On the structure of cellular algebras

by

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1 Introduction

Recently, Graham and Lehrer [17, 16] have introduced a new class of algebras, which they called cellular algebras. These are defined by the existence of a certain basis with very special properties motivated by the Kazhdan–Lusztig basis of Hecke algebras. Graham and Lehrer have shown that various important classes of algebras are cellular, in particular, Ariki–Koike Hecke algebras, and Brauer (centralizer) algebras.

The aim of this paper is to study the structure of cellular algebras, in particular in the case of the ground ring being a field (this is assumed in sections four to seven). We start by shortly describing the two classes of motivating examples just mentioned. In section three we review some of the results of Graham and Lehrer and give an equivalent form of their definition, and some examples.

In section four we investigate the cell ideals (which are the building blocks of cellular algebras). In particular we show that there are two different sorts of such ideals, one being familiar from (and thus providing a close connection to) the theory of quasi–hereditary algebras. In section five we show that the involution occuring in the definition of cellular algebras must fix isomorphism classes of simple modules. This is a strong restriction, as we illustrate by the example of Brauer tree algebras.

In section six we collect some homological properties for later use. In section seven we give an inductive construction of cellular algebras. This

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provides us with a method to produce many examples, which in particular show that there are no restrictions on the cell ideals e.g. with respect to endomorphism rings - which is a strong contrast to the situation for quasi–hereditary algebras. We also give a bound on the Loewy length of a cellular algebra in terms of the number of cell ideals. Moreover we determine the global dimension of certain cellular algebras.

This paper is the first in a series of three papers. In the subsequent paper [25] we define integral cellular algebras. This is a smaller class of algebras, still containing the examples of [17], but with much stronger properties than cellular algebras in general. The paper [26] contains a new characterization of cellular algebras which gives another inductive construction of cellular algebras and which leads to a generalization of Hochschild cohomology.

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2 Motivating examples

In representation theory of Lie algebras and algebraic groups several classes of finite dimensional associative algebras play important roles. Most of them are defined by generators and relations. The motivation for studying cellular algebras or related classes of algebras (like quasi–hereditary algebras) is to understand these special classes of examples by putting them into an axiomatic framework and thus to reveal hidden structure. In this section we give a list of those examples which at present seem to be the most interesting ones. Details are given only for those examples which are not already covered by the literature on quasi–hereditary algebras.

2.1 Ariki Koike Hecke algebras

Hecke algebras of type A are well–known deformations of the group algebra of the symmetric group (for a survey see e.g. [10]). They have been studied and used intensively in many areas of mathematics. For the other classical Coxeter groups (type B, C, and so on), Hecke algebras have been defined as well. Recently, several attempts have been made to define Hecke algebras in a more general context, for socalled complex reflection groups. The so–called cyclotomic Hecke algebras are closely related to a system of results and conjectures which has been built around Broué's conjecture (on derived equivalences for blocks with abelian defect group). For more information (on this topic and more general on the 'generic' approach to modular representation theory of finite groups of Lie type) the reader is referred to [7]. One large class of cyclotomic Hecke algebras has been studied by Ariki and Koike [5] in a direct way by deforming group algebras of wreath products.

Definition 2.1 *(Ariki and Koike [5]) Let R be the ring $\mathbb{Z}[q, q^{-1}, u_1, \ldots, u_r]$ for some natural number r. Then the Hecke algebra $\mathfrak{H}_{\mathfrak{n},\mathfrak{r}}$ (called an* **Ariki–Koike Hecke algebra***) is defined over R by the following generators and relations:*

Generators: $t = a_1, a_2, \ldots, a_n$
Relations: $(t - u_1)(t - u_2)\ldots(t - u_r) = 0$
$a_i^2 = (q-1)a_i + q (i = 2, \ldots, n)$
$ta_2ta_2 = a_2ta_2t$
$a_ia_j = a_ja_i (\mid i - j \mid \geq 2)$
$a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} (i = 2, 3, \ldots, n-1)$

Specializing $r = 1$ and $u_1 = 1$ respectively $r = 2$ and $u_1 = -1$ one gets back the classical Hecke algebras of types A and B, respectively.

This Hecke algebra can be seen as deformation of the group algebra of the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \Sigma_n$. Ariki and Koike show in [5] that $\mathfrak{H}_{n,r}$ is a free R–module of rank $n!r^n$, and they classify the simple representations of the semisimple algebra $frac(R) \otimes_R \mathfrak{H}_{n,r}$. A criterion deciding for which values of q and the u_i's the specialized Hecke algebra is semisimple is given by Ariki [3]. An important application of Ariki–Koike Hecke algebras is Ariki's proof of (a generalization of) the LLT–conjecture which gives an explicit way of computing characters of Hecke algebras (see [4] and [27]).

