Kit Algebras

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Abstract

One of the main tools in representation theory of finite-dimensional algebras is the one-point extension technique. It allows to determine the representation type of an algebra whose quiver is obtained by iterated addition of sinks or sources. For large algebras, however, it seems more appropriate to glue together several parts at once. We introduce such a technique here. The resulting algebras are called kit algebras. We present a criterion when a kit algebra is tame.

In this paper we introduce a method to construct certain tame algebras by "glueing" smaller algebras together. These are called "kit algebras". This method extends the well-known one-point extension technique and uses essentially the Gelfand-type matrix problems:

Let Q be a quiver and I an admissible ideal of the path algebra kQ over an algebraically closed field k. We consider the algebra $A = kQ/I$. If s is a source in Q, then A is called a one-point extension of the algebra A_0, the latter being obtained from A by deleting the vertex s. The A-modules can be interpreted as the category of subspaces of A_0-modules [21]. This technique has been applied successfully to determine the representation type of an algebra, first in [10] for representation-finite hereditary algebras.

Recall that an algebra A is tame if for any dimension d there are finitely many 1-parameter families of A-modules which cover (up to isomorphism) almost all indecomposable d-dimensional A-modules. If the minimal number of such families in each dimension d is bounded by a polynomial in d, then A is called of polynomial growth, otherwise of non-polynomial growth.

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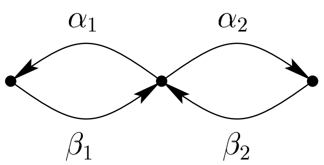
For non-polynomial growth tame algebras the one-point extension technique turns out to be inadequate and it seems more appropriate to glue together several algebras in one step instead of just adding one vertex at a time. But doing so, one has to replace the subspace category by a more general type of matrix problem.

This leads to the second ingredient, the Gelfand-type matrix problem. In 1970, I.M.Gelfand [13] reduced the study of Harish-Chandra-modules over $sl_2(\mathbb{C})$ to the classification of modules over the algebra given by the following quiver with relation $\alpha_1\beta_1 = \alpha_2\beta_2$:



Nazarova and Roiter [16] first showed that this algebra is tame (and of non-polynomial growth) using a class of "self-reproducing" matrix problems which we refer to as ones of "Gelfand-type". It is remarkable that this approach solved an entire class of matrix problems: The proof in [16] uses induction on the dimension of a module. The fact that the original problem is of non-polynomial growth leads in each induction step to a variety of new problems.

Crawley-Boevey [2] extended this result to a larger class of algebras, called "clannish algebras", and described their indecomposable representations. In his work, he used the functorial filtration method. The original approach of [16] was completed in [1] and [4], the latter including an abstract description of Gelfand-type matrix problems in terms of matrices over bimodules which turns out to be very useful for our approach.

We present in the first section a general method which reduces the study of modules over algebras to certain bimodule problems. In the second part we introduce the concept of a kit algebra and formulate the main theorem. The remaining two sections are devoted to the proof of this theorem and some applications.

We refer to [22] and [12] for general background on representations of finite-dimensional algebras.

1 From Algebras to Bimodules

1.1 Notation

Throughout the paper we fix an algebraically closed field k.

Algebras will be presented in the form $A = kQ/I$ where Q is a quiver and I is an admissible ideal of the path algebra kQ. The quiver $Q = (Q_0, Q_1, t, h)$ is described by its set of vertices Q_0, the set of arrows Q_1 and two maps $t, h : Q_1 \to Q_0$ which determine tail and head of each arrow.

By $\operatorname{mod} A$ we denote the category of finite-dimensional left A-modules. Its objects are the families $X = (X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where the X_i are finite-dimensional vector spaces, and the $X_\alpha : X_{t\alpha} \to X_{h\alpha}$ are k-linear maps such that $X_\xi = 0$ for all $\xi \in I$. Here X_ξ is defined in the following way: The element $\xi \in I$ is a linear combination of paths w in kQ, $\xi = \sum c_w w$ with $c_w \in k$. We set $X_w = X_{\alpha_n} \circ \ldots \circ X_{\alpha_1}$ for any path $w = \alpha_1 \ldots \alpha_n$ and define $X_\xi = \sum c_w X_w$.

Of course, it is sufficient to require $X_\xi = 0$ for all $\xi \in R$, where R is some set generating the ideal I.

We will see that modules over algebras are closely related to the following concept of B-matrices.

DEFINITION. Let $\mathcal{A}$ be a Krull-Schmidt k-category. A *bimodule over* $\mathcal{A}$ is a k-bilinear functor $B : \mathcal{A}^{op} \times \mathcal{A} \to \operatorname{mod} k$. By $\operatorname{mat} B$ we denote the category of *B-matrices.* Its objects are the pairs (X, f), where X is an object of $\mathcal{A}$ and $f \in B(X, X)$. The morphisms from (X, f) to (Y, g) are the $\phi \in \mathcal{A}(X, Y)$ such that $f\phi = \phi g$. We use here the notation $f\phi$ for the element $B(X, \phi)(f) \in B(X, Y)$ and analogously $\phi g = B(\phi, Y)(g)$.

1.2 Reduction

The following reduction replaces a chosen subalgebra by its module category. In this way we obtain an embedding of $\operatorname{mod} A$ into some $\operatorname{mat} B$.

