The Character Variety of Free Algebras and Free Groups

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Introduction

Let V a complex vector space of finite dimension d carrying a nondegenerate symmetric bilinear form which we denote by $<,>$. We use the standard notation $SO(<,>)$ for the special orthogonal group of this form. A function

$$f : \underbrace{V \oplus V \oplus ... \oplus V}_{n} \longrightarrow \mathbf{C}$$

which is polynomial (in terms of coefficients once a basis of V is fixed) is called an invariant if

$$f(\sigma v_1, \sigma v_2, ..., \sigma v_n) = f(v_1, v_2, ..., v_n)$$

for $v_i \in V$ arbitrary and $\sigma \in SO(<,>)$. The invariants form a commutative $\mathbf{C}-$algebra. Examples are, for $n = 2$, the bilinear form itself:

$$V \oplus V \ni (v_1, v_2) \longrightarrow < v_1, v_2 > \in \mathbf{C}$$

and, for $n = d$, the determinant:

$$\underbrace{V \oplus V \oplus ... \oplus V}_{d} \ni (v_1, v_2, ..., v_d) \longrightarrow det(v_1, v_2, ..., v_d) \in \mathbf{C}.$$

The first type of invariants and polynomials built up from them are called even invariants, the second type skew. We quote from Hermann Weyl's "Classical Groups" ([7]) that each $n-$variable invariant is a linear combination of monomials which are even or, if $n \geq d$, have the form $f(v_1, v_2, ..., v_n) \cdot det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ where f is again even and $\{i_1, i_2, ..., i_d\}$ is any choice of indices from $\{1, 2, ..., n\}$. The multiplicative structure of the ring of invariants is reflected by the identity

(1)

$$det(v_1, v_2, ..., v_d) \cdot det(w_1, w_2, ..., w_d) = det(< v_i, w_j >)_{1 \leq i,j \leq d}$$

where the determinant on V is normalized by being 1 for some orthonormal basis of V.

It is natural to ask whether or not (1) is the only type of relations in the ring of invariants. Parts of this paper concern this question, in a slightly modified version, in case the dimension of V is three. The context is classical: Let $\mathbf{C}^{2\times 2}$ the complex 2×2 matrix algebra be acted on by $PGL(2,\mathbf{C}) \cong PSL(2,\mathbf{C})$ via conjugation. Then

$$\mathbf{C}^{2\times 2} \cong \mathbf{C} \oplus V$$

splits into two invariant components: the multiples of the unit matrix and the subspace of trace 0 matrices. The trace function

$$\mathbf{C}^{2\times 2} \ni A \longrightarrow tr(A) \in \mathbf{C}$$

may be characterized by the requirements

(i)

$$tris\mathbf{C} - linear,$$

(ii)

$$tr(ABC)+tr(CBA)-tr(A)tr(BC)-tr(B)tr(AC)-tr(C)tr(AB)+tr(A)tr(B)tr(C)=0$$

for all triples $A, B, C \in \mathbf{C}^{2\times 2}$, and

(iii)

$$tr(E) = 2$$

for the unit matrix E.

It may be used to define a nondegenerate quadratic form and a determinant on V:

(2)

$$< T_1, T_2 >= tr(T_1 T_2)$$

and

(3)

$$< T_1, T_2, T_3 >= tr(T_1 T_2 T_3).$$

Relation (1) then reads

(4)

$$< T_1, T_2, T_3 > \cdot < T'_!, T'_2, T'_3 >= -\frac{1}{2} det(< T_i, T'_j >)_{1\le i,j\le 3}.$$

The definitions (2), (3) and the statement (4) can be translated to all of $\mathbf{C}^{2\times 2}$, (4) then reads

(5)

$$4tr((A_1A_2A_3)-tr(A_3A_2A_1))(tr(B_1B_2B_3)-tr(B_3B_2B_1))+det(2tr(A_iB_j)-tr(A_i)tr(B_j))_{1\leq i,j\leq 3}=0,$$

see [1], [3]

This identity alone allows to give the character variety of a free finitely generated $\mathbf{C}-$ algebra the structure of a complex affine algebraic set, defined over the integers. This fact does not seem to be present in the literature (see, e.g. [1], [2], [4], [6]):

Theorem 1. Let A a $\mathbf{C}-$algebra, free on $d\geq 3$ generators $\{a_1,a_2,...,a_d\}$. Let

$$x_i, 1\leq i\leq d,$$

$$y_{ij}, 1\leq i\leq j\leq d,$$

$$z_{ijk}, 1\leq i<j<k\leq d$$

be complex numbers. Then the following holds: There is a character s on A such that

$$s(a_i)=x_i, 1\leq i\leq d,$$

$$s(a_ia_j)=\frac{1}{2}(x_ix_j+y_{ij}), 1\leq i\leq j\leq d,$$

$$s(a_ia_ja_k)=\frac{1}{4}(x_ix_jx_k+x_iy_{jk}+x_jy_{ik}+x_ky_{ij}+z_{ijk}), 1\leq i<j<k\leq d$$

(and this character then is unique) if and only if these numbers satisfy

$$z_{i_1i_2i_3}z_{j_1j_2j_3}+det(y_{i_\mu j_\nu})_{1\leq\mu,nu\leq 3}=0$$

for all choices of the indices.

A similar statement is true for the character variety of a free group of finite rank, in which case we think of determinant 1 characters, only: Let G any group. A complex valued function s defined on G is called a character if it satisfies

(iv)

$$s(abc)+s(cba)-s(a)s(bc)-s(b)s(ac)-s(c)s(ab)+s(a)s(b)s(c)=0$$

for a, b, c arbitrary in G,

(v)

$$s(a^2) = s^2(a) - 2$$

for all $a \in G$, and

(vi)

$$s(e) = 2$$

for e the unit element of G.

Theorem 2. Let G a group, free on generators $\{a_1, a_2, ...a_d\}$. Let

$$x_i, 1 \leq i \leq d,$$

$$y_{ij}, 1 \leq i \leq j \leq d,$$

$$z_{ijk}, 1 \leq i < j < k \leq d$$

becomplex numbers. Then there is a character s on G (and it is then uniquely determined) such that

$$s(a_i) = x_i, 1 \leq i \leq d,$$

$$s(a_i a_j) = \frac{1}{2}(x_i x_j + y_{ij}), 1 \leq i \leq j \leq d,$$

$$s(a_i a_j a_k) = \frac{1}{4}(x_i x_j x_k + x_i y_{jk} + x_j y_{ik} + x_k y_{ij} + z_{ijk}), 1 \leq i < j < k \leq d$$

if and only if they satisfy

$$z_{i_1, i_2 i_3} z_{j_1 j_2 j_3} + det(y_{i_\mu j_\nu})_{1 \leq \mu, \nu \leq 3} = 0$$

and

$$y_{ii} = x_i^2 - 2$$

for all choices of indices.

These theorems say that Weyl's multiplicative relations (1) among even and skew invariants are the only ones in dimension three. Theorem 2 provides an explicit affine algebraic model for the character variety of a free group G of finite rank. It is defined over the integers. The automorphism group acts, sending a character s on G to $s \circ f$, f an automorphism of G, on the character variety of G. It is easy to see that this action is described, on this model, by polynomial transformations defined over $\mathbf{Z}[\frac{1}{2}]$. This is important for topology and hyperbolic geometry: A 3-manifold which fibers over the circle with fibre a (compact or not) oriented surface is a strong candidate for being hyperbolic, see [5]. In order to create a lattice in $SL(2, \mathbf{C})$ which faithfully represents its fundamental group one has to study the fixed point set of some $f \in \mathrm{Aut}G$ on the character variety. This subject will be dealt with in another paper, independently.

Some combinatorics

We use the following conventions and notations: S_n the symmetric group of degree n, $S_\infty = lim S_n$ the group of permutations of $\mathbf{N} = \{1, 2, 3, ...\}$ which fix almost all integers, $S_\infty^{(2)} = \{\pi \in S_\infty | \pi^2 = \varepsilon\}$ the set of permutations of $\mathbf{N}$ which are the product of (at most) finitely many pairwise commuting transpositions, $S_n^{(2)} = S_\infty^{(2)} \cap S_n$, $[n] = \{1, 2, 3, ..., n\}$.

For pairs $P_1, P_2 \subset \mathbf{N}$ we write $P_1 < P_2$ if $maxP_1 < minP_2$, similarly for triples $T_1, T_2 \subset \mathbf{N}$: $T_1 < T_2$ if $maxT_1 < minT_2$. For quadruples $Q_1, Q_2 \subset \mathbf{N}$ the meaning of $Q_1 < Q_2$ is the following: For $Q_1 = \{a_1 < b_1 < c_1 < d_1\}$, $Q_2 = \{a_2 < b_2 < c_2 < d_2\}$ we have $c_1 < a_2 \neq d_1 < b_2$.

We make use of the fact that each $\pi \in S_\infty^{(2)}$ may be viewed canonically as a finite (or empty) collection of pairwise disjoint pairs of integers, namely the set $\{\{l, \pi l\} | l \in \mathbf{N}$ is not a fixed point of $\pi\}$. We frequently visualize elements of $S_\infty^{(2)}$ and manipulations with them on what we call their diagram: The linear diagram of the transposition (l_1, l_2) where $l_1 < l_2$ is shown in figure 1 and

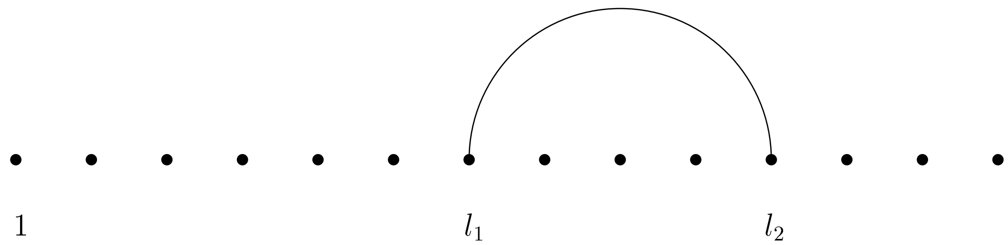


Figure 1

the diagram of a product of (pairwise commuting) transpositions is simply the superposition of the diagrams of the factors. So there are, e.g., three types of diagrams for the various π's in $S_\infty^{(2)}$ which have exactly four non fixed points (see figure 2).

We quite frequently omit the explicit labelling of the points. In a diagram two integers will be called adjacent or neighbours if between them there are only fixed points of the corresponding $\pi \in S_\infty^{(2)}$, if any at all. A triple $T = \{a < b < c\} \subset \mathbf{N}$ is called a triple of neighbours if there are only fixed points between a and c with the possible exception of b.

For fixed n it is practical to bend a diagram into a cyclic form. Figure shows a complete list of all types of cyclic diagrams from $S_6^{(2)}$ where no fixpoints are present. There are, up to cyclic rearrangement, exactly five such diagrams. The number of elements in $S_{2n}^{(2)}$ with no fixpoints is easily computed to equal $\frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}...\binom{4}{2}\binom{2}{2} = (2n-1)!!$, so it is equal to $3 \cdot 5 = 15$ if $n = 3$. The first column in figure 3 shows the number of π's with diagram in that row.

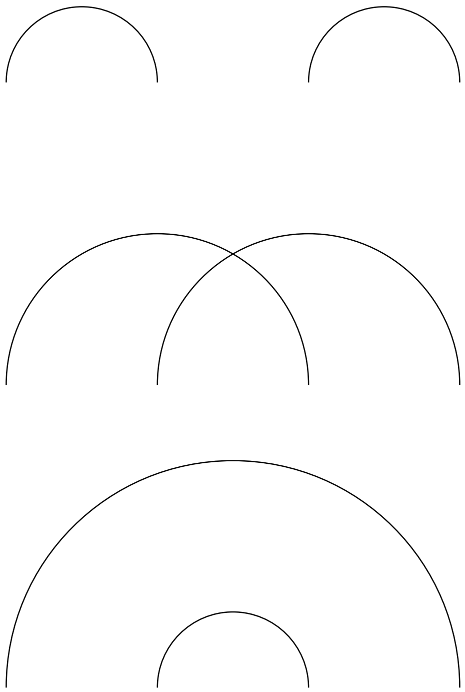


Figure 2

The second and third column contain the numbers c_π and v_π for $\pi \in S_6^{(2)}$ with diagram in that row. These numbers are defined in the following

Definition. For $\pi \in S_n^{(2)}$ we call c_π the number of intersections (of segments) in the (cyclic) diagram of π, and set $v_\pi = (-1)^{c_\pi}$. For $\pi \in S_n^{(2)}$ and $I = \{i < j < k\} \subset [n]$ such that $\pi i = i, \pi j = j, \pi k = k$ we define $v_{(\pi,I)} = v_{\pi(ij)}$.

We note that the number of fixedpointfree $\pi \in S_{2n}^{(2)}$ with $v_\pi = 1$ exceeds the number of those with $v_\pi = -1$ by one:

Proposition. $\sum v_\pi = 1$ where the summation runs over all $\pi \in S_{2n}^{(2)}$ which have no fixed point.

The proof is elementary: The statement is obvious for $n = 1$. By labelling each $\pi \in S_{2n+2}^{(2)}$ with no fixed point by $l_\pi = \pi(2n+2)$ we partition this set into $2n+1$ subsets of equal cardinality $(2n-1)!!$ whithin which, by induction hypothesis, the sum in question is 1. The segment joining $2n+2$ to l_π creates an odd number of additional intersections if l_π is even, and an even number if l_π is odd, so the sum for $n+1$ instead of n equals $-\sum_{l=1}^{2n+1}(-1)^l = 1$.

It should be expected that v_π for $\pi \in S_\infty^{(2)}$ also plays the rôle of the sign of some permutation:

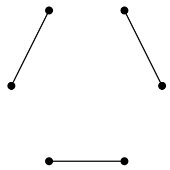
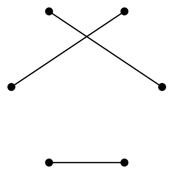
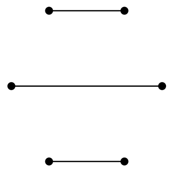
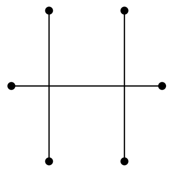
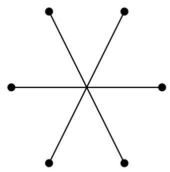
	2	0	1
	6	1	-1
	3	0	1
	3	2	1
	1	3	-1

Figure 3

Proposition. Let $\pi \in S_{2n}^{(2)}$ be fixpointfree, and $\pi = \tau_1\tau_2...\tau_n$ its decomposition into (pairwise commuting) transpositions. We write each pair $P_l \subset \mathbf{N}$ whose elements are interchanged by τ_l in its natural order (the one inherited from the ordering of $\mathbf{N}$). The juxtaposition $P_1P_2...P_n$ then represents an element $\pi^* \in S_{2n}$. We assert $v_\pi = sgn(\pi^*)$.

Proof. Although π^* heavily depends on the ordering of the transposition factors in π its sign does not: If $\tau_l = (a_l, b_l), \tau_{l+1} = (a_{l+1}, b_{l+1})$ interchanging these adjacent factors means to replace π^* by $(a_l, a_{l+1})(b_l, b_{l+1})\pi^*$ whose sign equals the sign of π^*. We again proceed by induction. For $n = 1$ we have to check only $\pi = (12)$. But in this case $\pi^* = \varepsilon$, so the proposition is true in this case. Assume it is true for n, and let $\pi \in S_{2n+2}^{(2)}$ with no fixpoints, $\pi = \tau_1\tau_2...\tau_{n+1}$ its decomposition into transpositions. Assume τ_{n+1} moves $2n+2$: $\tau_{n+1} = (l_\pi, 2n+2)$ with $1 \leq l_\pi \leq 2n+1$. We now delete τ_{n+1} from π and shift all integers l between l_π and $2n+2$ down by 1. We call ρ the resulting element of $S_{2n}^{(2)}$. By construction

$$\rho^* = (2n+1, 2n, ..., l_\pi)\pi^*.$$

So $sign\pi^* = (-1)^{2n+1-l_\pi} sgn\rho^*$. On the other hand deleting the segment joining l_π to $2n+2$ from the (cyclic) diagram of π has the effect of changing v_π by the factor $(-1)^{2n+1-l_\pi}$. So by induction hypothesis

$$sgn(\pi^*) = (-1)^{l_\pi+1} sgn(\rho^*) = (-1)^{l_\pi+1} v_\rho = v_\pi.$$

We plan to single out from the set $S_\infty^{(2)}$ a subset whose elements will be called standard. We do so in terms of (linear) diagrams on $\mathbf{N}$.

