Orthogonal Gelfand-Zetlin Algebras, I

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Abstract

We define a class of assosiative algebras those are similar to enveloping algebra of $gl(n,\mathbb{C})$. We construct a family of simple Verma and generalized Verma modules over such algebras. We also investigate two simpliest classes of such algebras in detailes. For the first class we construct all finite-dimensional modules and investigate an $\mathcal{O}$-category and for the second one we classify all weight simple modules.

1 Introduction

During last time several examples of algebras which are similar to the enveloping algebras of simple finite-dimensional complex Lie algebras appear as good as the representation theory of such algebras (see for example [1, 10, 12]). Till now all "good" examples were obtained by generalization of an algebra $sl(2,\mathbb{C})$ as it done in [1, 12]. Careful study of such algebras made it possible to obtain good results even for more difficult algebras ([2]) those are tenzor products of some copies of algebras similar to $sl(2,\mathbb{C})$. But there were no constructions which allows one to obtait some non-trivial generalization of Lie algebras of higher rank.

In the subsequent article we propose a construction of a new family of assosiative algebras that arose from the outstanding paper [8]. Using the formulas that describes an action of generating elements of $gl(n,\mathbb{C})$ on finite-dimensional modules we construct a new family of assosiative algebras that we call Orthogonal Gelfand-Zetlin algebras. It seems that the representation theory of such algebras has common features with the representation theory of $gl(n,\mathbb{C})$.

We show that the family of Orthogonal Gelfand-Zetlin algebras contains $gl(n,\mathbb{C})$, polinomial algebras and extended Heisenberg algebra. Note that the definition of our algebras is not too comfortable. We define Orthogonal Gelfand-Zetlin algebra as an operator algebra generated by some special operators in infinite-dimensional space. So it is not too simple to work with such algebras untill there were no analogue of PBW-theorem.

The positive side of our definition is that any Orthogonal Gelfand-Zetlin algebra appears both with a huge family of modules obtained by specialization of the formulas that defines the algebra itself. This allows us to investigate weight modules over our algebras in two simpliest cases: the first one is an analogue to the case of $sl(2,\mathbb{C})$ and the second one is the case of algebras for which GZ-formulas (formulas 1,2) has polinomial coefficients.

In the subsequent papers we will disscus some questions conserning the structure of Verma and generalized Verma modules over arbitrary Orthogonal Gelfand-Zetlin algebras, will try to develop the structure theory and describe all finite-dimensional simple modules in terms of highest weight.

Let us briefly describe the structure of the paper. In section 2 we collect all basic preliminaries. In section 3 we define Orthogonal Gelfand-Zetlin algebras and describe their basic properties. In section 4 we prove that $gl(n,\mathbb{C})$ is Orthogonal Gelfand-Zetlin algebra (theorem 1). In section 5 we obtain some identities in $\mathcal{U}$ and find a natural Harish-Chandra subalgebra in $\mathcal{U}$ (corollary 1). In section 6 we give some examples of Orthogonal Gelfand-Zetlin algebras. In sections 7-8 we construct two families of simple $\mathcal{U}$-modules which are analogues to Verma and generalized Verma modules over Lie algebra. In sections 9-11 we investigate first special class of algebras those are rather similar to $sl(2,\mathbb{C})$. In particular we describe for them all simple finite-dimensional modules (theorem 4) and prove the BGG-duality in the corresponding category $\mathcal{O}$ (theorem 5). In the last section 12 we classify weight simple modules for another class of Orthogonal Gelfand-Zetlin algebras (theorem 6).

2 Preliminaries

Let $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{Z}$ denotes a ring of integers, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{Z}_+$ denotes the set of all non-negative integers.

Throughout this paper field means field of zero characteristics and assosiative algebra means assosiative algebra with identity.

Fix $n \in \mathbb{N}$ and $r = (r_1, r_2, \dots, r_n) \in \mathbb{N}^n$. For a fixed field $\mathbb{F}$ and $k = |r| = \sum_{i=1}^{n} r_i$ consider a vectorspace $\mathcal{L} = \mathcal{L}(\mathbb{F}, r) = \mathbb{F}^k$. We will call the elements from $\mathcal{L}$ tableaux and will consider them as double indexed families

$$[\, l \,] = \{ l_{ij} \,|\, i = 1, \dots, n; j = 1, \dots, r_i \}.$$

For $[\, l \,] \in \mathcal{L}$ and $i \in \{1, 2, \dots, n\}$ we will denote by $[\, l \,]_i = \{ l_{ij} \,|\, j = 1, \dots, r_i \}$ the i-th row of $[\, l \,]$. An element r will be called signature of $[\, l \,]$. We also set $\mathrm{rank}([\, l \,]) = n - 1$.

We will denote by δ^{ij}, $1 \le i \le n$, $1 \le j \le r_i$ the Kronecker tableaux, i.e. $\delta^{ij}_{ij} = 1$ and other coordinates are zero.

Denote by $\mathcal{L}_0$ the subset $\mathcal{L}$ that consists of all $[\, l \,]$ satisfying following conditions:

1. $l_{nj} = 0$, $j = 1, \dots, r_n$;
2. $l_{ij} \in \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j \le r_i$.

3 Orthogonal Gelfand-Zetlin algebras

Fix some $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{N}^n$. Set $k = |r|$. Consider a field Λ of rational functions in k variables λ_{ij}, $1 \le i \le n$, $1 \le j \le r_i$. Let $[\, \mathfrak{l} \,] \in \mathcal{L}(\Lambda, r)$ be the tableaux defined by $\mathfrak{l}_{ij} = \lambda_{ij}$, $1 \le i \le n$, $1 \le j \le r_i$.

Consider a vectorspace M over Λ with the base $v_{[\mathfrak{i}]}$, $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$. For $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$, $1 \le i \le n-1$ and $1 \le j \le r_i$ denote

$$a_{ij}^{\pm}([\mathfrak{i}]) = \mp \frac{\prod_m (\mathfrak{i}_{i\pm 1\, m} - \mathfrak{i}_{ij})}{\prod_{m \ne j} (\mathfrak{i}_{im} - \mathfrak{i}_{ij})}.$$

Define a $\mathcal{L}$-linear operators $X_i^{\pm} : M \to M$, $1 \le i \le n-1$ by

$$X_i^{\pm} v_{[\mathfrak{i}]} = \sum_{j=1}^{r_i} a_{ij}^{\pm}([\mathfrak{i}]) v_{[\mathfrak{i}]+\delta^{ij}} \tag{1}$$

where $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$.

Let γ_{ij}, $1 \le i \le n$, $1 \le j \le r_i$ denots the j-th elementary symmetrical polinomial in $\lambda_{i1}, \ldots, \lambda_{ir_i}$. Define a $\mathcal{L}$-linear operators $d_{ij} : M \to M$, $1 \le i \le n$, $1 \le j \le r_i$ by

$$d_{ij} v_{[\mathfrak{i}]} = \gamma_{ij}(\mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ir_i}) v_{[\mathfrak{i}]}. \tag{2}$$

Consider an assosiative algebra $\mathcal{U}$, generated by elements $X_i^{\pm}$, $i = 1, \ldots, n-1$ and d_{ij}, $1 \le i \le n$, $1 \le j \le r_i$. We will call it Orthogonal Gelfand-Zetlin algebra (OGZ-algebra) of signature r. The reason of such a name would be explained in section 4.

