A projection property for buildings

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An object X of a category is said to have the *projection property* if the only idempotent morphisms $f : X \times X \to X$ are the projections. Here a morphism $f : X \times X \to X$ is called idempotent if $f \circ \Delta = id$ for the diagonal map $\Delta : X \to X \times X$.

There are two motivations for studying the question whether X has the projection property. Firstly, Arrow's "dictator theorem" states that the only maps of a product X^A to X with certain properties are the projections (see [A], [PRS]). Secondly, the projection property is closely related to the fixed point property (see [C]). For a detailed discussion of the projection property, its background and connections with other properties see the paper [P] in the same volume.

In this paper we prove that an irreducible building of spherical type and of rank at least 2 has the projection property. Actually, the theorem is more general. For a precise statement of the hypotheses see below. By contrast, every reducible building and every building of rank one does not have the projection property. We also give a counterexample concerning the finiteness assumption of the theorem.

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§ 1 Statement of the theorem

Let $\mathcal{C}$ be a building, which we think of as the set of its chambers endowed with a W –distance $\delta : \mathcal{C} \times \mathcal{C} \to W$ where W is the Weyl group of the building, see e.g. [Sch 3.3.12]. So W is a Coxeter group endowed with a set $\{s_i | i \in I\}$ of generators subject to the defining relations

$$s_i^2 = (s_i s_j)^{m_{ij}} = 1 \quad .$$

Here $(m_{ij})_{i,j\in I}$ is a Coxeter matrix, i.e. we have

$$\begin{aligned} m_{ii} &= 2 \\ 2 \leq m_{ij} &= m_{ji} \leq \infty \quad . \end{aligned}$$

The corresponding *Coxeter graph* has the set I for its set of vertices and the vertices i, j are joined by an $m_{ij} - 2$ –fold edge. So i and j are not joined by an edge if $m_{ij} = 2$. We say I is connected, if the Coxeter graph is connected. The number $r = \#I$ is called the *rank* of $\mathcal{C}$ or of W .

Two chambers x, y are called *i–adjacent* if $\delta(x, y) \in \{1, s_i\}$. We write $x \sim_i y$. For $J \subset I$ define the equivalence relation $\sim_J$ by $x \sim_J y$ iff $\delta(x, y)$ is in the subgroup of W generated by $\{s_j, j \in J\}$.

The building $\mathcal{C}$ is said to have the *projection property* if the only maps $f : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ with the following properties are the two projections $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$:

$$\begin{array}{ll} (1) & f(x, x) = x \\ (2a) & f(x, y) \sim_i f(x', y) \ \text{ if } \ x \sim_i x' \\ (2b) & f(x, y) \sim_i f(x, y') \ \text{ if } \ y \sim_i y' \quad . \end{array}$$

Let us call a map with these properties an idempotent morphism. This fits under the language of categories as follows. Regard $\mathcal{C}$ as a set with a family of equivalence relations $\sim_i$, indexed by elements i of I . In the category of sets with a family of equivalence relations $\sim_i, i \in I$, the set $\mathcal{C} \times \mathcal{C}$ together with the equivalence relations $(x, y) \sim_i (x', y')$ iff $x \sim_i x'$ and $y \sim_i y'$, is a product $\mathcal{C} \times \mathcal{C}$. Then $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a morphism in this category iff f satisfies $2a$ and $2b$, and f satisfies (1) iff $f \circ \Delta = id$ for the diagonal morphism $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

Theorem

Let $\mathcal{C}$ be a building. Suppose its Coxeter diagram is connected, its rank is at least two and all the Coxeter numbers are finite. Then $\mathcal{C}$ has the projection property.

In particular, every spherical building of rank at least 2 with connected diagram has the projection property. An example of such a building is the set of maximal flags in a vector space of dimension $n \geq 3$. Another example – actually the Coxeter complex of the preceding example – is the set of all maximal chains in the poset of all subsets of a finite set of cardinality at least 3. This special case proved in [PRS] was actually the starting point for the present paper. The equivalence relation i –adjacency here is the following: Two maximal chains $c_0 < c_1 < \cdots < c_n$, $d_0 < \cdots < d_n$ are i –adjacent if $c_j = d_j$ for every $j \neq i$.

In view of Arrow's theorem the following generalization is of interest.

Addendum

Let $\mathcal{C}$ be a building as in the theorem. Then for every power $\mathcal{C}^A$ of $\mathcal{C}$ with finite set A the only idempotent morphisms $f : \mathcal{C}^A \to \mathcal{C}$ are the projections.

