PROJECTION PROPERTIES AND BINARY RELATIONS

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Abstract. We establish general criteria of "projectivity" for reflexive binary relations. We obtain in particular, for each integer n, the existence of n-strongly projective graphs.

1. Introduction

Consider a structure $\mathfrak{M}$ for which constant mappings are endomorphisms. Given an integer n, the structure is *n-strongly projective* (for short, $\mathfrak{M}$ is n-SP) if projections are its only n-ary operations (regular with respect to each argument) which are identical on the diagonal ; it is *n-weakly projective* (for short, $\mathfrak{M}$ is n-WP) if projections are the only retractions of $\mathfrak{M}^n$ on its diagonal. Projective structures, and especially projective binary relational structures arise in diverse fields (eg. ordered sets [Cor], clone theory [P,R,S], social sciences [Pou], [P,R,S], algebraic compactness [Weh] ...) Weak projectivity has been studied in a universal algebra framework [P,R,S], [D,McK,N,P] : It was shown that 3-WP implies n-WP for all n's, and follows from 2-WP except for the 2-elementary groups. Strong projectivity does not behave in such a uniform way : for instance, a finite graph (reflexive and symmetric binary relation) can be n-SP only for finitely many n's [Del1].

In the following, we shall be concerned with binary relations, for which we establish criteria of projectivity, based on elementary techniques of algebraic topology. Here is how we proceed :

Given a integer n and a connected reflexive binary relation $\mathfrak{R}$ based on some set V, let $\delta : V \to V^n$ denote the diagonal injection and let, for each $\mathbf{a} \in V$ and $k \in \{1, \cdots, n\}$, $\iota_k^{\mathbf{a}} : V \to V^n$ denote the k^{th} marginal injection at $\mathbf{a}$. Consider $\mathfrak{R}^{(n)}$, the "smallest" of those relations $\mathfrak{P}$ based on V^n with the property that δ and the all the $\iota_k^{\mathbf{a}}$'s are morphisms from $\mathfrak{R}$ to $\mathfrak{P}$ (in particular, they induce homomorphisms δ_* and $\iota_{k*}^{\mathbf{a}} : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)}$ between the corresponding cycle spaces). We define $L_{\mathfrak{R}}^{\square n}$ and $T_{\mathfrak{R}}^{(n)}$, two subsets of $Z\mathfrak{R}^{(n)}$ such that (cf. lemma 1), the morphisms $\tau_{L_{\mathfrak{R}}^{\square n}} \circ \iota_{k*}^{\mathbf{a}} : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)}/\langle L_{\mathfrak{R}}^{\square n}\rangle$ depend only on k (with $\tau_{L_{\mathfrak{R}}^{\square n}}$ denoting the canonical surjection from $Z\mathfrak{R}^{(n)}$ on its quotient by its subgroup generated by $L_{\mathfrak{R}}^{\square n}$), and moreover, $\iota_{1\sharp} + \cdots + \iota_{n\sharp} = \tau_{L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)}} \circ \delta : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)}/\langle L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)}\rangle$ (each $\iota_{k\sharp}$ denotes any $\tau_{L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)}} \circ \iota_{k*}^{\mathbf{a}}$) : Indeed, let $\mathfrak{R}_*$ be the 1-dimentional complex associated to $\mathfrak{R}$, and let $\mathfrak{R}_*^{\otimes n}$ be its n^{th} tensorial power ; note that for each k, the $\iota_k^{\mathbf{a}}$'s define chain mappings from $\mathfrak{R}_*$ to $\mathfrak{R}_*^{\otimes n}$ which are homotopic with one another ; $L_{\mathfrak{R}}^{\square n}$ is then the set of boundaries of the 2-dimensional cells of $\mathfrak{R}_*^{\otimes n}$ (each such cell is the "product of 2 edges and $n-2$ vertices of $\mathfrak{R}$", and in particular, its length is equal to 4) ; as for $T_{\mathfrak{R}}^{(n)}$, it can be any subset of $Z\mathfrak{R}^{(n)}$ containing, for each diagonal edge, at least one cycle which is the "sum of this edge with a 1-chain of $\mathfrak{R}_*^{\otimes n}$" (the minimum length of such cycles is $n+1$). Now consider an n-ary operation F of $\mathfrak{R}$

such that $F \circ \delta = \mathrm{id}_V$, a relation $\mathfrak{P}$ on V^n containing all edges of $\mathfrak{R}^{(n)}$ (so that in particular, the identity mapping of V^n is a morphism from $\mathfrak{R}^{(n)}$ to $\mathfrak{P}$), a subset P of $Z\mathfrak{P}$ such that $L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)} \subset \langle P \rangle$ and a subset R of $Z\mathfrak{R}$. Assume that F is a morphism from $\mathfrak{P}$ to $\mathfrak{R}$ and, denoting by $F_* : Z\mathfrak{P} \to Z\mathfrak{R}$ the induced morphism, that $F_*(P) \subset \langle R \rangle$. Then the morphisms $\tau_R \circ F_* \circ \iota_{k*}^{\mathbf{a}} : Z\mathfrak{R} \to Z\mathfrak{R}/\langle R \rangle$ depend only on k (in that case denote them by $F_{k\sharp}$) ; moreover, they satisfy : $F_{1\sharp}+\cdots+F_{n\sharp} = \tau_R$. It is then the object of proposition 2 to conclude that some $F_{k\sharp}$ must be equal to τ_R, provided that $\mathfrak{R}$ satisfies some condition of stability with respect to R (which roughly expresses that $\langle R \rangle$ is not "too big" compared to R —in particular, $\mathfrak{R}$ is always $\varnothing$-stable—) : the key steps of the proof consist, given a "minimal" cycle γ outside R, in checking that for some k_0, $F_{k_0\sharp} \cdot \gamma = \tau_R \cdot \gamma$, and then in observing that, for any $k \neq k_0$, $F_{k\sharp}(Z\mathfrak{R}) \subset \mathbb{Z} \cdot \tau_R \cdot \sigma$. In particular, if moreover the identity mapping of V is the only endomorphism f of $\mathfrak{R}$ for which $\tau_R \circ f_* = \tau_R$ (in this case say that $\mathfrak{R}$ is R-tight), then the operation F had to be a projection (cf. corollary 2).

Summing up, given $\mathfrak{R}$, n, F, $\mathfrak{P}$, P and R, if the following four condition are met, then F is a projection :

— F is a morphism from $\mathfrak{P}$ to $\mathfrak{R}$,
— $L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)} \subset \langle P \rangle$,
— $F(P) \subset \langle R \rangle$,
— $\mathfrak{R}$ is R-stable and R-tight.

(Observe that, when $R = \varnothing$, the last condition is by no means restrictive for our purpose, as it is essentially required for $\mathfrak{R}$ to be projective -cf. fact 1 and lemma 3-, provided it is not a "cycle".)

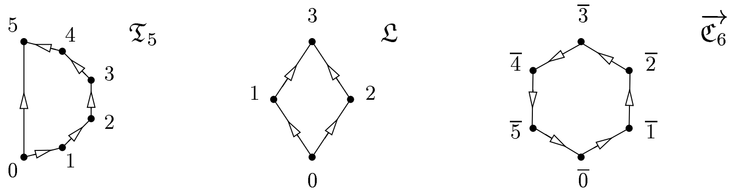
Now given a "class $\mathcal{F}$ of operations", the sufficient conditons that we get for an $\mathfrak{R}$ to be "projective with respect ot this class" express that $\mathfrak{R}$ is R-stable and R-tight for some R's related to $\mathcal{F}$ via some couples $(\mathfrak{P}, P)$'s. Note that, starting with the R's and proceeding backwards, one may eventually enlarge the given class $\mathcal{F}$. Double-fine projectivity, for instance, can be introduced in such a way starting from weak projectivity.

The sets $L_{\mathfrak{R}}^{\square n}$ and $T_{\mathfrak{R}}^{(n)}$ are introduced in section 3. Proposition 1 in the folowing section is a weak version of proposition 2, which does not require the developpements about tightness (indeed rigidity) and stability. Theorem 1, which is obtained as an immediate corollary, yields in particular the existence of strongly 3-projective graphs.