Definition. We say that $\pi \in S_\infty^{(2)}$ has property (P), the pair property, if, for any two adjacent nonfixpoints $l_1 < l_2$ of π we have $l_2 = \pi(l_1)$ or $\pi(l_1) < l_1 < l_2 < \pi(l_2)$:

$\{l_1, \pi(l_1)\} < \{l_2, \pi(l_2)\}$, or

$\{l_1, \pi(l_1)\} = \{l_2, \pi(l_2)\}$, or

$\{l_1, \pi(l_1)\} > \{l_2, \pi(l_2)\}$.

So in $S_{2n}^{(2)}$ there is exactly one element with no fixed points and property (P), namely $\pi = \prod_{l=1}^n (2l-1, 2l)$.

Definition. We say that $\pi \in S_\infty^{(2)}$ has property (T), the triple property, if, whenever $T = \{a < b < c\} \subset \mathbf{N}$ is a triple such that $\pi(T) = \{a' < b' < c'\} > T$ and b and c are not adjacent or a' and b' are not adjacent then there is a quadrupel $Q = \{a_1 < b_! < c_1 < d_1\} \supset T$ such that $\pi(Q) = \{a_1' < b_1' < c_1' < d_1'\} > Q$ and b_1 and c_1 are neighbours and b_1' and c_1' are neighbours.

So $\pi \in S_\infty^{(2)}$ not to have property (T) requires eight or more nonfixpoints. Inspection of $S_8^{(2)}$ yields that all elements do have property (T). The permutation $\pi = (1,10)(29)(34)(56)(78)$ with diagram shown in figure (4) does not have property (T): For $T = \{1,2,5\}$ we have $\pi(T) = \{6,9,10\} > T$. The only quadrupel $Q \supset T$ in which the pair in the middle consists of neighbours is $Q = \{1,2,3,5\}$. But in $\pi(Q) = \{4,6,9,10\} > Q$ the elements in the middle are no neighbours.

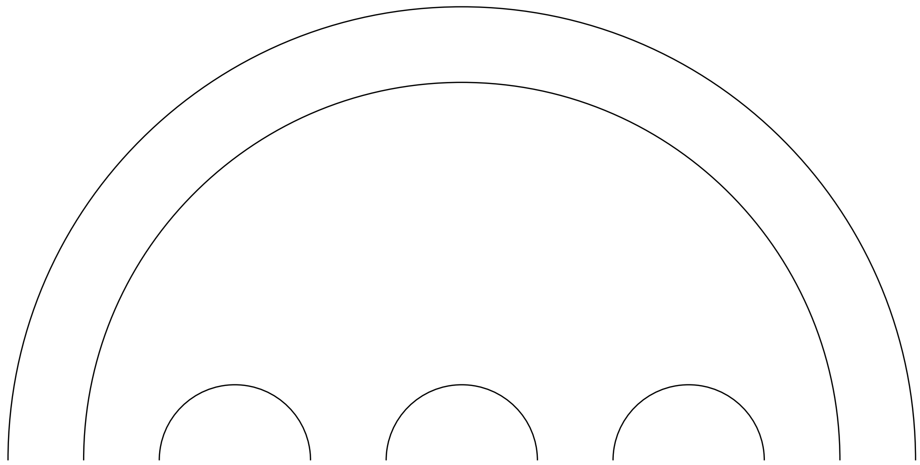


Figure 4

Definition. We say that $\pi \in S_\infty^{(2)}$ has property (Q), the quadrupel property, if for each quadrupel $Q \subset \mathbf{N}$ such that $\pi(Q) > Q$ we have

$$min\pi(Q) = \pi(maxQ).$$

$S_8^{(2)}$ contains $3 \cdot 5 \cdot 7 = 105$ elements with no fixed points, 69 of which do have property (Q) and 36 do not. The latter are described by

$$\pi\{1,2,3,4\} = \{5,6,7,8\}$$

or

$$\pi\{1,2,3,5\} = \{4,6,7,8\}$$

but

$$\pi 4 \neq 5.$$

From property (Q) we draw two consequences:

Lemma. Let $\pi \in S_\infty^{(2)}$ have property (Q), and let $T \subset \mathbf{N}$ a tripel such that $\pi(T) > T$. Write

$$T = \{a < b < c\}, \pi(T) = \{a' < b' < c'\}.$$

Then π preserves the intervall

$$\{l | c < l < a'\}.$$

Proof. Let l such that $c < l < a'$. Assume first $\pi(l) > l$. Then for $Q = \{a < b < c < l\}$ we have

$$\pi(Q) > Q.$$

So $\pi(maxQ) = \pi(l) = min(Q)$. This means $\pi(l) < a'$. If $\pi(l) < l$ we take $Q = \{l < a' < b' < c'\}$. Now $\pi(Q) < Q$. So $\pi(minQ) = \pi(l) = max(\pi(Q))$, so $\pi(l) > c$. This proves the lemma.

Lemma. Let $T = \{a < b < c\} \subset \mathbf{N}$ a triple such that $\pi(T) = \{a' < b' < c'\} > T$ where $\pi \in S_\infty^{(2)}$ has property (Q). Let l such that $b < l < c$. Then $\pi(l) > a' = minT$ or $\pi(l) = l$. Analoguously: If l such that $a' < l < b'$ then $\pi(l) < c = maxT$ or $\pi(l) = l$.

Proof. In the first case we assume l to be the maximal nonfixpoint of π between b and c. If $\pi(l) < a'$ then even $\pi(l) < c$ as, by the previous lemma, the intervall between c and a' is preserved by π. So even $\pi(l) < l$ as otherwise l would not have been maximal. So for $Q = \{l < a' < b' < c'\}$ we have $\pi(Q) = \{\pi(l), a, b, c\} < Q$. So from property (Q) we conclude $\pi(l) = max\pi(Q) = c$ which is a contradiction. So $\pi(l) = l$. The situation is analoguous in the second case: Assume l to be minimal, $a' < l < b', \pi(l) > c = max(T)$. Then $\pi(l) > l$ and for $Q = \{a < b < c < l\}$ we have $\pi(Q) = \{\pi(l), a', b', c'\} > Q$. So by property (Q) we have $\pi(l) = min\pi(Q) = a'$, again a contradiction.

The following is the main concept in this paragraph:

Definition. $\pi \in S_\infty^{(2)}$ is called standard if π has properties (T) and (Q) and, whenever $\pi T > T$ for some triple T it has property (P) on $\{l | maxT < l < min\pi T\}$. π is 'standard on $[n]$' if it is standard and contained in $S_n^{(2)}$. The set of all $\pi \in S_\infty^{(2)}$ which are standard on $[n]$ we denote by St_n.

Nonstandard permutations first show up in $S_8^{(2)}$, and there we have 36 of them, the ones which are listed after the definition of property (Q). We note that a standard $\pi \in S_\infty^{(2)}$ may become nonstandard after removing some transposition from its cycle decomposition: $(1,10)(29)(38)(45)(67)$ is an example. It is standard but does not allow to remove any one of the transpositions $(1,10), (29)$, or (38). On the other hand if $\pi = \pi_1 \cdot \pi_2$ with $\pi_1, \pi_2 \in S_\infty^{(2)}$ such that the two nonfixpoint sets lie in disjoint intervalls then π is standard if and only if π_1 and π_2 are. Being standard or not can be read from the combinatorial structure of the diagram of π; here two diagrams are called to have the same combinatorial structure if one is constructed from the other by deletion or addition of nonfixed points, only.

Definition. A pair (π, I) where $\pi \in S_\infty^{(2)}$ and $I = \{i < j < k\} \subset \mathbf{N}$ any triple we call standard if π has the form $\pi = \pi_1 \cdot \pi_2 = \pi_2 \cdot \pi_1$ with $\pi_1, \pi_2 \in S_\infty^{(2)}$ such that

(1) $\pi_1(l) = l$ for $l = i$ and $l \geq j$,

(2) $\pi_2(l) = l$ for $l \leq k$,

(3) $\pi_1(ij)$ is standard, and

(4) π_2 has property (P).

(π, I) is 'standard on $[n]$' if it is standard and $\pi \in S_n^{(2)}, I \subset [n]$.

Nonstandard pairs show up first on $[5]$. We list the standard ones in this case:

(ε, I) for $I \subset [5]$ arbitrary is standard; the remaining standard ones are $((12), \{3,4,5\}), ((13), \{2,4,5\}), ((23), \{1,4,5\}, ((45), \{1,2,3\})$.

We now define, for $n \geq 1$, a mapping

$\varphi_n : S_{n+1}^{(2)} \longrightarrow S_n^{(2)}$

which we call reduction. It will be shown later that reduction leads from St_{n+1} onto St_n.

$S_{n+1}^{(2)}$ splits into five classes, numbered $(I), (II), (III), (IV)$ and (V):

(I): $\pi(n+1) = n+1$. We set

$$\varphi_n(\pi) = \pi.$$

(II): $\pi(n+1) = n$. We set $\varphi_n(\pi) =$

$$\pi \cdot (n, n+1),$$

so the transposition $(n, n+1)$ is removed from π.

(III): $\pi(n+1) < n$ and $\pi(n) = n$. We set

$$\varphi_n(\pi) = \pi \cdot (\pi(n+1), n, n+1);$$

this means that in the decomposition of π into transpositions $n+1$ is replaced by n.

(IV): $\pi(n+1) < n, \pi(n) < n$, and all $l, max\{\pi(n), \pi(n+1)\} < l < n$ are fixed by π. We set $\varphi_n(\pi) = \pi \cdot (\pi(n+1), n, \pi(n), n+1)$. This means that the two transpositions in π which move n or $n+1$ are removed and the new transposition $(\pi(n), \pi(n+1))$ is added.

(V): $\pi(n+1) < n, \pi(n) < n$, and $\{l | max\{\pi(n), \pi(n+1)\} < l < n, \pi(l) \neq l\} \neq \emptyset$. Let p maximal in this set, and let $\pi\{p, n, n+1\} = \{i < j < k\}$. We then define $\varphi_n(\pi)$ to result from π by removing the three transpositions which move p, n, or $n+1$ and then add the two new transpositions $(ij)(pn)$.

Lemma. If $\pi \in S^{(2)}_{n+1}$ is standard so is $\varphi_n(\pi)$; more precisely

$$\varphi_n(St_{n+1}) = St_n.$$

Proof. Surjectivity of φ_n is clear since $St_n \subset St_{n+1}$ and φ_n is the identity on $S^{(2)}_n$. In case (II) the lemma is true since no triple T or quadrupel Q such that $\pi(T) < T$ or $\pi(Q) < Q$ can contain $n+1$. So standardness of π is checked on $\varphi_n(\pi)$. In case (III) the combinatorial type of the two diagrams of π and $\varphi_n(\pi)$ are the same; so there is no difference between π and $\varphi_n(\pi)$ from the point of view of standardness.

We discuss type (IV), see figure 5.

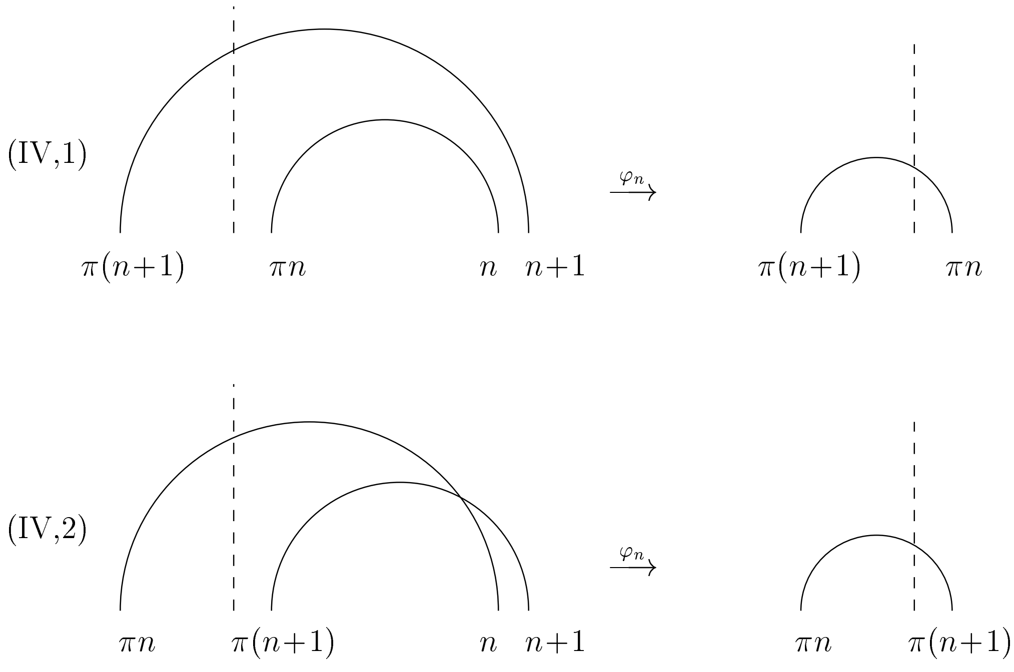


Figure 5

Components of diagrams on the left of the dotted lines are not drawn. A triple T moved down by π will not contain n in case $(IV,1)$ and will not contain $n+1$ in case $(IV,2)$. So, π will remain standard or nonstandard if, in the first case the transposition $(\pi n, n)$ and, in the second case the transposition $(\pi(n+1), n+1)$ is removed from π. But then we are dealing with a permutation whose diagram combinatorially equals the diagram of $\varphi_n(\pi)$. The same argument holds true for quadrupels Q instead of triples.

For π in class (V) the reasoning is more involved. We show the effect of φ_n on such π in diagrams; according to the six possibilities of orderings of $\{\pi p, \pi n, \pi(n+1)\}$ there are six types of diagrams from combinatorial point of view, see figure 6.

The meaning of the dotted lines is the following: In the diagrams for $\pi \in S_{n+1}^{(2)}$ of type (V) and its reduction there are no components which enter or are contained in the strip between the left most and the central dotted line except the ones which are drawn, and the same is true on the right of the right most dotted line. These properties of the diagram of π are guaranteed by standardness: π maps the intervall $\{l | k < l < p\}$ onto itself (and even has property (P) there); moreover the intervall $\{l | j < l < k\}$ consists of fixed points, only.

From the diagrams it is evident that a standard $\pi \in S_{n+1}^{(2)}$ remains standard under reduction: If a triple or quadruple is moved down by $\varphi_n(\pi)$ it cannot contain any point on the right of the center line as $\varphi_n(\pi)$ has property (P) there. The only point which it may contain on the right of the left most line is the point in the middle of the triple $\{\pi p, \pi n, \pi(n+1)\} = \{i < j < k\}$ as all other points on the left of the center line are fixed by π. So standardness of $\varphi_n(\pi)$ is equivalent to standardness of $\pi' \in S_{n+1}^{(2)}$ which arises from π by removing all parts of its diagram on the right of the left most line, except the arc from i to $\pi(i)$. But this diagram is combinatorially the same as the diagram of $\varphi_n(\pi)$, up to the irrelevant part on the right of the center line.

