On Approximation of Stepfunctions

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1 Introduction

Applications in VLSI modeling (see [1]) suggest the following problem. The results were partially announced in [2]. Let L_n be graph of the decreasing unit stepfunction with "corners" $(i, n-i+1)$ for $i = 1, ..., n$. The problem is to find good approximations of L_n by stepfunctions G with less than n corners. The following measures for the quality of approximation seem to be relevant.

- $s(G)$: the area between curves G and L_n (area of symmetric difference of G and L_n).
- $r(G)$: the area of the largest rectangle containing in the figure between the curves G and L_n.

Let $K \oplus M$ be the region bounded by stepfunctions K and M. A stepfunction G^* with less than n corners is called *s-optimal* (*r-optimal*) if it minimizes the area of the region $L_n \oplus G$ where G is a stepfunction with less than n corners (respectively, the area of the largest rectangle, contained in $L_n \oplus G$).

The goal of this paper is to specify the set of all solutions of these minimization problems. We prove the following statements.

Theorem 1 *It holds:*

$$\min_{G \text{ has } < n \text{ corners}} s(G) = 1. \tag{1}$$

Remark 1

(*i*) *The equality (1) is attained for some continual set* $\mathbf{N}_n$ *of approximating stepfunctions;*

(*ii*) *All optimal stepfunctions* $G \in \mathbf{N}_n$ *have* $n-1$ *corners and all but maybe one corners coincide with the corners of* L_n.

(*iii*) *At most one corner of optimal stepfunction* $G \in \mathbf{N}_n$ *has a one nonintegral coordinate. In this case just one of the coordinates of the corner is nonintegral.*

Theorem 2 *There exists a unique approximating stepfunction G_n^* with $n-1$ corners such that*

$$r(G_n^*) = \min_{G \text{ has } < n \text{ corners}} r(G).$$

Remark 2

(i) *All rectangles between L_n and G_n^* have the same area, which is the least positive zero of the polynomial*

$$T_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} t^k.$$

(ii) *For some $c > 0$*

$$\frac{1}{4} < r(G_n^*) \leq \frac{1}{4} + cn^{-2}.$$

(iii) *The corners of L_n and G_n^* alternate.*

Theorems 1 and 2 give a complete solution of the approximation problems.

2 Stepfunctions minimizing the area

First we establish some properties of s-optimal stepfunctions. We need some notions. A unit square with the diagonal points $(i+1, n-i+1)$ and $(i, n-i)$ for some i, $1 \leq i < n$, is called *adjacent* to L_n. We denote by F_n the union of closed squares adjacent to L_n and by $\hat{L}_n$ the stepfunction whose corners are $(i+1, n-i+1)$, $i = 1, 2, ..., n-1$. Denote by M_k the set of all non-increasing stepfunction with exactly k steps and put $M_{\hat{n}} = \cup_{k<n} M_k$.

Remark 3 *If we remove any unit step of L_n or add any unit square adjacent to L_n, we obtain a stepfunction with $n-1$ corners whose area differs from the area of L_n just on 1. Therefore, if G^* is an s-optimal approximating stepfunction, then*

$$s(G^*) \leq 1. \tag{2}$$

Lemma 1 *Let $G^* \in M_n$ be s-optimal. Then the following statements hold:*

(i) *G^* has exactly $n-1$ corners.*

(ii) *All corners of G^* belong to F_n.*

(iii) *If two corners of G^* are contained in some square adjacent to L_n, then they are corners of L_n.*

Proof.
We prove five properties from which statements (i)–(iii) follow.

(1) G^* has exactly $n-1$ corners.

Indeed, let G^* has less than $n-1$ corners. Then one can add at least one more corner to G^* and decrease the symmetric difference.

(2) For each corner (x_i, y_i) of G^* holds $x_i \le n$ and $y_i \le n$.

Indeed, cutting G^* along $x = n$ (along $y = n$), we obtain a stepfunction with smaller area of the symmetric difference.

(3) No corner of G^* is strictly above $\hat{L}_n$.

Otherwise the symmetric difference $L_n \oplus G^*$ would contain some square adjacent to L_n and have the area exceeding 1, which contradicts Remark 3.

(4) No corner of G^* is below Ł$_n$.

Otherwise take the topmost corner below L_n and move it up to the nearest horizontal line with integer y-coordinate. The area of the symmetric difference decreases.

(5) There are no two corners in a closed square adjacent to L_n s.t. at least one of the corners has a nonintegral coordinate.