2.2 Brauer algebras

These algebras (not to be confused with Brauer tree algebras) arise in the representation theory of orthogonal and symplectic groups in the following way (which is how Brauer came about studying them [6]): For analogy, let us first recall a basic property of the representation theory of the group $G := GL_n(k)$. It acts naturally on the n–dimensional k–vector space V, hence also on the r–fold tensor product $V^{\otimes r}$. The centralizer algebra of the G–action on the tensor product (that is, the set of all k–endomorphisms of $V^{\otimes r}$ which satisfy $\varphi(gv) = g\varphi(v)$ for all $v \in V^{\otimes r}$) is a quotient of (and in many cases equal to) the group algebra of the symmetric group Σ_r. This fact had been used by Schur for relating the representation theories (in characteristic zero) of the general linear and the symmetric group. Now it is natural to try what happens when one replaces $GL_n(k)$ by one of its classical semisimple subgroups, say an orthogonal or a symplectic one, which of course still acts on the tensor space $V^{\otimes r}$. Call the corresponding centralizer algebra $D_r(n)$ (where n is the number of the group in the series containing it). Now Brauer observed that this algebra is a quotient of (and in many cases isomorphic to) another algebra, say $B_r(x)$ (where x is specialised to a natural number in case of the orthogonal group and to a negative integer in the symplectic case) which is defined as follows, and which is now called the **Brauer algebra** (or the Brauer centralizer algebra).

Fix a field k, an indeterminate x, and a natural number r. Then $B_r(x)$ has a basis consisting of all diagrams, which consist of $2r$ vertices, divided into 2 ordered sets, the r top vertices and the r bottom vertices, and r edges such that each edge belongs to exactly 2 vertices and each vertex belongs to exactly one edge. Multiplication of basis elements is defined by concatenating diagrams: Assume we are given two basis elements, say a and b. First, draw an edge from bottom vertex i of a to top vertex i of b (for each $i = 1, \ldots, r$). This produces a diagram which is almost of the desired form except that there may be cycles not attached to any of the (new) top and bottom vertices. Denote the number of these cycles by d. Then delete all cycles; the result is a basis element, say c. Now the product ab is by definition $x^d c$.

Of course, for a field element, say n, the Brauer algebra $B_r(n)$ is defined by using n instead of x, that is, by forming the quotient of $B(x)$ modulo $x - n$.

For examples and more details the reader is referred to Wenzl's paper [33].

Brauer algebras have been studied extensively by Hanlon and Wales in a series of long papers with many explicit results [20, 21, 22]. Based on their computations they conjectured that $B_r(n)$ is semisimple if n is not an integer. Wenzl [33] proved this conjecture. He also studied applications to knot theory [34].

2.3 Other examples

There are other algebras which also can be defined by diagrams (and are of use in knot theory). We mention Temperley–Lieb algebras and Jones' annular algebras. The algebras in both classes are smaller (and thus more open to explicit computations) than the Brauer algebras.

Other important classes of examples which we want to cover are the blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of finite dimensional semisimple complex Lie algebras, and the (generalized) Schur algebras associated with rational (or polynomial) representations of semisimple algebraic groups (in describing characteristic). The reader is referred to [32] for more information on these two classes of algebras (which are the main examples of quasi–hereditary algebras).

3 Definitions and basic properties

First, we recall the original definition of Graham and Lehrer, and explain how they use it. Then we give an equivalent definition which we will use for looking at the structure of cellular algebras.

Definition 3.1 *(Graham and Lehrer, [17]) Let R be a commutative ring. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^\lambda_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^\lambda_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^\lambda_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

In the following, an R–linear anti–automorphism i of A with $i^2 = id$ will be called an involution.

Using this definition Graham and Lehrer could show that the following examples (mentioned in the previous section) fit into this context:

Theorem 3.1 *(Graham and Lehrer, [17]) The following algebras are cellular:*

(a) Ariki–Koike Hecke algebras;
(b) Brauer's algebras;
(c) Temperley–Lieb algebras;
(d) Jones' annular algebras.

The proofs for the first and the second class of algebras are quite tricky and complicated and use non–trivial combinatorial tools like Kazhdan–Lusztig basis and Robinson–Schensted algorithm; the remaining classes of examples are similar to (specializations of) the Brauer algebras.

Other examples of cellular algebras are discussed in [18, 31].

Let us give an easy example: The algebra $A = k[x]/(x^n)$ is cellular with cell chain $0 \subset (x^{n-1}) \subset (x^{n-2}) \subset \ldots \subset (x) \subset A$ and involution $i = id$. Note however that it is not cellular with respect to the involution $x \mapsto -x$. In fact, the cell chain would have to be the same as before, but one does not find a basis which is fixed under this involution (see section 5 for details). This indicates that the involution is not just an extra condition but there is a quite subtle interplay between the involution and the other conditions.

We remark that the finiteness of the index set I is not required in the definition in [17]. However, it is assumed in most of their results, and it is satisfied in all of their examples. Thus it seems to be convenient to include this condition in the definition, and thus to exclude infinite dimensional algebras like $k[x]$ which would be cellular otherwise. We will recall below the results of [17] on classifying simple modules of a cellular algebra. Roughly, they give a bijection between isomorphism classes of simple modules and those cell ideals which satisfy an additional condition. The example $k[x]$ shows that this cannot work without assuming I to be finite. There are, however, remarkable examples of quantum groups (namely, Lusztig's $\dot{U}$, see part four of [30]) which are cellular with infinite index set. In the general case of infinite index set it seems natural also to look at completions of cellular algebras (see [19]).