In the following proposition we observe the behaviour of the number v_π under reduction. One only has to compare the numbers of points of intersection in the diagrams of π and $\varphi_n(\pi)$:

Proposition. For $\pi \in St_{n+1}$ the following holds true:

π of type (I) : $v_{\varphi_n(\pi)} = v_\pi$,

π of type (II) : $v_{\varphi_n(\pi)} = v_\pi$,

π of type (III): $v_{\varphi_n(\pi)} = v_\pi$,

π of type $(IV,1)$: $v_{\varphi_n(\pi)} = v_\pi$,

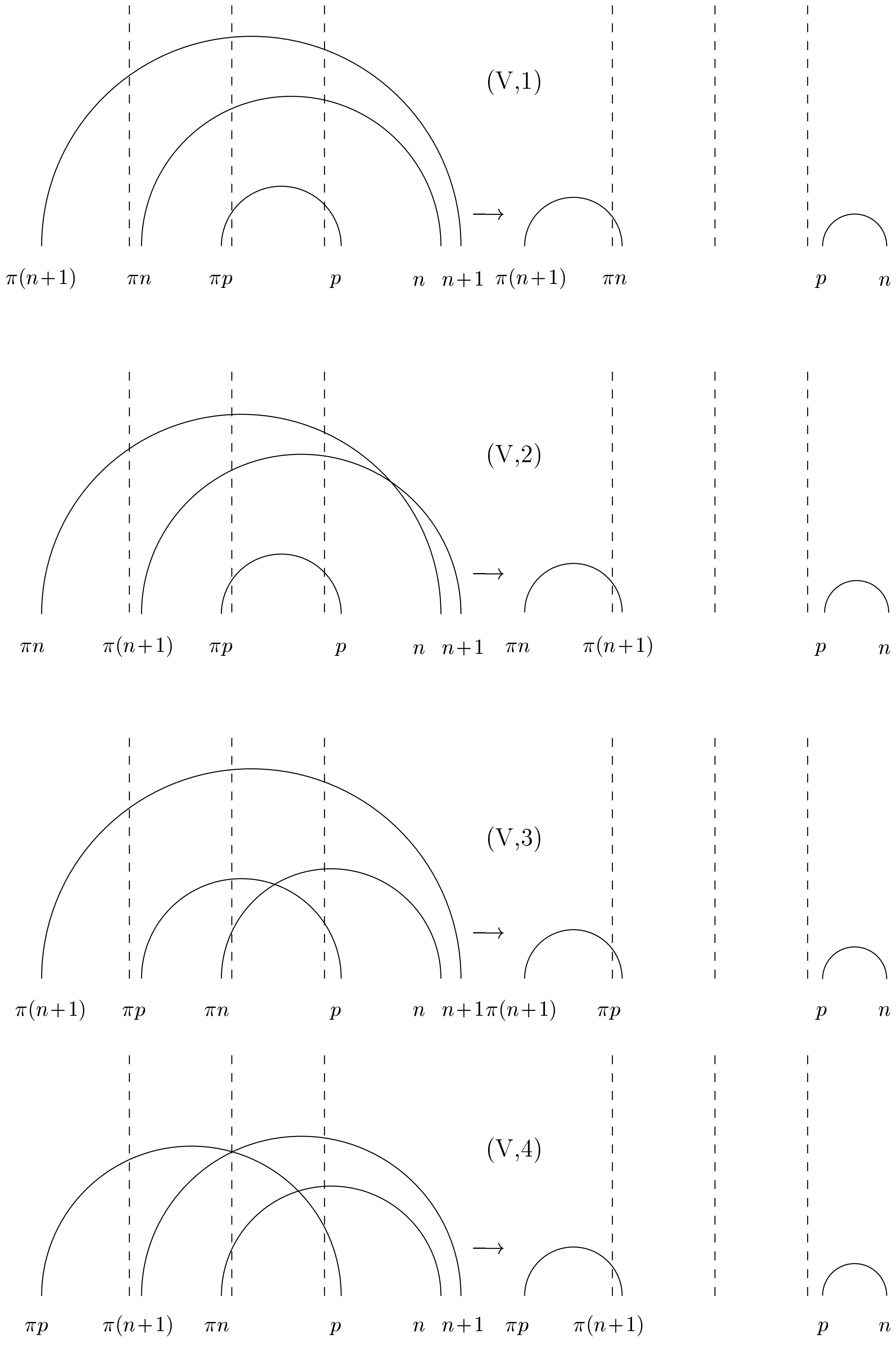
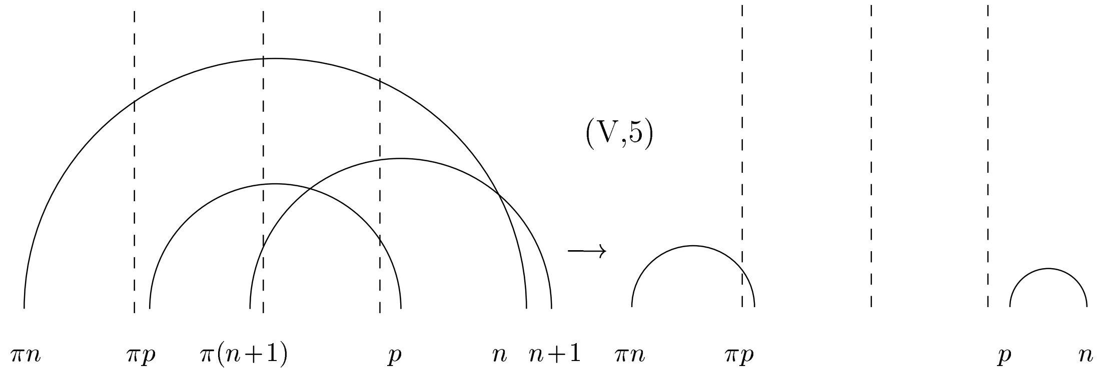

(V,1)
π(n+1) πn πp p n n+1 π(n+1) πn p n
(V,2)
πn π(n+1) πp p n n+1 πn π(n+1) p n
(V,3)
π(n+1) πp πn p n n+1 π(n+1) πp p n
(V,4)
πp π(n+1) πn p n n+1 πp π(n+1) p n

Figure 6



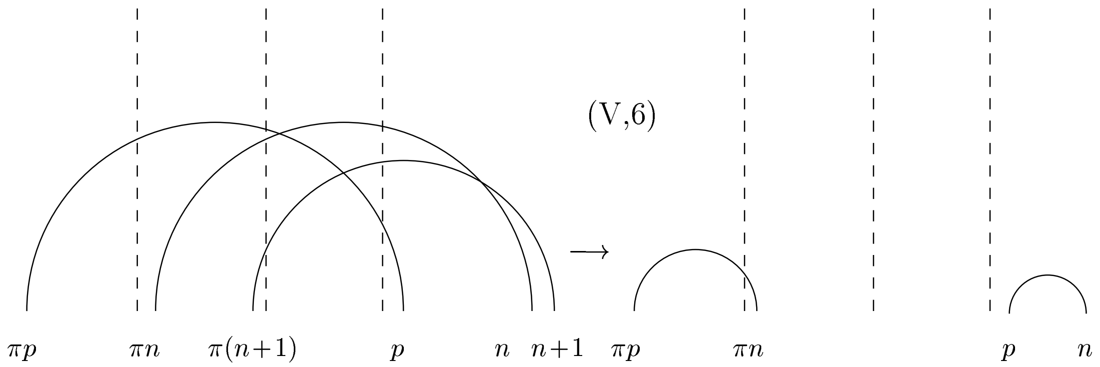


Figure 6 (cont'd)

π of type $(IV, 2)$: $v_{\varphi_n(\pi)} = -v_\pi$,

π of type $(V, 1)$: $v_{\varphi_n(\pi)} = v_\pi$.

π of type $(V, 2)$: $v_{\varphi_n(\pi)} = -v_\pi$,

π of type $(V, 3)$: $v_{\varphi_n(\pi)} = -v_\pi$,

π of type $(V, 4)$: $v_{\varphi_n(\pi)} = v_\pi$,

π of type $(V, 5)$: $v_{\varphi_n(\pi)} = v_\pi$,

π of type $(V, 6)$: $v_{\varphi_n(\pi)} = -v_\pi$.

The types $(I), (II), (III)$ are trivial; the reasoning in all cases under (IV) and (V) is the same: If in case $(IV, 1)$, e,g,, a point of intersection is lost under reduction this will be caused by deleting the arc from $\pi(n+1)$ to $n+1$. But a new intersection point will be created by drawing the arc from $\pi(n+1)$ to $\pi(n)$.

We now determine the preimage $\varphi_n^{-1}(\pi) \subset St_{n+1}$ for $\pi \in St_n$. It is easy to by looking at the diagram of π determine its preimage in $S(2)_{n+1}$. The proof that all these elements are standard on $[n+1]$ requires some combinatorial discussion. It is contained in the following four lemmata. (Trivially π itself is in its preimage.)

Lemma. Let π standard on $[n]$, $\pi n \neq n$. Then $\pi^* = \pi \cdot (n, \pi n, n+1)$ is standard on $[n+1]$. It is of type (III), and $\varphi_n(\pi^*) = \pi$.

The proof requires only a glance at the diagram of π: π^* arises from π by replacing the transposition $(\pi n, n)|\pi$ by $(\pi n, n+1)$.

Lemma. Let π standard on $[n]$, $\pi n = n$. Then $\pi^* = \pi \cdot (n, n+1)$ is standard on $[n+1]$. It is of type (II), and $\varphi_n(\pi^*) = \pi$.

This again is checked on the diagram of π. The diagram of π^* consists of the diagram of π plus the arc from n to $n+1$.

Lemma. Let $\pi \neq \varepsilon$ standard on $[n]$, $\pi n = n$. Let j the maximal nonfixed point of π. Let σ an involution which maps $\{\pi j, j\}$ to $\{n, n+1\}$ and is the identity otherwise. (So $\sigma = (jn)(\pi j, n+1)$ or $(\pi j, n)(j, n+1)$.) We define $\pi^* = \pi \cdot (\pi j, j) \cdot \sigma$ and assert: π^* is standard on $[n+1]$; it is of type (IV), and $\varphi_n(\pi^*) = \pi$.

Proof. π^* arises from π by removing the transposition $(\pi j, j)$ and adding π_0. We look at figure (5). The construction of π^* is equivalent to the inversion of the arrow which symbolizes φ_n, and it is clear that π^* will be of type (IV). We have to show that it is standard. We discuss the case of $\pi_0 = (jn)(\pi j, n+1)$ which corresponds to $(IV, 1)$ in figure (5). We observe that whenever $\pi^* T < T$ or $\pi^* Q < Q$ for some triple T or quadruple Q one has $n \notin T$ resp. $n \notin Q$. So one may delete the transposition (jn) from π^*:

$\pi^*(jn)T = \pi^* T < T$, resp.

$\pi^*(jn)Q = \pi^* Q < Q$.

But $\pi^*(jn)$ and π have diagrams which combinatorially coincide. This implies, making use of π being standard that the intervall $\{l | max \pi^* T < l < min T\}$ is invariant under $\pi^*(jn)$ and that $\pi^*(jn)$ has property (P) there. Of course, j is not in that intervall: there are only two nonfixed points of π^* above j. So π^* and π coincide on this intervall. So π^* has property (P) there. We check property (T) on π^*. If $T = \{a < b < c\}, \pi^* T = \{a' < b' < c'\} < T$ such that a is not a neighbour of b or b' is not a neighbour of c' in the diagram of π^* the same situation will be found in the diagram of π: Because of $n+1 \notin T$, i.e. $c \neq n+1$, we even have $c < j$, so the arc from j to n again may be removed without destroyng this situation; in case of $c = n+1 \in T$ removing this arc does not affect the statement that a is not a neighbour of b, and again the remaining diagram is combinatorially equivalent to that of π. In this case we simply replace $c = n+1$ by j in T. Call $T_0 = \{a < b < j\}$. Then

$\pi T_0 = \pi^* T_0 < T_0$. So there is a quadruple $Q = \{a_1 < b_1 < c_1 < d_1\} \subset [n]$ such that $Q \supset T$ resp. T_0, $\pi Q = \{a_1' < b_1' < c_1' < d_1'\} < Q$ and b_1 adjacent to c_1, b_1' adjacent to c_1'. It may happen that $j \in Q$, then, of course being maximal there. If so we replace it by $n+1$ getting a quadruple $Q^* = \{a_1 < b_1 < c_1 < n+1\} \subset [n+1]$ such that $Q^* \supset T, \pi^* Q^* = \pi Q < Q$ and b_1 is adjacent to c_1, b_1' adjacent to c_1'. This ends the checking of property (T) on π^*.

Checking property (Q) on π^* is again easy: If $\pi^* Q < Q$ for some quadruple Q the same will be true for $\pi^*(\pi j, j)$ instead of π^*, as mentioned at the very beginning of this proof. So in this respect the combinatorial structure of the diagrams of π and π^* is the same, so property (Q) for π^* is inferred from property (Q) for π.

The proof of the lemma for $\pi_0 = (\pi j, n)(j, n+1)$ is mutatis mutandis the same.

Lemma. Let π standard on $[n]$, $\pi \neq \varepsilon, \pi n = n$, and p the maximal nonfixed point of π. Let $(ij)|\pi$ such that $i < j < p-1, \pi(j+1) = j+1$, and π preserves $\{l | l > j\}$ and has property (P) there. Then the following holds: If $k > j$ is a fixed point of π, adjacent to j, and if σ is an involution which identifies $\{i, j, k\}$ $\{p, n, n+1\}$ and is the identity otherwise then $\pi^* = \pi \cdot (ij) \cdot \sigma$ is of type (V). It is standard on $[n+1]$ and $\varphi_n(\pi^*) = \pi$.

Proof. We first list the possibilities for π_0. There are six of them; they are ordered according to figure 6:

$$\sigma = (kp)(jn)(i, n+1),$$

$$\sigma = (kp)(in)(j, n+1),$$

$$\sigma = (jp)(kn)(i, n+1),$$

$$\sigma = (ip)(kn)(j, n+1),$$

$$\sigma = (jp)(in)(k, n+1),$$

$$\sigma = (ip)(jn)(k, n+1).$$

In order to prove standardness of π^* one has to discuss the situation "$\pi T = \{a' < b' < c'\} < T = \{a < b < c\}$ for $T \subset [n+1]$ a triple" under various aspects. We divide into eight cases:

Case 1) $T \cap \{p, n, n+1\} = \emptyset$. Then T is on the left of j as π^* has property (P) between k and p and is the identity between j and k as well as between p and n. In this case, so, $\pi^* T = \pi T$.

Case 2) $T \cap \{p, n, n+1\} = \{n+1\}$. So $T = \{a < b < n+1\}$. Again as π^* has property (P) between k and p we have $b < j$. So $\pi^* T = \pi\{a, b, j\}$.

Case 3) $T \cap \{p, n, n+1\} = \{n\}$. $\pi^* T < T$ is impossible, again by property (P) between k and p.

Case 4) $T \cap \{p, n, n+1\} = \{p\}$. This $\pi^* T < T$ is impossible by the same reason as in the previous case.

Case 5) $T \cap \{p, n, n+1\} = \{n, n+1\}$. So $T = \{a < n < n+1\}, \pi^* T = \{i < j < c'\}, k < c' < a < p$.

Case 6) $T \cap \{p, n, n+1\} = \{p, n\}$. So $T = \{a < p < n\}, \pi^* T = \{j < k < c'\}, k < c' < a < p$

Case 7) $T \cap \{p, n, n+1\} = \{p, n+1\}$. So $T = \{a < p < n+1\}, \pi^* T = \{i < k < c'\}, k < c' < a < p$.

Case 8) $T = \{p, n, n+1\}, \pi^* T = \{i, j, k\}$.

We concentrate on the first involution σ; the corresponding picture is $(V, 1)$ in figure 6. One should only replace the labellings $\pi(n+1), \pi n, \pi p$ by i, j, k, respectively.

We check the possibilities for a triple $T = \{a < b < c\} \subset [n+1]$ to be moved down by π^* case by case.