Otherwise one can truncate the coordinates making the area of the symmetric difference smaller.

¿From (1)–(5), the statement follows. □

The lemma shows that the corners of G^* belong to F_n. So we can use local coordinates taking the sides of squares adjacent to L_n as axes. These coordinates take values from the interval $[0, 1]$.

Proof of Theorem 1. The upper bound in (1) follows from Remark 3.

Now we prove the lower bound for $\min s(G)$. Let (x_i, y_i), $i = 1, ..., n-1$ be local coordinates of an approximating stepfunction G and let S_i be the area of the i-th rectangle of $L_n \oplus G$. Then

$$S_{2i} = x_i y_i, \quad S_{2i+1} = (1 - x_i)(1 - y_{i+1}), \tag{3}$$

for $i = 1, \ldots, n-1$ with $y_n = 0$. From (3) we obtain

$$\begin{aligned} s(G) &= 1 - y_1 + x_1 y_1 + (1 - x_1)(1 - y_2) + \ldots + x_{n-1} y_{n-1} + 1 - x_{n-1} \\ &= 1 + \sum_{i=1}^{n-1} (1 - x_i)(1 - y_i) + \sum_{i=1}^{n-1} x_i y_{i+1}. \end{aligned} \tag{4}$$

Since each term in (4) is nonnegative, it follows that $s(G) \ge 1$ and the equality holds iff each term in (4) is equal to 0. Hence if we choose the value of some variable x_i or y_i different from 0 and 1, then all others variables are either 0 or 1 and are uniquely defined by the following equations:

$$(1 - x_i)(1 - y_i) = x_i y_{i+1} = 0,$$

for $i = 1, 2, \ldots, n-1$ with $y_n = 0$. In the case when all the variables take values from the set $\{0, 1\}$, there are $2(n-1)$ possibilities to define a stepfunction G with $n-1$ corners such that $S(G) = 1$. They are just the functions which can be obtained from L_n by removing a step from L_n or adding a square adjacent to L_n.

This proves Theorem 1. □

3 Stepfunctions minimizing the area of the largest rectangle

We call a stepfunction with $n-1$ corners *normal* if the coordinates of its i-th corner satisfy the following inequalities:

$$0 \le x_i \le 1, \ \ 0 \le y_i \le 1, \ \ i = 1, 2, \dots, n-1.$$

Let (x_i, y_i), $i = 1, \dots, n-1$ be the local coordinates of a normal approximating stepfunction G and let S_i be the area of the i-th rectangle of $L_n \oplus G$. Recall that

$$S_{2i} = x_i y_i, \ \ S_{2i+1} = (1 - x_i)(1 - y_{i+1}), \tag{5}$$

for $i = 1, \dots, n-1$ with $y_n = 0$. If G is optimal then as above one can ensure that

$$S_1 = S_2 = \dots = S_{2n-1}. \tag{6}$$

The system (5) can be rewritten in the form

$$\begin{cases} 1 - y_1 &= x_1 y_1 \\ 1 - y_1 &= (1 - x_1)(1 - y_2) \\ & \dots \\ 1 - y_1 &= 1 - x_{n-1}. \end{cases}$$

Denoting the common area of rectangles by t we obtain the following system:

$$\begin{cases} t &= x_1(1 - t) \\ t &= (1 - x_1)(1 - y_2) \\ & \dots \\ t &= 1 - x_{n-1}. \end{cases} \tag{7}$$

The system (7) is equivalent to

$$\begin{cases} y_1 &= 1 - t, \quad x_1 = \dfrac{t}{1-t}, \\ x_i &= \dfrac{t}{y_i}, \quad y_i = 1 - \dfrac{t}{1 - y_{i-1}}, \quad i = 2, \dots, n-2, \\ t &= 1 - x_{n-1}. \end{cases} \tag{8}$$

Eliminating y_i's from (8) we obtain that

$$x_i = \frac{t}{y_i} = \frac{t}{1 - \frac{t}{1 - y_{i-1}}} = \frac{t}{1 - \frac{t}{1 - \frac{t}{x_{i-1}}}} = \frac{t(x_{i-1} - t)}{(1-t)x_{i-1} - t}. \tag{9}$$