2. Preliminaries

2.1. **Reflexive binary relations.** A *reflexive binary relation* on a set V is any couple $\mathfrak{R} = (V, \mathcal{E})$ such that $\Delta_V \subset \mathcal{E} \subset V \times V$, where Δ_V denotes the set $\{(x,x) : x \in V\}$; we shall write $x \sim_{\mathfrak{R}} y$ for $(x,y) \in \mathcal{E}$. By "relation", we shall always mean "reflexive binary relation". The relation $\mathfrak{R}$ is a *graph* when it is moreover *symmetric*, ie. when $x \sim_{\mathfrak{R}} y$ is equivalent to $y \sim_{\mathfrak{R}} x$. The *symmetric closure* of $\mathfrak{R}$ is the graph $S\mathfrak{R} = (V, S\mathcal{E})$, where $S\mathcal{E}$ denotes the set $\{(x,y) : (x,y) \in \mathcal{E} \text{ or } (y,x) \in \mathcal{E}\}$. Given any subset W of V, the relation *induced* on W by $\mathfrak{R}$ is $\mathfrak{R}(W) = \big(W, \mathcal{E} \cap (W \times W)\big)$. A relation on W is a *weak subrelation* of $\mathfrak{R}$, if all its edges are edges of $\mathfrak{R}(W)$. Given two sets V and V', to every $f : V' \to V$ is associated $\vec{f} : V' \times V' \to V \times V$, mapping each couple (x,y) to the couple $\big(f(x), f(y)\big)$. Given two relations $\mathfrak{R} = (V, \mathcal{E})$ and $\mathfrak{R}' = (V', \mathcal{E}')$, denote by $[\mathfrak{R}', \mathfrak{R}]$ the set of morphisms from $\mathfrak{R}'$ to $\mathfrak{R}$, ie. the set of those $f : V' \to V$ for which $\vec{f}(\mathcal{E}') \subset \mathcal{E}$. The relation $\mathfrak{R}$ *contains* (resp. *weakly-contains*) $\mathfrak{R}'$, if $\mathfrak{R}'$ is isomorphic to an induced (resp. to a weak) subrelation of $\mathfrak{R}$.

The *lozenge* is the relation $\mathfrak{L}$ on $\{0,1,2,3\}$ with edges $(0,1)$, $(0,2)$, $(1,3)$, $(2,3)$ -plus all possible loops-. Given an integer $h \geq 2$, the *transitivity* of heigth h is the relation $\mathfrak{T}_h$ on $\{0,\cdots,h\}$ the edges of which are all loops, the couple $(0,h)$ and the couples $(k,k+1)$, $0 \leq k < h$. For every integer $\ell \geq 3$, the *oriented circle* $\overrightarrow{\mathfrak{C}_\ell}$ of length ℓ is the relation on $\mathbb{Z}/\ell\mathbb{Z}$ with edge set $\{(\overline{k},\overline{k+1}) : k \in \mathbb{Z}\} \cup \Delta_{\mathbb{Z}/\ell\mathbb{Z}}$, where $\overline{k}$ denotes $k+\ell\mathbb{Z}$; the *circle* $\mathfrak{C}_\ell$ of length ℓ is its symmetric closure. A *circle* is any graph isomorphic to some $\mathfrak{C}_\ell$.



Given a relation $\mathfrak{R} = (V,\mathcal{E})$, a finite sequence $c = (c_0,\cdots,c_\ell)$ is an *ℓ-path* from a vertex a to a vertex b, if $a = c_0 \sim_{S\mathfrak{R}} c_1 \sim_{S\mathfrak{R}} \cdots \sim_{S\mathfrak{R}} c_\ell = b$; the *image* of c is the set $\{c_0,\cdots,c_\ell\}$; for any map f with domain V, $f \circ c$ will denote the sequence $\big(f(c_0),\cdots,f(c_\ell)\big)$. The relation $\mathfrak{R}$ is *connected*, if between any two of its vertices, there is a path. The *connected components* of $\mathfrak{R}$ are the maximal subsets W of V for which $\mathfrak{R}(W)$ is connected. A *circle* of $\mathfrak{R}$ is any $C \subset V$ such that $S\mathfrak{R}(C)$ is a circle. A *tree* of $\mathfrak{R}$ is any set of vertices W for which $\mathfrak{R}(W)$ has no circle ; it is a *dangling tree* if moreover, $S\mathcal{E} \cap \big((V \backslash W) \times W\big)$ has exactly one element. Note that when V is finite, $\mathfrak{R}$ has a dangling tree if and only if it has a dangling vertex.

2.2. **n-ary operations.** Given an integer $n \geq 2$, $k \in \{1,\cdots,n\}$ and an n-tuple $\mathbf{a} = (a_1,\cdots,a_n) \in V^n$, the identity mapping of V, the *diagonal injection* of V into V^n, the k^{th}*-projection* from V^n to V and the k^{th}*-marginal injection* of V into V^n associated with $\mathbf{a}$ are given by :

$$\begin{array}{llll}
\mathrm{id}_V & : & V \to V & x \to x \\
\delta & : & V \to V^n & x \to (x,\cdots,x) \\
\pi_k & : & V^n \to V & (x_1,\cdots,x_n) \to x_k \\
\iota_k^{\mathbf{a}} & : & V \to V^n & x \to (a_1,\cdots,a_{k-1},x,a_{k+1},\cdots,a_n)
\end{array}$$

Given an $F : V^n \to V$, the k^{th}*-partial mapping* of F *at* $\mathbf{a}$ is the composite $F_k^{\mathbf{a}} = F \circ \iota_k^{\mathbf{a}}$; F is called *idempotent* if $F \circ \delta = \mathrm{id}_V$, it is an *$n$-ary operation* of $\mathfrak{R}$ if the $F_k^{\mathbf{a}}$'s are all endomorphisms of $\mathfrak{R}$.

The n^{th}*-Hamming power* of $\mathfrak{R}$ is the "smallest" relation on V^n for which every marginal injection is a morphism ; denote it by $\mathfrak{R}^{\Box n}$. Hence $\mathbf{x} \sim_{\mathfrak{R}^{\Box n}} \mathbf{y}$ if and only if there is a k_0 in $\{1,\cdots,n\}$ such that $x_{k_0} \sim_{\mathfrak{R}} y_{k_0}$ and $x_k = y_k$ for every $k \neq k_0$; moreover the n-ary operations of $\mathfrak{R}$ are precisely the elements of $[\mathfrak{R}^{\Box n},\mathfrak{R}]$. We shall denote by $\mathfrak{R}^{(n)}$ the relation on V^n given by : $\mathbf{x} \sim_{\mathfrak{R}^{(n)}} \mathbf{y}$, if $\mathbf{x} \sim_{\mathfrak{R}^{\Box n}} \mathbf{y}$ or $\mathbf{x} = \delta(x)$ and $\mathbf{y} = \delta(y)$ for some x and y such that $x \sim_{\mathfrak{R}} y$; in particular id_{V^n} belongs to $[\mathfrak{R}^{\Box n},\mathfrak{R}^{(n)}]$ and every idempotent n-ary operation is in $[\mathfrak{R}^{(n)},\mathfrak{R}]$. The n^{th}*-categorial power* of $\mathfrak{R}$ is the "largest" relation on V^n for which every projection is a morphism ; denote it by $\mathfrak{R}^n$. Hence $\mathbf{x} \sim_{\mathfrak{R}^{\Box n}} \mathbf{y}$ if and only if $x_k \sim_{\mathfrak{R}} y_k$, for every $k \in \{1,\cdots,n\}$. As $\mathfrak{R}$ is reflexive, $\mathrm{id}_{V^n} \in [\mathfrak{R}^{(n)},\mathfrak{R}^n]$. The retractions of $\mathfrak{R}^n$ on its diagonal are the idempotent elements of $[\mathfrak{R}^n,\mathfrak{R}]$.

Definitions 1. The relation $\mathfrak{R}$ is *n-strongly projective* (for short, $\mathfrak{R}$ is n-SP) if the projections are its only idempotent n-ary operations ; it is *n-weakly projective* (for short, $\mathfrak{R}$ is n-WP) if the projections are the only retractions of $\mathfrak{R}^n$ on its diagonal.

Note that n-SP implies n-WP and that, for $m \leq n$, n-SP implies m-SP and n-WP implies m-WP.

Fact 1. *If $\mathfrak{R}$ is 2-weakly projective, then it is connected and it has no dangling tree.*

Proof. If $\mathfrak{R}$ is not connected, then it is easily seen not to be 2-WP. Now assume that it has a dangling tree W, and let (a, b) be the only element of $S\mathcal{E} \cap \big((V \setminus W) \times W\big)$. For each $x \in W \cup \{a\}$, consider $(c_0, \cdots, c_\ell)$ the shortest path from a to x and denote by $f_x : V \to V$ the mapping which fixes every point outside W and maps each point from W to the "nearest" element in $\{c_0, \cdots, c_\ell\}$. For x outside W, let f_x be equal to f_a. The mapping $F : V^2 \to V$ given by $F(x, y) = f_x(y)$ is easily seen to contradict $\mathfrak{R}$ being 2-WP. □

2.3. **The cycle space of $\mathfrak{R}$.** For every set X, we denote by $\widehat{X}$ the abelian group freely generated by X. For every subset B of a group A, $\langle B \rangle$ is the subgoup of A generated by B. For every groups A and A', $[[A', A]]$ denotes the set of morphisms from A' to A.

For each relation $\mathfrak{R}$, denote by $\mathfrak{R}_*$ the quotient of $\widehat{S\mathcal{E}}$ by all the "relations" $(x, y) + (y, x) = 0$ and $(x, x) = 0$, and denote by $\pi : \widehat{S\mathcal{E}} \to \mathfrak{R}_*$ the corresponding surjection. $\mathfrak{R}_*$ is a free abelian group ; its elements are the *chains* of $\mathfrak{R}$. Notice that $\mathfrak{R}$ and $S\mathfrak{R}$ have the same chains. For every path $c = (c_0, \cdots, c_\ell)$, the chain $\pi(c_0, c_1) + \cdots + \pi(c_{\ell-1}, c_\ell)$ is denoted by $\overline{c}$. An *ℓ-parametrization* of a chain σ is any ℓ-path c such that $\overline{c} = \sigma$. When $\mathfrak{R}$ is connected, every chain has a parametrization.