Case 1) $T \cap \{p, n, n+1\} = \emptyset$. Set $\pi^* T = \pi T = \{a' < b' < c'\} < T$. If b' and c' are not adjacent or a and b are not adjacent a quadruple $Q = \{a_1 < b_1 < c_1 < d_1\} \subset [n]$ exists such that $Q \supset T$, $\pi Q = \{a_1' < b_1' < c_1' < d_1'\} < Q$ and b_1 and c_1 are neighbours, b_1' and c_1' are neighbours. Q cannot contain k as k is fixed by π; by the same reason it cannot contain neither p nor n. If $Q \ni j$ then

$$Q = \{a < b < c < j\}, \pi Q = \{a' < b' < c' < d'\} < Q,$$

where one of the elements of πQ is i. We take $Q^* = \{a < b < c < n\}$. Then $\pi^* Q^* = \pi Q < Q^*$. Adjacency of b and c as well as of b' and c' is guaranteed by standardness of π.

Q cannot contain i: We have $\pi i = j > i$, but in a quardruple Q such that $\pi Q < Q$ the only element which may be moved up is the minimum. But πi is even greater than $maxT = c$.

If finally $Q \not\ni j$ we take $Q^* = Q = \{a_1 < b_1 < c_1 < d_1\}$ and have $\pi^* Q = \pi Q = \{a_1' < b_1' < c_1' < d_1'\}$. Because of standardness of π the pair (b_1, c_1) is a pair of neighbours in the diagram of π, and the same is true for (b_1', c_1'), and this fact cannot be destroyed by the existence of the nonfixed point i of π^* since i is also a nonfixed point of π

So property (T) is guaranteed for π^*.

We have to check property (P) for π^* on each intervall $\{l | max\pi T < l < minT\}$ whenever $T \subset [n+1]$ is a triple such that $\pi^* T < T$. But $\pi^* T = \pi T$, and moreover π^* acts on the intervall $\{l | max\pi^* T < l < minT\}$ exactly as π does with possibly the exception of i. But on the other hand i cannot lie in this intervall as π has property (P) there.

Finally, property (Q) has to be established on π^*. Let $Q = \{a < b < c < d\} \subset [n+1]$ a quadruple such that $\pi^* Q = \{a' < b' < c' < d'\} < Q$.

The triple $\{b < c < d\} \subset Q$ so is mapped to a triple $\pi^* T < T$. So by the previous discussion $d < j$. The situation that $i \in Q$ cannot occur because of $\pi^* Q < Q$, so we have $\pi^* Q = \pi Q$, so property (Q) for π^* follows from property (Q) for π.

Case 2) $T \cap \{p, n, n+1\} = \{n+1\}$. Call for $T = \{a < b < n+1\}, \pi^* T = \pi\{a, b, j\} = \{a', b', i\}$ where $a' < b'$ and assume $\pi^* T < T$. So $i < a < b < j$. If a and b are no neighbours or this happens to be the case for the upper pair in $\{a', b', i\}$ there is a quadruple $Q \supset T$ such that $\pi Q < Q$ and the middle terms of

Q and πQ are neighbours. Certainly $maxQ = j$ as π has property (P) above j. Replace $j \in Q$ by $n+1$, a manipulation which does not destroy the adjacency situation for the middle pairs. This new quadruple will guarantee property (T) for π^*.

Property (P) for π^* between $max\pi^*T$ and $minT$ is equivalent to property (P) for π there. Property (Q) follows on the same lines: If a quadruple $Q \subset [n+1]$ such that $\pi^*Q < Q$ actually contains $n+1$ then again $n+1$ should be replaced by j and property (Q) for π leads to property (Q) for π^*.

Case 5) $T \cap \{p, n, n+1\} = \{n, n+1\}, T = \{a < n < n+1\}, \pi^*T = \{i < j < c'\}, k < c' < a < p$. Now $a < p < n, j < k < c'$. Both b and k being nonfixed points of π^* neither a and b nor j and c' are adjacent. We take $Q = \{a < p < n < n+1\} \supset T$ and have $\pi^*Q = \{i < j < k < c'\} < Q$. p and n are neighbours by assumption on π, and j and k are neighbours by the choice of k. This establishes property (T) on π^*. The intervall between $max\pi^*T = c'$ and $minT = a$ consists entirely of fixed points of π^*, so property (P) is present there, and this also implies that there is no quadruple $Q = \{a < b < n < n+1\} > \pi^*Q$ if $b < p$. So property (Q) is trivial.

Case 6) $T \cap \{p, n, n+1\} = \{p, n\}, T = \{a < p < n\}, \pi^*T = \{j < k < c'\}, k < c' < a < p$. If a and p are no neighbours or k and c' are not we again take, as in case 5, $Q = \{a < p < n < n+1\}$. The reasoning then is the same.

Case 7) $T \cap \{p, n, n+1\} = \{p, n+1\}, T = \{a < p < n+1\}, \pi^*T = \{i < k < c'\}, k < c' < a < p$. This case is settled on the lines of the previous two cases.

Case 8) $T = \{p < n <, +1\}, \pi^*T = \{i < j < k\}$. Now p and n are neighbours, j and k are neighbours, so property (T) is trivially present, and by construction π^* has property (P) on the intervall between $max\pi^*T = k$ and $minT = p$. If $Q = \{a < p < n < n+1\}$ is moved down by π^* then $\pi^*a > j$, so $\pi^*a > k$, so $\pi^*a = max\pi^*Q$. This ends the proof of the lemma for $\sigma = (kp)(jn)(i, n+1)$. The remaining five possibilities for σ require the same arguments.

We collect everything in the following

Proposition. Let $n \geq 1$ and π standard on $[n]$. Then φ_n^{-1} consists of elements standard on $[+1]$. This set is the following:

a) $\pi = \varepsilon : \varepsilon, (n, n+1)$,

b) $\pi \neq \varepsilon, \pi n = n : \pi, \pi \cdot (n, n+1), \pi \cdot (ij)\sigma$

where j is the maximal nonfixed point of π, $i = \pi j$, and σ anyone of the involutions $(jn)(i, n+1), (in)(j, n+1)$.

c) $\pi n \neq n$ and πn no neighbour to n: $\pi, \pi \cdot (n, \pi n, n+1)$,

d) $\pi n \neq n$ and $\pi n = p$ is neighbour of n: $\pi, \pi \cdot (n, \pi n, n+1), \pi \cdot (ij) \cdot \sigma$ where $(ij)|\pi, i < j, \pi(j+1) = j+1, \pi$ has property (P) on $\{l | l > j\}$, and

σ is any one of the following six involutions:

$(kp)(jn)(i, n+1)$

$(kp)(in)(j, n+1)$

$(jp)(kn)(i, n+1)$

$(ip)(kn)(j, n+1)$

$(jp)(in)(k, n+1)$

$(ip)(jn)(k, n+1)$.

Here $k > j$ is any fixed point of π adjacent to j.

An auxiliary algebra

For $n \geq 3$ integral we define an associative $\mathbf{Q}$-algebra denoted by $\mathbf{Y}_n$ with unit element: The generators are symbols Y_τ indexed by the transpositions τ in S_n, the symmetric group of degree n, plus one additional symbol denoted Y_ε, ε the unit element in S_n. We impose the following relators:

If τ, τ_1, τ_2 are transpositions and τ_1 and τ_2 commute then

$$Y_{\tau_1} Y_{\tau_2} = Y_{\tau_2} Y_{\tau_1}, Y_\tau Y_\varepsilon = Y_\varepsilon Y_\tau = Y_\tau. \tag{1}$$

If $\{\tau_1, \tau_2, \tau_3\}$ forms the set of transpositions in some symmetric subgroup of degree 3 of S_n then

$$Y_{\tau_1} Y_{\tau_2} + Y_{\tau_2} Y_{\tau_1} = 2 Y_{\tau_3}, \tag{2}$$

$$Y_{\tau_3}(Y_{\tau_!} + Y_{\tau_2}) = Y_{\tau_1} + Y_{\tau_2}. \tag{3}$$

(It is easily deduced that $(Y_{\tau_1}+Y_{\tau_2})Y_{\tau_3} = Y_{\tau_1}+Y_{\tau_2}$, as well.)

We use the notation from the previous paragraph, especially $S_n^{(2)} = \{\pi \in S_n | \pi^2 = \varepsilon\}$.

For $\pi \in S_n^{(2)}$ and τ a transposition we write $\tau|\pi$ if τ arises in the cycle decomposition of π. So

$$\pi = \prod_{\tau|\pi} \tau.$$

We also write

$$Y_\pi = \prod_{\tau|\pi} Y_\tau$$

for $\pi \in S_n^{(2)}$ and identify the empty product Y_ε to the unit element 1 of $\mathbf{Y}_n$.

It is not intended in this paper to study the algebra $\mathbf{Y}_n$ for itself but to use it as a tool when computations with degree two characters (on an algebra or a group) have to be made. We only shall prove below that $\mathbf{Y}_n$ is finite dimensional.

We have an obvious action of S_n on $\mathbf{Y}_n$: For $\sigma \in S_n, \tau$ a transposition in S_n we define

$$\sigma(Y_\tau) = Y_{\sigma\tau\sigma^{-1}}. \tag{4}$$

We understand $\mathbf{Y}_n$ all the time to carry this $\mathbf{Q}S_n$-structure.

Again let $\{\tau_1, \tau_2, \tau_3\}$ a triple of transpositions generating a degree three symmetric subgroup of S_n (so $\tau_3 = \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$). Then

$$\tau_1\tau_2(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}) = Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}. \tag{5}$$

This is easy:

$\tau_1\tau_2(Y_{\tau_1}) = Y_{\tau_1\tau_2\tau_1\tau_2\tau_1} = Y_{\tau_2}$,

$\tau_1\tau_2(Y_{\tau_2}) = Y_{\tau_1\tau_2\tau_1\tau_2\tau_1} = Y_{\tau_3}$,

$\tau_1\tau_2(Y_{\tau_3}) = Y_{\tau_1\tau_2\tau_2\tau_1\tau_2\tau_2\tau_1} = Y_{\tau_1}$.

So

$$\begin{aligned}
\tau_1\tau_2(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}) &= Y_{\tau_2}Y_{\tau_3} - Y_{\tau_1} \\
&= Y_{\tau_2}(Y_{\tau_1} + Y_{\tau_3}) - Y_{\tau_2}Y_{\tau_1} - Y_{\tau_1} \\
&= Y_{\tau_1} + Y_{\tau_3} - Y_{\tau_2}Y_{\tau_1} \\
&= Y_{\tau_3} - (-Y_{\tau_1}Y_{\tau_2} + 2Y_{\tau_3}) \\
&= Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}.
\end{aligned}$$

Actually the identities (2) and (5) in $\mathbf{Y}_n$ allow to deduce (3): From

$$Y_{\tau_2}Y_{\tau_3} - Y_{\tau_1} = Y_{\tau_3}Y_{\tau_1} - Y_{\tau_2}$$

it follows, using (2), that

$$-Y_{\tau_3} \cdot Y_{\tau_2} + Y_{\tau_1} = Y_{\tau_3} \cdot Y_{\tau_1} - Y_{\tau_2},$$

and this is (3). In the applications below (5) is more natural than (3).

Proposition. For $\{\tau_1, \tau_2, \tau_3\}$ the set of transpositions in a degree three symmetric subgroup of S_n we have

$$\tau_1(Y_{\tau_1} \cdot Y_{\tau_2} - Y_{\tau_3}) = \tau_2(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}) = -(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}). \tag{6}$$

Proof. The preceding discussion says that τ_1 and τ_2 have the same effect on $Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}$. We compute

$$\begin{aligned}
\tau_1(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}) &= Y_{\tau_1}Y_{\tau_3} - Y_{\tau_2} \\
&= (\tau_1\tau_2)^2(Y_{\tau_1} \\
&= Y_{\tau_2}Y_{\tau_2} - Y_{\tau_3}) \\
&= -Y_{\tau_1}Y_{\tau_2} + Y_{\tau_3}.
\end{aligned}$$

We introduce some more notation. We write whenever i, j are different integers ≥ 1

$$Y_{ij} = Y_{ji} = Y_{(ij)},$$

(ij) to be understood as a transposition. To each triple I of pairwise different integers from $[n] = \{1, 2, ..., n\}$ we associate a specific element $Z_I \in \mathbf{Y}_n$ in the following way: Let $I = \{i < j < k\}$. Then $Z_I = Y_{ij}Y_{jk} - Y_{ik}$ We also write Z_{ijk} for Z_I if $I = \{i, j, k\}$ given in any order.

We let act S_n on the collection of triples canonically: $\sigma\{i, j, k\} = \{\sigma i, \sigma j, \sigma k\}$. For $i < j < k$ the triple $(\sigma i, \sigma j, \sigma k)$ in this order will require a certain permutation in order to give it its natural ordering. We call $sgn(\sigma, I)$ the sign of this element of the symmetric group on $\{\sigma i, \sigma j, \sigma k\}$. An immediate consequence of the preceding then is

Proposition. Let $I \subset [n] = \{1, 2, ..., n\}$ a triple of (pairwise different) integers and let $\sigma \in S_n$. Then

$$\sigma(Z_I) = sgn(\sigma, I) \cdot Z_{\sigma I}.$$

We now present a finite dimensional homomorphic image of $\mathbf{Y}_n$. Let $\{\tau_1, \tau_2, \tau_3\}$ be the transpositions in $S_3 \subset S_n$, and

$$\omega = \sum_{\pi \in S_3} (sgn\pi) \cdot \pi = \tau_1\tau_2 + \tau_2\tau_1 - \tau_1 - \tau_2 - \tau_3 + \varepsilon, \Omega_n = (\mathbf{Q}S_n)\omega(\mathbf{Q}S_n)$$

the twosided ideal generated by ω in $\mathbf{Q}S_n$. $\mathbf{Q}S_n$ is given the standard S_n-structure:

$$\mathbf{Q}S_n \ni \alpha \longrightarrow \sigma\alpha\sigma^{-1} \in \mathbf{Q}S_n \text{ for } \alpha \in S_n.$$

Ω_n is S_n-invariant, so this S_n-structure projects to an S_n-structure on the quotient $\mathbf{Q}S_n/\Omega_n$.

Theorem. Let $n \geq 3$. The assignment

$$\varphi_n(Y_\tau) \equiv 2\tau - \varepsilon \mod \Omega_n,$$

τ any transposition of S_n, extends to an $S_n$equivariant algebra epimorphism, again denoted φ_n from $\mathbf{Y}_n$ onto $\mathbf{Q}S_n/\Omega_n$.

So there is an exact sequence

$$0 \longrightarrow ker\varphi_n \longrightarrow \mathbf{Y}_n \longrightarrow \mathbf{Q}S_n/\Omega_n \longrightarrow 0.$$

Proof. We have to check that φ_n preserves the relations $(1), (2)$, and (3), everything else being trivial:

If τ_1 and τ_2 commute so do $2\tau_1 - \varepsilon$ and $2\tau_2 - \varepsilon$ in $\mathbf{Q}S_n$.

$$\begin{aligned}(7) \qquad & \varphi_n(Y_{\tau_1}Y_{\tau_2} + Y_{\tau_2}Y_{\tau_1} - 2Y_{\tau_3}) \\ &\equiv (2\tau_1 - \varepsilon)(2\tau_2 - \varepsilon) + (2\tau_2 - \varepsilon)(2\tau_1 - \varepsilon) - 2(2\tau_3 - \varepsilon) \\ &\equiv 4(\tau_1\tau_2 + \tau_2\tau_1 - \tau_1 - \tau_2 - \tau_3 + \varepsilon) \\ &\equiv 0 \mod \Omega_n,\end{aligned}$$

$$\begin{aligned}(8) \qquad & \varphi_n(Y_{\tau_3}Y_{\tau_1} + Y_{\tau_3}Y_{\tau_2} - Y_{\tau_1} - Y_{\tau_2}) \\ &\equiv (2\tau_3 - \varepsilon)(2\tau_1 - \varepsilon) + (2\tau_3 - \varepsilon)(2\tau_2 - \varepsilon) - (2\tau_1 - \varepsilon) - (2\tau_2 - \varepsilon) \\ &\equiv 4(\tau_3\tau_1 + \tau_3\tau_2 - \tau_1 - \tau_2 - \tau_3 + \varepsilon) \\ &\equiv 0 \mod \Omega_n.\end{aligned}$$

So the theorem is proven.