From the definition of cellular algebras it follows directly that they are filtered by a chain of ideals:

Proposition 3.2 *(Graham and Lehrer, [17]) Let A be cellular with cell datum (I, M, C, i). Fix an index $\lambda \in I$. Then the R–span of all basis elements $C^{\mu}_{S,T}$ for $\mu \leq \lambda$ is a two–sided ideal in A.*

The ideal in the proposition will be denoted by $J(\leq \lambda)$. If λ is minimal, this ideal is called a **cell ideal**. Varying λ produces a chain of ideals (called a **cell chain**). By $J(< \lambda)$ we denote the sum of the ideals $J(\mu)$ with $\mu < \lambda$.

Proposition 3.3 *(Graham and Lehrer, [17]) As a left module, the quotient $J(\lambda)/J(<\lambda)$ is a direct sum of copies of a module $\Delta(\lambda)$ which has basis $C^{\lambda}_{S,T}$ where S runs and T is fixed (in particular, the isomorphism class of this module does not depend on the choice of T).*

The module in the proposition will be called **standard module**, since we will see later on, that this notion extends the familiar notion of standard modules of a quasi–hereditary algebra. In [17] this module is called a **cell module**. Of course, there is a dual version of the proposition stating that as a right module $J(\lambda)/J(<\lambda)$ is a direct sum of copies of the module $i(\Delta(\lambda))$ (where the right action of A is via the involution i).

Although J is a direct sum of copies of Δ as a left module, it does not have to be generated by any of these copies as a two–sided ideal: Let A be the noncommutative algebra $k < a, b, c, d > /rad^2(k < a, b, c, d >)$ with four generators and radical square zero. Define i to send b to c and c to b and to fix a and d. Then the radical J of A is the four–dimensional space with basis a, b, c, d. In fact, J is a cell ideal as one can check by choosing $\Delta = ka \oplus kb$, hence $i(\Delta) = ka \oplus kc$. But because of $rad(A)J = 0 = Jrad(A)$ this cell ideal cannot be generated by any proper subspace.

The example $k[x]/(x^n)$ shows that standard modules $\Delta(i)$ and $\Delta(j)$ having different indices $i \neq j$ may be isomorphic. Moreover, there may be non–trivial extensions between them.

The main use Graham and Lehrer make of these notions is for constructing simple A–modules. (For the above classes of algebras this is usually a hard problem; see [17] for partial solutions.) Since the coefficients in expressing the product $C^{\lambda}_{S,T}C^{\lambda}_{U,V}$ as a linear combination of basis elements do not depend on the indices S and V, one can define a bilinear form ϕ_λ sending S, V to the coefficient of $C^{\lambda}_{S,V}$ in this expression. Graham and Lehrer show, that in case of ϕ_λ being non–zero, the standard module $\Delta(\lambda)$ has a unique simple quotient, say $L(\lambda)$. All simple A–modules arise in this way, and for different $\lambda \neq \mu$ one gets different simple modules. Since the radical of the standard module equals the radical of the associated bilinear form, the question of finding a basis of $L(\lambda)$ is reduced to a problem of linear algebra (so, one has the possibility of doing explicit computations in each individual case – which, of course, does not solve the problem of finding general 'formulae', e.g. for the dimensions of simple modules).

Now we rephrase the definition of cellular algebras and show that the new definiton is equivalent to the definition of [17].

Definition 3.2 *Let A be an R–algebra where R is a commutative Noetherian integral domain. Assume there is an antiautomorphism i on A with $i^2 = id$. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{tikzcd}
J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\
J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta)
\end{tikzcd}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is an R–module decomposition $A = J_1' \oplus J_2' \oplus \ldots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two–sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

From now on, we always will assume that the ring R is a commutative Noetherian integral domain. This is not necessary for the definitions. However, all the examples of cellular algebras, which we know of, satisfy this assumption. Moreover, the basic problems of the theory of cellular algebras, for example to compute decomposition numbers, make this assumption natural. In particular, the notion of rank will be available.

We note one immediate consequence of this definition: **A cellular algebra is finitely generated and free as an R–module.**

The two definitions of cellular algebras are equivalent. **Proof.** Assume that A is cellular in the sense of Graham and Lehrer. Fix a minimal index λ. Define $J(\lambda)$ to be the R–span of the basis elements $C_{S,T}^{\lambda}$. By (C3), this is a two–sided ideal. By (C2) it is fixed by the involution i. Fix any index T. Define Δ (as remarked above) as the R–span of $C^{\lambda}_{S,T}$ (where S varies). Defining α by sending $C^{\lambda}_{U,V}$ to $C^{\lambda}_{U,T} \otimes i(C^{\lambda}_{V,T})$ gives the required isomorphism. Thus $J(\lambda)$ is a cell ideal. Continuing by induction, it follows that A is cellular in the sense of the new definition.