Let Q'_1 be a set of arrows of Q and denote by $Q''_1 = Q_1 \backslash Q'_1$ its complement. We write A' for the subalgebra of A which is generated by Q'_1. More precisely, $A' = kQ'/I'$, where the quiver Q' has the same vertices as Q and arrows Q'_1. The ideal I' is defined as $I' = I \cap kQ'$.
We define a bimodule B over the Krull-Schmidt category $\operatorname{mod} A'$ by

$$B(X, Y) = \prod_{\alpha'' \in Q''_1} \operatorname{Hom}_k(X_{t\alpha''}, Y_{h\alpha''}) \, .$$

The multiplication with morphisms $\phi = (\phi_i : X_i \to Y_i)_{i \in Q_0}$ of $\operatorname{mod} A'$ is given by the composition of maps.

Lemma 1.1 *The functor $F : \operatorname{mod} A \to \operatorname{mat} B$ which maps $(X_i, X_\alpha)_{i\in Q_0,\alpha\in Q_1}$ to $((X_i, X_{\alpha'}), X_{\alpha''})_{i\in Q_0,\alpha'\in Q_1',\alpha''\in Q_1''}$ is full and faithful. Moreover, F is an equivalence if and only if I' generates I.*

Proof: That F is full and faithful follows directly from the construction. To show the second statement, let J denote the ideal of kQ generated by I' and set $\Lambda = kQ/J$. Since $I' \subset kQ'$, a kQ-module (X_i, X_α) lies in $\operatorname{mod}\Lambda$ precisely when the kQ'-module $(X_i, X_{\alpha'})$ lies in $\operatorname{mod} A'$. Hence the functor $G : \operatorname{mod}\Lambda \to \operatorname{mat} B, (X_i, X_\alpha) \mapsto ((X_i, X_{\alpha'}), X_{\alpha''})_{i\in Q_0,\alpha'\in Q_1',\alpha''\in Q_1''}$ is an equivalence. Since F is the composition of the canonical embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod}\Lambda$ with G, it is clear that F is an equivalence if and only if this embedding is an equivalence, i.e., when $J = I$. □

The lemma allows us to identify $\operatorname{mod} A$ with a full subcategory of $\operatorname{mat} B$. In general it is quite difficult to describe this subcategory by internal data of B.

We avoid these difficulties when F is an equivalence. This can be achieved by choosing a large subalgebra A'. But on the other hand, we want to keep A' small, because $\operatorname{mod} A'$ is supposed to be known when we investigate the bimodule B.

Example: One of the most frequently used examples is the one-point extension technique, first applied in [10]: Suppose $A = kQ/I$ admits a source $s \in Q_0$ and define Q_1'' to be the set of all arrows starting in s (and set $Q_1' = Q_1 \backslash Q_1''$). Then $\operatorname{mod} A'$ decomposes as $\operatorname{mod} A' = \operatorname{mod} k \times \operatorname{mod} A_0$, where A_0 is obtained from A by deleting the vertex s and all arrows $\alpha \in Q_1''$. Hence, the reduction bimodule B becomes

$$B((V,X),(W,Y)) = \prod_{\alpha\in Q_1''} Hom_k(V, Y_{t\alpha}) \cong \operatorname{Hom}_k(V, \operatorname{Hom}_{A_0}(P,Y)),$$

where the A_0-module P is defined as $P = \bigoplus_{\alpha\in Q_1''} P(t\alpha)$ and $P(t\alpha)$ denotes the projective module generated by the vertex $t\alpha$.

Consider $M = \operatorname{rad} P(s)$ as a submodule of P and define a subbimodule C of B by

$$C((V,X),(W,Y)) = \operatorname{Hom}_k(V, \operatorname{Hom}_{A_0}(M,Y)).$$

It is well known ([21]) that the image of $\operatorname{mod} A$ in $\operatorname{mat} B$ under the reduction-functor F is equivalent to $\operatorname{mat} C$, hence it is given by a suitable subbimodule of B.

There are many generalizations of this example; for instance, the case of a local extension: Here we fix a vertex s in $A = kQ/I$ which may have loops, but besides these only arrows starting in s. As before, let Q_1'' be the set of all arrows $s \xrightarrow{\alpha} t$ with $s \neq t$. The corresponding reduction-functor F is examined in [15] in detail and its image is given by a certain subbimodule of the reduction bimodule B. Unfortunately, we cannot expect such an easy description in general. See [6] for a more complicated situation.

REMARKS: As we have seen, if $A = kQ$ is hereditary (and Q' arbitrary), then F is an equivalence. Conversely, if B is a bimodule over a *semisimple* Krull-Schmidt category $\mathcal{A}$, we define a quiver Q in the following way: As vertices we take the isoclasses of indecomposable objects of $\mathcal{A}$ and the number of arrows from x to y equal to $\dim_k B(x,y)$. Then $\operatorname{mat} B$ is equivalent to $\operatorname{mod} kQ$.

Thus, the concepts of modules over hereditary algebras and of matrices over semisimple bimodules coincide. But they generalize in different directions: For algebras we consider subcategories given by an ideal of A, whereas for bimodules we enlarge the morphism spaces by passing to a more general $\mathcal{A}$.