We note, for $\{\tau_1, \tau_2, \tau_3\}$ the set of transpositions in some symmetric group of degree three:

$$\begin{aligned}\varphi_n(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3}) &\equiv (2\tau_1 - \varepsilon)(2\tau_2 - \varepsilon) - (2\tau_3 - \varepsilon) \\ &\equiv 4\tau_1\tau_2 - 2\tau_1 - 2\tau_2 - 2\tau_3 + 2\varepsilon \\ &\equiv 2(\tau_1\tau_2 - \tau_2\tau_1) \mod \Omega_n.\end{aligned}$$

E.g. for $1 \leq i < j < k \leq n$ we have $\varphi_n(Z_{i,j,k}) \equiv 2((ijk) - (kji)) \mod \Omega_n$. Furthermore: Let $\tau_i = (i, i+1), 1 \leq i \leq n-1$. Then

$$\varphi_n((1+Y_{\tau_1})(1+Y_{\tau_2})...(1+Y_{\tau_{n-1}})) \equiv 2^{n-1}\tau_1\tau_2...\tau_{n-1} \equiv 2^{n-1}(12...n) \mod \Omega_n.$$

The statement under line (5) is nothing but the fact that the 3-cycle (ijk) allows cyclic rearrangement of the entries.

We now remind of the standard matrix representation of the algebra $\mathbf{Q}S_n/\Omega_n$. From classical ordinary representation theory of the symmetric group one knows that

$$\mathbf{Q}S_n \cong \coprod_{\lambda} M_\lambda(\mathbf{Q})$$

where λ runs over the partitions of n and $M_\lambda(\mathbf{Q})$ is a full matrix algebra over $\mathbf{Q}$. The element ω which generates Ω_n is, in the component indexed by λ, represented by 0 if and only if λ has at most two rows. So

$$\mathbf{Q}S_n/\Omega_n \cong \coprod_{\lambda_2} M_{\lambda_2}(\mathbf{Q})$$

where λ_2 runs over all partitions of n with one or two rows.

Proposition. For $\tau \in S_n$ a transposition the element $Y_\tau^2 + 2Y_\tau - 3 = (Y_\tau - 1)(Y_\tau + 3)$ is contained in the kernel of φ_n; it is at the same time central in $\mathbf{Y}_n$.

Proof. We have

$$Y_\tau^2 + 2Y_\tau - 3 = (Y_\tau + 1)^2 - 4,$$

so

$$\varphi_n(Y_\tau^2 + 2Y_\tau - 3) \equiv (2\tau - \varepsilon + \varepsilon)^2 - 4\varepsilon \equiv 0 \mod \Omega_n.$$

Let $\sigma \in S_n$ a transposition which does not commute with τ . We set $\tau_1 = \tau, \tau_2 = \sigma, \tau_3 = \tau_1\tau_2\tau_1$. Then

$$\begin{aligned} Y_{\tau_1}Y_{\tau_2} &= Y_{\tau_1}Y_{\tau_2} - Y_{\tau_3} + Y_{\tau_3} \\ &= Y_{\tau_3}Y_{\tau_1} - Y_{\tau_2} + Y_{\tau_3}. \end{aligned}$$

So

$$(Y_{\tau_1} + 1)Y_{\tau_2} = Y_{\tau_3}(Y_{\tau_1} + 1).$$

We repeat, getting

$$\begin{aligned} (Y_{\tau_1} + 1)^2 Y_{\tau_2} &= (Y_{\tau_1} + 1)Y_{\tau_3}(Y_{\tau_1} + 1) \\ &= Y_{\tau_2}(Y_{\tau_1} + 1)^2. \end{aligned}$$

This proves the proposition.

Remark. One can prove a similar statement for

$$[(1 + Y_{\tau_1})(1 + Y_{\tau_2})...(1 + Y_{\tau_{p-1}})]^p - 2^{(p-1)p}$$

which is, for $p = 2$, the element of the proposition: It is contained in the kernel of φ_n and central in $\mathbf{Y}_n$, the corresponding assumption being that $\tau_1\tau_2...\tau_{p-1}$ is a p-cycle in S_n.

At this point a discussion of the algebra structure of $\mathbf{Y}_n$ is inserted.

Theorem. $\mathbf{Y}_n$ is finite dimensional.

It suffices to prove this theorem for $n = 3$: $\mathbf{Y}_3$ maps canonically into $\mathbf{Y}_n$, and this subalgebra generates $\mathbf{Y}_n$ as a $\mathbf{Q}S_n$-algebra.

Proof. We replace the generators $Y_\varepsilon = 1, Y_1 = Y_{(23)}, Y_2 = Y_{(13)}, Y_3 = Y_{(12)}$ by new ones which are easier to handle:

$$U = Y_1 + 1, V = Y_2 + 1, c = Y_1 + Y_2 + Y_3 + 3.$$

We define

$$T = \frac{1}{2}(UV - VU), u = U^2, v = V^2.$$

The relations in $\mathbf{Y}_3$ imply that c, u, and v are central, and that

$$UV + VU = 2(c-2),$$

so

$$UV = c - 2 + T, \tag{9}$$

$$VU = c - 2 - T, \tag{10}$$

and, furthermore

$$cU = 2c + u - 4, \tag{11}$$

$$cV = 2c + v - 4, \tag{12}$$

both being contained in the center of $\mathbf{Y}_3$. We draw consequences:

$$cT = \frac{1}{2}((cU)V - V(cU)) = 0. \tag{13}$$

From $(cU)U = cU^2$ we get

$$(u-4)U = (u-4)(c-2), \tag{14}$$

and, similarly, from $(cV)V = cV^2$:

$$(v-4)V=(v-4)(c-2). \tag{15}$$

From $c(UV)=(cU)V=U(cV)$ we get

$$(u-4)V=c(c-6)-2(v-4), \tag{16}$$

$$(v-4)U=c(c-6)-2(u-4). \tag{17}$$

From (14) and (17) we have

$$(u-4)T=\frac{1}{2}(((u-4)U)V-V((u-4)U))=0, \tag{18}$$

$$(v-4)T=\frac{1}{2}(((v-4)U)V-V((v-4)U))=0. \tag{19}$$

The relations

$$(u-4)(UV)=((u-4)U)V=(U(u-4))V$$

and

$$(v-4)(UV)=((v-4)U)V=U(v-4)V$$

lead to

$$(c-2)(u-4)=(c-2)(v-4)=\frac{1}{3}c(c-2)(c-6). \tag{20}$$

Furthermore, from $cUT=cVT=0$ it follows that

$$uv=(c-2)(2c-4-u)=(c-2)(2c-4-v) \tag{21}$$

which together with (11) leads to

$$uv = -\frac{1}{3}(c-2)(c^2-12c+24). \tag{22}$$

From $U(UT) = U^2T$ and $V(VT) = V^2T$ we get

$$UT = -TU = 2U + 4V - (u-4) - 2(v-4) + c(c-8), \tag{23}$$

$$VT = -TV = -4U - 2V + 2(u-4) + (v-4) - c(c-8). \tag{24}$$

Moreover $U(VT) = (UV)T$ leads to

$$T^2 = \frac{1}{3}(c-2)(c-3)(c-6), \tag{25}$$

so, from $cT = 0$ (see (13)), we infer

$$c(c-2)(c-3)(c-6) = 0. \tag{26}$$

Finally $(u-4)U^2 = ((u-4)U)U = (c-2)(u-4)U = (c-2)^2(u-4)$ implies

$$u^2 - 4u = \frac{1}{3}c(c-2)^2(c-6), \tag{27}$$

and similarly we obtain

$$v^2 - 4v = \frac{1}{3}c(c-2)^2(c-6). \tag{28}$$

So, altogether we conclude that the subalgebra of $\mathbf{Y}_3$ generated by c, u, and v is contained in

$$Z = \mathbf{Q} + \mathbf{Q}c + \mathbf{Q}c^2 + \mathbf{Q}c^3 + \mathbf{Q}u + \mathbf{Q}v$$

see $(20), (22), (26), (27), (28)$. The formulae $(9), (23), (24), (25)$ plus the definition of u and v show that any product of two factors from $\{U, V, T\}$ is in Z.

$$ZU \subset Z + \mathbf{Q}U + \mathbf{Q}V + \mathbf{Q}T$$

follows from $(11), (14), (17)$, and similarly

$$ZV \subset Z + \mathbf{Q}U + \mathbf{Q}V + \mathbf{Q}T$$

from $(12), (15), (16)$. Finally $ZT \subset \mathbf{Q} + \mathbf{Q}T$ results from $(13), (18), (19)$. This proves that $Z + \mathbf{Q}U + \mathbf{Q}V + \mathbf{Q}T$ is multiplicatively closed. Its dimension is nine or less. This proves the theorem.

Remark: $(UV)^3$ is central in $\mathbf{Y}_3$ which we know already. This may be checked here directly:

$$UV = c - 2 + T, cT = 0, T^2 = \frac{1}{3}(c-2)(c-3)(c-6)$$

allows to compute

$$\begin{aligned}
(UV)^3 &= (c-2)^3 + 3(c-2)^2T + 3(c-2)T^2 + T^3 \\
&= (c-2)^3 + 3(c-2)T^2 + 12T - 12T \\
&= (c-2)^2(c-4)^2.
\end{aligned}$$

We recall the definition of $Z_I \in \mathbf{Y}_n$ for $I \subset [n]$ an arbitrary triple: If $I = \{i < j < k\}$ then

$$Z_I = Y_{ij}Y_{jk} - Y_{ik}$$

.

It follows from (6) that, whenever σ permutes I then

$$\sigma Z_I = sgn\sigma Z_I. \tag{29}$$

We use this quite frequently in the following form: Whenever $\{i,j,k\}$ is a triple of integers then

$$Y_{ij}Y_{jk} = Y_{ik} + sgn\sigma Z_{i,j,k}$$

where σ is the permutation of $\{i,j,k\}$ which orders (i,j,k) according to their magnitude.

Proposition. Let $\zeta \in S_n$ an $n-$cycle, $n \geq 2$. We write ζ as a product of $n-1$ transpositions:

$$\zeta = \tau_1\tau_2...\tau_{n-1}$$

with

$$\tau_i = (\zeta^{i-1}1, \zeta^i 1), 1 \leq i \leq n-1.$$

Let

$$P_n = (1+Y_{\tau_1})(1+Y_{\tau_2})...(1+Y_{\tau_{n-1}}) \in \mathbf{Y}_n. \tag{30}$$

Then $\zeta P_n = P_n$.

Proof. We again argue by induction on n: The case $n=2$ is trivial, and the case $n=3$ is contained in (29): Any $3-$cycle may be written as the product of two non commuting transpositions τ_1, τ_2. We so have

$$\tau_1\tau_2(1+Y_{\tau_1})(1+Y_{\tau_2})$$

$$= \tau_1\tau_2(1+Y_{\tau_1}+Y_{\tau_2}+Y_{\tau_1\tau_2\tau_1}) + \tau_1\tau_2(Y_{\tau_1}Y_{\tau_2} - Y_{\tau_1\tau_2\tau_1})$$

$$= (1+Y_{\tau_1}+Y_{\tau_2}+Y_{\tau_1\tau_2\tau_1}) + (Y_{\tau_1}Y_{\tau_2} - Y_{\tau_1\tau_2\tau_1})$$

$$= (1+Y_{\tau_1})(1+Y_{\tau_2}).$$

We assume with no loss of generality

$$\zeta = (1, 2, 3, ..., n, n+1) = \tau_1\tau_2\tau_3...\tau_n \in S_{n+1}$$

where $\tau_i = (i, i+1), 1 \leq i \leq n$, and we write $\zeta_0 = \zeta\tau_n = (1, 2, 3, ..., n) \in S_n$. We compute

$$\begin{aligned} \zeta P_{n+1} &= \zeta(1+Y_{12})(1+Y_{23})...(1+Y_{n,n+1}) \\ &= (1+Y_{23})(1+Y_{34})...(1+Y_{n-1,n})(1+Y_{n,n+1})(1+Y_{n+1,1}). \end{aligned}$$

Using the statement for $n = 3$ again on the two rightmost factors we get

$$(1+Y_{n,n+1})(1+Y_{n+1,1}) = (n, 1, n+1)(1+Y_{n,n+1})(1+Y_{n+1,1}) = (1+Y_{n,1})(1+Y_{n,n+1}).$$

So

$$\begin{aligned} \zeta P_{n+1} &= (1+Y_{23})(1+Y_{34})...(1+Y_{n-1,n})(1+Y_{n,1})(1+Y_{n,n+1}) \\ &= (\zeta_0 P_n)(1+Y_{n,n+1}) = P_n(1+Y_{n,n+1}) \end{aligned}$$

by induction hypothesis. So

$\zeta P_{n+1} = P_{n+1}$.

The main goal of this paragraph is a formula which allows to canonically express P_n from (30) as a linear combination of terms $Y_\pi, \pi \in S_n^{(2)}$ and $Y_\pi Z_I = Z_I Y_\pi$ with $\pi \in S_n^{(2)}, I \subset [n]$ some triple. This formula, for $n = 3$, reads

$$(1+Y_{(12)})(1+Y_{(23)}) = Y_\varepsilon + Y_{(12)} + Y_{(13)} + Y_{(23)} + Z_{\{1,2,3\}},$$

which is nothing but the definition of $Z_{\{1,2,3\}}$. For $n = 4$ it is given in the next proposition; for $n \geq 5$ it will be derived by induction on n:

$$P_{n+1} = P_n(1+Y_{n,n+1}) = P_n + P_n Y_{n,n+1}.$$

(The underlying $n-$cycle $\zeta \in S_n$ which is used to define P_n in (30) is all the time assumed to be $(1,2,3,...,n)$, similarly for $n+1$ instead of n; this creates, of course, no loss of generality.)

Proposition. We have

$$P_4 = (1+Y_{(12)})(1+Y_{(23)})(1+Y_{(34)}) = \sum v_\pi Y_\pi + \sum Z_I.$$

Here the first sum extends over all $\pi \in S_4^{(2)}$, and the second over all triples $I \subset \{1,2,3,4\} = [4]$.

Proof. We several times apply the previous lemma for $n=3$ and $n=4$, and also the formula

$$(1+Y_{\tau_1})(1+Y_{\tau_2}) = -(1+Y_{\tau_1})(1+Y_{\tau_3}) + 2(1+Y_{\tau_1}+Y_{\tau_2}+Y_{\tau_3}) \tag{31}$$

for $\{\tau_1, \tau_2, \tau_3\}$ the set of transpositions in some symmetric degree three group; it is an easy consequence of the relations defining $\mathbf{Y}_n$. We have

$$\begin{aligned} P_4 &= [(1+Y_{12})(1+Y_{23})](1+Y_{34}) \\ &= -(1+Y_{12})(1+Y_{13})(1+Y_{34}) + 2(1+Y_{12}+Y_{13}+Y_{23})(1+Y_{34}) \end{aligned}$$

by (31). The indices in the triple product multiply up to $(12)(13)(34) = (1342)$ which is a $4-$cycle. So this summand is invariant under (1342):

$$P_4 = -[(1+Y_{13})(1+Y_{34})](1+Y_{24}) + 2(1+Y_{12}+Y_{13}+Y_{23})(1+Y_{34}).$$

We use again (31) on the bracketed product and get

$$\begin{aligned} P_4 = \ & (1+Y_{13})(1+Y_{14})(1+Y_{24}) \\ & -2(1+Y_{13}+Y_{14}+Y_{34})(1+Y_{24}) \\ & +2(1+Y_{12}+Y_{13}+Y_{23})(1+Y_{34}). \end{aligned}$$

Here again the indices of the triple product multiply up to a $4-$cycle: $(13)(14)(24) = (1423)$. We use invariance under this $4-$cycle:

$$\begin{aligned}P_4 &= [(1+Y_{14})(1+Y_{24})](1+Y_{23}) \\ &\quad -2(1+Y_{13}+Y_{14}+Y_{34})(1+Y_{24}) \\ &\quad +2(1+Y_{12}+Y_{13}+Y_{23})(1+Y_{34}).\end{aligned}$$

Again (31) applied to the bracketed product gives

(32)

$$\begin{aligned}P_4 &= -(1+Y_{14})(1+Y_{12})(1+Y_{23}) \\ &\quad +2(1+Y_{12}+Y_{14}+Y_{24})(1+Y_{23}) \\ &\quad -2(1+Y_{13}+Y_{14}+Y_{34})(1+Y_{24}) \\ &\quad +2(1+Y_{12}+Y_{13}+Y_{23})(1+Y_{34}).\end{aligned}$$

The final step is to observe invariance of the triple product under $(14)(12)(23) = (1234)$. Applying this 4−cycle to this product yields P_4 with coefficient -1. (32) then immediately leads to the formula of the proposition.