Conversely, if a cell ideal, say J, in the sense of the new definition is given, we choose any R–basis, say $\{C_S\}$, of Δ, and denote by $C_{S,T}$ the inverse image under α of $C_S \otimes i(C_T)$. Since Δ is a left module, (C3) is satisfied. (C2) follows from the required commutative diagram. This finishes the proof for those basis elements occuring in a cell ideal. Induction (on the length of the chain of ideals J_j) provides us with a cellular basis of the quotient algebra A/J. Choosing any preimages in A of these basis elements together with a basis of J as above we produce a cellular basis of A. ■

At this point, one may ask the following questions: Is there a universal property of the standard modules Δ? Are standard modules always indecomposable? Do their endomorphism rings have special properties? Later on, we will see, that the answer to each of these questions is no in general (but yes in the examples in which one is interested, as will be shown in [25].

Since the definition of cellular algebras uses induction, one has to ask which algebras provide the induction start.

Proposition 3.4 *Let A be an algebra with a cell ideal J which is equal to A. Then A is isomorphic to a full matrix ring over the ground ring R.*

Proof. The assumption says that A has an involution i and can be written as $\Delta \otimes_R i(\Delta)$ for some left ideal Δ. Hence there is an R–isomorphism $A \simeq Hom_A(A,A) \simeq Hom_A(\Delta \otimes_R i(\Delta), A) \simeq Hom_R(i(\Delta), Hom_A(\Delta, A))$. Denote the R–rank of the free R–module Δ by m. Then A has R–rank m^2, and as left module, A is isomorphic to m copies of Δ. Hence, $Hom_A(\Delta, A)$ (which is a submodule of the R–free module $Hom_R(\Delta, A)$, hence torsionfree) has R–rank at least m. But by the above isomorphism it cannot have larger rank. Thus, the A–endomorphism ring E (which again is torsionfree) of Δ has rank one and contains R. Now E is a subring of $frac(R)$ (which equals $End_{frac(R)\otimes_R A}(frac(R) \otimes_R \Delta))$ which sends Δ into itself by multiplication, hence E is equal to R. ■

The proof shows that for any cellular algebra, the standard modules of maximal index are indecomposable and have endomorphism ring R. Of course, the full matrix ring over R is cellular, so the converse of the proposition also is true. We remark that the proposition also follows from corollary 2.6' in [17]. Conversely, our argument reproves this result.

We finish this section by giving another class of examples:

Proposition 3.5 *Let A be a finite dimensional commutative algebra over an algebraically closed field k. Then A is cellular with respect to the involution $i = id$.*

Proof. Since A is commutative, its maximal semisimple quotient is commutative as well. Thus the simple A–modules are one–dimensional. The left socle of A coincides of course with the right socle, thus any one–dimensional direct summand of it is a cell ideal. Now we can proceed by induction. ∎

4 Cell ideals

In this section we have a closer look at cell ideals. It turns out that they are of two different kinds, one of them being familiar from the theory of quasi–hereditary algebras. Part of this information (phrased differently) is contained in sections two and three of [17]. **From now on we assume R to be a field.**

Proposition 4.1 *Let A be an R–algebra ($R = k$ any field) with an involution i and J a cell ideal. Then J satisfies one of the following (mutually exclusive) conditions:*

(A) J has square zero.

(B) There exists a primitive idempotent e in A such that J is generated by e as a two–sided ideal. In particular, $J^2 = J$. Moreover, eAe equals $Re \simeq R$, and multiplication in A provides an isomorphism of A–bimodules $Ae \otimes_R eA \simeq J$.

Proof. By assumption, J has an R–basis $C_{S,T}$ whose products satisfy the rule (C3). If all the products $C_{S,T}C_{U,V}$ are zero, then we are in situation (A). Thus we may assume that there is one such product which is not zero. Since the coefficients do not depend on S or V, the product $C_{U,T}C_{U,T}$ also is not zero. But by [17], 1.7 (or a direct comparison of the two ways writing this product as a linear combination of basis elements, using (C3) and its dual), this product is a scalar multiple of $C_{U,T}$. Hence there is an idempotent in J, which thus cannot be nilpotent.

So, J contains a primitive idempotent, say e, and Ae is a left ideal which is contained in J. The cell ideal J as a left A–module is a direct sum of copies of a standard module Δ. But Ae is a submodule, hence a direct summand of the left ideal $J = Ae \oplus J(1-e)$. It follows that Ae is a direct summand of Δ which we can decompose into $Ae \oplus M$ for some A–module M. Because of $J \simeq \Delta \otimes_R i(\Delta)$ we can decompose J as left module into $Ae^m \oplus M^m$ where m is the R–dimension of Δ (which equals the R–dimension of $i(\Delta)$). Of course, Ae^m is contained in the trace X of Ae inside J (that is, the sum of all images of homomorphisms $Ae \to J$). This trace X is contained in the trace AeA of Ae in A. But the dimension of AeA is less than or equal to the product of the dimension of Ae with the dimension, say n, of eA. The number n equals the dimension of $Ai(e)$, which (by the same argument) also is a direct summand of Δ. Since m is less than or equal to n, there must be equality $i(\Delta) = i(e)A$, hence also $Ae = \Delta$ and there must be equality in all of the above inequalities. In particular, J equals AeA and also $Ae \otimes_R eA$. Since multiplication $Ae \otimes_R eA$ always is surjective, it must be an isomorphism. As this multiplication factors over $Ae \otimes_{eAe} eA$ it follows that eAe must be equal to $Re \simeq R$. ∎

We note that here we need the assumption that $R = k$ is a field.