The abelian group $\mathfrak{R}_*$, seen as a module over $\mathbb{Z}$, is endowed with a norm : Given a chain σ, $\|\sigma\|$ is the least integer k for which σ is equal to a sum $\overline{(x_1, y_1)} + \cdots + \overline{(x_k, y_k)}$.

The boundary operator $d \in [[\widehat{S\mathcal{E}}, \widehat{V}]]$, defined by $d(x, y) = y - x$, gives rise to $\partial \in [[\mathfrak{R}_*, \widehat{V}]]$ satisfying $\partial \circ \pi = d$. The *cycle space of* $\mathfrak{R}$ is the kernel of ∂ ; denote it by $Z\mathfrak{R}$ and denote by $Z_\ell\mathfrak{R}$ its subset consisting of those cycles with norm less or equal to ℓ.

For each subset C of $Z\mathfrak{R}$, $\tau_C : Z\mathfrak{R} \to Z\mathfrak{R}/\langle C \rangle$ will denote the canonical surjection.

A cycle γ is *essential* if it has a $\|\gamma\|$-parametrization the image of which is a circle of $\mathfrak{R}$; such parametrizations of γ are called *simple* ; all the simple paramatrizations of γ have the same image, which we shall denote by $\widetilde{\gamma}$ and term the *support* of γ. A circle of $\mathfrak{R}$ is the support of exactly two essential cycles : they are opposite to one another.

Fact 2. *Let γ be an essential circle of $\mathfrak{R}$.*
— Given $f \in [\mathfrak{R}(\widetilde{\gamma}), \mathfrak{R}]$, if $f_* \cdot \gamma = \gamma$, then f is a rotation of $S\mathfrak{R}(\widetilde{\gamma})$.
— For every path c, if the image of c is a subset of $\widetilde{\gamma}$, then $\overline{c}$ is a multiple of γ, ie. it is an element of $\mathbb{Z} \cdot \gamma = \{n \cdot \gamma : n \in \mathbb{Z}\}$; if moreover, the length of c does not exceed the norm of γ, then $c \in \{\gamma, 0, -\gamma\}$. □

A cycle σ is *principal* if it cannot be decomposed as a sum of strictly smaller cycles ; principal cycles are essential. The set of principal cycles generates the cycle space $Z\mathfrak{R}$. The *girth* of $\mathfrak{R}$, denoted by $\mathfrak{g}\mathfrak{R}$, is the smallest non-zero norm of elements of $Z\mathfrak{R}$. The cycles with norm equal to the girth are all principal.

Given a mapping f in $[\mathfrak{R}', \mathfrak{R}]$, $\pi \circ \vec{f}$ can be factored out by π', yielding an element f_* of $[[\mathfrak{R}'_*, \mathfrak{R}_*]]$ which maps $Z\mathfrak{R}'$ into $Z\mathfrak{R}$. Note that f_* is non-expansive, ie. for every $\sigma \in \mathfrak{R}'_*$, $\|f_*(\sigma)\| \leq \|\sigma\|$. We shall denote by $Z_{\mathfrak{R}'}\mathfrak{R}$ the set $\{f_* \cdot \sigma : f \in [\mathfrak{R}', \mathfrak{R}], \sigma \in Z\mathfrak{R}'\}$.

Given subsets C of $Z\mathfrak{R}$ and C' of $Z\mathfrak{R}'$ such that $f_*(C') \subset \langle C \rangle$, we shall denote by $f_{C',C}$ the element of $[[Z\mathfrak{R}'/\langle C' \rangle, Z\mathfrak{R}/\langle C \rangle]]$ obtained from $\tau_C \circ f_*$ by $\tau_{C'}$-factorisation. The sets C' and C can be thought of as sets of 2-dimentional cells filling some cycles of the "1-dimentional complexes" $\mathfrak{R}'_*$ and $\mathfrak{R}_*$; then $Z\mathfrak{R}'/\langle C' \rangle$ and $Z\mathfrak{R}/\langle C \rangle$ are the resulting homology groups.

These conventions are collected in the following commutative diagram :

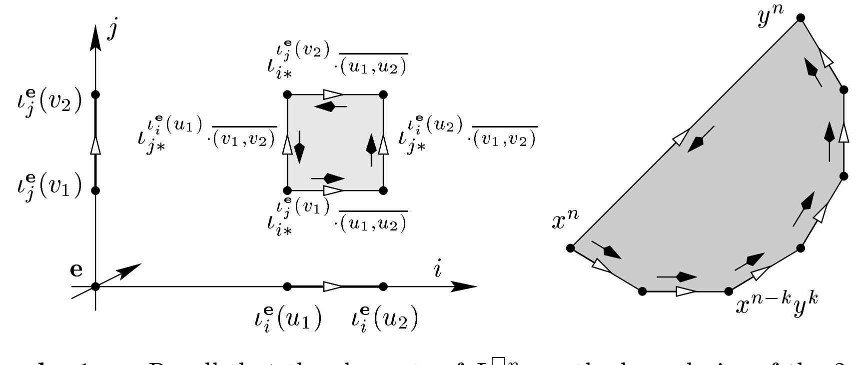
$$\begin{tikzcd}
\widehat{V'} \arrow[d, "f"'] & \widehat{S\mathcal{E}'} \arrow[l, "d'"'] \arrow[r, "\pi'"] \arrow[d, "\vec{f}"'] & \mathfrak{R}'_* \arrow[ll, bend right, "\partial'"'] \arrow[d, "f_*"] & \supset & Z\mathfrak{R}' \arrow[r, "\tau_{C'}"] \arrow[d, "f_*"'] & Z\mathfrak{R}'/\langle C' \rangle \arrow[d, "f_{C',C}"] \\
\widehat{V} & \widehat{S\mathcal{E}} \arrow[l, "d"] \arrow[r, "\pi"] & \mathfrak{R}_* \arrow[ll, bend left, "\partial"] & \supset & Z\mathfrak{R} \arrow[r, "\tau_C"] & Z\mathfrak{R}/\langle C \rangle
\end{tikzcd}$$

3. Reduction

In the sequel, $\mathfrak{R}$ denotes a connected relation and $\mathfrak{P}$ denotes a relation on V^n such that $\mathrm{id}_{V^n} \in [\mathfrak{R}^{(n)}, \mathfrak{P}]$. Consider the following sets of cycles of $\mathfrak{R}^{(n)}$ (hence of $\mathfrak{P}$ as well) :

$$L_{\mathfrak{R}}^{\square n} = \left\{[\mathbf{e}; (u_1, u_2), (v_1, v_2); i, j] : (u_1, u_2), (v_1, v_2) \in \mathcal{E}, 1 \leq i, j \leq n, i \neq j\right\} \text{ and}$$
$$T_{\mathfrak{R}}^{(n)} = \left\{\overline{\xi(x, y)} - \overline{(\delta(x), \delta(y))} : (x, y) \in \mathcal{E}\right\}, \text{ with}$$

$$[\mathbf{e}; (u_1, u_2), (v_1, v_2); i, j] = [\iota_{j*}^{\iota_i^{\mathbf{e}}(u_2)} - \iota_{j*}^{\iota_i^{\mathbf{e}}(u_1)}] \cdot \overline{(v_1, v_2)} - [\iota_{i*}^{\iota_j^{\mathbf{e}}(v_2)} - \iota_{i*}^{\iota_j^{\mathbf{e}}(v_1)}] \cdot \overline{(u_1, u_2)} \text{ and}$$
$$\xi(x, y) = (x^n, \cdots, x^{n-k}y^k, \cdots, y^n), \text{ where } x^{n-k}y^k \text{ denotes } (\underbrace{x, \cdots, x}_{n-k}, \underbrace{y, \cdots, y}_{k}).$$



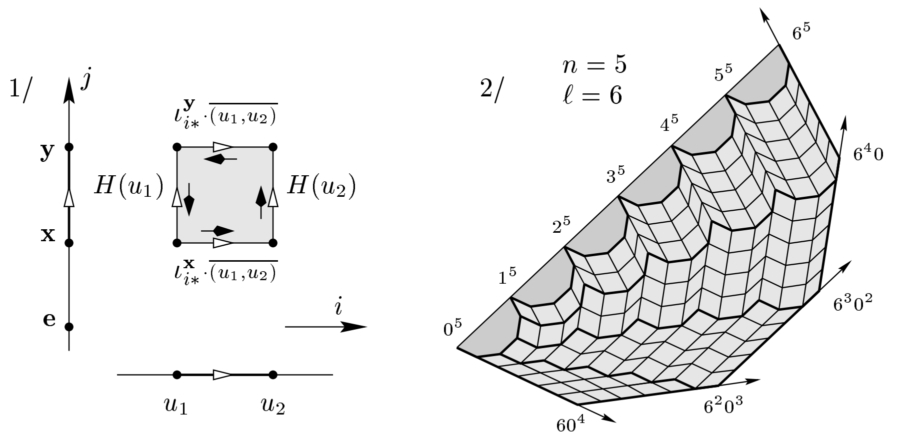
Remarks 1. — Recall that the elements of $L_{\mathfrak{R}}^{\square n}$ are the boundaries of the 2-cells of $\mathfrak{R}_*^{\otimes n}$, the n^{th}-tensorial power of $\mathfrak{R}_*$ (thought of as a 1-dimentional complex).