Taking into account that x_i, $i = 1, \dots, n-1$, are rational functions of t and putting $x_i = R_i/Q_i$ with some polynomials P_i, Q_i we get from the last equation of (8) that

$$\frac{R_{n-1}}{Q_{n-1}} = x_{n-1} = 1 - t. \tag{10}$$

Thus the area of the least rectangle in $L_n \oplus G$ denoted by t is the smallest positive zero of

$$T_n = (1 - t)Q_{n-1} - R_{n-1}. \tag{11}$$

¿From (9)-(11) one can easily obtain the following recurrent relation for T_n:

$$T_{n+2} = (1-2t)T_{n+1} - t^2 T_n. \tag{12}$$

¿From (7) applied for $n = 2, 3$ we obtain that

$$T_1 = 1 - 3t + t^2 \text{ and } T_2 = 1 - 5t + 6t^2 - t^3. \tag{13}$$

¿From (12) and (13) it follows by induction that

$$T_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} t^k \tag{14}$$

Our nearest goal is to find the least positive zero of T_n. It is seems to be easier to deal with some simpler polynomial P_n. We define a sequence of polynomials P_n by the recurrence:

$$P_{n+2}(t) = P_{n+1}(t) - tP_n(t), \quad P_1(t) = 1, \quad P_2(t) = 1 - t. \tag{15}$$

By induction from (15) it follows that

$$P_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^k. \tag{16}$$

So $T_n(t) = P_{2(n+1)}(t),\ n = 1, 2, \ldots$. Thus we shall determine the least positive zero τ_n of the polynomial $P_n(t)$.

Lemma 2

$$\frac{1}{4} < \tau_n \leq \frac{1}{4} + O(n^{-2}).$$

as $n \to \infty$.

Proof.
Lower bound:
Let us consider the generating function

$$F(z,t) = \sum_n P_n(t) z^n.$$

Using (15) we obtain

$$F(z,t) = \frac{P_1(t) - z(P_1(t) - P_2(t))}{1 - z + tz^2} = \frac{1 - zt}{1 - z + tz^2}.$$

It can be rewritten in the form

$$F(z,t) = \frac{1 - zt}{(1 - \lambda_1 z)(1 - \lambda_2 z)} = \frac{a}{1 - \lambda_1 z} + \frac{b}{1 - \lambda_2 z}, \tag{17}$$

where

$$a = \frac{1}{2} + \frac{1 - 2t}{2\sqrt{1-4t}}, \quad b = \frac{1}{2} - \frac{1 - 2t}{2\sqrt{1-4t}}, \quad \text{and} \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1-4t}}{2}.$$

¿From (17) we conclude that

$$P_n(t) = \frac{1}{2}\left(\lambda_1^n + \lambda_2^n\right) + \frac{1-2t}{2\sqrt{1-4t}} \cdot \left(\lambda_1^n - \lambda_2^n\right). \tag{18}$$

Note that $\lambda_1^n > \lambda_2^n > 0$ for $0 < t < 1/4$. Moreover, $P_n(0) = 1$ and $P_n(1/4) = 2^{-n}$. Therefore

$$P_n(t) > 0 \tag{19}$$

for all $0 \le t \le 1/4$. Thus $\tau_n > \frac{1}{4}$.

Upper bound:
Let us choose a small positive ϵ and put $t = \frac{1}{4}(1+\epsilon^2)$. With this notation we have $\lambda_{1,2} = (1 \pm i\epsilon)/2$. Now choose r and φ s.t. $1 \pm \epsilon = r(\cos\varphi \pm i\sin\varphi)$. We have

$$\begin{aligned} P_n(t) &= \frac{1}{2}r^n\left(2\cos n\varphi + \frac{1-2t}{2\sqrt{1-4t}} i\sin n\varphi\right) \\ &= r^n\left(\cos n\varphi - \frac{(1-\epsilon)^2}{2\epsilon}\sin n\varphi\right). \end{aligned} \tag{20}$$

Let us choose ϵ such that $\frac{\pi}{4} < n\varphi = n\arcsin(\frac{\epsilon}{\sqrt{1+\epsilon^2}}) < 1$. Under this choice the right-hand side of (20) is negative. Hence the least zero of $P_n(t)$ is less than $\frac{1}{4}(1+\epsilon^2) = \frac{1}{4}(1+O(n^2))$.

Remark 4 *Note, that a "natural" hypothesis that for the minimization of the area it necessary to minimize the largest rectangle is far from the truth. The optimal stepfunction which minimizes the largest rectangle has the area of the symmetric difference asymptotically equal to $n/2$. At the same time this area can be equal to 1.*

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References

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