The subalgebra $\Gamma \subset \mathcal{U}$ generated by elements d_{ij}, $1 \le i \le n$, $1 \le j \le r_i$ will be called basic Gelfand-Zetlin subalgebra (basic GZ-subalgebra) of $\mathcal{U}$. Clearely Γ is commutative polinomial algebra in k variables.

We will also denote by Γ_i, $i = 1, \ldots, n$ the subalgebra of Γ generated by elements d_{ij}, $j = 1, \ldots, r_i$. Clearely Γ_i is a polinomial algebra in r_i variables.

Consider the polinomial algebra Ω in k variables χ_{ij}, $1 \le i \le n$, $1 \le j \le r_i$. Identify Ω with the algebra of polinomial functions on $\mathcal{L}(\Lambda, r)$ putting $\chi_{ij}([\mathfrak{i}]) = \mathfrak{i}_{ij}$ for all possible i and j. Define an homomorphism $\phi : \Gamma \to \Omega$ by $d_{ij} \mapsto \gamma_{ij}(\chi_{i1}, \ldots, \chi_{ij})$ for all possible i and j.

The symmetric group S_{r_i} acts on Ω in natural way permuting χ_{ij} with fixed i. Thus the direct product $S = \prod_{i=1}^{n} S_{r_i}$ acts on Ω. Clearely ϕ is a monomorphism and its image coincides with an algebra of S-invariants in Ω. We would identify Γ with $\phi(\Gamma)$.

We will also denote by $\mathcal{U}^{\pm}$ the subalgebras of $\mathcal{U}$ generated by $X_i^{\pm}$, $1 \le i \le n-1$.

4 $U(gl(n, \mathbb{C}))$ as orthogonal Gelfand-Zetlin algebra

In this section we will prove that the calss of OGZ-algebras is big enough to contain all enveloping algebras $U(gl(n, \mathbb{C}))$. To prove this we need some auxilary notations and lemmas.

Let $\mathcal{U}$ be OGZ-algebra of signature $(1, 2, \ldots, n)$.

Let $\mathfrak{G} = gl(n, \mathbb{C})$ be simple finite-dimensional complex Lie algebra of $n \times n$ matrices, $U(\mathfrak{G})$ be its universal enveloping algebra and $\Gamma(\mathfrak{G})$ its Gelfand-Zetlin subalgebra ([7]). We will denote by e_{ij}, $i, j \in \{1, \ldots, n\}$ standard matrix units.

Lemma 1. *A map $e_{ii+1} \mapsto X_i^+$, $e_{i+1\,i} \mapsto X_i^-$, $i = 1, \ldots, n-1$; $e_{11} \mapsto d_{11} + 1$ defines on M a structure of $\mathfrak{G}$-module ($= U(\mathfrak{G})$-module).*

Proof. Follows from [7, Theorem 30]. □

Denote by Λ' the $\mathbb{C}$-subalgebra of Λ, generated by λ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$ and $(\lambda_{ij} - \lambda_{im} - t)^{-1}$, $1 \leq i \leq n-1$, $1 \leq j < m \leq r_i$, $t \in \mathbb{Z}$. For $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$, $l_{im} - l_{ij} \notin \mathbb{Z}$ for $1 \leq i \leq n-1$, $1 \leq m < j \leq r_i$ consider a homomorphism $\Phi : \Lambda' \to \mathbb{C}$ defined as follows $\Phi(\lambda_{ij}) = l_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq r_i$. Clearly Φ induces a homomorphism from a free Λ'-module M' with the base $v_{[\mathfrak{i}]}$, $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ in $\mathcal{L}(\mathbb{C}, r)$, moreover, $\Phi([\,\mathfrak{l}\,]) = [\,l\,]$. Since $\Phi|_{[\,l\,]+\mathcal{L}_0}$ is a monomorphism it follows that it induces a homomorphism from the algebra of all Λ'-linear operators on M' into the algebra of $\mathbb{C}$-linear operators on the $\mathbb{C}$-space $M_{[\,l\,]}$ with the base $v_{[\,t\,]}$, $[\,t\,] \in [\,l\,] + \mathcal{L}_0$. We will call it specialization of M with respect to $[\,l\,]$ and denote it by $M_{[\,l\,]}$. Clearly Φ defines on $M_{[\,l\,]}$ a structure of $\mathcal{U}$-module.

Lemma 2. *Φ defines a structure of $\mathfrak{G}$-module on $M_{[\,l\,]}$.*

Proof. Follows immediately from lemma 1. □

The following statement is the key observation of this chapter. One can consider it as some analogue of the well-known Harish-Chandra theorem [6, Theorem 2.5.7].

Lemma 3. *For any $0 \neq u \in U(\mathfrak{G})$ there exists $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$, $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \leq i \leq n-1$, $1 \leq j < m \leq r_i$ such that an action of u on $M_{[\,l\,]}$ is non-trivial.*

Proof. Let $0 \neq u \in U(\mathfrak{G})$. Let $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$ be the triangular decomposition of $\mathfrak{G}$ with respect to standard Cartan subalgebra $\mathfrak{H}$. Let $\Delta = \Delta_- \cup \Delta_+$ be the decomposition of the root system of $\mathfrak{G}$ with respect to some base π. We will also denote by X_α, $\alpha \in \Delta$ the elements from the fixed Weyl-Chevalle basis.

We can assume that there exists $\mu \in \mathfrak{H}^*$ such that $[h, u] = \mu(h)u$ for all $h \in \mathfrak{H}$. Clearly there exists a sequence $X_{\alpha_1}, \ldots, X_{\alpha_t}$ such that $\mu + \alpha_1 + \cdots + \alpha_t = \sum_{\beta \in \Delta_+} n_\beta \beta$ where $n_\beta \in \mathbb{Z}_+$.

By Harish-Chandra theorem [6, Theoram 2.5.7] an element $u' = uX_{\alpha_1} \ldots X_{\alpha_t}$ acts non-trivially on some finite-dimensional simple $\mathfrak{G}$-module F.

By [8] each simple finite-dimensional module has a basis consisting of special tableaux of signature r and the action of generating elements is defined by the same formulas as (1),(2). By choosing of $\alpha_1, \ldots, \alpha_t$ we obtain that there exists $v \in F$, $uv \neq 0$ such that action of u on v is defined only by GZ-formulas from [8].

Thus each rational function that defines this action is non-trivial and should have non-trivial value in some specialization $M_{[\,l\,]}$ since the set $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$, $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \leq i \leq n-1$, $1 \leq j < m \leq r_i$ obviousely can not be the zero set of a non-trivial rational function.

This completes the proof. □

Theorem 1. $\mathcal{U} \simeq U(\mathfrak{G})$.