Proposition. For $1 \le i < j < k < l \in \mathbf{N}$ we have

$$Z_{ijk}Y_{kl} = -Y_{ik}Y_{jl} + Y_{il}Y_{jk} + Z_{ijl},$$

$$Y_{kl}Z_{ijk} = Y_{ik}Y_{jl} - Y_{il}Y_{jk} + Z_{ijl},$$

and $Y_\pi Z_I = Z_I Y_\pi$ for $\pi \in S_\infty^{(2)}, I \subset \mathbf{N}$ any triple such that π is the identity on I. (There are more formulae of this type according to the various different orderings of the indices.)

Proof. With no loss of generality we take $i = 1, j = 2, k = 3, l = 4$.

Then

$Z_{123}Y_{34} = Y_{12}Y_{23}Y_{34} - Y_{13}Y_{34}$

$= (1+Y_{12})(1+Y_{23})(1+Y_{34}) - Y_{12}Y_{23} - Y_{23}Y_{34} - Y_{13}Y_{34} - Y_{(12)(34)} - Y_{12} - Y_{23} - Y_{34} - 1.$

We then use the previous lemma, and the result drops out. The two other statements are then obvious.

Proposition. The following formula holds true:

(33)

$$\begin{aligned} P_5 &= (1+Y_{12})(1+Y_{23})(1+Y_{34})(1+Y_{45}) \\ &= \sum_{\pi \in S_5^{(2)}} v_\pi Y_\pi + \sum_{(\pi,I)\text{standard on}[5]} v_{(\pi,I)} Y_\pi Z_I . \end{aligned}$$

This is an easy consequence of the formula for P_4; one only has to multiply on the right by $1+Y_{45}$.

As has been shown above the cycle (12345) leaves invariant P_5; one notices that the same thing is true for the first sum in (33). So

$$(\varepsilon - (12345)^2) \sum_{(\pi,I) \text{ standard on } [5]} v_{(\pi,I)} Y_\pi Z_I = 0.$$

We note this fact for arbitrary i,j,k,l,m instead of $1,2,3,4,5$:

Proposition. For $1 \leq i < j < k < l < m \in \mathbf{N}$ we have

$$Y_{im}Z_{jkl} - Y_{jm}Z_{ikl} + Y_{km}Z_{ijl} - Y_{lm}Z_{ijk} = 0,$$

$$Y_{il}Z_{jkm} - Y_{jl}Z_{ikm} + Y_{kl}Z_{ijm} - Y_{lm}Z_{ijk} = 0,$$

$$Y_{ik}Z_{jlm} - Y_{jk}Z_{ilm} + Y_{kl}Z_{ijm} - Y_{km}Z_{ijl} = 0,$$

$$Y_{ij}Z_{klm} - Y_{jk}Z_{ilm} + Y_{jl}Z_{ikm} - Y_{jm}Z_{ikl} = 0,$$

$$Y_{ij}Z_{klm} - Y_{ik}Z_{jlm} + Y_{il}Z_{jkm} - Y_{im}Z_{jkl} = 0.$$

Lemma. Let $I = \{i_1 < i_2 < i_3\}, J = \{j_1 < j_2 < j_3\}$ triples of integers such that $I \cap J = \emptyset$. Then

$$Z_I Z_J + det(Y_{ij})_{i \in I, j \in J} = 0.$$

(The rows and columns in the determinant are understood to be ordered according to the ordering of I and J; observe that the entries of this determinant commute pairwise.)

The proof is an easy consequence of the previous lemma. We take, for simplicity, $I = \{1, 2, 3\}, J = \{4, 5, 6\}$. We know

$$Y_{45}Z_{123} = Y_{15}Z_{234} - Y_{25}Z_{134} + Y_{35}Z_{124}.$$

We multiply by Y_{46} first from the right, then from the left, and take the difference:

$$(Y_{45}Y_{46} - Y_{46}Y_{45})Z_{123} \tag{34}$$

$$= Y_{15}(Z_{234}Y_{46} - Y_{46}Z_{234}) - Y_{25}(Z_{134}Y_{46} - Y_{46}Z_{134}) + Y_{35}(Z_{124}Y_{46} - Y_{46}Z_{124}).$$

We have

$$Y_{45}Y_{46} = -Z_{456} + Y_{56},$$

$$Y_{46}Y_{45} = Z_{456} + Y_{56},$$

and for $1 \leq i < j \leq 3$;

$$Z_{ij4}Y_{46} = -Y_{i4}Y_{j6} + Y_{i6}Y_{j4} + Z_{ij6},$$

$$Y_{46}Z_{ij4} = Y_{i4}Y_{j6} - Y_{i6}Y_{j4} + Z_{ij6}.$$

So (34) leads to

$$-2Z_{456}Z_{123} = 2Y_{15}(-Y_{24}Y_{36}+Y_{26}Y_{34})-2Y_{25}(-Y_{14}Y_{36}+Y_{16}Y_{34})+2Y_{35}(-Y_{14}Y_{26}+Y_{16}Y_{24}),$$

and this is the desired formula.

We now state and prove the basic technical fact.

Theorem. Let $n \geq 3$ integral and

$$P_n = (1+Y_{12})(1+Y_{23})...(1+Y_{n-1,n}).$$

Then

$$P_n = \sum_{\pi \text{ standard on } [n]} v_\pi Y_\pi + \sum_{(\pi,I) \text{ standard on } [n]} v_{(\pi,I)} Y_\pi Z_I.$$

The proof goes by induction on n. For $n=3$ the formula is nothing but the definition of Z_{123}; for $n=4$ and $n=5$ it has been proven, already.

We compute $Y_\pi Y_{n,n+1}$ and $Y_{pi} Z_I Y_{n,n+1}$ for π respectively (π, I) standard 0n $[n]$. For π standard on $[n]$ we distinguish

(a_1) $\pi n = n$,

(a_2) $\pi n \neq n$,

and for

(π, I) standard on $[n]$

(b_1) $\pi n = n$ and $n \notin I$,

(b_2) $n \in I($ so $\pi n = n)$,

(b_3) $\pi n \neq n (son \notin I)..$

Case (a_1).

$$Y_\pi Y_{(n,n+1)} = Y_{\pi(n,n+1)}$$

which is contained in

$\varphi_n^{-1}(\pi)$ in St_{n+1}.

Case (a_2). We write $\pi = \pi_0(\pi_n, n)$ with $\pi_0 \in St_n$ fixing n. Then

$$Y_\pi Y_{n,n+1} = Y_{\pi_0} Y_{\pi_n,n} Y_{n,n+1} = Y_{\pi_0(\pi n,n+1)} + Y_{\pi_0} Z_{\pi n,n,n+1}.$$

The first index $\pi_0(\pi n, n+1)$ again is contained in $\varphi_n^{-1}(\pi) \subset St_{n+1}$. We claim that the pair $(\pi_0, \{\pi n, n, n+1)\}$ is standard on $[n+1]$:

$$\pi_0 = \pi_0 \cdot \varepsilon$$

is a decomposition with properties:

1) $\pi_0 l + l$ for $l = n$ and $l \geq n$,

2) $\varepsilon l = l$ for $l \leq n+1$,

3) $\pi_0 \cdot (\pi n, n)$ is standard,

4) ε has property (P).

Case (b_1).

$$Y_\pi Z_I Y_{n,n+1} = Y_{\pi\cdot(n,n+1)} Z_I.$$

We write $I = \{i < j < k\}$. Let $\pi_1, \pi_2 \in S_n^{(2)}$ such that $\pi = \pi_1 \cdot \pi_2 = \pi_2 \cdot \pi_1$ and

1)$\pi_1 l = l$ for $l = i$ and $l \geq k$,

2)$\pi_2 l = l$ for $l \leq k$,

3)$\pi_1 \cdot (ij)$ is standard,

4)π_2 has property (P).

Then

$$\pi \cdot (n, n+1) = \pi_1 \cdot (\pi_2 \cdot (n, n+1))$$

is the decomposition of $\pi \cdot (n, n+1)$ which allows to conclude on $(\pi \cdot (n, n+1), I)$ to be standard on $[n+1]$.

Case (b_2). Let $I = \{i < j < n\}$ and (π, I) standard on $[n]$.

$$Y_\pi Z_{ijn} Y_{n,n+1} = Y_\pi(-Y_{(in)(j,n+1)} + Y_{(jn)(i,n+1)} + Z_{i,j,n+1}) = -Y_{\pi(in)(j,n+1)} + Y_{\pi(jn)(i,n+1)} + Y_\pi Z_{i,j,n+1}.$$

We see that each summand is indexed by a standard element: (π, I) to be standard means that

1) $\pi l = l$ for $l = i$ and $l \geq j$,

2) $\pi \cdot (ij)$ is standard.

So j is the maximal nonfixed point of $\pi \cdot (ij)$. Lemma then says that both $\pi(in)(j, n+1)$ and $\pi(jn)(i, n+1)$ are standard on $[n+1]$. The pair $(\pi, \{i, j, n+1\})$ is standard, trivially.

Case (b_3). Let $\pi = \pi_0 \cdot (pn), I = \{i < j < k\}$. Let $\pi = \pi_1\pi_2 = \pi_2\pi_1$ with $\pi_1, \pi_2 \in S_n^{(2)}$ such that

1)$\pi_1 l = l$ for $l = i, l \geq j$,

2)$\pi_2 l = l$ for $l \leq k$,

3)$\pi_1 \cdot (ij)$ is standard,

4)π_2 has property (P).

Then (pn) is a factor of π_2, so p is the maximal nonfixed point of π below n, and $\geq k$.

$$Y_\pi Z_I Y_{n,n+1} = Y_{\pi_0} Z_I Y_{pn} Y_{n,n+1}$$

$$= Y_{\pi_0} Z_I (Y_{p,n+1} + Z_{p,n,n+1}) = Y_{\pi_0} Z_I Z_{p,n,n+1} + Y_{\pi_0 \cdot (p,n+1)} Z_I$$

$$= Y_{\pi_0}(-Y_{(ip)(jn)(k,n+1)} - Y_{(jp)(kn)(i,n+1)} - Y_{(kp)(in)(j,n+1)}$$

$$+T_{(ip)(kn)(j,n+1)} + Y_{(jp)(in)(k,n+1)} + Y_{(kp)(jn)(i,n+1)}) + Y_{\pi_0(p,n+1)} Z_I.$$

$$= -Y_{\pi_0(ip)(jn)(k,n+1)} - Y_{\pi_0(jp)(kn)(i,n+1)} - Y_{\pi_0(kp)(in)(j,n+1)}$$

$$+Y_{\pi_0(ip)(kn)(j,n+1)}+Y_{\pi_0(jp)(in)(k,n+1)}+Y_{\pi_0(kp)(jn)(i,n+1)}+Y_{\pi_0(p,n+1)}Z_I.$$

Lemma. says that the indices of the first six summands are standard on $[n+1]$, and

$$\pi_0(p,n+1)=\pi_1\cdot(\pi_2(n,\pi_2 n,n+1))$$

is the decomposition of $\pi_0(p,n+1)$ which shows that $(\pi_0(p,n+1),I)$ is standard on $[n+1]$.

Finally, it has to be noted that $P_n+P_nY_{n,n+1}$ contains each summand $Y_\pi, Y_\pi Z_I$ with $\pi,(\pi,I)$ standard on $[n+1]$ with multiplicity v_π, or $v_{(\pi,I)}$, respectively. This is guaranteed by the existence of the reduction φ_n and proposition.

The character variety

We now state and prove the main result of this paper. Let A an associative $\mathbf{C}$-algebra with unit element which is free on finitely many generators the number of which we assume to be at least three. So

$$A=\bigotimes V=\coprod_{k\geq 0}V^{\otimes k}$$

is the tensor algebra of some complex vector space of finite dimension $d\geq 3$. We once for all fix a basis $\{b_1,b_2,...,b_d\}$ of V.

To each character s on A we associate its coordinates which form an $N-$tuple of complex numbers with $N=d+d+\binom{d}{2}+\binom{d}{3}=\frac{1}{6}d(d^2+11)$:

$$\begin{aligned}
&x_i=s(b_i), 1\leq i\leq d,\\
&y_{ij}=2s(b_ib_j)-s(b_i)s(b_j), 1\leq i\leq j\leq d,\\
&z_{ijk}=2(s(b_ib_jb_k)-s(b_kb_jb_i)), 1\leq i<j<k\leq d.
\end{aligned}$$

(Multiplication in A is written "ab" instead of "$a\otimes b$".

For $I=\{i_1<i_2<i_3\}\subset[n], J=\{j_1<j_2<j_3\}\subset[n]$ let

$$F_{IJ}=z_Iz_J+det(y_{i_\mu j_\nu})_{1\leq\mu,\nu\leq 3}$$

as an element of the affine coordinate ring of $\mathbf{C}^N$. Let I_d the ideal in the coordinate ring of $\mathbf{C}^N$ generated by the collection of all F_{IJ}, Z_d the set of common zeros in $\mathbf{C}^N$ of all $F\in I_d$, $S(A)$ the set of (ordinary degree 2) characters of A.

Theorem. The map p which assigns to each character $s \in S(A)$ its coordinates in $\mathbf{C}^N$ is a bijection between $S(A)$ and Z_d.

Proof. A character s on any $\mathbf{C}-$algebra A (associative, with unit element) satisfies

$$4(s(a_1a_2a_3)-s(a_3a_2a_1))(s(b_1b_2b_3)-s(b_3b_2b_1))+det(2s(a_ib_j)-s(a_i)s(b_j))_{1\leq i,j\leq 3}=0$$

for arbitrary $a_1, a_2, a_3, b_1, b_2, b_3 \in A$, so the mapping p leads into Z_d. It is also injective: Any character is determined by its values on some set of generators plus the values on their twofold and threefold products. The main part of the proof consists in showing surjectivity. We proceed as follows:

Let p a point on Z_d. Let, for $k \geq 0$

$$A_k = \coprod_{0\leq\nu\leq k} V^{\otimes\nu}$$

as a finite dimensional vector space with partial multiplicative structure inherited from $A = \coprod_{\nu\geq 0} V^{\otimes\nu}$. We take $A_2 = \mathbf{C} \oplus V \oplus V^{\otimes 2}$ and define on it, using $p \in Z_d$, a symmetric bilinear form B_p; the radical we call R_p. The quotient vector space

$$\overline{A(p)} = A_2/R_p$$

will then carry a natural structure as an associative $\mathbf{C}-$algebra with unit element enjoying the following properties:

(1) There is an algebra homomorphism

$$A \longrightarrow \overline{A(p)}$$

denoted $a \to \overline{a}$ which extends the canonical projection $A_2 \longrightarrow \overline{A(p)}$ to all of A.

(2) There is a character $\overline{s}$ on $\overline{A(p)}$ such that

$$\overline{s}(\overline{b_i}) = x_i, \qquad 1 \leq i \leq d,$$

$$\overline{s}(\overline{b}_i\overline{b}_j) = \frac{1}{2}(x_ix_j + Y_{ij}), \qquad 1 \leq i \leq j \leq d,$$

$$\overline{s}(\overline{b}_i\overline{b}_j\overline{b}_k) = \frac{1}{4}(x_ix_jx_k + x_iy_{jk} + x_jy_{ik} + x_ky_{ij} + z_{ijk}), \qquad 1 \leq i < j < k \leq d.$$

We then define

$$s : A \longrightarrow \mathbf{C}$$

by $s(a) = \overline{s}(\overline{a})$. This is a character on A with coordinates p on Z_d.