Here again $\mathcal{C}^A$ is the power set with the indexed family $\{\sim_i , i \in I\}$ of equivalence relations, where $(x_\alpha)_{\alpha\in A} \sim_i (y_\alpha)_{\alpha\in A}$ iff $x_\alpha \sim_i y_\alpha$ for every $\alpha \in A$. A morphism $f : \mathcal{C}^A \to \mathcal{C}$ is a map such that $x \sim_i y$ implies $f(x) \sim_i f(y)$ for every $i \in I$. The map f is called idempotent if $f \circ \Delta = id$ for the diagonal map $\Delta : \mathcal{C} \to \mathcal{C}^A$.

Recall from [PRS] that the projection property for a finite number n of factors is not implied by the projection property for $n = 2$ factors, in general, but it is implied for arbitrary n by the case $n = 3$. So the important — in fact the only not obvious — step in the proof 3.7 of the addendum is the case $n = 3$.

Remark

If A is infinite, there are examples of idempotent morphisms $f : \mathcal{C}^A \to \mathcal{C}$ different from projections.

E.g. if $\mathcal{C}$ is finite and $\mathcal{F}$ is an ultrafilter on A then for every element $\overline{x} = (x_\alpha)_{\alpha\in A}$ of $\mathcal{C}^A$ there is exactly one element $B \in \mathcal{F}$ such that all the components $x_\alpha, \alpha \in B$, are equal, say $x_\alpha = x$ for $\alpha \in B$. So $x = \lim\limits_{\mathcal{F}} \overline{x}$, if we regard the elements of $\mathcal{C}^A$ as maps from A into the discrete topological space $\mathcal{C}$. Then $f(\overline{x}) = \lim\limits_{\mathcal{F}} \overline{x}$ defines an idempotent morphism different from any projection unless $\mathcal{F}$ is a principal ultrafilter. Recall that an ultrafilter on a set A is called principal if it consists of all subsets of A containing a certain element α .

§ 2 Counterexamples

In the following examples $\mathcal{C}$ does not have the projection property. We show that every reducible building (Example 2) and every building of rank 1 (Example 1) does not have the projection property and give examples (Example 3) of buildings with one infinite Coxeter number which do not have the projection property. So our theorem does not hold if we drop one of the hypotheses for the building.

Example 1 Let the diagram of W consist of one vertex only, i.e. rank $\mathcal{C} = 1$. Then $W = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{C}$ is a set of cardinality ≥ 2 together with the δ –distance

$$\delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} .$$

Obviously, $\mathcal{C}$ does not have the projection property.

Example 2 Suppose the diagram of W is not connected. Then we shall define a map $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as above which is not a projection. Let J_1 and J_2 be non empty subsets of I such that $I = J_1 \dot{\cup} J_2$ and $m_{ij} = 2$ if $i \in J_1$ and $j \in J_2$. Let $W_\rho = \langle s_j | j \in J_\rho \rangle, \rho = 1,2$. Then $W = W_1 \times W_2$. Now I claim (i) for every pair x, y of chambers there is a unique chamber z such that $x \sim_{J_1} z$ and $y \sim_{J_2} z$. (ii) The map $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by $f(x,y) = z$ has the properties (1) and (2) and is not a projection.

Proof (i) Let $\delta(x,y) = w_1 \cdot w_2$, $w_\rho \in W_\rho$. So there is a gallery from x to y composed of two galleries Γ_1 , Γ_2 where the type of Γ_ρ gives a reduced word for w_ρ . So the common chamber of Γ_1 and Γ_2 is a z with $x \sim_{J_1} z$ and $y \sim_{J_2} z$. To see uniqueness let z' be a chamber such that $z' \sim_{J_1} x$ and $z' \sim_{J_2} y$. Let Γ_1 and Γ_2 be shortest galleries from x to z' and z' to y , respectively. Then the composed gallery $\Gamma = (\Gamma_1, \Gamma_2)$ is of minimal length from x to y , since otherwise the exchange property would yield a relation of the form $s_1 w s_2 = w$ with $s_\rho \in J_\rho$ contradicting $W = W_1 \times W_2$. Hence $\delta(x,y) = \delta(x,z') \cdot \delta(z',y)$ and hence $\delta(x,z')$ and $\delta(z',y)$ are the two components w_1, w_2 of $\delta(x,y)$. Take an apartment $\mathcal{A}$ containing x and y . Then Γ is contained in $\mathcal{A}$. For any pair $(\mathcal{A}, x)$ consisting of an apartment $\mathcal{A}$ and a chamber $x \in \mathcal{A}$ the map $z \mapsto \delta(x,z)$ is injective, so z is unique.