Proof. By lemma 1 we have $\mathcal{U} \simeq U(\mathfrak{G})/I$, where I is an annihilator of M. By lemma 3 we obtain $I = 0$ and the statement follows. □

5 Some identeties in $\mathcal{U}$

It can be obtained directly from the definition of an algebra $\mathcal{U}$ that the generating elements satisfy several canonical identities. We will describe some of them in this section.

Lemma 4. *For any* $1 \leq i \leq n-1$ *holds* $[X_i^+, X_i^-] \in \Gamma$.

Proof. Fix some $i \in \{1, \ldots, n-1\}$. Consider an element $[X_i^+, X_i^-]v_{[\mathfrak{i}]}$ for $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$. By definition

$$[X_i^+, X_i^-]v_{[\mathfrak{i}]} = (X_i^+ X_i^- - X_i^- X_i^+)v_{[\mathfrak{i}]} =$$
$$= \sum_{m=1}^{r_i}\sum_{j=1}^{r_i} a_{im}^+([\mathfrak{i}] - \delta^{ij})a_{ij}^-([\mathfrak{i}])v_{[\mathfrak{i}]-\delta^{ij}+\delta^{im}} - \sum_{j=1}^{r_i}\sum_{m=1}^{r_i} a_{ij}^-([\mathfrak{i}] + \delta^{im})a_{im}^+([\mathfrak{i}])v_{[\mathfrak{i}]-\delta^{ij}+\delta^{im}} =$$
$$= \sum_{j=1}^{r_i}(a_{ij}^+([\mathfrak{i}] - \delta^{ij})a_{ij}^-([\mathfrak{i}]) - a_{ij}^-([\mathfrak{i}] + \delta^{ij})a_{ij}^+([\mathfrak{i}]))v_{[\mathfrak{i}]}.$$

Set

$$f(\mathfrak{i}_{i-1\,1}, \ldots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \ldots, \mathfrak{i}_{i+1\,r_{i+1}}) =$$
$$= \frac{f_1(\mathfrak{i}_{i-1\,1}, \ldots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \ldots, \mathfrak{i}_{i+1\,r_{i+1}})}{f_2(\mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ir_i})} =$$
$$= \sum_{j=1}^{r_i}(a_{ij}^+([\mathfrak{i}] - \delta^{ij})a_{ij}^-([\mathfrak{i}]) - a_{ij}^-([\mathfrak{i}] + \delta^{ij})a_{ij}^+([\mathfrak{i}])).$$

Clearly $f_2 = \prod_{1 \leq j < k \leq r_i} (\mathfrak{i}_{ik} - \mathfrak{i}_{ij})$.

By direct calculations we obtain that

$$f_1(\mathfrak{i}_{i-1\,1}, \ldots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ij-1}, \mathfrak{i}_{im}, \mathfrak{i}_{ij+1}, \ldots, \mathfrak{i}_{im-1}, \mathfrak{i}_{im}, \ldots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \ldots, \mathfrak{i}_{i+1\,r_{i+1}}) = 0$$

and thus f is a polinomial function in $\mathfrak{i}_{i-1\,1}, \ldots, \mathfrak{i}_{i+1\,r_{i+1}}$. One can obtain that f does not change after elementary transpositions of variables $\mathfrak{i}_{i-1\,j} \leftrightarrow \mathfrak{i}_{i-1\,m}$, $\mathfrak{i}_{ij} \leftrightarrow \mathfrak{i}_{im}$, $\mathfrak{i}_{i+1\,j} \leftrightarrow \mathfrak{i}_{i+1\,m}$.

The statment of the lemma now follows from the main theorem on symmetric polinomials. □

Lemma 5. *For* $1 \leq i \leq n-1$ *holds*

1. $[d_{i1}, X_i^\pm] = \pm X_i^\pm$.

2. $[d_{ij}, X_i^{\pm}] = f_{ij}^{\pm}(\gamma_{i1}, \dots, \gamma_{ij-1})X_i^{\pm}$.

Proof. For $1 \le i \le n-1$ and $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ we have

$$[d_{i1}, X_i^{\pm}]v_{[\,\mathfrak{i}\,]} = \sum_{j=1}^{r_i} \left((\mathfrak{i}_{i1} + \dots + \mathfrak{i}_{ir_i} \pm 1)a_{ij}^{\pm}([\,\mathfrak{i}\,]) - (\mathfrak{i}_{i1} + \dots + \mathfrak{i}_{ir_i})a_{ij}^{\pm}([\,\mathfrak{i}\,])\right) v_{[\,\mathfrak{i}\,]+\delta^{ij}} =$$

$$= \pm \sum_{j=1}^{r_i} a_{ij}^{\pm}([\,\mathfrak{i}\,])v_{[\,\mathfrak{i}\,]+\delta^{ij}} = \pm X_i^{\pm} v_{[\,\mathfrak{i}\,]}$$

and the first statement follows.

The second follows analogously from the fact that for any non-trivial polinomial $f \in \mathbb{C}[x]$ and for any $a \in \mathbb{C}$ holds $\deg(f(x-a) - f(x)) < \deg f(x)$. □

The following identities are obtained at the same way as [7, Lemma 25].

Any $x \in M$ can be written in the form $x = \sum_{[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0} x_{[\,\mathfrak{i}\,]} v_{[\,\mathfrak{i}\,]}$ where $x_{[\,\mathfrak{i}\,]} \in \Lambda$. For $u \in \mathcal{U}$ and $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ denote $\mathcal{L}(u, [\,\mathfrak{i}\,])$ the set of all such tableaux $[\,\tau\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ that $(uv_{[\,\mathfrak{i}\,]})_{[\,\tau\,]} \neq 0$.

Lemma 6. *The set* $\mathcal{L}(u, [\,\mathfrak{i}\,]) - [\,\mathfrak{i}\,]$ *does not depend on* $[\,\mathfrak{i}\,]$.

Proof. By the definition of $\mathcal{U}$ all $(uv_{[\,\mathfrak{i}\,]})_{[\,\tau\,]}$ are non-trivial elements in Λ. Then the statement follows from the fact that by the shift of variables a non-trivial rational function maps to a non-trivial rational function. □

We will denote the set $\mathcal{L}(u, [\,\mathfrak{i}\,]) - [\,\mathfrak{i}\,]$ by $\mathcal{L}_u$. Clearly $\mathcal{L}_u$ is finite and $\mathcal{L}_u \subset \mathcal{L}_0$. Thus we have

$$uv_{[\,\mathfrak{i}\,]} = \sum_{[\,\tau\,] \in \mathcal{L}_u} \theta(u, [\,\mathfrak{i}\,], [\,\tau\,])v_{[\,\mathfrak{i}\,]+[\,\tau\,]},$$

where $\theta(u, [\,\mathfrak{i}\,], [\,\tau\,]) = (uv_{[\,\mathfrak{i}\,]})_{[\,\mathfrak{i}\,]+[\,\tau\,]} \neq 0$.