The presentation of the material is made much simpler if one allows index pairs and triples at y_{ij} and z_{ijk}, respectively, not to stand in their canonical order. We shall treat y_{ij} symmetrically and z_{ijk} alternatingly with respect to rearrangements; in particular z_{iij} is understood to be 0.

Lemma. Let $(\mu_1, \mu_2, \mu_3, \mu_4), (\nu_1, \nu_2, \nu_3, \nu_4)$ any quadruples of indices in $[d]$. We assert: $det(y_{\mu_i \nu_j})_{1 \le i,j \le 4}$ is in the radical of I_d.

Proof. We for short denote the $4 \times 4-$matrix in question by M. The algebraically adjoint matrix $M^\sim$ equals, by the polynomial relations defining Z_d:

$$M^\sim = -(m_{ij}^\sim)_{1 \le i,j \le 4}, m_{ij}^\sim = z_{I_i} z_{J_j}$$

where I_i is the ordered triple of indices from $(\mu_1, \mu_2, \mu_3, \mu_4)$ which remains after having removed μ_i; similarly J_j the triple in $(\nu_1, \nu_2, \nu_3, \nu_4)$ where ν_i is omitted. So

$$M^\sim = - \begin{pmatrix} z_{I_1} \\ -z_{I_2} \\ z_{I_3} \\ -z_{I_4} \end{pmatrix} \cdot \begin{pmatrix} z_{J_1} & -z_{J_2} & z_{J_3} & -z_{J_4} \end{pmatrix}$$

has rank at most one, so M is singular on Z_d. One can make this a little bit more precise: From

$$M \cdot M^\sim = (detM)E$$

(E the 4×4 unit matrix) we conclude

$$(detM)^4 \in I_d.$$

Lemma. Let $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ a quadruple of indices from $[d]$. Set

$$t_{\nu_1\nu_2\nu_3\nu_4\nu_5} = y_{\nu_1\nu_5} z_{\nu_2\nu_3\nu_4} - y_{\nu_2\nu_5} z_{\nu_1\nu_3\nu_4} + y_{\nu_3\nu_5} z_{\nu_1\nu_2\nu_4} - y_{\nu_4\nu_5} z_{\nu_1\nu_2\nu_3}.$$

We assert that $t_{\nu_1\nu_2\nu_3\nu_4\nu_5}$ is in the radical of the ideal I_d.

Proof. We expand the determinant of the previous lemma along the first row:

$$detM = -z_{\nu_2\nu_3\nu_4} t_{\mu_1\mu_2\mu_3\mu_4\nu_1}.$$

We use this for $(\nu_2,\nu_3,\nu_4)=(\mu_2,\mu_3,\mu_4),(\mu_1,\mu_3,\mu_4),(\mu_1,\mu_2,\mu_4)$, and (μ_1,μ_2,μ_3) and produce

$$(y_{\mu_1\nu_1}z_{\mu_2\mu_3\mu_4}-y_{\mu_2\nu_1}z_{\mu_1\mu_3\mu_4}+y_{\mu_3\nu_1}z_{\mu_1\mu_2\mu_4}-y_{\mu_4\nu_1}z_{\mu_1\mu_2\mu_3})\cdot t_{\mu_1\mu_2\mu_3\mu_4\nu_1}=t^2_{\mu_1\mu_2\mu_3\mu_4\nu_1}.$$

This so is in the radical of I_d. So the lemma is proven.

For $p\in Z_d$ fixed with coordinates

$$\begin{aligned} x_i, &\qquad 1\le i\le d,\\ y_{ij}, &\qquad 1\le i\le j\le d,\\ z_{ijk}, &\qquad 1\le i<j<k\le d \end{aligned}$$

we define, for $0\le k\le 8$ an element of the dual of $V^{\otimes k}$, denoted s_k the following way:

$$\begin{aligned} s_0(e)&=2,\\ s_1(b_i)&=x_i, \qquad 1\le i\le d,\\ s_2(b_ib_j)&=\frac{1}{2}(x_ix_j+y_{ij}), \qquad 1\le i,j\le d,\\ s_3(b_ib_jb_k)&=\frac{1}{4}(x_ix_jx_k+x_iy_{jk}+x_jy_{ik}+x_ky_{ij}+z_{ijk}), \qquad 1\le i,j,k\le d. \end{aligned}$$

For $4\le k\le 8$ we use the formula of the last chapter: Let $(\nu_1,\nu_2,...,\nu_k)$ any $k-$tuple of indices in $[d]$:

$$\begin{aligned} 2^{k-1}s_k(b_{\nu_1}b_{\nu_2}...b_{\nu_k}) = &\sum_{\pi \text{ standard on } [k]} v_\pi \prod_{\pi i=i} x_{\nu_i}\prod_{(ij)|\pi} y_{\nu_i\nu_j}\\ &+\sum_{(\pi,I) \text{ standard on } [k]} v_{(\pi,I)}\prod_{\pi i=i, i\notin I} x_{\nu_i}\prod_{(ij)|\pi} y_{\nu_i\nu_j}\ z_I \end{aligned}$$

where, for $I=\{i<j<k\}\subset[d]$ we have abbreviated $z_{\nu_i\nu_j\nu_k}$ by z_I.

Lemma. For $k\le 8$ we have

$$s_k(b_{\nu_1}b_{\nu_2}...b_{\nu_k})=s_k(b_{\nu_2}b_{\nu_3}...b_{\nu_k}b_{\nu_1}).$$

Proof. For $k\le 3$ this is clear immediately from the definition of s_k. For the remaining cases we have to deal with the formula from the theorem of part 2 explicitly. The reasoning would be trivial if the two sums from there had to be extended over all $\pi\in S_k^{(2)}$ and all pairs (π,I), respectively, with no standardness restrictions, as then the sums would be formally invariant under cyclic permutation of the indices. We show that the missing terms (the nonstandard ones) constitute elements from the radical of I_d. We restrict ourselves to $k = 5$ as here, for the first time, there are nonstandard terms of type (π, I), and to $k = 8$, as here, for the first time we have nonstandard involutions π.

In the case $k = 5$ the second sum reads

$$\sum_{(\pi,I) \text{ standard on } [5]} v_{(\pi,I)} \prod_{\pi=i, i\notin I} x_{\nu_i} \prod_{(ij)|\pi} y_{\nu_i\nu_j}\ z_I$$
$$= (x_{\nu_1}x_{\nu_2} + y_{\nu_1\nu_2})z_{\nu_3\nu_4\nu_5} + (x_{\nu_1}x_{\nu_3} - y_{\nu_1\nu_3})z_{\nu_2\nu_4\nu_5}$$
$$+(x_{\nu_2}x_{\nu_3} + y_{\nu_2\nu_3})z_{\nu_1\nu_4\nu_5} + (x_{\nu_4}x_{\nu_5} + y_{\nu_4\nu_5}z_{\nu_1\nu_2\nu_3}$$
$$+x_{\nu_1}x_{\nu_4}z_{\nu_2\nu_3\nu_5} + x_{\nu_2}x_{\nu_4}z_{\nu_1\nu_3\nu_5} + x_{\nu_3}x_{\nu_4}z_{\nu_1\nu_2\nu_5}$$
$$+x_{\nu_1}x_{\nu_5}z_{\nu_2\nu_3\nu_4} + x_{\nu_2}x_{\nu_5}z_{\nu_1\nu_3\nu_4} + x_{\nu_3}x_{\nu_4}z_{\nu_1\nu_2\nu_5}.$$

To show that this polynomial allows cyclic replacement of $(1, 2, 3, 4, 5)$ requires only to look at the formally not cyclically invariant partial sum

$$y_{\nu_1\nu_2}z_{\nu_3\nu_4\nu_5} - y_{\nu_1\nu_3}z\ z_{\nu_2\nu_4\nu_5} + y_{\nu_2\nu_3}z_{\nu_1\nu_4\nu_5} + y_{\nu_4\nu_5}z_{\nu_1\nu_2\nu_3}.$$

This expression is invariant under $(12345)^2 = (13524)$ by part 2. The discussion of the cyclicity of the second sum in the cases $k = 6, 7, 8$ is similar.

For $k = 8$ the first sum differs from the sum extended over all elements $\pi \in S_8^{(2)}$ additively by the sum over the nonstandard ones. This is a sum of 36 summands. They are collected in the following polynomial:

$$det \begin{pmatrix} y_{\nu_1\nu_5} & y_{\nu_1\nu_6} & y_{\nu_1\nu_7} & y_{\nu_1\nu_8} \\ y_{\nu_2\nu_5} & y_{\nu_2\nu_6} & y_{\nu_2\nu_7} & y_{\nu_2\nu_8} \\ y_{\nu_3\nu_5} & y_{\nu_3\nu_6} & y_{\nu_3\nu_7} & y_{\nu_3\nu_8} \\ y_{\nu_4\nu_5} & y_{\nu_4\nu_6} & y_{\nu_4\nu_7} & y_{\nu_4\nu_8} \end{pmatrix} - det \begin{pmatrix} y_{\nu_1\nu_4} & y_{\nu_1\nu_6} & y_{\nu_1\nu_7} & y_{\nu_1\nu_8} \\ y_{\nu_2\nu_4} & y_{\nu_2\nu_6} & y_{\nu_2\nu_7} & y_{\nu_2\nu_8} \\ y_{\nu_3\nu_4} & y_{\nu_3\nu_6} & y_{\nu_3\nu_7} & y_{\nu_3\nu_8} \\ y_{\nu_4\nu_5} & y_{\nu_5\nu_6} & y_{\nu_5\nu_7} & y_{\nu_5\nu_8} \end{pmatrix}$$

which is zero by part 2.

We define $s' = \sum_{0\leq k\leq 8} s_k$ the element of the dual of A_8 which is s_k on each summand $V^{\otimes k}, 0 \leq k \leq 8$. We have by the preceding discussion

$$s'(ab) = s'(ba)$$

whenever a, b, and ab are in A_8.

Let

$$B_p = B : A_2 \times A_2 \longrightarrow \mathbf{C}$$

be defined by $B(a,b) = s'(ab)$. Note that $ab \in A_4$ whenever $a, b \in A_2$. So B is a bilinear symmetric form on A. We denote its radical by R_p, and set

$$\overline{A(p)} = A_2/R_p.$$

B_p will induce a bilinear symmetric form $\overline{B_p}$ on $\overline{A(p)}$ which now is nondegenerate.

Lemma. Let $r \in R_p$. Then $s'(rab) = 0$ for all $a, b, \in A_2$. So the value of s' on any triple product of elements from A_2 depends only on the classes of the factors mod R_p.

Proof. We write

$$r = \alpha_0 + \sum_{\mu} \alpha_\mu b_\mu + \sum_{\mu,\nu} \alpha_{\mu\nu} b_\mu b_\nu$$

with $\alpha_0, \alpha_\mu, \alpha_{\mu\nu} \in \mathbf{C}$. r to be an element of the radical of B is equivalent to the collection of three types of identities:

$$\begin{aligned} I &= s'(r) = 0, \\ I_i &= s'(b_i r) = 0, 1 \leq i \leq d, \\ I_{ij} &= s'(b_i b_j r) = 0, 1 \leq i, j \leq d. \end{aligned}$$

These identities are the following:

$$I = 4\alpha_0 + 2\sum_{\mu} \alpha_\mu x_\mu + \sum_{\mu,\nu} \alpha_{\mu\nu}(x_\mu x_\nu + y_{\mu\nu}),$$

$$I_i = 4\alpha_0 x_i + 2\sum_{\mu} \alpha_\mu (x_i x_\mu + y_{i\mu}) + \sum_{\mu,\nu} \alpha_{\mu\nu}(x_i x_\mu x_\nu + x_i y_{\mu\nu} + x_\mu y_{i\nu} + x_\nu y_{i\mu} + z_{i\mu\nu}),$$

$$I_{ij} = 4\alpha_0(x_i x_j + y_{ij}) + 2\sum_{\mu} \alpha_\mu (x_i x_j x_\mu + x_i y_{j\mu} + x_j y_{i\mu} + y_{ij} x_\mu + z_{ij\mu})$$

$$+ \sum_{\mu,\nu} \alpha_{\mu\nu}(x_i x_j x_\mu x_\nu$$

$$+y_{ij}x_\mu x_\nu + x_i x_\mu y_{j\nu} + x_i x_\nu y_{j\mu}$$
$$+x_j x_\mu y_{i\nu} + x_j x_\nu y_{i\mu} + x_i x_j y_{\mu\nu}$$
$$+y_{ij}y_{\mu\nu} - y_{i\mu}y_{j\nu} + y_{i\nu}y_{j\mu}$$
$$+x_i z_{j\mu\nu} + x_j z_{i\mu\nu} + x_\mu z_{ij\nu} + x_\nu z_{ij\mu}).$$

In case $ab \in A_2$ there is nothing to prove. Besides that one has to discuss only the cases $a = b_i b_j, b = b_k$ and $a = b_i b_j, b = b_k b_l$. We discuss the first case; the second is similar but lengthier. We have

$$
\begin{aligned}
&2^4 s'(b_i b_j b_k r)\\
&= 4\alpha_0(x_i x_j x_k + x_i y_{jk} + x_j y_{ik} + x_k y_{ij} + z_{ijk}\\
&+ 2\sum_\mu \alpha_\mu (x_i x_j x_k x_\mu + x_i x_j y_{k\mu} + x_i x_k y_{j\mu} + x_j x_k y_{i\mu}\\
&+ x_i y_{jk} x_\mu + x_j y_{ik} x_\mu + x_k y_{ij} x_\mu + y_{ij} y_{k\mu} - y_{ik} y_{j\mu} + y_{jk} y_{i\mu}\\
&+ z_{ijk} x_\mu + x_i z_{jk\mu} + x_j z_{ik\mu} + x_k z_{ij\mu})\\
&+ \sum_{\mu,\nu} \alpha_{\mu\nu} (x_i x_j x_k x_\mu x_\nu\\
&+ x_i x_j x_\mu y_{k\nu} + x_i x_j x_\nu y_{k\mu} + x_i x_k x_\mu y_{j\nu} + x_i x_k x_\nu y_{j\mu} + x_j x_k x_\mu y_{i\nu} + x_j x_k x_\nu y_{i\mu}\\
&+ x_i x_j x_k y_{\mu\nu} + x_i y_{jk} x_\mu x_\nu + x_j y_{ik} x_\mu x_\nu + x_k y_{ij} x_\mu x_\nu\\
&+ x_i y_{jk} y_{\mu\nu} + x_j y_{ik} y_{\mu\nu} + x_k y_{ij} y_{\mu\nu}\\
&+ y_{ij} x_\mu y_{k\nu} + y_{ij} x_\nu y_{k\mu} - y_{ik} x_\mu y_{j\nu} - y_{ik} x_\nu y_{j\mu}\\
&- x_i y_{j\mu} y_{k\nu} + x_i y_{j\nu} y_{k\mu} - x_j y_{i\mu} y_{k\nu} + x_j y_{i\nu} y_{k\mu}\\
&- x_k y_{i\mu} y_{j\nu} + x_k y_{i\nu} y_{j\mu} + y_{jk} x_\mu y_{i\nu} + y_{jk} x_\nu y_{i\mu}\\
&+ z_{ijk}(x_\mu x_\nu + y_{\mu\nu})\\
&+ x_k x_\mu z_{ij\nu} + x_k x_\nu z_{ij\mu} + x_j x_\mu z_{ik\nu} + x_j x_\nu z_{ik\mu} + x_i x_\mu z_{jk\nu} + x_i x_\nu z_{jk\mu}\\
&+ (x_i x_j + y_{ij}) z_{k\mu\nu} + (x_i x_k - y_{ik}) z_{j\mu\nu} + (x_j x_k + y_{jk}) z_{i\mu\nu}\\
&= x_i I_{jk} + x_j I_{ik} + x_k I_{ij}\\
&- (x_i x_j - y_{ij}) I_k - (x_i x_k + y_{ik}) I_j - (x_j x_k - y_{jk}) I_i\\
&+ (x_i x_j x_k - x_i y_{jk} + x_j y_{ik} - x_k y_{ij} + z_{ijk}) I\\
&= 0.
\end{aligned}
$$

We also need a similar lemma on fourfold products.The proof runs on the same lines as the previous one:

Lemma. Let $r \in R_p$. Then $s'(rabc) = 0$ for arbitrary $a, b, c \in A_2$.