One may try to unify these methods to obtain bimodules (resp. biquivers) with relations. The general framework is that of a BOCS, and in that situation there exists also a reduction with respect to a chosen substructure ([14],[8],[24]; a survey of the various methods is given in [3]).

But since bimodules with relations (or equivalently, "non-free" BOCSes) have been investigated only rarely so far ([18],[9]), we preferred to present the reduction of Lemma 1.1 in an elementary way.

2 Garlands and the main Theorem

2.1 Hammocks

An algebra A is said to be *representation-directed* if its Auslander-Reiten quiver Γ_A is finite and directed.

For any vertex a of a representation-directed algebra $A = kQ/I$, we denote by $H(a)$ the full subquiver of Γ_A formed by the A-modules X with $X_a \neq 0$. The quiver $H(a)$ is equipped with a translation $\tau_{H(a)}$ (which is induced by the Auslander-Reiten-translate τ_A on Γ_A) and a function $h_{H(a)}$ which maps each vertex X of $H(a)$ to $\dim_k X_a$.

REMARK: The triple $(H(a), \tau_{H(a)}, h_{H(a)})$ is called a *hammock*. We refer to [23] for a general definition and recall from there some basic properties:

(i) Each hammock $H = (H, \tau, h)$ has a unique sink and source. In the case $H = H(a)$ considered here, the source is the projective A-module $P(a)$, whereas the sink is the injective module $I(a)$.

(ii) The map h is additive on short exact sequences. In particular, if some $X \in H(a)$ has a unique direct predecessor in $H(a)$, then τX is not contained in $H(a)$.

2.2 Garlands

Definition. Let $H = (H, \tau, h)$ be a hammock. We call H a *garland* provided

(G1) any $x \in H_0$ has at most two direct successors (in H) and

(G2) if x has two direct successors y_1 and y_2, then $\tau^- x$ is the unique direct successor of y_1 and y_2 in H.

$$\begin{tikzcd} & y_1 \arrow[dr] & \\ x \arrow[ur] \arrow[dr] & & \tau^- x \\ & y_2 \arrow[ur] & \end{tikzcd}$$

Remarks:

1) We borrowed the name garland from the theory of partially ordered sets, where it refers to the situation when any point has at most one incomparable point (these posets are also called semichains or rods). Accordingly, we call the points y_1 and y_2 in (G2) *incomparable.*

2) If H is a garland, then the hammock-function h has value 1 on all vertices (See [23]).

3) Since hammocks have a unique source, any garland also satisfies the dual conditions:

$(G1)^*$ any $x \in H_0$ has at most two direct predecessors and

$(G2)^*$ if x has two direct predecessors y_1 and y_2, then τx is the unique direct predecessor of y_1 and y_2 in H.

Example: Let Λ_1 be the algebra given by the quiver Q of Figure 2 with relation $\alpha_1\beta_1 = \alpha_2\beta_2$. Then $H(a)$ is a garland with one pair of incomparable points. Note that $\tau^- P(a)$ has 3 direct successors in Γ_A , but among them, only $I(a)$ lies in $H(a)$.

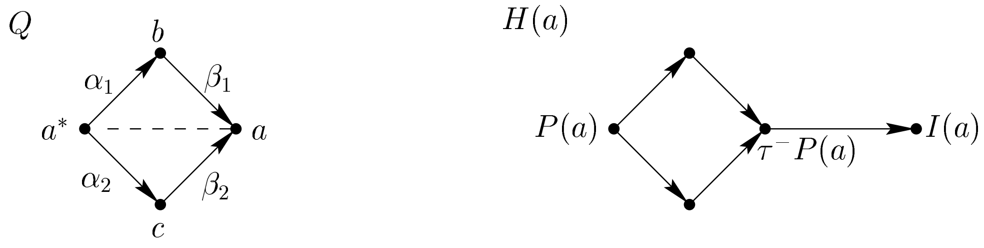


Figure 2.

2.3 Glueing of algebras

Let $Q = (Q_0, Q_1, t, h)$ be a quiver. We do not assume Q to be connected.

The following procedure formally describes how to obtain a new quiver Q^* from Q by glueing pairs of vertices. For that purpose, fix an involution $a \mapsto a^*$ on Q_0. Using the notation $\overline{a} = \{a, a^*\}$, the quiver $Q^* = (Q_0^*, Q_1^*, t^*, h^*)$ is defined as follows:

$$Q_0^* = \{\overline{a} : a \in Q_0\}\,,$$

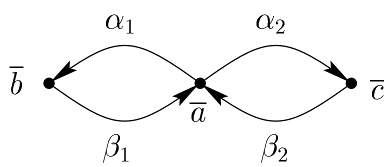
$$Q_1^* = Q_1\,,$$

$$h^*(\alpha) = \overline{h(\alpha)} \text{ and } t^*(\alpha) = \overline{t(\alpha)} \text{ for any } \alpha \in Q_1.$$

Now let $A = kQ/I$ be an algebra. From the definition of the tail and head of an arrow $\alpha \in Q_1^*$ it follows that any path in Q is also a path in Q^*. Hence we can regard I as a subset of kQ^* (but it will not be an ideal in general). Let I^* be the ideal of kQ^* that is generated by I and set $A^* = kQ^*/I^*$.

