak{R} = (A; \mathcal{R})$, a cardinal κ and a κ-retraction f of $\mathfrak{R}$. Assume that $\mathfrak{R}$ is finitely-projective and that its diameter is finite, but that f is not proquotientable by the ultrafilter $\mathcal{U}_f$. Then some endomorphism φ of $\mathfrak{R}$ distinct from id_A satisfies $\varphi \sim \mathrm{id}_A$ or $\mathrm{id}_A \sim \varphi$.*

Proof. By assumption, there exists $U \in \mathcal{U}_f$ and $\mathbf{a} \in A^{\complement U}$ such that $f(\mathbf{a} \otimes a^U) \neq a$ for some $a \in A$, ie. such that $f^{\mathbf{a}}_{(U)} \neq \mathrm{id}_A$. Besides, $f^{\mathbf{b}}_{(U)} = \mathrm{id}_A$ for every constant $\mathbf{b} \in A^{\complement U}$ (ie. of the form $b^{\complement U}$ for some $b \in A$). But as $\mathfrak{R}$ (is reflexive and) has a finite diameter, $\mathfrak{R}^{\complement U}$ is connected. Hence there must exist $\mathbf{a}'$ and $\mathbf{b}'$ in $A^{\complement U}$ such that $f^{\mathbf{a}'}_{(U)} \neq \mathrm{id}_A$, $f^{\mathbf{b}'}_{(U)} = \mathrm{id}_A$ and $\mathfrak{R}^{\complement U} \models r(\mathbf{a}',\mathbf{b}') \vee r(\mathbf{b}',\mathbf{a}')$. Now observe that $f^{\mathbf{x}}_{(U)} \sim f^{\mathbf{y}}_{(U)}$ for any $\mathbf{x}$ and $\mathbf{y}$ such that $\mathfrak{R}^{\complement U} \models r(\mathbf{x},\mathbf{y})$. □

Remark : One could state a version of this lemma for binary relational structures in general, in order to handle the case of metric spaces (cf. [Del2]).

Corollary 5. *Consider a reflexive binary relation $\mathfrak{R} = (A; \mathcal{R})$ which is finitely-projective. In each of the folowing cases, $\mathfrak{R}$ is prequotientable :*

— It is a graph of finite diameter (see corolary 4).

— It has a finite diameter and has no transitivity (see 6.1.4).

— It is a poset of finite diameter and with no endomorphism comparable to id_A *(and distinct from* id_A*). In particular,*

— It is a poset of finite diameter which is not upward directed or has maximal elements, and which is not downward directed or has minimal elements (see proposition 1). □

6.2.1. *Remark.* The above condition about the non-existence of any endomorphism comparable to the identity mapping is in fact necessary for a finitely-projective poset to be prequotientable : Given a poset $\mathfrak{P} = (A; \leq)$ and a cardinal κ, denote by $\downarrow \Delta_{A^\kappa}$ the set of those κ-uples $(a_\alpha : \alpha < \kappa)$ for which the set $\{a_\alpha : \alpha < \kappa\}$ has an upper bound. Consider an endomorphism $\varphi \geq \mathrm{id}_A$, and define a κ-retraction f by :

$$f(a_\alpha : \alpha < \kappa) = \begin{cases} \varphi(a_0) & \text{if } (a_\alpha : \alpha < \kappa) \notin \downarrow \Delta_{A^\kappa} \\ a_0 & \text{if } (a_\alpha : \alpha < \kappa) \in \downarrow \Delta_{A^\kappa} \end{cases}$$

It is not a projection if and only if some $(a_\alpha : \alpha < \kappa) \notin \downarrow \Delta_{A^\kappa}$ satisfies $\varphi(a_0) \neq a_0$. This occurs precislely when $\varphi \neq \mathrm{id}_A$ and $\mathfrak{P}$ is not κ-upward directed (ie. some subset of A with cardinality less or equal to κ has no upper bound) -cf proposition 1 b/-. Now, note that when $\mathfrak{P}$ is finitely-projective, on the one hand the ultrafilter $\mathcal{U}_f$ is principal, and on the other hand $\mathfrak{P}$ is κ-upward directed for no $\kappa \geq |A|$, as it must have no greater element (cf. proposition 1 a/).