— Obviously, $L_{\mathfrak{R}}^{\square n} \subset Z_{\mathfrak{L}}\mathfrak{P}$ and $T_{\mathfrak{R}}^{(n)} \subset Z_{\mathfrak{T}_n}\mathfrak{P}$.

— For $(x, y) \in \mathcal{E}$, $\xi(x, y)$ is an "oriented" path of $\mathfrak{R}^{\square n}$.

Lemma 1. *Let* $P \subset Z\mathfrak{P}$.
1/ If $L_{\mathfrak{R}}^{\square n} \subset \langle P\rangle$*, then the* $\tau_P \circ \iota^{\mathbf{x}}_{k*}$*'s depend only on* k *; in that case denote them by* $\iota_{k\sharp}$.
2/ If moreover $T_{\mathfrak{R}}^{(n)} \subset \langle P\rangle$*, then* $\iota_{1\sharp} + \cdots + \iota_{n\sharp} = \tau_P \circ \delta$.

Proof.



1/ Let us check that $\iota^{\mathbf{x}}_{i*} = \iota^{\mathbf{y}}_{i*}$, for every $i \in \{1, \cdots, n\}$ and $\mathbf{x}$, $\mathbf{y}$ in V^n such that $\mathbf{x} \sim_{\mathfrak{R}^{\square n}} \mathbf{y}$ (as $\mathfrak{R}$ is connected, $\mathfrak{R}^{\square n}$ is connected too). Given $\mathbf{e} \in V^n$, $j \in \{1, \cdots, n\}$ and $(v_1, v_2) \in \mathcal{E}$, let $\mathbf{x} = \iota^{\mathbf{e}}_j(v_1)$ and $\mathbf{y} = \iota^{\mathbf{e}}_j(v_2)$. If $j = i$, then $\iota^{\mathbf{x}}_i = \iota^{\mathbf{y}}_i$; now assume that $j \neq i$, then consider the following homotopy operator between $\iota^{\mathbf{x}}_i$ and $\iota^{\mathbf{y}}_i$:

$$H : V \to \mathfrak{P}_*, \qquad u \to \iota_{j*}^{\iota_i^{\mathbf{e}}(u)} \cdot \overline{(v_1, v_2)}.$$

(for every $\sigma \in \mathfrak{R}_*$, $(\iota^{\mathbf{y}}_{i*} - \iota^{\mathbf{x}}_{i*} - H \circ \partial) \cdot \sigma \in L_{\mathfrak{R}}^{\square n}$).

2/ Let us check that, given any ℓ-path c of $\mathfrak{R}$, $(\iota^{\mathbf{e}_1}_{1*} + \cdots + \iota^{\mathbf{e}_n}_{n*} - \delta_*) \cdot \overline{c} \in \langle L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)} \rangle$ for some $\mathbf{e}_1, \cdots, \mathbf{e}_n$ in V^n. Let $\mathfrak{Q}$ be the relation $\big(\{0, \cdots, \ell\}, \Delta_{\{0,\cdots,\ell\}} \cup \{(k-1, k) : 0 < k \leq n\}\big)$. Consider the following commutative diagrams (with $\mathbf{p} \in \{0, \cdots, \ell\}^n$, $k \in \{1, \cdots, n\}$ and $\mathbf{c}$ implicitly defined) :

$$\begin{tikzcd}
\mathfrak{Q} \arrow[r, "\delta", "\iota^{\mathbf{p}}_k"'] \arrow[d, "c"'] & \mathfrak{Q}^{(n)} \arrow[d, "\mathbf{c}"] \\
S\mathfrak{R} \arrow[r, "\delta", "\iota^{\mathbf{c}(\mathbf{p})}_k"'] & S(\mathfrak{R}^{(n)})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{Q}_* \arrow[r, "\delta*", "\iota^{\mathbf{p}}_{k*}"'] \arrow[d, "c_*"'] & \mathfrak{Q}^{(n)}_* \arrow[d, "\mathbf{c}_*"] & \supset L_{\mathfrak{Q}}^{\square n}, T_{\mathfrak{Q}}^{(n)} \arrow[d, "\mathbf{c}_*"] \\
\mathfrak{R}_* \arrow[r, "\delta_*", "\iota^{\mathbf{c}(\mathbf{p})}_{k*}"'] & \mathfrak{P}_* & \supset L_{\mathfrak{R}}^{\square n}, T_{\mathfrak{R}}^{(n)}
\end{tikzcd}$$

Now consider the cycle of $\mathfrak{Q}^{(n)}$, $\sigma = (\iota^{\mathbf{p}_1}_{1*} + \cdots + \iota^{\mathbf{p}_n}_{n*} - \delta_*) \cdot \overline{(0, \cdots, \ell)}$, with $\mathbf{p}_k = \ell^{k-1} 0^{n-(k-1)}$. According to the diagrams above, we have just to check that $\sigma \in \langle L_{\mathfrak{Q}}^{\square n} \cup T_{\mathfrak{Q}}^{(n)} \rangle$, but :

$$\sigma = \overbrace{(\iota^{\mathbf{p}_1}_{1*} + \cdots + \iota^{\mathbf{p}_n}_{n*}) \cdot \overline{(0, \cdots, \ell)} - \sum_{1 \le k \le \ell} \overline{\xi(k-1,k)}}^{\text{cycle of } \mathfrak{Q}^{\square n}} + \sum_{1 \le k \le \ell} \underbrace{\overline{\xi(k-1,k)} - \overline{(\delta(k-1), \delta(k))}}_{\text{element of } T^{(n)}_{\mathfrak{Q}}}$$

Finally, observe that, as the cycles of $\mathfrak{Q}_*$ are all trivial (ie. $Z\mathfrak{Q} = \{0\}$), so are the 1-cycles of $\mathfrak{Q}_*^{\otimes n}$; then recall remarks 1 (first dash). One can also proceed with the computation of the first term, explicitly decomposing it as a sum of elements of $L^{\square n}_{\mathfrak{Q}}$. □

Corollary 1. *Given $R \subset Z\mathfrak{R}$, for every idempotent $F \in [\mathfrak{P}, \mathfrak{R}]$ such that $F_*(L^{\square n}_{\mathfrak{R}} \cup T^{(n)}_{\mathfrak{R}}) \subset \langle R \rangle$, the $\tau_R \circ F^{\mathbf{x}}_{k*}$'s depend only on k (in that case denote them by $F_{k\sharp}$) ; moreover, they satisfy : $F_{1\sharp} + \cdots + F_{n\sharp} = \tau_R$.* □

Let $T_{\mathfrak{R},n} = \{f_* \cdot \sigma : \sigma \in Z\mathfrak{T}_n, f \in [\mathfrak{T}_n, \mathfrak{R}], f(0) \neq f(n)\}$ (notice that $T_{\mathfrak{R},n} \subset Z_{\mathfrak{T}_2}\mathfrak{R}$). As $L^{\square n}_{\mathfrak{R}} \subset Z_{\mathfrak{L}}\mathfrak{P}$, for each $F \in [\mathfrak{P}, \mathfrak{R}]$, $F_*(L^{\square n}_{\mathfrak{R}}) \subset Z_{\mathfrak{L}}\mathfrak{R}$; when moreover F is idempotent, it "contracts no diagonal edge", and then $F_*(T^{(n)}_{\mathfrak{R}}) \subset T_{\mathfrak{R},n}$.

4. "Loose criteria"

4.1. Consider an idempotent element F of $[\mathfrak{P}, \mathfrak{R}]$.

Proposition 1. *Assume that each $F^{\mathbf{x}}_{k*|Z\mathfrak{R}}$ depends only on k, and denote it by F_{k*}. If*

$$F_{1*} + \cdots + F_{n*} = \mathrm{id}_{Z\mathfrak{R}}, \tag{1}$$

then there is a k_0 such that $F_{k_0*} = \mathrm{id}_{Z\mathfrak{R}}$.

Proof of the proposition. Consider a cycle γ with minimal norm $\mathfrak{g}\mathfrak{R}$. Recall that, being principal, γ has a simple parametrization c.