Any $[\,\tau\,] \in \mathcal{L}(\Lambda, r)$ defines an automorphism $\chi \mapsto \chi^{[\,\tau\,]}$ of Ω where $\chi_{ij}^{\tau} = \chi_{ij} + \tau_{ij}$ for all possible i and j. For any $z \in \Gamma$ and $u \in \mathcal{U}$ set

$$F_{u,z}(T, \chi) = \prod_{[\,\tau\,] \in \mathcal{L}_u} (T - z^{\tau}).$$

Clearly $F_{u,z} \in \Gamma(T)$.

Lemma 7. *Let* $z \in \Gamma$ *and* $F_{u,z} = \sum_{i \in I} T^i g_i$, $g_i \in \Gamma$. *Then* $\sum_{i \in I} z^i u g_i = 0$.

Proof. For $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$ we have

$$\sum_{i\in I} z^i u g_i v_{[\mathfrak{i}]} = \sum_{i\in I} z^i u g_i([\mathfrak{i}]) v_{[\mathfrak{i}]} = \sum_{i\in I} z^i g_i([\mathfrak{i}]) \sum_{[\tau]\in\mathcal{L}_u} \theta(u,[\mathfrak{i}],[\tau]) v_{[\mathfrak{i}]+[\tau]} =$$
$$\sum_{[\tau\in\mathcal{L}_u]} \theta(u,[\mathfrak{i}],[\tau]) \sum_{i\in I} z([\mathfrak{i}]+[\tau])^i g_i([\mathfrak{i}]) v_{[\mathfrak{i}]+[\tau]} =$$
$$\sum_{[\tau]\in\mathcal{L}_u} \theta(u,[\mathfrak{i}],[\tau]) F_{u,z}(z^{[\tau]}([\mathfrak{i}]),[\mathfrak{i}]) v_{[\mathfrak{i}]+[\tau]} = 0.$$

Thus $\sum_{i\in I} z^i u g_i = 0$. □

Corllary 1. *Γ is a Harish-Chandra subalgebra in $\mathcal{U}$ in the sence of [7] (i.e. $\Gamma u \Gamma$ is finitely generated both as right and left Γ-module for any $u \in \mathcal{U}$).*

Proof. Let $s = \deg F_{u,z}$. By lemma 7 we have $z^s \in \sum_{i=1}^{s-1} z^i u \Gamma$ and thus $\Gamma u \Gamma$ is finitely generated right Γ module since Γ is finitely generated.

The map $X_i^{\pm} \mapsto X_i^{\mp}$ can be continued to the involution on $\mathcal{U}$ which maps Γ to Γ. Thus $\Gamma u \Gamma$ is also finitely generated as left Γ-module. This completes the proof. □

6 Examples

In this section we give some trivial but interesting examples of OGZ-algebras.

6.1 $U(gl(n,\mathbb{C}))$

As it was shown in section 4 an algebra $U(gl(n,\mathbb{C}))$ is OGZ-algebra of signature $r = (1,2,\ldots,n)$.

6.2 Polinomial algebra

Polinomial algebra is an OGZ-algebra of rank 0. By the main theorem on symmetric polinomials the OGZ-algebra $\mathcal{U}$ of signature $r = (r_1)$ is a polinomial algebra in r_1 variables. Clearely any simple irreducible module over $\mathcal{U}$ is defined (up to the action of S_{r_1}) by the vector $(l_j)_{j=1}^{r_1} \in \mathbb{C}^n$ from which one could obtain the eigenvalues of generating elements d_{1j}, $1 \le j \le r_1$.

6.3 Extended Heisenberg algebra

One can obtain that in the case $r = (1,1)$ an algebra $\mathcal{U}$ is a trivial central extension of an algebra $\mathcal{U}'$ generated by X, Y, H with relations $[X,Y] = -1$, $[H,X] = X$, $[H,Y] = Y$ that is nothing more than extended Heisenberg algebra. It is obvious but it would also follows easely from section 10 that this algebra have no simple finite-dimensional modules.

7 Verma modules in general position

The most comfortable thing in investigation of OGZ-algebras is that they appear both with a big family of own modules. In this section we construct some analogue of Verma modules over OGZ-algebras but only those which could be obtained directly from the definition of an algebra. For the first step we define a specialization of $\mathcal{U}$.

At the same way as at section 4 to each $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$, $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j < m \le r_1$ corresponds some $\mathcal{U}$-module $M_{[\,l\,]}$ with $\mathbb{C}$-base $v_{[\,t\,]}$, $[\,t\,] \in [\,l\,] + \mathcal{L}_0$ and the action of generated elements given by (1),(2) with the only substitution l_{ij} instead of $\mathfrak{l}_{ij}$ for all possible i and j. We will call such $[\,l\,]$ quasi-generic.

Lemma 8. *Let $K \subset [\,t\,] + \mathcal{L}_0$ for some quasi-generic $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ be such a subset that for any $[\,t\,] \in K$ and any $1 \le i \le n-1$ holds*

$$X_i^{\pm} v_{[\,t\,]} = \sum_{[\,s\,] \in K} a_{[\,s\,]} v_{[\,s\,]}.$$

Then the space N spaned by $v_{[\,s\,]}$, $[\,s\,] \in K$ is an $\mathcal{U}$-submodule in $M_{[\,l\,]}$.

Proof. Follows from the definition of $\mathcal{U}$. □

An element v from $\mathcal{U}$-module V will be called primitive provided $X_i^+ v = 0$ for all $1 \le i \le n-1$. In this section we construct a huge family of simple $\mathcal{U}$-modules generated by primitive element as submodules in some $M_{[\,l\,]}$ for different $[\,l\,]$.

From now on we suppose that the signature vector r saticfies the condition $r_j \ge r_k$ for all possible $j > k$.

Consider a map $\Phi : \mathbb{C}^{r_n} \to \mathcal{L}(\mathbb{C}, r)$ defined as follows $\Phi((a_1, \dots, a_{r_n})) = [\,t\,]$ where $t_{ij} = a_j$, $1 \le i \le n$, $1 \le j \le r_i$. Since $r_j \ge r_k$ for $j > k$ this is a will-defined map.

Let $[\,l\,] = \Phi((a_1, \dots, a_{r_n}))$, $a_j - a_m \notin \mathbb{Z}$ for $1 \le j < m \le r_n$. Consider a specialization $M_{[\,l\,]}$.

Let $K = K([\,l\,])$ be a subset in $[\,l\,] + \mathcal{L}_0$ consisting of all those $[\,t\,]$ which satisfies the following conditions:

1. $l_{ij} - t_{ij} \in \mathbb{Z}_+$, $1 \le i \le n-1$, $1 \le j \le r_i$;
2. $t_{ij} \le t_{i+1\,j}$, $1 \le i \le n-1$, $1 \le j \le r_i$.

Lemma 9. *Let $[\,t\,] \in K$ and $a_{ij}^+([\,t\,]) \ne 0$ for some possible i and j. Then $[\,t\,] + \delta^{ij} \in K$.*

Proof. Since $a_{ij}^+([\,t\,]) \ne 0$ it follows that $t_{ij} \ne t_{i+1\,m}$ for all possible m and thus $t_{ij} < t_{i+1\,j}$. This implies that $[\,t\,] + \delta^{ij} \in K$. □

Theorem 2. *$M([\,l\,]) = \langle v_{[\,t\,]} \,|\, [\,t\,] \in K \rangle$ is simple $\mathcal{U}$-submodule of $M_{[\,l\,]}$, generated by primitive element $v_{[\,l\,]}$.*

Proof. N is $\mathcal{U}$-submodule by lemma 9 and lemma 8.