EXAMPLES:

1) Let Λ_1 be the algebra of the preceding example, with involution as indicated in Figure 2: The vertices a and a^* are interchanged, whereas b and c are fixed. Then Λ_1^* is given by the following quiver with relation $\alpha_1\beta_1 = \alpha_2\beta_2$:



2) The algebra Λ_2, given by the quiver Q^* in Figure 3 with relation $\beta_1\beta_2\beta_3 = 0$ is obtained by glueing a quiver Q with 3 components, one of them consisting of β_1, β_2 and β_3, in the vertices $\overline{a}$ and $\overline{b}$.

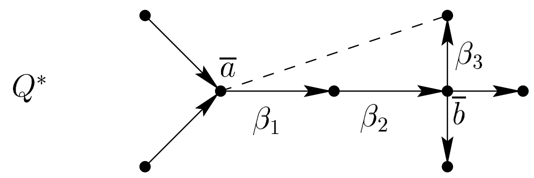


Figure 3

The vertices a of A with $a \neq a^*$ are those where the quiver Q is glued together; we therefore call them *connecting* vertices.

Theorem 2.1 *Let A be an algebra together with an involution on the vertices, and A^* the algebra constructed from A as above. Then A^* is tame provided*

(K0) The algebra A is representation-directed;

(K1) The hammock $H(a)$ of any connecting vertex a is a garland;

(K2) Whenever a and b are different connecting vertices of A, the intersection $H(a) \cap H(b)$ contains no arrows.

In this case, we call A^* a kit algebra.

2.4 Examples

It is easy to check that the algebras Λ_1 and Λ_2 of the preceding example are kit algebras. In fact, both are known to be tame: We discussed already at the beginning that Λ_1 is a clannish algebra ([2]). The second algebra Λ_2 is a pg-critical algebra ([17]).

Note that one may use the various components to build larger tame algebras which are neither clannish nor pg-critical. Such an example is given in Figure 4. For more examples we refer to the last section.

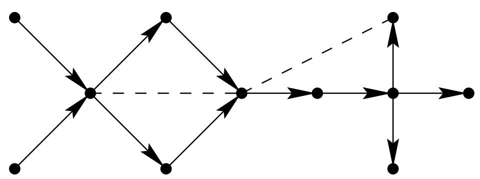


Figure 4

2.5 More on garlands

Lemma 2.2 *Suppose A is an algebra with an involution on the vertices satisfying condition (K0),(K1) and (K2) of the theorem. Then it also satisfies*

(K3) If for some connecting vertex a of A the hammock $H(a)$ contains a pair of incomparable points Y_1, Y_2, then neither Y_1 nor Y_2 are contained in $H(b)$, for any connecting vertex $b \neq a$.

Proof: Suppose $a \neq b$ are connecting vertices in A such that $H(a)$ contains a "mesh" formed by points $X, Y_1, Y_2, \tau^- X$ as in Figure 5 and Y_1 is contained in $H(b)$. Of course, Y_1 is no isolated vertex in $H(b)$, so we may suppose that Y_1 has a direct successor Z in $H(b)$ (otherwise, there has to be a direct predecessor, and we apply the dual arguments).

By (K2) the point Z is not contained in $H(a)$. Therefore, it cannot be projective, since the irreducible map from Y_1 to Z leaves $H(a)$. Hence τZ exists, and by the additivity of the hammock function $h_{H(a)}$, it lies in $H(a)$. But on the other hand, $\tau Z \neq X$ (otherwise, $Z = \tau^- X \in H(a)$), hence Y_1 does not satisfy property $(G2)^*$ in the definition of a garland. $\square$

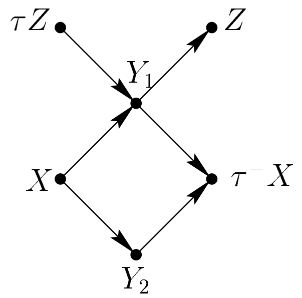


Figure 5

Remark: Under the assumption that (K0) and (K1) hold, the properties (K2) and (K3) are in fact equivalent.

Lemma 2.3 *Suppose A is an algebra with an involution on the vertices satisfying condition (K0) and (K2) of the theorem. Then it also satisfies*

*(K4) Let a, b, c be pairwise different connecting vertices.
Then the intersection $H(a) \cap H(b) \cap H(c)$ is empty.*

Proof: Suppose X lies in $H(a) \cap H(b) \cap H(c)$. Since hammocks have no common sink, there is some direct successor Z of X with, say, $Z \in H(a)$. By (K2), Z is not contained in $H(b) \cup H(c)$. Hence, Z is not projective and $\tau Z \in H(b) \cap H(c)$. Thus, $H(b) \cap H(c)$ contains a common arrow which contradicts Lemma 2.3. □

Remark: The intersection of two hammocks $H(a)$ and $H(b)$ may contain several points in general, even if all conditions (K0) to (K4) are satisfied. To illustrate this, we consider the algebra given by the quiver of Figure 6 with relations $\alpha_1\beta_1 = 0 = \alpha_2\beta_2$. In this case $H(a) \cap H(b)$ contains two points.