Claim 1. *For each k in $\{1, \cdots, n\}$, $F_{k*} \cdot \gamma \in \{0, \gamma, -\gamma\}$.*

As $Z\mathfrak{R}$ has no torsion, from claim 1 and equality (1), it follows that for some $k_0 \in \{1, \cdots, n\}$,

$$F_{k_0*} \cdot \gamma = \gamma. \tag{2}$$

Claim 2. *For every $\mathbf{x} = (x_1, \cdots, x_n) \in V^n$, if $x_{k_0} \in \widetilde{\gamma}$, then $F(\mathbf{x}) \in \widetilde{\gamma}$.*

Claim 3. *For every $x \in \widetilde{\gamma}$, $F^{\delta(x)}_{k_0}$ fixes $\widetilde{\gamma}$ pointwise.*

Now consider some cycle σ in $Z\mathfrak{R}$. We shall prove that $F_{k_0*} \cdot \sigma = \sigma$.

Claim 4. *For each $k \in \{1, \cdots, n\} \backslash \{k_0\}$, $F_{k*} \cdot \sigma \in \mathbb{Z} \cdot \gamma$.*

According to claim 4 and to equality (1), there is an integer p for which

$$F_{k_0*} \cdot \sigma = \sigma + p \cdot \gamma. \tag{3}$$

We have to check that p is null.

Claim 5. *There is a chain* $\nu \in \mathfrak{R}_*$ *satisfying :*

$$\sigma - \nu \in \mathfrak{R}(\widetilde{\gamma})_* \tag{4}$$

$$\|\alpha \cdot \gamma + \beta \cdot \nu\| = |\alpha| \cdot \|\gamma\| + |\beta| \cdot \|\nu\|, \textit{ for all } \alpha \textit{ and } \beta \textit{ in } \mathbb{Z}. \tag{5}$$

Take some x in $\widetilde{\gamma}$. According to equality (3) and to claim 3 together with (4),

$$\sigma + p \cdot \gamma = F^{\delta(x)}_{k_0 *} \cdot \sigma = F^{\delta(x)}_{k_0 *} \cdot (\sigma - \nu) + F^{\delta(x)}_{k_0 *} \cdot \nu = \sigma - \nu + F^{\delta(x)}_{k_0 *} \cdot \nu$$
$$\text{ie.} \qquad F^{\delta(x)}_{k_0 *} \cdot \nu = p \cdot \gamma + \nu$$

but then, as $F^{\delta(x)}_{k_0 *}$ is non-expansive, (5) yields

$$\|\nu\| \geq \|F^{\delta(x)}_{k_0 *} \cdot \nu\| = p \cdot \|\gamma\| + \|\nu\|,$$

hence $p = 0$, as was to be shown. □

To complete the proof of the proposition, there remains to check the claims.

Proof of claim 1. — Assume that the cycle $F_{k*} \cdot \gamma$ is not zero. Then, by minimality of $\|\gamma\|$ and as F_{k*} is non-expansive, $\|F_{k*} \cdot \gamma\| = \|\gamma\|$; in particular, $F_{k*} \cdot \gamma$ is principal.

— Now it is enough to show that $\widetilde{\gamma} \subset \widetilde{F_{k*} \cdot \gamma}$ (for then, $\gamma \in \mathbb{Z} \cdot (F_{k*} \cdot \gamma)$ with $\|\gamma\| = \|F_{k*} \cdot \gamma\| \neq 0$). But any element x from $\widetilde{\gamma}$ is contained in the image of $F^{\delta(x)}_k \circ c$, which is a simple parametrization of $F_{k*} \cdot \gamma$. □

Proof of claim 2. $F(\mathbf{x})$ is contained in the image of $F^{\mathbf{x}}_{k_0} \circ c$, which is a simple parametrization of γ. □

Proof of claim 3. Recall that $\mathfrak{R}(\widetilde{\gamma})$ is a circle. As $F^{\delta(x)}_{k_0 *} \cdot \gamma = \gamma$, $F^{\delta(x)}_{k_0 | \widetilde{\gamma}}$ is a rotation of $\mathfrak{R}(\widetilde{\gamma})$ (note that $F^{\delta(x)}_{k_0}(\widetilde{\gamma}) \subset \widetilde{\gamma}$ could be deduced from claim 2). But this rotation has a fixed point, namely x. □

Proof of claim 4. Take any $\mathbf{x} = (x_1, \cdots, x_n)$ such that $x_{k_0} \in \widetilde{\gamma}$ and any parametrization s of σ. Now consider the path $F^{\mathbf{x}}_k \circ s$, and observe that it is a parametrization of $F^{\mathbf{x}}_{k*} \cdot \sigma$, and that, according to claim 2, its image is a subset of $\widetilde{\gamma}$. □

Proof of claim 5. Just decompose the chain σ according to the decomposition of $\mathfrak{R}_*$: $\mathfrak{R}_* = \mathfrak{R}(\widetilde{\gamma})_* \oplus \pi(\mathcal{E}^{\setminus \widetilde{\gamma}})$, where $\mathcal{E}^{\setminus \widetilde{\gamma}}$ is the set of those edges from $\mathfrak{R}$ which do not figure in $\mathfrak{R}(\widetilde{\gamma})$. □

Lemma 2. *The set* $Z_{\mathfrak{L}}\mathfrak{R}$ *is reduced to* $\{0\}$ *if and only if* $\mathfrak{R}$ *weakly contains neither* $\mathfrak{L}$ *nor* $\mathfrak{T}_2$. *The set* $T_{\mathfrak{R},n}$ *is reduced to* $\{0\}$ *if and only if* $\mathfrak{R}$ *weakly contains* $\mathfrak{T}_h$ *for no* $h \leq n$ *and weakly contains* $\overrightarrow{\mathfrak{C}_\ell}$ *for no* $\ell < n$.

Note that $\mathfrak{R}$ may weakly contain $\overrightarrow{\mathfrak{C}_n}$.

Proof. Easy. □

Lemma 3. *If* $\mathfrak{R}$ *is connected with no dangling tree and if* $S\mathfrak{R}$ *is not a circle, then* id_V *is the only endomorphism* f *of* $\mathfrak{R}$ *for which* $f_* = \mathrm{id}_{Z\mathfrak{R}}$.

Proof. Observe that the hypothesis precisely mean that V is a bunch (and is not a circle). Now see fact 4 of section 6. □

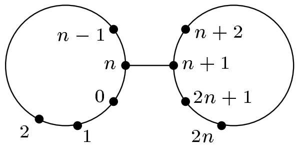
4.2. **Stong projectivity.** Here, consider $\mathfrak{P}$ equal to $\mathfrak{R}^{(n)}$.

Theorem 1. *Given an integer $n \geq 2$, let $\mathfrak{R}$ be a reflexive binary relation which is connected with no dangling tree and such that $S\mathfrak{R}$ is not a circle. If $\mathfrak{R}$ weakly contains neither $\mathfrak{L}$, nor $\mathfrak{T}_h$ for any $h \leq n$, nor $\overrightarrow{\mathfrak{C}_\ell}$ for any $\ell < n$, then it is n-strongly projective.*

In particular, a connected graph with no dangling tree and which is not a circle is n-strongly projective, as soon as it contains no circle of length less or equal to 4 or to $n+1$.

Proof. Considering R equal to $Z_{\mathfrak{L}}\mathfrak{R} \cup T_{\mathfrak{R},n}$, successively invoke corollary 1, lemma 2, proposition 1 and lemma 3. □

Remarks 2. 1/ When $\mathfrak{R}$ satisfies all the hypothesis of the theorem but the one about $S\mathfrak{R}$ not being a circle, it can easily be deduced from the above statements that, given F an idempotent binary operation, every $F_1^{(y,y)}$ (for instance) is a rotation of $S\mathfrak{R}$ with a fixed point (namely y) ; hence, for every (x,y), $F(x,y) = F_1^{(y,y)}(x) = x = \pi_1(x,y)$.
2/ According to this theorem, the graph represented below is $(n-1)$-strongly projective. But it is not n-strongly projective : observe that, as $k \sim k'$ implies $|k - k'| \leq n$, M is an n-ary idempotent operation which obviously, is not a projection ($\lfloor x \rfloor$ denotes the lower integral part of x).



$$M(k_1, \cdots, k_n) = \left\lfloor \frac{k_1 + \cdots + k_n}{n} \right\rfloor$$

3/ The circles are n-strongly projective for no $n \geq 3$: consider $F(x_1, \cdots, x_n) = x_1 - x_2 + x_3$, see [P,R,S].

4.3. **p-fine projectivitiy.**

Definitions 2. The n-ary operation F *respects duels* if $x \sim_{\mathfrak{R}} F(\mathbf{z}) \sim_{\mathfrak{R}} y$, for all $(x,y) \in \mathcal{E}$ and $\mathbf{z} \in \{x,y\}^n$. Given $p \in \{1, \cdots, n\}$, F is *p-fine* if $F(\mathbf{x}) \sim_{\mathfrak{R}} F(\mathbf{y})$, as soon as $\mathbf{x} \sim_{\mathfrak{R}^n} \mathbf{y}$ and $x_k = y_k$ for all but at most p values of k.

The relation $\mathfrak{R}$ is *n-SFP* (resp. *n-DFP*) if the projections are its only n-ary operations which are idempotent (resp. idempotent and 2-fine) and which respect duels.