Case (B) just says that J is a heredity ideal (generated by a primitive idempotent). Conversely, a heredity ideal J which is fixed by an involution i and generated by a primitive idempotent e, clearly is a cell ideal. The following corollary is already known (see e.g. [17, 14]).

Corollary 4.2 *Let A be a quasi–hereditary algebra with an involution i fixing a complete set of primitive orthogonal idempotents. Then A is cellular. Conversely, a cellular algebra with all cell ideals being idempotent (i.e. of type (B)) is quasi–hereditary.*

An ideal J fixed by an involution i and generated by a primitive idempotent e fixed by i is a cell ideal if and only if it is a heredity ideal.

There are two subtle points in this context which we would like to mention. A cell ideal which is a heredity ideal must be generated by a primitive idempotent. But it can happen that no such idempotent is fixed by i. Moreover, a heredity ideal which is fixed by an involution i need not be a cell ideal. Examples (which in fact use simple algebras) are given in [26].

Here, a warning is necessary: The indexing of cell ideals of a cellular algebra unfortunately is precisely the converse of the usual indexing for ideals in a heredity chain of a quasi–hereditary algebra, so passing from one theory to the other one should replace $\geq$ by $\leq$.

An alternative proof (and a generalization of this result) has been given already in the recent preprint [14] by Du and Rui. A similar result for integral quasi–hereditary algebras is given in [24]. The latter can be used to see that integral Schur algebras are cellular.

In general, a quasi–hereditary algebra does not have to admit any involution. But there is a large class of examples which are both quasi–hereditary and cellular. This includes in particular blocks of category $\mathcal{O}$ and Schur algebras. More examples can be found in [11].

With a quasi–hereditary algebra A, there come two series of smaller quasi–hereditary algebras, having the form A/J_i or eAe, respectively. Here, the idempotents e have to be chosen in a special way (which is prescribed by the partial order $\leq$). For cellular algebras, there is a much larger class of idempotents with eAe cellular:

Proposition 4.3 *Let A be cellular with respect to an involution i. Let e be an idempotent in A which is fixed by i. Then eAe is cellular with respect to the restriction of i.*

Proof. Clearly, i is an involution of eAe. For a cell ideal J of A we have to show that eJe is a cell ideal of eAe. Putting $\Delta' = e\Delta$, hence $i(\Delta') = i(e\Delta)$, one gets the isomorphism $eJe \simeq \Delta' \otimes_k i(\Delta')$. ■

We note that this result does not need the assumption of k being a field. An application of this result is an alternative proof that Hecke algebras of type A, in particular the group algebras of the symmetric groups, are cellular:

Corollary 4.4 *Hecke algebras of type A_n are cellular.*

Proof. Integral Schur algebras are quasi–hereditary, and there is an involution i fixing the ideals in the usual heredity chain. Now, the Hecke algebras of type A_n can be written as eAe for A some integral Schur algebra and e some idempotent fixed by i. ■

5 The involution

The involution occuring in the definition of cellular algebras plays a crucial role. It is not just an additional datum but it makes the other data in the definition much more subtle. In this section we show in particular that the involution necessarily fixes isomorphism classes of simple modules. As an application we classify the cellular Brauer tree algebras. This implies that for the known classes of cellular algebras which are connected to finite group theory, like the Hecke algebras, only a special kind of Brauer tree can occur. In the case of Hecke algebras, this reproves a result of Geck [15].

A generalization of cellular algebras, called standardly based, obtained by omitting the involution in the definition has been studied by Du and Rui, see [14]; the class of algebras obtained in this way is much larger than the class of cellular algebras. In fact, any finite dimensional algebra A over a perfect field is standardly based: A minimal (with respect to inclusion) two–sided ideal I of A is a minimal $A \otimes_k A^{op}$ submodule of A. Hence I is simple as $A \otimes_k A^{op}$–module, and thus I has the form $I \simeq S \otimes T$ for some simple left A–module S and some simple right A–module T.

We start with an easy example showing that it depends on the choice of the involution whether an algebra is cellular or not. In fact, for $A = k[x]/(x^2)$, k any field of characteristic different from two, we may consider two involutions, $i_1 = id$ and $i_2 : x \mapsto -x$. In the first case, A is clearly cellular (the cell chain being $0 \subset (x) \subset A$). In the second case, however, A is not cellular. Assume to the contrary, that J is a non–zero cell ideal with respect to i_2. Then the dimension of J has to be a square, thus it equals one. So, J must be the ideal generated by x, and Δ equals J and also $i_2(\Delta)$. But the square

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "y \otimes z \mapsto i(z) \otimes i(y)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

cannot be made commutative by any α, since the left hand vertical map sends x to $-x$ whereas the right hand vertical map sends a generator $y \otimes z$ to itself.