It follows from direct calculation with (1),(2) that $M([\,l\,])$ is generated by $v_{[\,l\,]}$ and that $v_{[\,l\,]}$ is primitive.

Since $\mathcal{U}^+v$ is finite-dimensional for any $v \in M([\,l\,])$ it follows that each submodule of $M([\,l\,])$ contains primitive element. But one can see that by the construction of K the only primitive elements in $M([\,l\,])$ have the form $const \cdot v_{[\,t\,]}$ and thus $M([\,t\,])$ is simple. □

It is rather natural to call $M([\,l\,])$ Verma module over $\mathcal{U}$. One can see that in the case when $\mathcal{U} \simeq U(gl(n,\mathbb{C}))$ the module $M([\,l\,])$ is a Verma module over $gl(n,\mathbb{C})$. We note that we do not construct the complete family of Verma module but only some class of simple ones arising directly from the definition of the $\mathcal{U}$. The rest would be done in subsequent papers.

8 Generalized Verma modules in general position

In this section we construct at the same way as in section 7 the family of simple modules over $\mathcal{U}$ those are similar to generalized Verma modules induced from parabolic subalgebra of a Lie algebra (see [5, 11]).

Fix $m < n$ and let $\mathcal{U}'$ be a subalgebra in $\mathcal{U}$ generated by $X_i^{\pm}$, $1 \le i \le m-1$ and d_{ij}, $1 \le i \le m$, $1 \le j \le r_i$. We will construct a family of $\mathcal{U}$-modules those can be wieved as modules induced from $\mathcal{U}'$.

Set $r' = (r_1, \dots, r_{m-1})$. Consider a map $\Phi : \mathbb{C}^{r_n} \oplus \mathcal{L}(\mathbb{C}, r') \to \mathcal{L}(\mathbb{C}, r)$ defined as follows: $\Phi((a_1, \dots, a_{r_n}), [\,t\,]) = [\,l\,]$ where $[\,t\,] \in \mathcal{L}(\mathbb{C}, r')$ and $l_{ij} = t_{ij}$, $1 \le i \le m-1$, $1 \le j \le r_i$; $l_{ij} = a_j$, $m \le i \le n$, $1 \le j \le r_i$.

Let $[\,l\,] = \Phi(a, [\,t\,])$ for some $a \in \mathbb{C}^{r_i}$, $[\,t\,] \in \mathcal{L}(\mathbb{C}, r')$. For the rest of the section we suppose that $[\,l\,]$ satisfies the following conditions:

1. $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j < m \le r_i$;
2. $l_{ij} - l_{i+1\,m} \notin \mathbb{Z}$, $1 \le i \le m-1$, $1 \le j \le r_i$, $1 \le m \le r_{i+1}$.

Consider a subset $K = K([\,l\,]) \subset [\,l\,] + \mathcal{L}_0$ consisting from all such $[\,t\,]$ that

1. $l_{ij} - t_{ij} \in \mathbb{Z}$, $1 \le i \le m-1$, $1 \le j \le r_i$;
2. $l_{ij} - t_{ij} \in \mathbb{Z}_+$, $m \le i \le n-1$, $1 \le j \le r_i$;
3. $t_{ij} \le t_{i+1\,j}$, $m \le i \le n-1$, $1 \le j \le r_i$.

Lemma 10. *Let $[\,t\,] \in K$ such that $a_{ij}^+([\,t\,]) \neq 0$ then $[\,t\,] + \delta^{ij} \in K$.*

Proof. For $1 \le i \le m-1$ the statement is obvious by definition of K. For the rest of i one can prove it at the same way as it done in lemma 9. □

Theorem 3. *The subspace $M([\,l\,]) \subset M_{[\,l\,]}$ generated by $v_{[\,t\,]}$, $[\,t\,] \in K$ is simple $\mathcal{U}$-submodule in $M_{[\,l\,]}$. Moreover, $M([\,l\,])$ is generated by $v_{[\,l\,]}$.*

Proof. Analogouse to that of theoerm 2. □

It is natural to call the modules constructed in this section generalized Verma modules. In the case $\mathcal{U} \simeq U(gl(n,\mathbb{C}))$ the module $M([\,l\,])$ is a generalized Verma module induced from generic Gelfand-Zetlin modules ([11]). We also note that the family of modules constructed in this section is not a complete family of generalized Verma modules over $\mathcal{U}$ which would be investigated in subsequent papers.

9 Orthogonal Gelfand-Zetlin algebras of signature $(1, m)$, basic properties

For the following three sections we assume that $n = 2$ and the signature vector r has the form $r = (1, m)$. In this case it is possible to constract a theory of $\mathcal{U}$-modules almost at the same way as it is done for a Lie algebra $sl(2,\mathbb{C})$. And the most interesting that for $m > 2$ there appears some non-trivial effects just as in the case of another algebras considered in [10, 12].

It would be shown that OGZ-algebras of signature $(1, m)$ is non-trivial extensions of the algebras similar to enveloping algebras [10, 12]. But it happens that almost all problems of finite-dimensional modules over OGZ-algebras could be solved in simple way using our terminology of tableaux and ommiting difficult calculations from [12]. We also note that both algebras from [12] and OGZ-algebras of signature $(1, m)$ are the so-called generalized Weyl algebras (in sence of [1]). For the algebras from [12] the ground ring was $\mathbb{C}[H]$ and for our algebras it was $\mathbb{C}[d_{11}, d_{21}, \ldots, d_{2m}]$ which is more difficult ring. We note that our classification is obtained in easier terms than it done in [1, 12] and, moreover, we show that the modules over OGZ-algebras have clear geometrical structure.

Let A_i, $i = 1, \ldots, m+3$ be the generating elements $X_1^{\pm}$, d_{11}, d_{2j}, $j = 1, \ldots, m$ written in some order. For non-negative integers $i_1, \ldots, i_{m+3}$ set $F(i_1, \ldots, i_{m+3}) = A_1^{i_1} \cdot \cdots \cdot A_{m+3}^{i_{m+3}}$.

To simplify the notations we will also write $X = X_1^+$ and $Y = X_1^-$.