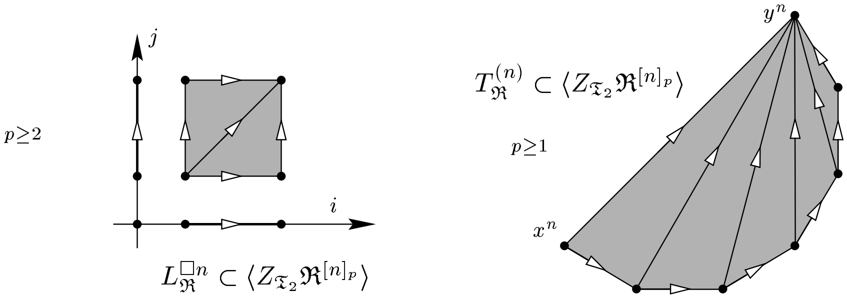
Remarks 3. — Idempotent n-ary operations are always 1-fine, and those which are n-fine are precisely the retractions of $\mathfrak{R}^n$ on its diagonal.
— One could only demand, given x and y such that $x \sim_{\mathfrak{R}} y$, that for instance, $F(\mathbf{z}) \sim_{\mathfrak{R}} y$, for every duel $\mathbf{z}$ between x and y.

The idempotent p-fine n-ary operations respecting restricted duels are precisely the idempotent morphisms from $\mathfrak{R}^{[n]_p}$ to $\mathfrak{R}$, with $\mathfrak{R}^{[n]_p}$ the obvious relation on V^n. Recall that the natural objects to consider are $\mathfrak{R}^{[n]_2}$ and $\mathfrak{R}^{[n]_1}$ (together with their "2-cells"), and that definitions 2 just makes explicit what their idempotent morphisms to $\mathfrak{R}$ are.

Theorem 2. *Given an integer $n \geq 2$, let $\mathfrak{R}$ be a reflexive binary relation which is connected with no dangling tree and such that $S\mathfrak{R}$ is not a circle. If $\mathfrak{R}$ does not weakly-contain $\mathfrak{T}_2$ then it is n-DFP (hence it is n-weakly projective). If moreover,* *it does not weakly-contain $\mathfrak{L}$ then it is n-SFP (hence it is n-affine projective : cf.* [P,R,S]*).*

In particular, a connected graph with no dangling tree and which is not a circle is n-DFP, as soon as it has no triangle. When moreover, it has no square, then it is n-SFP.

Proof. For n-DFP, observe that, for $\mathfrak{P} = \mathfrak{R}^{[n]_2}$, $L^{\square n}_{\mathfrak{R}} \cup T^{(n)}_{\mathfrak{R}} \subset Z_{\mathfrak{T}_2}\mathfrak{P}$; then consider $R = Z_{\mathfrak{T}_2}\mathfrak{R}$. For n-DFP, observe that, for $\mathfrak{P} = \mathfrak{R}^{[n]_1}$, $T^{(n)}_{\mathfrak{R}} \subset Z_{\mathfrak{T}_2}\mathfrak{P}$; then consider $R = Z_{\mathfrak{L}}\mathfrak{R} \cup Z_{\mathfrak{T}_2}\mathfrak{R}$, recalling that $L^{\square n}_{\mathfrak{R}} \subset Z_{\mathfrak{L}}\mathfrak{P}$. Now proceed as for theorem 1, noticing that $Z_{\mathfrak{T}_2}\mathfrak{R}$ is $\{0\}$ if and only if $\mathfrak{R}$ does not weakly contain $\mathfrak{T}_2$.



□

Remarks 4. — The circles are not 3-SFP, but they are n-DFP for every n (except the triangle), see. [Del2].

— Given a connected relation $\mathfrak{R} = (V, \mathcal{E})$, with no dangling tree, distinct from any circle, and with girth greater than 4, we know that $\mathfrak{R}$ is n-strongly projective for every $n < \mathfrak{g}\mathfrak{R} - 1$. More generally, we could state that, for each integer p such that $\mathfrak{g}\mathfrak{R} > p + 1$ and every integer $n \leq (\mathfrak{g}\mathfrak{R} - 2) \times p$, the projections are the only idempotent n-ary operations F of $\mathfrak{R}$ which satisfy the following condition with respect to "p-closed duels" : for each $\{x, y\} \in \mathcal{E}$, $F(\mathbf{x}) \sim F(\mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\{x, y\}^n$ which disagree at most on p coordinates.

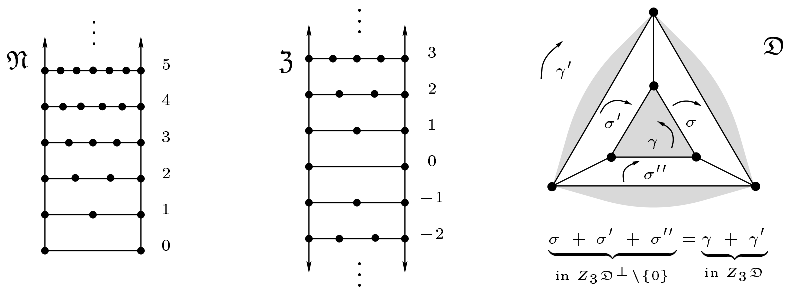
5. R-stability and R-rigidity

Definitions 3. Given a subset R of $Z\mathfrak{R}$, an *R-principal* cycle is any principal cycle which does not belong to $\langle R\rangle$, an *R-minimal* cycle is any R-principal cycle of minimal length. Denote by $R^{\perp}$ the subgroup of $Z\mathfrak{R}$ generated by the R-principal cycles.

The relation $\mathfrak{R}$ is *R-stable* if $\langle R\rangle \cap R^{\perp} = \{0\}$, it is *$R$-rigid* if the identity mapping id_V is the only endomorphism f of $\mathfrak{R}$ for which f_* fixes $R^{\perp}$ pointwise. For R equal to $Z_\ell\mathfrak{R}$, we shall speak of ℓ-stability and ℓ-rigidity.

Remarks 5. — As principal cycles generate $Z\mathfrak{R}$, $\mathfrak{R}$ is R-stable if and only if $Z\mathfrak{R} = \langle R\rangle \oplus R^{\perp}$, or if and only if $R^{\perp}$ and $Z\mathfrak{R}/\langle R\rangle$ are isomorphic via τ_R.
— A 0-rigid relation is connected with no dangling tree (cf. lemma 3). The converse is true for graphs (except for circles).

Examples 1. The *ladders* $\mathfrak{N}$ and $\mathfrak{Z}$ pictured below are ℓ-stable for every ℓ. The *prism* $\mathfrak{D}$ is not 3-stable.



Lemma 4. *Assume that $\mathfrak{R}$ is R-stable and let σ and γ be two R-principal cycles such that $\|\gamma\| < \|\sigma\|$. For any cycle σ' such that $\|\sigma'\| \leq \|\sigma\|$ and any p in $\mathbb{Z}$, if $\tau_R \cdot \sigma = \tau_R \cdot (\sigma' + p \cdot \gamma)$, then $\sigma = \sigma' + p \cdot \gamma$.*

Proof. First note that σ' cannot lie in $\langle R \rangle$, as $\tau_R \cdot \sigma'$ is equal to $\tau_R(\sigma + p \cdot \gamma)$, which is not zero since $\sigma + p \cdot \gamma$ belongs to $R^{\perp} \backslash \{0\}$. Now it is enough to check that σ' is principal, since then, σ and $\sigma' + p \cdot \gamma$ will be two elements of $R^{\perp}$ on which τ_R agrees.

Assume that σ' is not principal and consider principal cycles with norms strictly less than $\|\sigma\|$ adding up to σ, namely $\sigma' = \sum_{i \in I} \rho_i + \sum_{j \in J} \gamma_j$, with the ρ_i's in $\langle R \rangle$ and the γ_j's out of it. Then σ and $p \cdot \gamma + \sum_{j \in J} \gamma_j$ must be equal, since they both lie in $R^{\perp}$ and τ_R agrees on them. But this contradicts σ being principal. □

Corollary 2. *If $\mathfrak{R}$ is R-stable, then for every endomorphism f of $\mathfrak{R}$ such that $\tau_R \circ f_* = \tau_R$, f_* fixes $R^{\perp}$ pointwise.*

Proof. For each R-principal cycle σ, $\|f_* \cdot \sigma\| \leq \|\sigma\|$ and $\tau_R \cdot \sigma = \tau_R \cdot (f_* \cdot \sigma)$. □

Remark 6. The property of R-tightness (cf. introduction) is obviously stronger than the property of R-rigidity. But according to the above lemma, these properties become equivalent under the hypothesis of R-stability.

Proposition 2. *Given $R \subset Z\mathfrak{R}$, assume that each $\tau_R \circ F^{\mathbf{x}}_{k*|Z\mathfrak{R}}$ depends only on k and denote it by $F_{k\sharp}$. If*

$$F_{1\sharp} + \cdots + F_{n\sharp} = \tau_R, \tag{6}$$

then there is a k_0 such that $F_{k_0\sharp} = \tau_R$.