(ii) $f(x,x) = x$, clearly, and if $x \sim_i x' \sim_j y$ with $i \in J_1$, $j \in J_2$, we have $f(x,y) = x'$, so f is not a projection. To see $2a$ suppose $x \sim_i x'$ and put $z = f(x,y)$. If $x' \sim_{J_1} z$ then $f(x',y) = z$. If not, we have $i \in J_2$ and

$$\delta(x',z) = s_i \delta(x,z) = \delta(x,z) s_i \quad .$$

So there is a z' with $z' \sim_i z$ and $\delta(x',z') = \delta(x,z) \in W_1$, thus $z' \sim_{J_1} x'$ and $z' \sim_i z \sim_{J_2} y$, so $z' \sim_{J_2} y$. We have obtained $f(x',y) \sim_i f(x,y)$ in any case. The property $2b$ is proved the same way.

Example 3 Suppose W has the diagram $\underset{0}{\bullet}\overset{\infty}{\text{———}}\underset{1}{\bullet}$. Then $W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is the infinite dihedral group. We identify its Coxeter complex $\mathcal{C}$ with the set of intervals $I_n = [n, n+1]$ in $\mathbb{R}$. Two such intervals are adjacent of type 0 and 1 respectively, if their intersection contains an even (odd) number. Then $f(I_n, I_m) = I_{\sup(n,m)}$ is a map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the properties 1 and $2a$, $2b$ and is not a projection. More generally, a building with this Weyl group W is a numbered tree without finite branches. One can comb the tree and thus regard it as a poset. Then the infimum gives a map f as above.

§ 3 Proof

We begin the proof of the theorem with an observation we shall use frequently .

3.1

Suppose we have four chambers a, b, c, d *with adjacencies as indicated*

$$\begin{array}{ccc} a & \sim_i & b \\ \wr_j & & \wr_j \\ c & \sim_i & d \end{array} \quad .$$

If $m_{ij} > 2$ *then the square degenerates in the sense that* $a = b$ *and* $c = d$ *, or* $a = c$ *and* $b = d$ *.*

Proof If $a \neq b$ and $b \neq d$, then the gallery (a,b,d) yields $\delta(a,d) = ij$. The gallery (a,c,d) gives $\delta(a,d) \in \{1,j,i,ji\}$, a contradiction. If $a = b$, say, then $d \sim_j a \sim_j c$ and $c \sim_i d$ imply $c = d$. Similarly $b = d$ implies $a = c$.

The case $r = 2$ Suppose $r = 2$ and the off diagonal Coxeter number m is finite and larger than 2 . Let $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be an idempotent morphism.

3.2
We have

$$d(f(x,y),z) \leq \max(d(x,z)\,,\ d(y,z)) \quad .$$

Proof by induction on $M = \max(d(x,z)\,,\ d(y,z))$. The case $M = 0$ follows from (1). If $d(x,z) \neq d(y,z)$, say $d(x,z) < d(y,z) = M$, take a chamber y' adjacent to y such that $d(y',z) = d(y,z) - 1$. Then $d(f(x,y'),z) \leq d(y',z) \leq M - 1$, hence $d(f(x,y),z) \leq M$ since $f(x,y)$ is adjacent to y' . If $d(x,z) = d(y,z)$, let $y' \sim_i y$ and $x' \sim_j x$ such that $d(y',z) = d(x',z) = M - 1$. We obtain the square

$$\begin{array}{ccc} f(x',y') & \sim_j & f(x',y) \\ \wr_i & & \wr_i \\ f(x,y') & \sim_j & f(x,y) \end{array} \quad .$$

Then $f(x,y)$ is adjacent to $f(x',y')$, which is clear for $i = j$ and follows from 3.1 for $i \neq j$. This proves 3.2.

3.3
We have $f(x,y) \in \{x,y\}$.

Proof Suppose first $d(x,y)$ is even, say $2r$. Take an apartment $\mathcal{A}$ containing x and y . Then there is a chamber z in $\mathcal{A}$ such that $d(x,z) = d(y,z) = r$. Let z' be the chamber of $\mathcal{A}$ opposite of z . Then $d(x,z') = d(y,z') = m - r$. So $f(x,y)$ has distance $\leq r$ from z and distance $\leq m - r$ from z' , by 3.2. It follows that $f(x,y)$ is on a shortest gallery from z to z' , hence in $\mathcal{A}$, and the inequalities imply our claim for this case.