Proposition 1. *The elements $F(i_1, \ldots, i_{m+3})$ span the $\mathbb{C}$-space $\mathcal{U}$.*

Proof. Follows from lemma 4 and lemma 5 by standard arguments. □

Lemma 11. *The senter Z of $\mathcal{U}$ coinsides with Γ_2.*

Proof. Since $[X, Y] \in \Gamma$ by lemma 4 it follows immediately that $Z \subset \Gamma$. But clearely $\Gamma_2 \subset Z$ and any polinomial in d_{11} does not commute with X. Thus the lemma follows. □

Lemma 12. *$\mathcal{U}$ is somewhat commutative, i.e. there is a standard filtration on $\mathcal{U}$ such that the assosiated graded algebra is comutative.*

Proof. One can prove this using the following $\mathbb{Z}$-grading on $\mathcal{U}$: $X^{i_1} Y^{i_2} d_{11}^{i_3} \ldots d_{2m}^{i_{m+3}} \in \mathcal{U}_{i_1 - i_2}$. □

10 Orthogonal Gelfand-Zetlin algebras of signature $(1, m)$, finite-dimensional modules

In this section we establish complete theory of Verma modules over $\mathcal{U}$ and describe all finite-dimensional modules. We give an answer in terms of some group action as it done in classical case of Verma modules over finite-dimensional simple Lie algebra over $\mathbb{C}$.

For an $\mathcal{U}$-module V and $\chi \in \Gamma^*$ we set

$$V_\chi = \{v \in V \mid zv = \chi(z)v \text{ for all } z \in \Gamma\}.$$

In the case $V_\chi \neq 0$ we will call it the weight space of V corresponding to the weight χ. A module V will be called weight module if

$$V = \bigoplus_{\chi \in \Gamma^*} V_\chi.$$

For a weight module V we will denote by $\operatorname{supp} V$ the set of all weights of V. Each element from V_χ will be called weight element. Since we identify Γ with $\phi(\Gamma)$ (see section 3) it is possible to parametrize $\chi \in \Gamma^*$ by elements $[\,t\,] \in \mathcal{L}(\mathbb{C}, r)$ in natural way. Thus if χ is parametrized by $[\,t\,]$ wi will also write $V_\chi = V_{[t]}$ and say that an element $v \in V_{[t]}$ has tableaux $[\,t\,]$. For a weight module V we will also denote by $\operatorname{Tsupp} V$ the set of all tableaux which parametrize elements from $\operatorname{supp} V$. Note that by definition $\operatorname{Tsupp} V$ is invariant under the action of S.

We recall that an element $v \in V_\chi$ is called primitive if $Xv = 0$.

For $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ we will denote $M(([\,l\,]))$ the Verma module ([6]) generated by primitive element which has tableaux $[\,l\,]$.

Lemma 13. *Let F be simple finite-dimensional $\mathcal{U}$-module. Then F is a weight module generated by some primitive element.*

Proof. Since $[X, [d_{11}, X]] = [X, X] = 0$ it follows from Klienike-Shirokov theorem that X is nilpotent on F and thus there exists some non-trivial primitive element v in F. Since F is simple it follows that v generates F. Clearely $Y^i v$, $i \geq 0$ span some $\mathcal{U}$-submodule in F which coinsides with F since F is simple. One can show by induction in i that $Y^i v$ is a weight element for each $i \geq 0$ and the lemma follows. □

Corllary 2.

1. *Each finite-dimensional $\mathcal{U}$-module F is a quotient of some Verma module.*
2. *The unique irreducible quotient of a Verma module is finite-diminsional if and only if Verma module is not simple.*

Proof. The first statement fiollows from proposition 1 which can be reformulated in standard universal property of Verma modules: The module generated by primitive element is a quotient of some Verma module.

The second is obvious. □

By corollary 2 to classify all finite-dimensional module we need only to classify all maximal submodules in non-simple Verma modules.

Recall the construction of Verma modules from section 7. Let $\Phi : \mathbb{C}^m \to \mathcal{L}(\mathbb{C}, r)$ be a map defided as follows: $\Phi((a_1, \ldots, a_m)) = [\, l \,]$ where $l_{ij} = a_j$, $1 \leq i \leq 2$, $1 \leq j \leq r_i$. A group S_m acts on $\mathbb{C}^m$ in natural way $\sigma(a_1, \ldots, a_m) = (a_{\sigma(1)}, \ldots, a_{\sigma(m)})$ for $(a_1 \ldots, a_m) \in \mathbb{C}^m$ and $\sigma \in S_m$. Clearely this induces an action of S_m on $P = \Phi(\mathbb{C}^m)$.

For $[\, l \,], [\, t \,] \in P$ we will say $[\, t \,] < [\, l \,]$ if there exists $\sigma \in S_m$ such that $[\, l \,]_2 = \sigma([\, t \,]_2)$ and $l_{11} - t_{11} \in \mathbb{N}$.

Let $W \simeq S_{m-1}$ be a subgroup in S_m which fix the first element.

For a given $[\, l \,] \in P$ set $K = K([\, l \,]) = \{[\, l \,] - j\delta^{11} \,|\, j \geq 0\}$. Then the sabspace $M([\, l \,]) \subset M_{[l]}$ spaned by $v_{[t]}$, $[\, t \,] \in K$ is $\mathcal{U}$-submodule in $M_{[l]}$ which we call Verma module corresponding to $[\, l \,] \in P$.

Lemma 14. *For $[\, l \,] \in P$ holds $M([\, l \,]) \simeq M(([\, l \,]))$.*

Proof. Follws from proposition 1. □

Now we show that we construct a complete family of Verma modules.

Lemma 15. *Each $\mathcal{U}^-$-free module generated by primitive element is of the form $M([\, l \,])$ for some $[\, l \,] \in P$.*

Proof. Follows from formulas (1),(2). □

Lemma 16. *Let $[\, l \,] \in P$ and $\sigma \in W$ then $M([\, l \,]) \simeq M(\sigma([\, l \,]))$.*

Proof. Follows from definition of d_{2j}, $1 \leq j \leq m$. □

Lemma 17. *Let $[\, l \,] \in P$. A module $M([\, l \,])$ is irreducible if and only if $l_{21} - l_{2j} \notin \mathbb{N}$ for all $2 \leq j \leq m$ or equivalently $M([\, l \,])$ is irreduceble if and only if $\sigma[\, l \,] \not< [\, l \,]$ for any $\sigma \in S_m$.*

Proof. Follows from formula (1) and construction of module $M([\, l \,])$. □

Set P^+ be the subset of P consicting from all such $[\, l \,]$ for which there exists $j \in \{2, \ldots, m\}$ such that $l_{21} - l_{2j} \in \mathbb{N}$.

Theorem 4. *1. Let $[\, l \,] \in P^+$. There exists the unique finite-dimensional module $L([\, l \,])$ which is generated by primitive vector that has tableaux $[\, l \,]$.*

2. Each finite-dimensional module F is isomorphic to $L([\, l \,])$ for some $[\, l \,] \in P^+$.

3. $L([\, l \,]) \simeq L([\, t \,])$, $[\, l \,], [\, t \,] \in P^+$ if and only if there exists $\sigma \in W$ such that $[\, l \,] = \sigma([\, t \,])$.

Proof. the first statement follows from the uniqueness of the simple quotient in a Verma module. The second follows from lemma 15. The last follows from lemma 16. □

We will also denote by $L([\, l \,])$, $[\, l \,] \in P$ the unique simple quotient of $M([\, l \,])$. We immediately obtain the following BGG-criterion for an existense of a submodule in Verma module:

Corllary 3. *The following statements are equivalent:*

1. $M([\,t\,]) \subset M([\,l\,])$.
2. $L([\,t\,])$ *is a subquotient in composition series of* $M([\,l\,])$.
3. *There exists* $\sigma \in S_m$ *such that* $[\,t\,] = \sigma([\,l\,]) < [\,l\,]$.