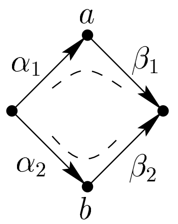


Figure 6

Suppose the algebra A satisfies (K0) to (K4) and there are distinct connecting vertices a, b such that $H(a) \cap H(b)$ contains two different points X and Y. By (K3), the points X and Y are comparable inside each hammock, hence they are connected by a path in $H(a)$ and a path in $H(b)$. Since A is representation-directed, these paths point in the same direction, say from X to Y and by (K2) they share no common arrow. It follows from [23] that these paths correspond to non-zero radical morphisms from X to Y which are linearly independent: The only possibility for them being congruent is that Γ_A contains a subquiver with source X and sink Y which consists only of Auslander-Reiten-sequences with 2 middle terms. But the remark (ii) in section 2.1 shows that a garland containing X and Y cannot be embedded into such a subquiver.

3 Proof of the Theorem

The proof of theorem 2.1 proceeds in three steps: First, we replace A^* by some slightly larger algebra $\overline{A}$. The reason for this enlargement is that we separate the relations this way and thus are able to apply the reduction described in the first section. So in step 2 we obtain a bimodule B which is shown to be tame in the last step. Then the assumptions (K0),(K1) and (K2) enter the proof.

3.1 The algebra $\overline{A}$

As in section 2.3, let Q be a quiver with involution $a \mapsto a^*$ on Q_0, and $A = kQ/I$ an algebra. To obtain A^*, we glued each set $\{a, a^*\}$ of connecting vertices to one vertex. Instead, in case of $\overline{A}$ we introduce a new vertex $\overline{a} = \{a, a^*\}$ together with arrows

$$a \xleftarrow{(\overline{a},a)} \overline{a} \xrightarrow{(\overline{a},a^*)} a^* \,.$$

Thus the quiver $\overline{Q}$ of $\overline{A}$ has vertices $\overline{Q}_0 = Q_0 \coprod \{\overline{a} \,|\, a \neq a^*\}$ and arrows $\overline{Q}_1 = Q_1 \coprod \{(\overline{a}, a) \,|\, a \in Q_0, a \neq a^*\}$. As for the construction of A^*, the ideal $\overline{I}$ of $\overline{A} = k\overline{Q}/\overline{I}$ is generated by I.

Obviously, $\mathrm{mod}\, A^*$ is equivalent to the full subcategory of $\mathrm{mod}\, \overline{A}$ consisting of those modules (X_i, X_α) whose maps X_α are bijective whenever α is one of the newly introduced arrows $(\overline{a}, a), (\overline{a}, a^*)$. To proof Theorem 2.1 it is therefore sufficient to show that the algebra $\overline{A}$ is tame.

3.2 The bimodule B

Let A' be the subalgebra of $\overline{A}$ generated by the "old" arrows $\alpha \in Q_1$. Hence A' is obtained from A by adding an isolated vertex $\overline{a}$ for each set $\{a, a^*\}$ with $a \neq a^*$. Of course, the ideal I' of $A' = kQ'/I'$ coincides with I and generates $\overline{I}$. Therefore, by section 1, the categories $\mathrm{mod}\, \overline{A}$ and $\mathrm{mat}\, B$ are equivalent, where the bimodule B over $\mathrm{mod}\, A'$ is defined by

$$B(X, Y) = \prod_{\substack{a \in Q_0 \\ a \neq a^*}} \mathrm{Hom}_k(X_{\overline{a}}, Y_a) \,.$$

For any $v \in Q'_0 = Q_0 \coprod \{\overline{a} \,|\, a \neq a^*\}$, let $H_v : \mathrm{mod}\, A' \to \mathrm{mod}\, k$ be the functor mapping X to $\mathrm{Hom}_{A'}(P(v), X)$. As in the example of 1.2, $P(v)$ denotes the projective A'-module with simple top in v. Since $H_v(X)$ is naturally isomorphic to X_v, we may define B also as

$$B(X, Y) = \prod_{\substack{a \in Q_0 \\ a \neq a^*}} \mathrm{Hom}_k(H_{\overline{a}}(X), H_a(Y)) \,.$$

3.3 A tame bimodule

We recall the following Theorem from [4]:

Let $\mathcal{A}$ be a Krull-Schmidt category and let M_i^-, M_i^+ be k-linear functors from $\mathcal{A}$ to $\mathrm{mod}\, k$, for $i = 1, \ldots, n$. Define a bimodule M over $\mathcal{A}$ by

$$M(X, Y) = \prod_{i=1}^{n} \mathrm{Hom}_k(M_i^-(X), M_i^+(Y)) \,.$$

We denote by $\mathcal{R}_{\mathcal{A}}$ the radical of $\mathcal{A}$ and by $\mathcal{R}_i^\epsilon(x,y)$ the set of all $f \in \mathrm{Hom}_k(M_i^\epsilon(x), M_i^\epsilon(y))$ satisfying $f(N(x)) \subset \mathcal{R}_{\mathcal{A}}N(y)$ for each subfunctor N of M_i^ϵ.