Now suppose $d(x,y)$ is odd, say $2r+1$. We shall derive this case from the case of even distance. First suppose x and y are opposite, i.e. $2r+1 = m$. Then there are chambers $x_i \sim_i x$ and $x_j \sim_j x$ with $d(x_i, y) = d(x_j, y) = m - 1 = 2r$. It follows from the even case that $f(x, y)$ is i –adjacent to x_i or to y and j –adjacent to x_j or to y . Thus $x_i \sim_i f(x, y) \sim_j x_j$ or $y \sim_i f(x, y) \sim_j y$, since the other cases $x \sim_i x_i \sim_i f(x, y) \sim_j y$ or $x \sim_j x_j \sim_j f(x, y) \sim_i y$ would imply $m = d(x, y) \leq 2$. In the second case $f(x, y) = y$ and in the first case $f(x, y) = x$ since x is the only chamber z such that $x_i \sim_i z \sim_j x_j$.

In the remaining case we have $d(x, y) = 2r + 1 < m$, say $\delta(x, y) = ij \cdots i$. Let $x' \neq x$ be j –adjacent to x . Then $\delta(x', y) = j \cdot \delta(x, y)$, since $\ell(j\, ij \cdots i) = 2r + 2 \leq m$. So $f(x, y)$ is j –adjacent to x' or to y , by the case of even distance. But $f(x, y)$ has distance $\leq 2r + 1$ from both x and y , by 3.2, hence $f(x, y) \in \{x, y\}$.

3.4 The case $r = 2$, end of proof

Put $M = \{(x, y) \in \mathcal{C} \times \mathcal{C} \mid f(x, y) = x\}$. We have the following facts.

a) Suppose $(x, y) \in M$, $\delta(x, x') = i$, $x \not\sim_i y$. Then $(x', y) \in M$.

b) Suppose $(x, y) \in M$, $\delta(y, y') = i$, $x \not\sim_i y'$. Then $(x, y') \in M$.

c) If $(x, y) \in M$, $x \neq y$, then $x \times \mathcal{A} \subset M$ for every apartment $\mathcal{A}$ containing x and y .

d) If $(x, y) \in M$, $x \neq y$, then $\mathcal{A} \times \mathcal{A} \subset M$ for every apartment $\mathcal{A}$ containing x and y .

a) and b) follow from 3.3 and b) easily implies c). We prove d). We first have $x \times \mathcal{A} \subset M$ by c), then $(x', z) \in M$ for x' adjacent to x and z properly chosen, e.g. opposite of x , by a), then $x' \times \mathcal{A} \subset M$ by c). Continuing in this way we obtain d).

e) If $(x, y) \in M$, $x \neq y$, then $\mathcal{C} \times \mathcal{C} = M$.

Note that M is symmetric by d), i.e. $(x, y) \in M$ iff $(y, x) \in M$. So it suffices to show that $(x, y) \in M$, $x \neq y$, implies $x \times \mathcal{C} \subset M$. By d) it now suffices to show that $(x, x') \in M$ for every x' adjacent to x . Say $\delta(x, x') = i$. Let $\mathcal{A}$ be an apartment containing x and y . For $j \neq i$, let z be the chamber of $\mathcal{A}$ such that $\delta(x, z) = j$. There is an apartment $\mathcal{A}'$ containing x' and z . Then $\mathcal{A}'$ contains every shortest gallery from x' to z , hence the gallery (x', x, z) , so $x \in \mathcal{A}'$. Now $(x, z) \in M$ by d), since $(x, y) \in M$ and $x, y, z \in \mathcal{A}$. Then $(x, x') \in M$ again by d), since $(x, z) \in M$ and $x, x', z \in \mathcal{A}'$.