Proof. Follows from lemma 17, theorem 4 and definition of the S_m-action. □

The last theorem describes the set of simple finite-dimensional modules over $\mathcal{U}$. We emphasize three different cases:

$m = 1$. In this case there were no finite-dimensional modules over $\mathcal{U}$.

$m = 2$. Since $\mathcal{U} \simeq U(gl(2, \mathbb{C}))$ than by Weyl theorem each finite-dimensional module decomposes into direct some of simple modules.

$m > 2$. In this case an anologue of the Weyl theorem does not take place. For the rest of this section we will investigate this case more carefully. So tilll the end of this section we assume that $m > 2$.

Denote by (xy), $1 \leq x < y \leq m$ the elementary transposition in S_m.

Proposition 2. *Let* $[\,l\,] \in P$, $T = \{l_{2j} \,|\, 2 \leq j \leq m, l_{21} - l_{2j} \in \mathbb{N}\}$. *Set* $t = |T|$ *and* $T = \{p_1, \ldots, p_t\}$, $p_1 > p_2 > \cdots > p_t$. *Assume that* $p_j = l_{2\,j+1}$, $1 \leq j \leq t$. *Then the length of* $M([\,l\,])$ *equals* $t + 1$. *Moreover the composition series of* $M([\,l\,])$ *has the form*

$$0 \subset M((1\,t+1)([\,l\,])) \subset M((1t)([\,l\,])) \subset \cdots \subset M((12)[\,l\,]) \subset M([\,l\,]).$$

Proof. Follows from the construction of $M([\,l\,])$ and formula (1). □

Lemma 18. *Let* $L([\,l\,])$, $L([\,t\,])$, $[\,l\,], [\,t\,] \in P^+$ *be two non-isomorphic simple modules. Suppose that*

$$0 \to L([\,t\,]) \to X \to L([\,l\,]) \to 0$$

is a non-split extension. Then there exists $\sigma \in S_m$ *such that* $[\,l\,]_2 = \sigma([\,t\,]_2)$.

Proof. Follows from the fact that Γ_2 is a center of $\mathcal{U}$ (lemma 11) and Fittings lemma. □

Lemma 19. *Let* $L([\,l\,])$, $L([\,t\,])$ *be distinct finite-dimensional modules,* $l_{11} - t_{11} \in \mathbb{N}$. *Then the following statements are equivalent:*

1. $\mathrm{Ext}_{\mathcal{U}}(L([\,l\,]), L([\,t\,])) \neq 0$.
2. $L([\,t\,])$ *is a subquotient of* $M([\,l\,])$.
3. $\dim \mathrm{Ext}_{\mathcal{U}}(L([\,l\,]), L([\,t\,])) = 1$.
4. $\dim \mathrm{Ext}_{\mathcal{U}}(L([\,t\,]), L([\,l\,])) = 1$.

Proof. Let $0 \to L([\,t\,]) \to X \to L([\,l\,]) \to 0$ be a non-split sequence. Then by lemma 18 there exists a homomorphism $\phi : M([\,l\,]) \to X$ which should be epimorphism. Conversly one can obtain a non-split extension of $L([\,t\,])$ and $L([\,l\,])$ under the lemma conditions as a quotient of $M([\,l\,])$. This proves that the first and second statements are equivalent.

The rest follows from fomula (1). □

Proposition 3. *Let $L([\,l\,])$, $[\,l\,] \in P^+$ be simple finite-dimensional $\mathcal{U}$-module. Assume that $l_{21} - l_{22} \in \mathbb{N}$ and $l_{21} - l_{2j} \in \mathbb{N}$, $3 \leq j \leq m$ implies $l_{2j} \leq l_{22}$. Then the following holds:*

1. $\dim \mathrm{Ext}_{\mathcal{U}}(L([\,l\,]), L([\,l\,])) \leq 1$.
2. $\dim \mathrm{Ext}_{\mathcal{U}}(L([\,l\,]), L([\,l\,])) = 1$ *if and only if*

$$\prod_{j=3}^{m}(l_{2j} - l_{21}) = -\prod_{j=3}^{m}(l_{2j} - l_{22}).$$

Proof. Let I be an ideal in $\mathcal{U}$ generated by $d_{2j} - \gamma_{2j}(l_{21}, \ldots, l_{2j})$, $1 \leq j \leq m$.

Consider an algebra $\mathcal{U}_I = \mathcal{U}/I$. Clearely $L([\,l\,])$ is $\mathcal{U}_I$-module. Obviousely it is sufficient to prove our result for $\mathcal{U}_I$-modules by lemma 11.

Consider a polinomial $g(x) = \prod_{j=1}^{m}(l_{2j} - x + 1)$. One can see that $[X, Y] = (g(d_{11} + 1) - g(d_{11}))$ mod I.

Thus $\mathcal{U}_I$ is a quotient of some algebra similar to the enveloping algebra of $sl(2)$ by the central character ([12]).

Now the statement of the theorem follows from [10, Theorem 3.1] by direct calculation. □

From last two statements we obtain that representation theory of our algebras is rather similar to that of algebras that are close to $U(sl(2))$. We obtain that the category of finite-dimensional modules is not semi-simle for $m > 3$, moreover, some of finite-dimensional simples has non-trivial self-extensions.

11 Orthogonal Gelfand-Zetlin algebras of signature $(1, m)$, category $\mathcal{O}$

In this section we construct an analogue of the well-known category $\mathcal{O}$ for an algebra $\mathcal{U}$ ([4]).

We define $\mathcal{O}$ to be a full subcategory of such $\mathcal{U}$-modules V that satisfies the following conditions:

1. V is d_{11}-diagonalizable;
2. V is finitely generated;

3. $\mathcal{U}^+ v$ is finite-dimensional for any $v \in V$.

Set $\mathcal{L}^+$ to be a subset of $\mathcal{L}_0$ that consist of all such $[\,t\,] \in \mathcal{L}_0$ that $t_{11} \in \mathbb{N}$.
Basic properties of $\mathcal{O}$ is described in the following lemma:

Lemma 20. *Let $V \in \mathcal{O}$. Then*

1. $\operatorname{Tsupp} V \subset \bigcup_{j=1}^{N} ([\,l_j\,] - \mathcal{L}^+)$.

2. *All d_{11}-weight spaces of V is finite-dimensional.*

3. *V has finite length.*

4. $\dim \hom(V, N) < \infty$ *for any $N \in \mathcal{O}$.*

Proof. Follows from section 10. □

Lemma 21. *1. $\mathcal{O}$ is closed under the operation of taking of submodules, quotients and finite direct sums.*

2. $\{L([\,l\,]) \,|\, [\,l\,] \in P\}$ is a complete set of simples in $\mathcal{O}$.