Proposition 5.1 *Let A be a cellular algebra with involution i and e a primitive idempotent of A. Then $i(e)$ is equivalent to e.*

Proof. (See [17], 3.7 for a similar argument.) Fix a chain of cell ideals $J_1 \subset \ldots \subset J_n \subset A$ and assume that e lies in J_j but not in J_{j-1}. Since the ideals in a cell chain are fixed by i, $i(e)$ also lies in J_j but not in J_{j-1}, hence we can pass to a cellular quotient algebra of A and assume that e and $i(e)$ both are contained in a cell ideal, say J which has the form $\Delta \otimes_R i(\Delta)$. Since J contains an idempotent, it is a heredity ideal and Δ equals Ae by proposition 4.1. But $Ai(e)$ is a direct summand of J, hence also of Δ, which implies that Ae and $Ai(e)$ must be isomorphic. ■

Corollary 5.2 *Let A be directed (that is, the isomorphism classes of simple A–modules can be ordered in such a way that $Ext^1_A(L(i), L(j)) \neq 0$ implies $i < j$). Then A is cellular (for some involution) if and only if it is split semisimple.*

Proof. We know already that split semisimple algebras are cellular (for any involution fixing simples). Conversely, let A be cellular with respect to some involution, say i. Then i fixes simple modules, hence for any two simple modules L and L', there are extensions in one direction if and only there are extensions in the other direction. By the definition of directedness, extensions in one direction are zero. Hence there are no non–trivial extensions at all, and A is semisimple. Thus, A is a direct sum of its cell ideals, hence it is split semisimple. ■

As an application we consider the following question: When is a Brauer tree algebra cellular? (For definition and properties of Brauer tree algebras we refer to [2], chapter five.)

Proposition 5.3 *A Brauer tree algebra (with an arbitrary number of exceptional vertices) is cellular if and only if the Brauer tree is a straight line (that is, at each vertex there are at most two edges, with arbitrary multiplicities).*

Proof. If there is a vertex with more than two edges meeting there, then (by definition of Brauer tree algebra) there are simple modules L_1, L_2, $\dots$, L_m (corresponding to these edges) for some $m \geq 3$ such that $Ext^1(L_j, L_{j+1}) \neq 0$ (to be read cyclically), and all other first extensions between two of these simples vanish. On the other hand, being cellular implies the existence of an involution i fixing isomorphism classes of simple modules and hence providing vector space isomorphisms between $Ext^1(L_j, L_{j+1})$ and $Ext^1(L_{j+1}, L_j)$ which is a contradiction.

Thus it remains to show that in the case of a straight line the algebra is cellular. This can be done easily using the well–known description by quiver and relations. We describe the procedure in the basic case. The involution i fixes the vertices and reverses the arrows. The cell chain can be chosen in such a way that J_1 is one– dimensional, the unique socle of a projective module $P(1)$ which belongs to an edge with one neighbour only. Then J_2/J_1 is four–dimensional and intersects non–trivially with $P(1)$ and its neighbour $P(2)$. If the vertex has multiplicity one, then J_2/J_1 is a heredity ideal. Otherwise, one continues to factor out four–dimensional ideals containing composition factors $L(1)$ and $L(2)$ until the idempotent at 1 has been factored out. Then one continues inductively. ∎

It follows that for example Brauer trees of Ariki–Koike Hecke algebras are straight lines (this has been proved already by Geck, [15], also using an involution argument – however, his result is more general).

In [25] we will see that for the examples we are interested in (that is, the integral cellular algebras) the involution i has a rather special shape. In general, however, it seems to be an open problem to find all involutions on a given finite dimensional algebra.

6 Homological properties

In proposition 4.1, we have seen that (over a field) for a cell ideal J there are precisely two possibilities. Either it is a heredity ideal or it has square zero. In the first case, the homological properties are well understood, for example there is a homological epimorphism $A \to A/J$, and A–cohomology can, to a large extent, be read off from A/J (see [32, 13]). In the second case this is not true; cohomology of A and of A/J can be rather different as one can see already from the example $k[x]/(x^2)$. In this section we give a list of homological properties of J in the second case; in particular we show how the cell basis can be read off naturally from certain Ext or Tor groups. We restrict to algebras over a field in this section in order to have the above dichotomy available.

We start by recalling an exercise of Cartan–Eilenberg ([8], VI-19): If A is a ring and I is a right ideal and J is a left ideal then the following isomorphisms are easily shown by using long exact sequences:

$$Tor_1^A(A/I, A/J) \simeq (I \cap J)/IJ,$$

$$Tor_2^A(A/I, A/J) \simeq kernel(I \otimes_A J \overset{mult}{\to} IJ),$$

$$Tor_n^A(A/I, A/J) \simeq Tor_{n-2}^A(I, J), n > 2.$$

Proposition 6.1 *For any ideal J in a k–algebra A, the following two assertions are equivalent:*
(I) $J^2 = 0$,
(II) $Tor_2^A(A/J, A/J) \simeq J \otimes_A J$.