6.2.2. From corollary 2, we deduce that, if $\mathfrak{R} = (A; \mathcal{R})$ reflexive binary relation which is finitely-projective and satisfies one of the properties of corollary 5, then it is ω-projective, provided it is not ω-admissible (which will be the case whenever A is countable and $\mathfrak{A}$ is not $[\omega^-]$-atomic compact (cf. 5.1.4).

6.3. **Examples.** see [Del2]

— The graphs $\mathfrak{D}_\omega$ and $\mathfrak{H}_\omega$ (see 4.4 and 3.2) are the comparability graphs of two-levels posets $\overrightarrow{\mathfrak{D}_\omega}$ and $\overrightarrow{\mathfrak{H}_\omega}$. The ω-projectivity of $\mathfrak{D}_\omega$, $\mathfrak{H}_\omega$, $\overrightarrow{\mathfrak{D}_\omega}$ and $\overrightarrow{\mathfrak{H}_\omega}$ immediately follows from 6.2.2 (see also 6.1.4).

— Let us now give an example of a countable graph which is ω-projective and $[\omega^-]$-atomic compact.

Consider the graph $\mathfrak{G}$ with vertex set the disjoint union of $\mathbb{N}$ and $\mathbb{Z}/3\mathbb{Z}$ (respectively the set of non-negative integers and the set of integers modulo 3), and for which any two distinct vertices are linked if and only if one is an integer n and the other is either its successor $n+1$ in $\mathbb{N}$ or its class $\overline{n}$ in $\mathbb{Z}/3\mathbb{Z}$ (cf. [Del3]). This graph is finitely-projective (see 2.2.1) ; as its diameter is 3, it is prequotientable (see corollary 5) ; moreover, it is not ω-admissible (consider the ω-system $\Phi(x_k : k < \omega) = \{x_k \sim x_{k+1} : k < \omega\} \cup \{x_k \sim -\overline{k} : k < \omega\}$) ; hence it is ω-projective (see corollary 2). But it is also $[\omega^-]$-atomic compact, as it has only three vertices of infinite degree (cf. 3.2).

— Consider the graph $\mathfrak{L}$ obtained from the shift-graph of some countable chain $\mathfrak{P}$, after the deletion of the isolated or dangling vertices (in fact, at most 3 vertices are concerned). This graph is finitely-projective and prequotientable (see 2.2.1 and corollary 5). Moreover, one shows that it is always $[\omega^-]$-atomic compact (see [Del2]). Furthermore, it is ω-admissible if and only if the chain $\mathfrak{P}$ is isomorphic to $2+\omega^*+\omega+2$, to $\omega+2$ or to $2+\omega^*$ (in which cases $\mathfrak{L}$ is in fact topologically compact). So, except in these three cases, the graph $\mathfrak{L}$ is ω-projective.

— The countable homogeneous graphs (see [L,W]) are finitely-projective, except the disjoint-sums of complete graphs (which trivially are not). Moreover, they are ω-projective, except the complete-sums of cliques of a given finite size (which are topologically compact).

Finally, let us mention that the countable homogeneous structures (see [Fra]) $\mathfrak{G}=(A;\mathcal{R})$ and $\mathfrak{M}_{(\delta)}=(A,d)$ which are universal with respect to all finite posets, resp. to all finite metrics spaces with rational (or integer) distances bounded above by δ ($\delta\in]0,\infty]$) are finitely-projective. Moreover, the metric space $\mathfrak{M}_{(\delta)}$ -as well as its completion $\overline{\mathfrak{M}_{(\delta)}}$- is ω-projective if and only if $\delta<\infty$; as for the poset $\mathfrak{P}$, it is neither prequotientable nor ω-projective (it is easily seen to admit an endomorphism $\varphi>\mathrm{id}_A$) -see 6.2.1-, nor [1]-atomic compact (it has no upper bound).