Theorem 3.1 ([4]) *The bimodule M is tame provided the following conditions are satisfied:*

(R0) $\mathcal{A}$ has only finitely many indecomposable objects (up to isomorphism);

(R1) For each i, the lattices of subfunctors of M_i^- and M_i^+ are garlands;

(R2) For indecomposable objects $x, y \in \mathcal{A}$, the canonical map

$$\theta(x,y) : \mathcal{R}_{\mathcal{A}}(x,y) \to \prod_{i=1}^{n} \mathcal{R}_i^-(x,y) \times \mathcal{R}_i^+(x,y)$$

is surjective;

(R3) If for some indecomposable objects $x, y \in \mathcal{A}$ the subfunctors of M_i^ϵ generated by elements $\xi \in M_i^\epsilon(x)$ and $\eta \in M_i^\epsilon(y)$ are incomparable for some i and some $\epsilon \in \{-,+\}$, then

$$\sum_{j,\eta} \dim_k M_j^\eta(x) = 1 = \sum_{j,\eta} \dim_k M_j^\eta(y)\,;$$

(R4) For each indecomposable object $x \in \mathcal{A}$,

$$\sum_{i=1}^{n} (\dim_k M_i^-(x) + \dim_k M_i^+(x)) \leq 2\,.$$

3.4 Checking the conditions

Since the bimodule B of 3.2 is already given by pairs of functors from $\mathcal{A} = \mathrm{mod}\, A'$ to $\mathrm{mod}\, k$, we only have to show that the conditions (R0) to (R4) from Theorem 3.1 are satisfied.

Of course, (R0) follows from the property (K0) of Theorem 2.1 and we show now that also the remaining conditions (Ri), $i = 1, \ldots, 4$ are implied by the corresponding (Ki) from section 2. In the notation of 3.3, the functors M_i^- are of the form $H_{\overline{a}}$, and the M_i^+ are of the form H_a. Each $H_{\overline{a}}$ appears twice, once for a and a second time for a^*, whereas the H_a occur only once for each a.

The Auslander-Reiten quiver $\Gamma_{A'}$ of A' is obtained from Γ_A by adding an isolated vertex $\overline{a}$ for any set $\{a, a^*\}$ with $a \neq a^*$ (we identify the vertex $\overline{a}$ of $\overline{Q}$ with the simple module concentrated in $\overline{a}$).

3.4.1

We first show that (R1) to (R4) hold whenever one of the isolated vertices $\overline{a}$ is involved: The functor $H_{\overline{a}}$ is simple, hence its lattice of subfunctors is a garland and the situation of (R3) cannot occur when $M_i^\epsilon = H_{\overline{a}}$. Also, $\mathcal{R}_i^-(X,Y) = 0$ for any $X, Y \in \Gamma_{A'}$ and $\mathcal{R}_i^+(X,Y) = 0$ whenever X or Y equals $\overline{a}$.

We already discussed that $H_{\overline{a}}$ appears twice, hence

$$\sum_{i=1}^{n}(\dim_k M_i^-(\overline{a}) + \dim_k M_i^+(\overline{a})) = 2$$

and so (R4) holds for $X = \overline{a}$.

3.4.2

For the rest of this section we fix some $X, Y \in \Gamma_A$ and a connecting vertex $a \in Q_0$. It remains to show that these data satisfy (R1) to (R4).

On (R1): We pointed out in section 2.2, remark 2 that $\dim H_a(X) \leq 1$ for all $X \in \Gamma_A$. Hence the functor H_a is completely determined by its support, which is just the hammock $H(a)$. By (K1), we know that $H(a)$ is a garland. Therefore, the subfunctors of H_a are generated either by a single $X \in H(a)$ or by two incomparable modules $Y_1, Y_2 \in H(a)$. The sublattice of all subfunctors of H_a with one generator is isomorphic to the hammock $H(a)$. The additional subfunctors with 2 generators are comparable to all other subfunctors, hence the lattice of H_a is a garland.

On (R4): This is a direct consequence of (K4).

On (R2): As already discussed, we just have to consider the map

$$\theta(X,Y) : \mathcal{R}_{\mathcal{A}}(X,Y) \longrightarrow \prod_{\substack{a \in Q_0 \\ a \neq a^*}} \mathcal{R}_a^+(X,Y)\,, \quad \rho \mapsto (H_a(\rho))_a\ .$$

By (R4), at most two of the factors $\mathcal{R}_a^+(X,Y)$ are non-zero and each factor is at most one-dimensional. Consider first one single factor: If N denotes the subfunctor of H_a generated by X, we have $\mathcal{R}N(Y) \neq 0$. Therefore, there is a $\rho \in \operatorname{rad}(X,Y)$ with $H_a(\rho) \neq 0$ and (R2) holds for one non-trivial factor.

Now suppose there are 2 non-trivial factors: There are distinct connecting vertices a, b with $X, Y \in H(a) \cap H(b)$. But then we know from the discussion in 2.5 that (K2) implies the existence of two linearly independent radical morphisms with non-zero image under $H(a)$ resp. $H(b)$.

On (R3): As discussed above, incomparable subfunctors N_1, N_2 are generated by indecomposable modules $Y_1, Y_2 \in H(a)$. By condition (K3) we know that Y_1, Y_2 are contained in no other $H(b)$, hence (R3) holds. □

3.5 Tameness

We finally show that the tameness of B implies that the algebra $\overline{A}$ is tame and Theorem 2.1 holds.

Definition. Let M be a bimodule over a Krull-Schmidt-category $\mathcal{A}$. For a given $X \in \mathcal{A}$ and elements $e, f \in M(X,X)$ we call the affine subspace $L = e + kf$ of the vector space $M(X,X)$ a *line* of M-matrices.