3.6 The general case Suppose first the rank $r = \#I$ is finite and $r > 2$. There is a vertex k in the Coxeter diagram I of $\mathcal{C}$ such that $J = I \setminus \{k\}$ is connected, e.g. let k be a vertex of maximal distance in the graph I from a given vertex i_0 of I. Let $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be an idempotent morphism. For every J–equivalence class $\mathcal{D} \subset \mathcal{C}$ the restriction $f_{\mathcal{D}} = f|\mathcal{D} \times \mathcal{D}$ has image in $\mathcal{D}$ and is an idempotent morphism. Hence $f_{\mathcal{D}}$ is a projection by our inductive hypothesis. Suppose $f_{\mathcal{D}}$ is the projection p_1 to the first component. Let $M = \{(x,y) \in \mathcal{C} \times \mathcal{C} | f(x,y) = x\}$. I claim that $\mathcal{D} \times \mathcal{C} \subset M$. It suffices to prove that the set $N = \{y \,|\, \mathcal{D} \times y \subset M\}$ has the following property if y' is adjacent to y and $y \in N$ then $y' \in N$. So suppose $y \in N$ and $\delta(y, y') = i \in N$. Then there is a $j \in J$ such that $m_{ij} > 2$. This is true for $i \in J$ since J is connected and for $i = k$ since there is an edge from k to some vertex of J. For $x \in \mathcal{D}$ let $x' \in \mathcal{C}$ be such that $\delta(x, x') = j$. Then $x' \in \mathcal{D}$ and we have a square

$$\begin{array}{ccc} f(x,y) = x & \sim_i & f(x,y') \\ \int_j & & \int_j \\ f(x',y) = x' & \sim_i & f(x',y') \end{array} ,$$

since $y \in N$. Hence $f(x, y') = x$ by 3.1, since $x \neq x'$. Since this holds for every $x \in \mathcal{D}$, we have $y' \in N$. So we have $\mathcal{D} \times \mathcal{C} \subset M$ if $\mathcal{D} \times \mathcal{D} \subset M$. If now $\mathcal{D}'$ is some other J–equivalence class we have $\mathcal{D}' \times \mathcal{C} \subset M$, since $f|\mathcal{D}' \times \mathcal{D}' = p_2$ would imply $f|\mathcal{C} \times \mathcal{D}' = p_2$ by an analogous argument as above. So we have proved that $f = p_1$.

If I is infinite the restriction of an idempotent morphism $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to $\mathcal{D} \times \mathcal{D}$ is a projection for any J–equivalence class $\mathcal{D}$ for any finite connected J of cardinality ≥ 2. If $f|\mathcal{D} \times \mathcal{D} = p_1$ say, then $f|\mathcal{D}' \times \mathcal{D}' = p_1$ for any J'–equivalence class $\mathcal{D}' \supset \mathcal{D}$, where J' is as above and contains J. We get $f = p_1$ since $\mathcal{C}$ is the union of such $\mathcal{D}'$ containing $\mathcal{D}$.

3.7 Proof of the addendum by induction on $n = \#A$. The cases $n \leq 2$ are settled. Let $\Delta_i : \mathcal{C}^{n-1} \to \mathcal{C}^n$ be $\Delta_i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_i, x_i, \ldots, x_{n-1})$ for $i < n$. Then, by the inductive hypothesis, $f \circ \Delta_i$ is a projection, say p_k to the k–th component. If $k = i$ then $(x, y) \longmapsto f(x_1, \ldots, x_{i-1}, x, y, x_{i+2}, \ldots, x_n)$ is a projection, by the case $n = 2$, which implies our claim. So there remains the case that $f \circ \Delta_i = p_{k(i)}$ with $k(i) \neq i$ for every i. If $n \geq 4$, assume $k(1) = 3$. Then

$$f(x_1, x_2, x_3, x_4, \dots) = \begin{cases} x_3 & \text{if } x_1 = x_2 \\ x_{k(3)} & \text{if } x_3 = x_4 \text{ with } k(3) \neq 3, 4 \end{cases}$$

gives a contradiction for $x_1 = x_2$ and $x_3 = x_4 \neq x_{k(3)}$. So there remains the case that $n = 3$ and

$$f(x, y, z) = \begin{cases} z & \text{if } x = y \\ y & \text{if } x = z \\ x & \text{if } y = z. \end{cases}$$

In Arrow's interpretation this means: If there is a vote between two alternatives take the minority vote. Now take three pairwise different chambers $0, i, j$ with $j \underset{j}{\sim} 0 \underset{i}{\sim} i$ and $m_{ij} > 2$, i.e. i and j are adjacent in the Coxeter graph. Then

$$\begin{array}{ccc} f(j,j,i) = i & \underset{i}{\sim} & f(j,j,0) = 0 = f(0,i,i) \\ \int_j & & \int_j \\ (f(j,0,i) & \underset{i}{\sim} & f(j,i,i) = j \end{array}$$

gives a contradiction to 3.1.

References

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