Proof. Applying $J \otimes_A -$ to the exact sequence $0 \to J \to A \to A/J \to 0$ produces the exact sequence

$$0 \to Tor_1^A(J, A/J) \to J \otimes_A J \to J \otimes_A A \to J \otimes_A A/J \to 0$$

Now, the last term equals J/J^2. Hence, if $J^2 = 0$, the first two terms must be isomorphic. Dimension shift then proves that (II) is valid. Conversely, again by dimension shift, condition (II) implies that the first two terms, hence also the last two terms are isomorphic, thus (I) is valid. ■

Now assume A has a nilpotent cell ideal, say J, which is isomorphic to $\Delta \otimes_k i(\Delta)$, where i is an involution on A and Δ is a left ideal inside J.

Since J is isomorphic to $\Delta \otimes_k i(\Delta)$ we get an isomorphism of k vector spaces $J \otimes_A J \simeq \Delta \otimes_k (i(\Delta) \otimes_A \Delta) \otimes_k i(\Delta)$, thus the Tor space in the previous proposition will be quite large, provided $i(\Delta) \otimes_A \Delta$ is not zero. But the latter space is the k–dual of $Hom_A(\Delta, Hom_k(i(\Delta), k))$ which is non–zero since it contains the map $\Delta \to top(\Delta) \simeq socle(Hom_k(i(\Delta), k)) \to Hom_k(i(\Delta), k)$.

Corollary 6.2 *Let J be a nilpotent cell ideal in the k–algebra A. Then the space $Tor_2^A(A/J, A/J)$ is not zero.*

This has the following consequence: If A has a cell chain which contains nilpotent cell ideals, then this chain of ideals does not give a recollement of derived module categories, and the chain of ideals does not make A into a stratified algebra in the sense of Cline, Parshall, and Scott [9]. Hence their notion of stratified algebra is quite different from the notion of cellular algebra (although both classes of algebras contain the quasi–hereditary algebras).

Another consequence of the exercise in Cartan–Eilenberg is the following (here, we do not need that k is a field, it is enough to assume that k is a noetherian integral domain):

If J can be written as $\Delta \otimes_k i(\Delta)$ and m is the k–rank of Δ, then there are m linearly independent embeddings of Δ into J, and similarly for $i(\Delta)$. So,

J can be written as a direct sum $J = \oplus(\Delta_j) \cap i(\Delta_l)$ where j and l run from 1 to m each. But $\Delta \cap i(\Delta)$ is isomorphic to $Tor_1^A(A/i(\Delta), A/\Delta)$, which in particular must have rank one over k. Thus **the choice of the cell basis can be seen as choosing a decomposition of the above Tor–space**.

The involution i on A induces an antiequivalence $\epsilon : A-mod \simeq mod-A$. Its effect on standard modules can be described explicitly:

Proposition 6.3 *Let A be cellular with respect to an involution i and denote by ϵ the antiequivalence $A - mod \simeq mod - A$ induced by i. Then for a standard module Δ of minimal index there is an isomorphism of left A–modules $\epsilon(A/\Delta) \simeq Hom_k(A/i(\Delta), k)$.*

Proof. We first recall the definition of the module structures on the two modules: On $Hom_k(A/i(\Delta), k)$ an element $a \in A$ acts by sending a linear form α to the form which maps x to $\alpha(xa)$. On $\epsilon(A/\Delta)$ (which as a vector space coincides with $Hom_k(A/\Delta, k)$) the action is by sending f to $af : x \mapsto f(i(a)x)$.

Now the isomorphism is straightforward: Send f to $\alpha_f : x \mapsto f(i(x))$. ■

7 An inductive construction

For a quasi–hereditary algebra, there are two inductive constructions, due to Parshall and Scott [32] and to Dlab and Ringel [13]. (One should note, that there are too many quasi–hereditary algebras, so one cannot use these constructions in order to classify them all, but one can inductively check properties or produce examples in this way.)

The first of these constructions [32] takes as input an algebra B with n simples and produces as output an algebra A with $n + 1$ simples and a heredity ideal J of A such that $A/J = B$. One can use this construction as well in case (A) of our situation (i.e. for heredity ideals); one just has to add the existence of an involution as an additional condition.

The aim of this section is to look at an inductive construction (similar to that of [32]) which works in the second case of a nilpotent cell ideal. In particular we will use this construction for producing examples which give negative answers to some of the questions asked in the third section. As positive results we get a statement about global dimensions and bounds for the Loewy length of a cellular algebra.

Proposition 7.1 *Let B be a cellular algebra with an involution i. Let Δ be any B–module. Let $\hat{\Delta}$ be the image of Δ under the antiequivalence $\epsilon : B-mod \to mod-B$ defined by i (thus, as vector space $\hat{\Delta}$ equals Δ). Denote by J the B–bimodule $\Delta \otimes_R \hat{\Delta}$. Define i on J by sending $x \otimes y$ to $y \otimes x$. Pick a Hochschild cocycle $\phi \in H^2(B, J)$ which satisfies in addition the equation $\phi(i(x), i(y)) = i\phi(y, x)$ for all $x, y \in B$.*

Then $A = J \oplus B$ with multiplication defined by $(j, b)(j', b') = (jb' + j'b + \phi(b, b'), bb')$ is a cellular algebra with nilpotent cell ideal J.