The bimodule M is *tame* if for any $X \in \mathcal{A}$ there are lines $L_1, \ldots, L_n \subset M(X,X)$ of M-matrices such that any indecomposable M-matrix (X,f) is isomorphic to some M-matrix lying in one of the lines L_i.

The corresponding definition for algebras is the following: Let $\Lambda = kQ/I$ be an algebra. For a Q_0-graded vector space $X_0 = (X_i)_{i \in Q_0}$ we denote by $\mathrm{mod}_\Lambda(X_0) \subset \prod_{\alpha \in Q_1} \mathrm{Hom}_k(X_{t\alpha}, X_{h\alpha})$ the affine variety of Λ-modules with underlying vector space X_0.

Then the algebra Λ is *tame* if for any Q_0-graded finite-dimensional vector space X_0 there are lines $L_1, \ldots, L_n \subset \mathrm{mod}_\Lambda(X_0)$ of Λ-modules such that any indecomposable Λ-module $X \in \mathrm{mod}_\Lambda(X_0)$ is isomorphic to some Λ-module lying in one of the lines L_i.

We return to the algebra $\overline{A} = k\overline{Q}/\overline{I}$ of 3.1 and fix a $\overline{Q}_0$-graded finite-dimensional vector space $X_0 = (X_i)_{i \in \overline{Q}_0}$. Since A is representation-finite, there are only finitely many isomorphism classes of A'-modules X with underlying vector space X_0. As the bimodule B over $\mathrm{mod}\, A'$ is tame, there are lines $L_1, \ldots L_n, \subset B(X,X)$ for any $X \in \mathrm{mod}\, A'$ such that almost all indecomposable B-matrices over X are isomorphic to ones lying in the L_i.

By the definition of B, we can regard these lines as lines in $\mathrm{mod}_{A'}(X_0)$ and the equivalence $\mathrm{mod}\, \overline{A} \to \mathrm{mat}\, B$ yields that almost all indecomposable $\overline{A}$-modules over a fixed vector space X_0 are isomorphic $\overline{A}$-modules lying in these lines.

Remarks:

1) Theorem 3.1 yields in fact more than a family of lines, namely a complete classification of all indecomposable B-matrices for an arbitrary field.

2) As discussed in [11], the lines $L = e + kf \subset \mathrm{mod}_\Lambda(X_0)$ correspond just to the traditional $\Lambda - k[t]-$bimodules: Set $V_i = X_i \otimes_k k[t]$ and define a $k[t]$-linear map $V_\alpha : X_{t\alpha} \to X_{h\alpha}$ for any $\alpha \in Q_1$ by $V_\alpha = e_\alpha \otimes 1 + f_\alpha \otimes t$. Here we write $e = (e_\alpha) \in \prod_{\alpha \in Q_1} \mathrm{Hom}_k(X_{t\alpha}, X_{h\alpha})$.

This defines a $\Lambda - k[t]-$bimodule structure on $V = (V_i, V_\alpha)$ such that V is free of rank $\dim_k X_0$ over $k[t]$ and the functor $V \otimes_{k[t]} - : \operatorname{mod} k[t] \to \operatorname{mod} \Lambda$ maps the simple $k[t]$-modules onto the line L.

4 Applications

4.1 pg-critical algebras

Definition. Let Λ be a tame algebra. For $d \in \mathbb{N}$ we denote by $\mu_\Lambda(d)$ the minimal number of lines $L \subset \operatorname{mod}_\Lambda(X_0)$ such that almost all indecomposable d-dimensional Λ-modules are isomorphic to modules in some of the lines L.

We say Λ is of *polynomial growth* if there is a natural number m such that $\mu_\Lambda(d) \leq d^m$ for all $d \in \mathbb{N}$. Otherwise, Λ is of *non-polynomial growth.*

In [17] the pg-critical (i.e., minimal non-polynomial growth) simply connected algebras are classified and it is easy to see that all completely separating algebras among the list of [17] are kit algebras, formed by glueing pieces from Figure 7.

Remember that an algebra $\Lambda = kQ/I$ is called *completely separating schurian* if for each convex subquiver Q' of Q and any vertex $s \in Q'$ the following holds: The indecomposable direct summands of the radical of the projective module $P(s)$ over the algebra $\Lambda' = kQ'/I \cap Q'$ are pairwise non-isomorphic and lie in different connected components of the quiver obtained from Q' by deleting all predecessors of s (see [5]). Among the 31 "frames" in the list of [17], the first 11 give rise to completely separating algebras.

The algebras in Figure 7 are given by quivers with relations, where the relations are indicated by dotted lines. For the algebra (2), one may choose any orientation. The vertices are distributed into two types, small and big ones, where only the big vertices may be used as connecting vertices.

Definition. Let Λ be a tame algebra. We say Λ is *sincere* if there is an indecomposable representation X of Λ with $X_i \neq 0$ for all $i \in Q_0$. If there exists a line $L \subset \operatorname{mod}_\Lambda(X_0)$ of pairwise non-isomorphic indecomposable Λ-modules with $X_i \neq 0$ for all $i \in Q_0$, then we call Λ *infinitely sincere.*

Of course, there are plenty of possibilities how to glue algebras from Figure 7 together. For instance, we used the algebra (2) in our very first example. From the knowledge of the Auslander-Reiten quivers of the algebras in Figure 7 the following statement is easily seen:

Let Λ be a disjoint union of algebras from Figure 7. Choose any involution on the big vertices of Λ and denote by Λ^* the corresponding kit algebra. Then Λ^* is infinitely sincere if and only if any big vertex of Λ is connecting.