6.4. **Questions.** Is every countable ω-admissible graph atomic compact ? In particular, is every countable, finitely-projective and non-atomic compact graph of finite diameter ω-projective ? In case of a positive answer, the above three conditions would in fact be equivalent, for a countable graph, to being ω-projective. More generally, is every non-ω-atomic compact, finitely-projective and prequotientable structure ω-projective ?

References

[Ab] H. Abels *The projection property for buildings.* Manuscrit, 1991.

[Chi] G. Chichilnisky. *Social Choice and the Topology of Preferences.* Advances in Mathematics 37 (1980), pp. 165-176.

[C,N] W.W. Comfort, S. Negrepontis. *The Theory of Ultrafilters.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 211, Springer Verlag 1974.

[Cor] E. Corominas. *Sur les ensembles ordonnés projectifs et la propriété du point fixe.* C.R. Acad. Sci. Paris, t. 311, série I (1990) 199-204.

[D,McK,N,P] B.A. Davey, J.B. Nation, R.N. Mac Kenzie, P.P. Palfy. *Braids and their monotone clones.* Algerbra Universalis, 32 (1994), pp 153-176.

[Del1] C. Delhommé. *Propriétés de projection des graphes sans triangle ni carré.* Europ. J. Combinatorics (1996) 17, pp. 15-22.

[Del2] C. Delhommé. *Propriétés de projection.* Thèse, Université Claude Bernard, Lyon I, 1995.

[Del3] C. Delhommé. *Atomic compactness and reflexive graphs.*

[Fra] R. Fraissé. *Theory of Relations.* Studies in Logic and the Foundations of Mathematics, vol. 118, 1986, North Holland.

[Haz] S. Hazan. *On triangle-free projective graphs.* Algerbra Universalis, 35 (1996), pp 185-196.

[J,M,P] M. Jawhari, D. Misane, M. Pouzet. *Retracts : graphs and ordered sets from the metric point of view.* Contemporary Mathematics, vol. 57, 1986, pp. 175-224.

[J,P,R] M. Jawhari, M. Pouzet, I. Rival. *A classification of reflexive graphs, the use of holes.* Canadian Journal of Mathematics, Vol. XXXVIII, n. 6, 1986, pp. 1299-1328.

[K,P] K. Kunen, K. Prikry. *On descendingly incomplete ultrafilters.* The Journal of Symbolic Logic, Vol 36, n. 4, Dec. 1971, pp 650-652.

[L,W] R.H. Lachlan, R.W. Woodrow. *Countable ultrahomogeneous undirected graphs.* Transactions of the American Mathematical Society, vol. 262, n. 1, nov 1980, pp 51-94.

[Myc] J. Mycielski. *Some compactifications of general algebras.* Colloquium Mathematicum XIII (1964) pp. 1-9.

[Myc,Ry-Na] J. Mycielski, C. Ryll-Nardzewski. *Equationnally compact algebras II.* Fundamenta Mathemeticae, LXI 1968, pp. 271-281.

[Poi] B. Poizat. *Théorie de Galois pour les algèbres de Post infinitaires.* Zeitsch. f. math. Logik und Grundlagen d. Math., Bd. 27, S. 31-44 (1981).

[P,R,S] M. Pouzet, I.G. Rosenberg, M.G. Stone. *A projection property.* Research paper n. 716, University of Calgary, April 1991, 44 pp.

[Rosb] I.G. Rosenberg. *Strongly rigid relations.* Rocky Mountain Journal of Mathematics, volume 3, number 4, Fall 1973, pp 631-639.

[Ury] P. Urysohn. *Sur un espace métrique universel.* Bull. Sci. Math. (2) 51, pp. 43-64 (1927).

[Weg] B. Weglorz. *Equationally compact algebras (I).* Fundamenta Mathematicae LIX (1966), pp. 289-298.

[Weh] F. Wehrung. *Equational compactness of bi-frames and projection algebras.* Algerbra Universalis, 33 (1995), pp 478-515.

Christian Delhommé
Universität Bielefeld
SFB 343
Diskrete Strukturen in der Mathematik
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany
E-mail address: Delhomme@Mathematik.Uni-Bielefeld.de