Proof of the proposition.

Claim 6. *Consider an R-minimal cycle γ.*
— 1/ For each $k \in \{1, \cdots, n\}$ such that $F_{k\sharp} \cdot \gamma \neq 0$, the $F^{\mathbf{x}}_{k*} \cdot \gamma$ are R-principal and do not depend on $\mathbf{x}$ (in that case, denote them by F_{k*}).
— 2/ If $F_{k\sharp} \cdot \gamma \neq 0$ then $F_{k*} \cdot \gamma \in \{\gamma, -\gamma\}$.
— 3/ There is a k in $\{1, \cdots, n\}$ such that $F_{k*} \cdot \gamma = \gamma$

Now fix an R-minimal cycle γ, and consider k_0 in $\{1, \cdots, n\}$ such that $F_{k_0*} \cdot \gamma = \gamma$. Note that claims 2 and 3 still hold here along the very same lines, as well as their proofs (see section 4). Hence, given any cycle σ, property (7) below follows from the proof of claim 4 ; this property, together with equality (6), yields relation (8).

$$\text{Each } F_{k\sharp} \cdot \sigma \text{ is an element of } \tau_R \cdot (\mathbb{Z} \cdot \gamma). \tag{7}$$

$$F_{k_0\sharp} \cdot \sigma = \tau_R \cdot (\sigma + p \cdot \gamma) \text{ for some } p \text{ in } \mathbb{Z}. \tag{8}$$

Let us check that, for every R-principal cycle σ, $F_{k_0\sharp}\cdot\sigma=\tau_R\cdot\sigma$.

If σ is an R-minimal cycle, then it follows from claim 6 that $F_{k\sharp}\cdot\sigma=\tau_R\cdot\sigma$ for some k ; but when moreover γ is not a multiple of γ, it follows from (6) and from τ_R being one-to-one on $R^\perp$, that this k must be equal to k_0.

If σ is R-principal but not R-minimal, then it follows from (8) and lemma 4 (with $\sigma'=F^{\mathbf{x}}_{k_0*}\cdot\sigma$), that there is a p such that, for any $\mathbf{x}\in V^n$, $F^{\mathbf{x}}_{k_0*}\cdot\sigma=\sigma+p\cdot\gamma$. Now proceed exactly as in the proof of proposition 1 starting from equality (3) onwards. $\square$

Proof of claim 6.
— 1/ If the cycle $F^{\mathbf{x}}_{k*}\cdot\gamma$ is not principal, then it is the sum of finitely many principal cycles strictly shorter than γ, hence it must lie in $\langle R\rangle$. So if $F_{k\sharp}\cdot\gamma\neq 0$, then the $F^{\mathbf{x}}_{k*}\cdot\gamma$'s do not lie in $\langle R\rangle$ and are principal, so they all belong to $R^\perp$ and must coincide with one another, since they get the same image by τ_R.
— 2/ Proof of claim 1, second dash.
— 3/ Let α, resp β be the number of those k's in $\{1,\cdots,n\}$ for which $F_{k\sharp}\cdot\gamma\neq 0$ and $F_{k*}\cdot\gamma=\gamma$, resp. $F_{k\sharp}\cdot\gamma\neq 0$ and $F_{k*}\cdot\gamma=-\gamma$. According to equality (6), $(\alpha-\beta)\cdot\tau_R\cdot\gamma=\tau_R\cdot\gamma$, so $(\alpha-\beta)\cdot\gamma=\gamma$; hence $\alpha>0$. $\square$

From the above proposition, corollary 1 and corollary 2, it follows :

Theorem 3. *Given an integer $n\geq 2$, let $\mathfrak{R}$ be a connected reflexive binary relation. If $\mathfrak{R}$ is R-stable and R-rigid for some R containing $Z_{\mathfrak{L}}\mathfrak{R}\cup T_{\mathfrak{R},n}$, resp. $Z_{\mathfrak{T}_2}\mathfrak{R}$, resp. $Z_{\mathfrak{T}_2}\mathfrak{R}\cup Z_{\mathfrak{L}}\mathfrak{R}$, then it is n-strongly projective, resp. n-DFP, resp, n-SFP.In particular, if $\mathfrak{R}$ is ℓ-stable and ℓ-rigid for ℓ equal to* $\max(4,n+1)$, *resp to* 3, *resp to* 4, *then ist is n-SP, resp. n-DFP, resp, n-SFP.* $\square$

6. Rigidity

6.1. **Bunch decomposition.** Given $\mathfrak{R}$, consider the finest partition of V for which every circle is included in a class. The classes of this partition which are not reduced to a single point are the *clusters* of $\mathfrak{R}$. A circle of $\mathfrak{R}$ is *isolated* if it meets no other circle, ie. if it is equal to a cluster. An *arc* of a circle C is any set of "consecutive" vertices on C.

Fact 3. *For every non-isolated circle C of $\mathfrak{R}$, there exists a circle C' such that $C\cap C'$ is a proper arc of both C and C'.*

Proof. Given two essential cycles γ and σ with respective simple parametrizations $c=(c_0,\cdots,c_\ell)$ and $s=(s_0,\cdots,s_m)$ such that $c_0=s_0$, $c_1\notin\widetilde{\sigma}$ and $s_1\notin\widetilde{\gamma}$, assume that $\widetilde{\gamma}\cap\widetilde{\sigma}$ has at least two elements and consider a minimal couple (ℓ',m') in $\{1,\cdots,\ell\}\times\{1,\cdots,m\}$ for which $c_{\ell'}\sim_{S\mathfrak{R}}s_{m'}$; observe that $\ell'<\ell-1$ and $m'<m-1$, and consider the cycle $\theta=\overline{(c_0,\cdots,c_{\ell'},s_{m'},\cdots,s_0)}$; it is essential, moreover $\widetilde{\theta}\cap\widetilde{\gamma}$, resp. $\widetilde{\theta}\cap\widetilde{\sigma}$ is a proper non-empty arc of both $\widetilde{\theta}$ and $\widetilde{\gamma}$, resp. $\widetilde{\theta}$ and $\widetilde{\sigma}$. $\square$

Two distinct clusters K and K' are called *adjacent* if there is a path joining K to K' with only its initial and final points belonging to a circle ; in this case, the path is unique and indeed, it is the only path of minimal length (resp. the only injective path) which joins K to K' ; denote it by $j_{K,K'}$; its image is the *join* of K and K'.

Clusters lie in the same connected component of $\mathfrak{R}$ if and only if they belong to the same adjacency class (indeed, given $\mathcal{K}$ and $\mathcal{K}'$ two non-empty disjoint sets of clusters which lie in the same connected component of $\mathfrak{R}$, consider a path of minimal length joining $\cup\mathcal{K}$ to $\cup\mathcal{K}'$ and observe that it is some $j_{K,K'}$). The *bunch* of a connected component W is the set of those points in W which belong either to a cluster lying in W or to a join between two such clusters. Bunches, maximal dangling trees and isolated trees constitute a partition of V.

We are now able to state (cf. fact 2) :

Fact 4. *Let f be an endomorphism of $\mathfrak{R}$ such that $f_* = \mathrm{id}_{Z\mathfrak{R}}$. One gets successively :*
— For every circle C, f_C is a rotation of $S\mathfrak{R}$ (hence f fixes C pointwise as soon as it globally fixes a proper arc of it).
— For every cluster K, $f(K) \subset K$; moreover, if f fixes some point in K, then it fixes K pointwise.
— Every join is fixed pointwise by f.
— f fixes pointwise every bunch which is not reduced to a circle. □

6.2. **A criterion of rigidity.** We shall state a criterion for a weak form of R-rigidity, which in practice, may enable to establish projectivity, either via theorem 3, or directly from proposition 2 and corollaries 1 and 2.

Definitions 4. Given $\mathfrak{R}$ and $R \subset Z\mathfrak{R}$.
— The set of cycles R is *hereditary* if for every endomorphism f of $\mathfrak{R}$, $f_*(R) \subset \langle R\rangle$.
— Denote by V_R and $\mathcal{E}_R$ the sets of those vertices and edges of $\mathfrak{R}$ which occur in $\mathfrak{R}(\widetilde{\gamma})$ for some R-principal cycle γ. The weak subrelation of $\mathfrak{R}$ equal to $(V_R, \mathcal{E}_R)$ is denoted by $\mathfrak{R}_R$ (denote it by $\mathfrak{R}_\ell = (V_\ell, \mathcal{E}_\ell)$ when R is equal to $Z_\ell\mathfrak{R}$).
— The relation $\mathfrak{R}$ is *stongly R-stable* if $Z\mathfrak{R}_R \cap \langle R\rangle = \{0\}$.
— The relation $\mathfrak{R}$ is *well R-connected* if for every isolated circle C of $\mathfrak{R}_R$, there exists another circle C' of $\mathfrak{R}_R$, a vertex a in C and a vertex a' in C' such that only finitely many paths of minimal length join a to a' in $\mathfrak{R}$.