Conversely, any cellular algebra with a nilpotent cell ideal can be written in this form.

Proof. First we assume that A is a cellular algebra with a nilpotent cell ideal J. Then $J^2 = 0$ implies that the A–module structure of J factors over the quotient algebra $B = A/J$. Hence Hochschild cohomology can be applied. It is easy to check that the involution i on A imposes the above condition on the Hochschild cocycle.

Conversely, it is well–known that the above data define an associative R–algebra structure on A and that J is an ideal. Since i is defined both on B and on J, it is defined on A as well and the condition on the cocycle implies that i actually is an involutory antiautomorphism of A. Clearly, J is an ideal of A. Its A–bimodule structure is given by its B–module structure, thus it is a cell ideal. ■

An easy special case of Hochschild extensions are trivial extensions where one chooses the cocycle to be zero. One should not expect our examples to be trivial extensions in general. But trivial extensions can be used to construct cellular algebras which provide negative answers to the questions mentioned after definition 3.2: In the theorem, Δ can be chosen arbitrary. As a B–module, it is not distinguished by any property. Since its A–module structure coincides with its B–module structure, this implies that Δ does not have to be indecomposable. Moreover, there is no restriction on its endomorphism ring, on its composition factors and so on.

We will see in [25] that the situation is much better if we restrict to a smaller class of algebras still containing all the examples provided by Graham and Lehrer.

Now we apply the inductive construction to get more information on the ring structure of cellular algebras. For an algebra A, we denote by $LL(A)$ its Loewy length.

Proposition 7.2 *Let A be a cellular algebra with a nilpotent cell ideal J and $B = A/J$ the quotient algebra. Then the following assertions hold true:*

(a) $rad(A) = rad(B) \oplus J$;

(b) there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & rad^2(A) & \longrightarrow & rad^2(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & rad(A) & \longrightarrow & rad(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & rad(A)/rad^2(A) & \longrightarrow & rad(B)/rad^2(B) & \longrightarrow & 0
\end{array}$$

where $X = J \cap rad^2(A) = rad(B)J + Jrad(B) + \{\phi(b,b'), b, b' \in rad(B), bb' = 0\}$ and $Y = (rad^2(B) \oplus J)/(rad(B) \times_A rad(B) + rad(B)J + Jrad(B))$ (with $+$ indicating a non–direct sum and $\times_A$ indicating multiplication in A). In particular, the quiver of A contains the quiver of B as a subquiver, and the additional arrows correspond to a basis of Y;

(c) there is an inequality $LL(A) \leq 2LL(B)$.

Proof. Since J is nilpotent, it is contained in $rad(A)$; this implies (a). Multiplying two elements in $rad(A)$ gives a product of the form $(j'b + bj' + \phi(b,b'), bb')$, which implies exactness of the first row in the diagram in (b). The rest of the diagram is then clear. Statement (c) follows since J is a B–module. ∎

In the case of heredity ideals there is a similar bound as in (c) (see [12]), thus we get:

Corollary 7.3 *Let A be a cellular algebra with a cell chain consisting of n ideals. Then the Loewy length of A is bounded above by 2^n.*

Quasi–hereditary algebras alway have finite global dimension. This is not true for cellular algebras (for example, there are many local cellular algebras which are not simple).

Corollary 7.4 *If A is a trivial extension of an algebra B by a nilpotent cell ideal J, then the global dimension of A is infinite.*

Proof. For a trivial extension, the decomposition $A = J \oplus B$ is even a decomposition of B–bimodules which induces B–bimodule decompositions

$rad(A) = J \oplus rad(B)$ and $rad^2(A) = rad^2(B) \oplus (Jrad(B) + rad(B)J)$. Thus $rad(A)/rad^2(A)$ decomposes into $rad(B)/rad^2(B) \oplus J/(rad(B)J + Jrad(B))$. Since J is fixed by the involution i, the space$J/(rad(B)J + Jrad(B))$ is fixed by i as well. In particular, there is an idempotent e in A and a non–zero element, say x, in this space such that $ex \neq 0 \neq xe$. Hence the quiver of A contains a loop, and by [28] the global dimension of A is infinite. ■

The examples we have checked seem to indicate that there is a much larger class of cellular algebras having infinite global dimension (although not necessarily containing loops in their quiver); moreover, the results of [28] seem to be applicable in a more general context. We note that the above result also follows from a general criterion [29] deciding when a trivial extension has finite global dimension. We also note that 'typical' examples of cellular alegbras, like the group algebras of symmetric groups, are self–injective, hence either semisimple or of infinite global dimension.

Problem. When does an extension of an algebra B by a nilpotent cell ideal J have infinite global dimension? In particular: When does a cellular algebra have infinite global dimension?

A different inductive construction of cellular algebras (not distinguishing between the two sorts of cell ideals and valid over any ground ring) will be given in [26].

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