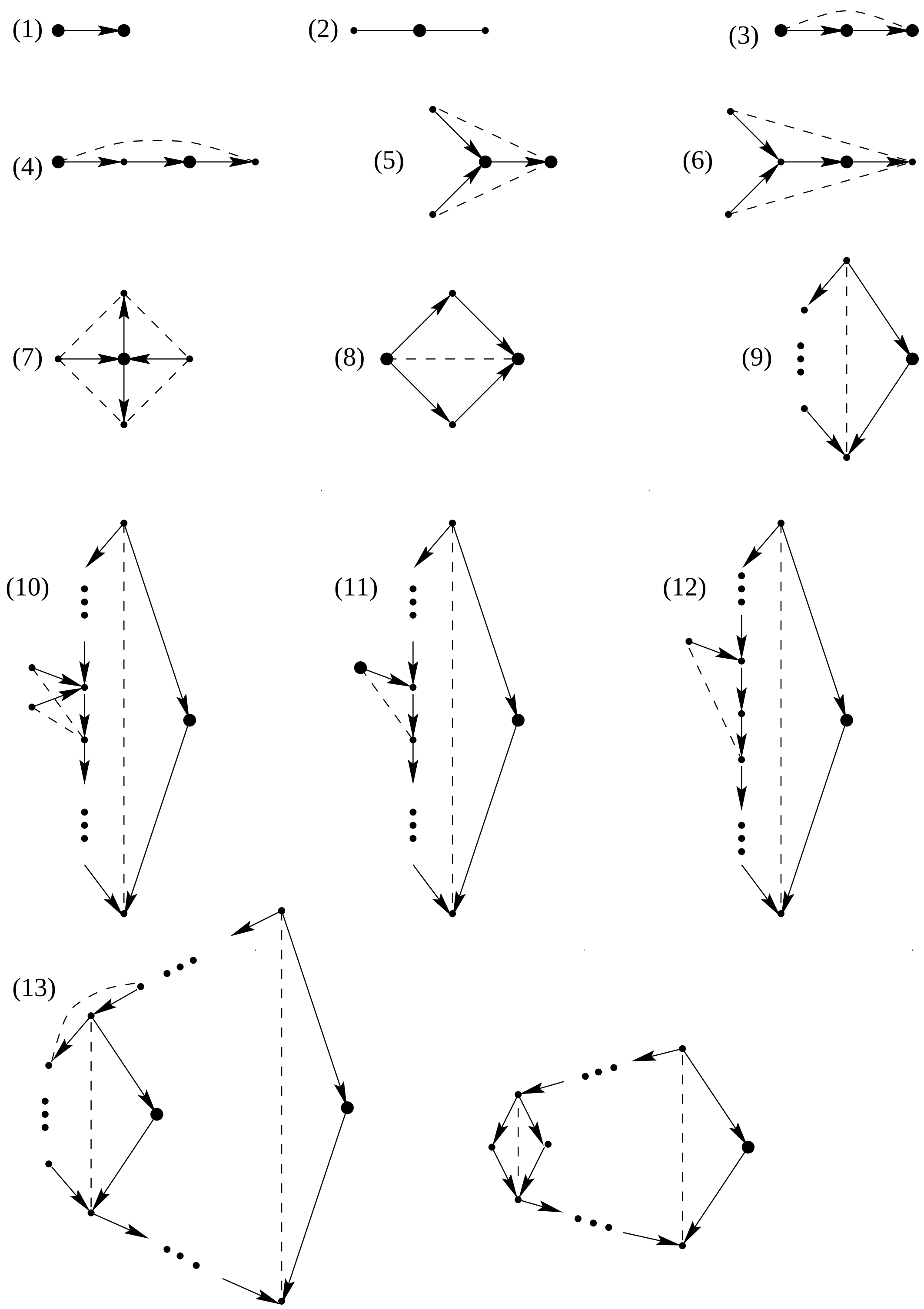


Figure 7: The kit for pg-critical completely separating algebras

4.2 Tame tree algebras

We concentrate now on tree algebras, i.e., those algebras $T = kQ/I$ whose underlying quiver Q is a tree. There are handy criteria to decide when a tree algebra T is tame of polynomial growth and these T are investigated quite well ([19],[20]). In the non-polynomial growth case, however, there are only two classes of tree algebras known so far: The clannish ones ([2]) and those of global dimension at least 2 whose Tits-form is non-negative ([7]), the latter containing the pg-critical tree algebras.

It is easy to see that the algebras in both classes are (quotients of) kit algebras and that there are much more tree kit algebras than these. In fact, we expect that the following questions can be answered positively:

Problem 1. Is any sincere regular tree algebra quotient of a kit algebra ?

Problem 2. Is any sincere non-polynomial growth tame tree algebra regular?

Here we call a tree algebra T *regular* (or *type E-free*) if no convex subalgebra of T admits a quotient which is tilted of type E_6.

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