Remarks 7. — Every $Z_{\mathfrak{R}'}\mathfrak{R}$ and every $Z_\ell\mathfrak{R}$ is hereditary.
— If $\mathfrak{R}$ is locally finite then it is well R-connected for every R such that $Z\mathfrak{R}_R$ is not a cyclic group.
— The connected components of $\mathfrak{R}_R$ coincide with its clusters.
— As $R^\perp \subset Z\mathfrak{R}_R$ and $Z\mathfrak{R} = \langle R\rangle + R^\perp$, $\mathfrak{R}$ is strongly R-stable if and only if it is R-stable and $R^\perp = Z\mathfrak{R}_R$.
— Consider the graph $\mathfrak{G}$ made up of four "disjoint" paths of lengths $\ell_1 \leq \ell_2 < \ell_3 \leq \ell_4$ joining two distinct vertices. This graph is $\ell_1 + \ell_2$-stable, but it is not strongly $\ell_1 + \ell_2$-stable.
— By identifying all vertices belonging to the support of a same essential cycle in R, one gets a relation on the cycle space of which there is a canonical surjection with domain $Z\mathfrak{R}_R$; if that surjection is an isomorphism, then $\mathfrak{R}$ is R-strongly stable. For example, the ladders $\mathfrak{N}$ and $\mathfrak{Z}$ above are strongly ℓ-stable for every ℓ.

Lemma 5. *If $\mathfrak{R}$ is connected, R-stable, well R-connected and if R is hereditary, then an endomorphism f of $\mathfrak{R}$ fixes V_R pointwise, as soon as f_* fixes $Z\mathfrak{R}_R$ pointwise.*

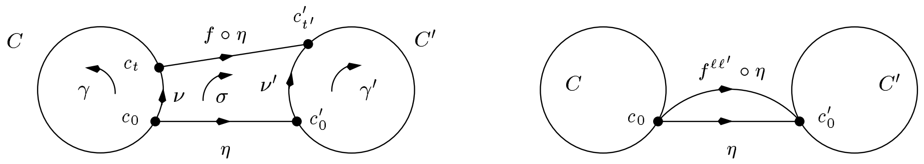
Proof. Given f such that f_* fixes $Z\mathfrak{R}_R$ pointwise, first note that $f_{|V_R}$ is an endomorphism of $\mathfrak{R}_R$ (it follows from fact 2 that for each R-principal cycle γ, $f_{|\widetilde{\gamma}}$ is an endomorphism of $\mathfrak{R}(\widetilde{\gamma})$; so each vertex (resp. edge) of $\mathfrak{R}_R$ is mapped to a vertex (resp. edge) of $\mathfrak{R}_R$). Now, according to fact 4, and as every vertex of $\mathfrak{R}_R$ belongs to a cluster of $\mathfrak{R}_R$, there remains to check that every isolated circle of $\mathfrak{R}_R$ is fixed pointwise.

For such a circle C, consider C', a and a' as in definitions 4, then $\eta = (\eta_0, \cdots, \eta_p)$ a minimal path from a to a', an isomorphism c from $\mathfrak{C}_\ell$ to $S\mathfrak{R}(C)$, an isomorphism c' from $\mathfrak{C}_{\ell'}$ to $S\mathfrak{R}(C')$, t in $\{0, \cdots, \ell-1\}$ and t' in $\{0, \cdots, \ell'-1\}$ such that (c_k and $c_{k'}$ denoting respectively $c(k+\ell\mathbb{Z})$ and $c'(k+\ell'\mathbb{Z})$) :

$$c_0 = \eta_0,\ c'_0 = \eta'_0 \text{ and for every } k \in \mathbb{Z},\ f(c_k) = c_{k+t},\ f(c'_k) = c'_{k+t'}.$$

We have to prove that t is equal to zero. Denote by ν and ν' the paths $(c_0, \cdots, c_t)$ and $(c'_0, \cdots, c'_{t'})$, and consider the cycles

$$\sigma = \overline{\nu} + \overline{f \circ \eta} - \overline{\nu'} - \overline{\eta}, \qquad \gamma = \overline{(c_0, \cdots, c_\ell)} \text{ and } \gamma' = \overline{(c'_0, \cdots, c'_{\ell'})}.$$



One would easily check that, for every integers α_1 and α_2 :

$$\sum_{r=\alpha_1\ell\ell'}^{\alpha_2\ell\ell'} f_*^r \cdot \sigma = (\alpha_2 - \alpha_1)\cdot(\ell' t \cdot \gamma - \ell t' \cdot \gamma') + \overline{f^{\alpha_2\ell\ell'} \circ \eta} - \overline{f^{\alpha_1\ell\ell'} \circ \eta}.$$

As the $f^{\alpha\ell\ell'}$'s are all paths of minimal length from a to a', there are α_1 and α_2 such that $\alpha_1 < \alpha_2$ and

$$\sum_{r=\alpha_1\ell\ell'}^{\alpha_2\ell\ell'} f_*^r \cdot \sigma = (\alpha_2 - \alpha_1)\cdot(\ell' t \cdot \gamma - \ell t' \cdot \gamma'). \tag{9}$$

Let X denote the set of R-principal cycles distinct from γ and from γ'. Consider $p \in \mathbb{Z}, \sigma' \in \langle X \rangle$ and $\rho \in \langle R \rangle$ such that $\sigma = p\cdot\gamma + \sigma' + \rho$. As f_* fixes $Z\mathfrak{R}_R$ pointwise and fixes $\langle R \rangle$ globally, and as τ_R is null on R, for every integer r, $\tau_R \cdot f_*^r \cdot \sigma = p \cdot \tau_R \cdot \gamma + \tau_R \cdot \sigma'$. Now, as τ_R is one-to-one on $R^\perp$, this, together with equality (9), yields :

$$(\alpha_2 - \alpha_1)\ell\ell' p \cdot \gamma + \sigma' = (\alpha_2 - \alpha_1)\cdot(\ell' t \cdot \gamma - \ell t' \cdot \gamma').$$

Finally, observe that $\mathbb{Z}\cdot\gamma \cap \langle X \rangle = \{0\}$ and hence, that $t = p\ell$. □

Remarks 8. — When the hypothesis of well R-connectedness is trivially satisfied (ie. when $\mathfrak{R}_R$ has no isolated cycle -which is often the case in practice-), the conclusion of the lemma is a direct consequence of facts 2 and 4. However,
— The conclusion of the lemma may fail if $\mathfrak{R}$ is not well R-connected : For p and q two integers such that $4 < p < 2q$, define the graph $\mathfrak{I}_\mathbb{Z}\square\mathfrak{I}_p$ on $\mathbb{Z}\times\{0, \cdots, p\}$ by $(n,m) \sim (n',m') \Leftrightarrow |n-n'|+|m-m'| \leq 1$, and denote by σ its shift automorphism which maps each (n,m) to $(n+1,m)$. Now consider the graph $\mathfrak{G}$ obtained from $\mathfrak{I}_\mathbb{Z}\square\mathfrak{I}_p$ by identifying, for each $k \in \{1, \cdots, p\}$, all the elements of $(k+q\mathbb{Z})\times\{0\}$ with one another and all the elements of $(k+q\mathbb{Z})\times\{p\}$ with one another (the strip $\mathfrak{I}_\mathbb{Z}\square\mathfrak{I}_p$ is rolled, and then stuck along its borders). The set $Z_4\mathfrak{G}$ is hereditary, $\mathfrak{G}$ is strongly 4-stable and $\mathfrak{G}_4$ contains two circles ; but σ defines an automorphism of $\mathfrak{G}$ which does not fix V_4 pointwise, while σ_* fixes $Z\mathfrak{G}_4$ pointwise.

Examples 2. — The two-ends ladder $\mathfrak{Z}$ is n-strongly projective for every n (recall that such a graph must be infinite). Indeed, let m denote $\max(4, n+1)$, recall that $\mathfrak{Z}$ is strongly m-stable ; its being m-rigid is then a consequence of the above lemma together with the following fact : The vertex set V of $\mathfrak{Z}$ is equal to the union $\cup_{k\in\mathbb{N}} V_m(k)$, where $V_m(0)$ is the vertex set of $\mathfrak{Z}_m$ and, for each k, $V_m(k+1)$ is the union of the images of the totally geodesic paths with both their extremities in $V_m(k)$ (a path is totally geodesic if it is the only path of minimal length between its extremities) —here $V = V_m(2)$—.

— The one-end ladder $\mathfrak{N}$ is strongly m-stable for evey m, but it is m-rigid only for the m's stricly smaller than its girth. It is however n-strongly projective for every n ; this can be checked (given proposition 2, corolary 2 and lemma 5), by considering a decomposition $V = \cup_{k\in\mathbb{N}} V_m(k)$ with $V_m(0)$ equal to the vertex set, and each $V_m(k+1)$ equal to the union of the images of all injective paths $(c_1, \cdots, c_n)$ with both their extremities in $V_m(k)$ and such that for some k, the paths $(c_1, \cdots, c_k)$ and $(c_k, \cdots, c_n)$ are totally geodesic.

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