Proof. Follows from lemma 15. □

For an $\mathcal{U}$-module V and $\chi \in \Gamma_2^*$ set

$$V^\chi = \{v \in V \,|\, (z - \chi(z))^s v = 0 \text{ for all } z \in \Gamma,\ s >> 0\}.$$

Clearly $M = \oplus_\chi M^\chi$ and therefore $\mathcal{O} = \oplus_\chi \mathcal{O}_\chi$ where $\mathcal{O}_\chi$ consists of those $M \in \mathcal{O}$ such that $(z - \chi(z))^s M = 0$, $s >> 0$ for all $z \in \Gamma$.

Lemma 22. *Each object in $\mathcal{O}$ is a quotient of a projective object in $\mathcal{O}$.*

Proof. Clearly it is enough to prove this for each $\mathcal{O}_\chi$. Fix $[\,t\,] \in \mathcal{L}(\mathbb{C}, r)$.

One can see that there exists such $s > o$ that $X^s V_{[\,t\,]} = 0$ for any $V \in \mathcal{O}_\chi$. Set I to be a left ideal of $\mathcal{U}$ generated by $d_{11} - t_{11}$ and X^s. Then $Q([\,t\,]) = \mathcal{U}/I \in \mathcal{O}$. One can see that the map $\hom_{\mathcal{U}}(Q([\,t\,]), V) \to V_{[\,t\,]}$ is an isomorphism and thus the functor $V \mapsto \hom_{\mathcal{U}}(Q([\,t\,])^\chi, V)$ is exact.

We conclude that $Q([\,t\,])$ is projective which completes the proof since V is finitely generated. □

Corllary 4. *There is a 1-1 correspondence between simples in $\mathcal{O}$ and indecomposable projectives in $\mathcal{O}$.*

For $[\,l\,] \in P$ we will denote by $Pr([\,l\,])$ the projective cover of $L([\,l\,])$.

Lemma 23. *Each $Pr([\,l\,])$, $[\,l\,] \in P$ has a Verma flag (in a sence of [4]).*

Proof. Follows from the fact that $Pr([\,l\,])$ is $\mathcal{U}^-$-free. □

For an $\mathcal{U}$-module V we turn V^* into $\mathcal{U}$-module by setting $(X\psi)(v) = \psi(Yv)$, $(Y\psi)(v) = \psi(Xv)$, $(z\psi)(v) = \psi(zv)$, $\psi \in V^*$, $v \in V$, $z \in \Gamma$. Let $\delta V = \{\psi \in V^* \mid \dim(\Gamma\psi) < \infty\}$. Then δ is left exact contravariant functor on $\mathcal{O}$. The following lemma is clear:

Lemma 24. *1.* $\delta^2 V = V$, $V \in \mathcal{O}$.

2. $\delta L([\,t\,]) \simeq L[\,t\,]$, $[\,t\,] \in P$.

Proof. The first is obvious and the second follows from the uniques of simple highest weight module with fixed highest weight. □

For a fixed $\chi \in \Gamma_2^*$ let $M([\,t_j\,])$, $1 \le j \le s$ be a complete set of Verma modules in $\mathcal{O}_\chi$. Set $Pr = \oplus_{j=1}^s Pr([\,t_j\,])$. Then $A = \hom_{\mathcal{U}}(Pr, Pr)$ is finite-dimensional algebra and $\mathcal{O}_\chi$ is equivalent to the category of left A-modules of finite length.

For $[\,l\,], [\,t\,] \in P$ let $[Pr([\,l\,]) : M([\,t\,])]$ denotes the number of subquotients in Verma flag of $Pr([\,l\,])$ isomorphic to $M([\,t\,])$. Also set $(M([\,t\,]) : L([\,l\,]))$ to be the number of subquotients in composition series of $M([\,t\,])$ isomorphic to $L([\,l\,])$.

Theorem 5. *A is BGG-algebra in the sence of [9], in particular*

$$[Pr([\,l\,]) : M([\,t\,])] = (M([\,t\,]) : L([\,l\,]))$$

for all $[\,t\,], [\,l\,] \in P$.

Proof. We only need to prove that any $V \in \mathcal{O}_\chi$ with $\operatorname{Rad} V \simeq L([\,l\,])$ and any subquotients of the form $L([\,t\,])$, $[\,t\,] \in [\,l\,] - \mathcal{L}^+$ is a quotient of a Verma module. This follows from the universal propety of Verma modules.

The rest follows from [9]. □

12 Weight modules over OGZ-algebras of signature $(1, 1, \ldots, 1)$

In this section we investigate one more family of OGZ-algebras and describes all irreducible weight modules over them.

We assume that $r = (1, 1, \ldots, 1)$. We will use notation of the weight module from section 10.

Set $X_{ii}^{\pm} = X_i^{\pm}$, $1 \le i \le n-1$ and define for $1 \le i < j \le n-1$ by induction $X_{ij}^{\pm} = [X_{ii}^{\pm}, X_{i+1\,j}^{\pm}]$.

Lemma 25. *The following identities holds:*

1. $[X_m^{\pm}, X_{ij}^{\pm}] = 0$, $1 \le i < j \le n-1$, $1 \le m \le n-1$, $m \ne i-1$, $m \ne j+1$.

2. $[X_m^{\pm}, X_{ij}^{\mp}] = 0$, $1 \le i < j \le n-1$, $1 \le m \le n-1$, $m \ne i+1$, $m \ne j-1$.

3. $[X_1^-, X_{ij}^+] = 0$, $1 \leq i < j \leq n-1$;

4. $[X_1^-, X_1^+] = 1$.

Proof. One can obtain all the statements by direct calculations. □

Consider the set T consisting of all elements $X_{ij}^{\pm}$, $1 \leq i \leq j \leq n-1$, d_{i1}, $1 \leq i \leq n$. Rename the elements from T at the following way: $T = \{t_1, t_2, \ldots, t_p\}$. For $i_1, \ldots, i_p \in \mathbb{Z}_+$ the monomial

$$F(i_1, \ldots, i_p) = t_1^{i_1} \cdot \cdots \cdot t_p^{i_p}$$

will be called standard monomial.

Corllary 5. *The standard monomials span a* $\mathbb{C}$*-space* $\mathcal{U}$.

Proof. Follows from lemma 25 by standard arguments (see for example [6, Chapter 2]). □

For $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ consider a specialization $M_{[l]}$. Clearely $\dim(M_{[l]})_{[l]} = 1$. Let $M([\,l\,])$ be a submodule of $M_{[l]}$ generated by $(M_{[l]})_{[l]}$.

Lemma 26. *1.* $M([\,l\,])$ *has a unique maximal submodule and thus the unique simple quotient* $L([\,l\,])$.

2. $\dim L([\,l\,])_{[l]} = 1$.

Proof. Proof is analogues to the proof of the first part of [7, Theorem 30]. □

Theorem 6. *1. The set* $L([\,l\,])$, $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ *is a complete set of simple weight modules over* $\mathcal{U}$.

2. $L([\,l\,]) \simeq L([\,t\,])$ *if and only if* $(L([\,l\,]))_t \neq 0$.

Proof. Using lemma 7 and [7, Proposition 31] one can prove this statement at the same way as [7, Theorem 30]. □

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