We are now in the position to give $\overline{A(p)}$ the structure of an associative $\mathbf{C}-algebra$. The class $\overline{1}$ of 1 mod R_p will be the unit element. Let $a_1, a_2 \in A_2$. The mapping

$$A_2 \ni c \longrightarrow s'(a_1 a_2 c) \in \mathbf{C}$$

is an element of the dual of A_2, and it induces an element of the dual of $\overline{A(p)}$. So there is an element $a_{12} \in A_2$ such that

$$s'(a_1 a_2 c) = B_p(a_{12}, c)$$

for all $c \in A_2$, and $\overline{a_{12}} = a_{12}\ modR_p$ is unique. By the previous $\overline{a_{12}}$ depends on $\overline{a_1}, \overline{a_2}$, only. We write

$$\overline{a_{12}} = \overline{a}_1 \overline{a}_2,$$

and show that this creates an algebra structure on $\overline{A(p)}$.

Distributivity is clear,

$$\overline{1}\overline{a} = \overline{a}$$

as well. We prove associativity. We have, for $a, b, c \in A_2$ arbitrary

$$\overline{B}(\overline{a}\overline{b}, \overline{c}) = s'((ab)c) = s'(a(bc)) = \overline{B}(\overline{a}, \overline{b}\overline{c}).$$

Let $a_1, a_2, a_3, c \in A_2$. Then

$$\overline{B}((\overline{a}_1 \overline{a}_2)\overline{a_3}, \overline{c}) = \overline{B}(\overline{a}_1 \overline{a_2}, \overline{a}_3 \overline{c}) = s'(a_1 a_2 a_3 c).$$

Similarly

$$\overline{B}(\overline{a_1}(\overline{a}_2 \overline{a_3}), \overline{c}) = \overline{B}(\overline{c}, \overline{a_1}(\overline{a}_2 \overline{a_3})) = \overline{B}(\overline{c}\overline{a}_1, \overline{a}_2 \overline{a_3}) = s'(c a_1 a_2 a_3) = s'(a_1 a_2 a_3 c)$$

for all $c \in A_2$ which proves associativity.

Proposition. Let

$$\overline{s} : \overline{A(p)} \longrightarrow \mathbf{C}$$

be defined by $\overline{s}(\overline{a}) = s'(a)$. Then $\overline{s}$ satisfies
(1)

$$\overline{s}(\overline{a}\overline{b}\overline{c}) + \overline{s}(\overline{c}\overline{b}\overline{a}) - \overline{s}(\overline{a})\overline{s}(\overline{b}\overline{c}) - \overline{s}(\overline{b})\overline{s}(\overline{a}\overline{c}) - \overline{s}(\overline{c})\overline{s}(\overline{a}\overline{b}) + \overline{s}(\overline{a})\overline{s}(\overline{b})\overline{s}(\overline{c}) = 0,$$

(2)

$$\overline{s}(\overline{e}) = 2.$$

The proof is elementary, although it requires some lengthy computations. We present it for $\overline{a} = \overline{b_i}, \overline{b} = \overline{b_j}, \overline{c} = \overline{b}_k \overline{b}_l$ with b_i, b_j, b_k, b_l from the basis of V.

$$2^3[s'(b_i b_j (b_k b_l)) + s'((b_k b_l) b_j b_i) - s'(b_i) s'(b_j b_k b_l) - s'(b_j) s'(b_i b_k b_l) - s'(b_i b_j) s'(b_k b_l) + s'(b_i) s'(b_j) s'(b_k b_l)]$$

$$
\begin{aligned}
&= x_i x_j x_k x_l \\
&+ x_i x_j y_{kl} + x_i x_k y_{jl} + x_i x_l y_{jk} \\
&+ x_j x_k y_{il} + x_j x_l y_{ik} + x_k x_l y_{ij} \\
&+ y_{ij} y_{kl} - y_{ik} y_{jl} + y_{il} y_{jk} \\
&+ x_i z_{jkl} + x_j z_{ikl} + x_k z_{ijl} + x_l z_{ijk} \\
&+ x_i x_j x_k x_l \\
&+ x_i x_j y_{kl} + x_i x_k y_{jl} + x_i x_l y_{jk} \\
&+ x_j x_k y_{il} + x_j x_l y_{ik} + x_k x_l y_{ij} \\
&+ y_{ij} y_{kl} - y_{il} y_{kj} + y_{ik} y_{jl} \\
&+ x_i z_{jkl} + x_j z_{ikl} - x_k z_{ijl} - x_l z_{ijk} \\
&- 2x_i (x_j x_k x_l + x_j y_{kl} + x_k y_{jl} + x_l y_{jk} + z_{jkl}) \\
&- 2x_j (x_i x_k x_l + x_i y_{kl} + x_k y_{il} + x_l y_{ik} + z_{ikl}) \\
&- 2(x_i x_j + y_{ij})(x_k x_l + y_{kl}) + 4 x_i x_j (x_k x_l + y_{kl}) \\
&= 0.
\end{aligned}
$$

The second statement of the proposition is obvious..

We have a natural epimorphism

$$A = \bigotimes V \ni a \longrightarrow \overline{a} \in \overline{A(p)};$$

it is defined by

$$\overline{b_{\nu_1} b_{\nu_2} ... b_{\nu_k}} = \overline{b}_{\nu_1} \overline{b_{\nu_2}} ... \overline{b_{\nu_k}}$$

for any $k \geq 0$ and any index-k-tuple $(\nu_1, \nu_2, ..., \nu_k)$ from $[d]$.

We then have proven the following

Theorem. Let p a point on Z_d and

$$s : A \longrightarrow \mathbf{C}$$

be defined by

$$s(a) = \overline{s}(\overline{a}).$$

Then s is a character on A.

There is, of course, a group theoretical analogue to the algebra case treated here. We state the result, omitting details, and give an application. We deal with unimodular characters, only.

Definition. A (degree two unimodular ordinary) character on a group G is a function

$$s : G \longrightarrow \mathbf{C}$$

satisfying

(1)

$$s(g_1g_2g_3)+s(g_3g_2g_1)-s(g_1)s(g_2g_3)-s(g_2)s(g_1g_3)-s(g_3)s(g_1g_2)+s(g_1)s(g_2)s(g_3) = 0,$$

(2)

$$s(g^2) = s^2(g) - 2,$$

(3)

$$s(e) = 1$$

for arbitrary $g, g_1, g_2, g_3 \in G$ and e the neutral element in G.

We may pass over to the complex group algebra $\mathbf{C}G$ and use the results from the algebra case in order to show that

$$4(s(g_1g_2g_3)-s(g_3g_2g_1))(s(h_1h_2h_3)-s(h_3h_2h_1))+det(2s(g_ih_j)-s(g_i)s(h_j))_{1\leq i,j\leq 3} = 0$$

for any group character s. We add some easy observation:

Lemma. Let s a character on the group G. Then

a)

$$G \ni g \longrightarrow s(g) \in \mathbf{C}$$

is invariant under inversion of g: $s(g^{-1}) = s(g)$.

b)

$$G \times G \ni (g_1, g_2) \longrightarrow 2s(g_1g_2) - s(g_1)s(g_2) \in \mathbf{C}$$

depends symmetrically on the pair (g_1, g_2) and changes its sign if g_1 (equivalently g_2) is replaced by its inverse.

c)

$$G \times G \times G \ni (g_1, g_2, g_3) \longrightarrow s(g_1g_2g_3) - s(g_3g_2g_1) \in \mathbf{C}$$

depends skewly on the group triple (g_1, g_2, g_3); it changes its sign if (exactly) one of the entries is inverted.

All this is classical: a) follows from (1) by setting $g_1 = g_2 = g, g_3 = g^{-1}$, b) by setting $g_2 = g_1, g_3 = g_2^{-1}$, and c) from a combination of these cases.

We deal, in analogy to the algebra case with a free group G of finite rank $d \geq 3$ and fix a set $\{g_1, g_2, ..., g_d\}$ of generators. In $\mathbf{C}^N$ with $N = \frac{1}{6}d(d^2+11)$ we again denote the coordinates by

$$\begin{gathered} x_i, 1 \leq i \leq d, \\ y_{ij}, 1 \leq i \leq j \leq d, \\ z_{ijk}, 1 \leq i < j < k \leq d. \end{gathered}$$

For $I = \{i_1 < i_2 < i_3\}, J = \{j_1 < j_2 < j_3\} \subset [d]$ we take, as above

$$F_{IJ} = z_I z_J + det(y_{i_\mu j_\nu})_{1 \leq \mu,\nu \leq 3}$$

and additionally

$$F_i = y_{ii} - x_i^2 + 4$$

for $1 \leq i \leq d$. Let I_d the ideal in the coordinate ring of $\mathbf{C}^N$ generated by the collection of all F_{IJ}, F_i, and Z_d the algebraic set defined by I_d. The coordinates of a character s are the coordinates of the point $p \in \mathbf{C}^N$ with

$$\begin{gathered} x_i = s(g_i), 1 \leq i \leq d, \\ y_{ij} = 2s(g_i g_j) - s(g_i)s(g_j), 1 \leq i \leq j \leq d, \\ z_{ijk} = 2(s(g_i g_j g_k) - s(g_k g_j g_i)), 1 \leq i < j < k \leq d. \end{gathered}$$

This point lies on Z_d. (It should be noted that at this point there is no advantage in eliminating y_{ii} by $x_i^2 - 4$.) Then we have the

Theorem. The mapping p described in () identifies the set of (ordinary unimodular degree two) characters on the group G which is free on $\{g_1, g_2, ..., g_d\}$ to the affine algebraic set $Z_d \subset \mathbf{C}^N$.

The proof goes on the lines of the proof in the algebra case after having passed over to the group algebra $\mathbf{C}G$ of G. This algebra is, of course, not free on the generators $g_1, g_2, ..., g_d$. Fortunately lemma () allows to observe the effect on some character value on a product which contains negative exponents. So, the formula () of the preceding paragraph may, in an obvious way, be used again to compute such values.

We present an example. Free groups (of finite rank $d \geq 2$) play a rôle in topology, Teichmüller theory, hyperbolic geometry, a.s.o. Let G be free on

$\{g_1, g_2, g_3, g_4\}$. So $G \cong \pi_1(F_2 \bullet)$, the fundamental group of a closed oriented surface punctured once. The generators may be realized topologically in a way such that the element $\lambda = g_1 g_2 g_3 g_4 g_1^{-1} g_2^{-1} g_3^{-1} g_4^{-1}$ is represented by a loop around the puncture' One then is interested in (discrete faithful) $SL_2(\mathbf{C})-$representations of G in which λ becomes a parabolic matrix L with trace -2. This is an algebraic condition on the characters s on G.

We have computed, using the 8-fold product formula, for s any character on G, $s(\lambda)$ in terms of the coordinates of s. This is the result:

$$
\begin{gathered}
2^7 s(g_1 g_2 g_3 g_4 g_1^{-1} g_2^{-1} g_3^{-1} g_4^{-1}) = \\
x_1^2 x_2^2 x_3^2 x_4^2 \\
-x_2^2 x_3^2 x_4^2 y_{11} - x_1^2 x_3^2 x_4^2 y_{22} - x_1^2 x_2^2 x_4^2 y_{33} - x_1^2 x_2^2 x_3^3 y_{44} \\
-8 x_1 x_2 x_3 x_4 y_{12} y_{34} + 8 x_1 x_2 x_3 x_4 y_{14} y_{23} \\
+2(x_3^2 x_4^2 y_{12}^2 + x_2^2 x_4^2 y_{13}^2 + x_2^2 x_3^2 y_{14}^2 + x_1^2 x_4^2 y_{23}^2 + x_1^2 x_3^2 y_{24}^2 + x_1^2 x_2^2 y_{34}^2) \\
-x_3^2 x_4^2 y_{11} y_{22} - x_2^2 x_4^2 y_{11} y_{33} - x_2^2 x_3^2 y_{11} y_{44} - x_1^2 x_4^2 y_{22} y_{33} - x_1^2 x_3^2 y_{22} y_{44} - x_1^2 x_2^2 y_{33} y_{44} \\
+4(-x_2 x_3 x_4^2 y_{11} y_{23} - x_2 x_3^2 x_4 y_{11} y_{24} - x_2^2 x_3 x_4 y_{11} y_{34} + x_1 x_3 x_4^2 y_{22} y_{13} + x_1 x_3^2 x_4 y_{22} y_{14} - x_1^2 x_3 x_4 y_{22} y_{34}) \\
+4(-x_1 x_2 x_4^2 y_{33} y_{12} + x_1 x_2^2 x_4 y_{33} y_{14} + x_1^2 x_2 x_4 y_{33} y_{24} - x_1 x_2 x_3^2 y_{44} y_{12} - x_1 x_2^2 x_3 y_{44} y_{13} - x_1^2 x_2 x_3 y_{44} y_{23}) \\
+4(-x_1 x_3^2 x_4 y_{12} y_{24} + x_2 x_3 x_4^2 y_{12} y_{13} + x_2 x_3^2 x_4 y_{12} y_{14} - x_1 x_2^2 x_4 y_{13} y_{34} + x_2^2 x_3 x_4 y_{13} y_{14} + x_1 x_2 x_4^2 y_{13} y_{23}) \\
+4(x_1 x_2 x_3^2 Y_{14} y_{24} + x_1 x_2^2 x_3 y_{14} y_{34} - x_1 x_3 x_4^2 y_{12} y_{23} - x_1^2 x_2 x_4 y_{23} y_{34} + x_1^2 x_3 x_4 y_{23} y_{24} + x_1^2 x_2 x_3 y_{24} y_{34}) \\
+4(x_4^2 y_{12} y_{13} y_{23} + x_3^2 y_{12} y_{14} y_{24} + x_2^2 y_{13} y_{14} y_{34} + x_1^2 y_{23} y_{24} y_{34}) \\
-2(x_4^2 y_{12}^2 y_{33} + x_4^2 y_{23}^2 y_{11} + x_4^2 y_{13}^2 y_{22} + x_3^2 y_{12}^2 y_{44} + x_3^2 y_{24}^2 y_{11} + x_3^2 y_{14}^2 y_{22}) \\
-2(x_2^2 y_{13}^2 y_{44} + x_2^2 y_{34}^2 y_{11} + x_2^2 y_{14}^2 y_{23} + x_1^2 y_{23}^2 y_{44} + x_1^2 y_{34}^2 y_{22} + x_1^2 y_{24}^2 y_{33}) \\
+x_4^2 y_{11} y_{22} y_{33} + x_3^2 y_{11} y_{22} y_{44} + x_2^2 y_{11} y_{33} y_{44} + x_1^2 y_{22} y_{33} y_{44} \\
+8(x_3 x_4 y_{12} y_{14} y_{23} - x_1 x_4 y_{12} y_{23} y_{34} + x_2 x_3 y_{12} y_{14} y_{34} + x_1 x_2 y_{14} y_{23} y_{34}) \\
+4(-x_3 x_4 y_{13} y_{14} y_{22} - x_3 x_4 y_{23} y_{24} y_{11} + x_1 x_4 y_{12} y_{24} y_{33} + x_1 x_4 y_{13} y_{34} y_{22} - x_2 x_3 y_{12} y_{13} y_{44} - x_2 x_3 y_{24} y_{34} y_{11}) \\
+4(-x_1 x_2 y_{13} y_{23} y_{44} - x_1 x_2 y_{14} y_{24} y_{33} - x_1 x_3 y_{12} y_{23} y_{44} + x_1 x_3 y_{14} y_{34} y_{22} - x_2 x_4 y_{23} y_{34} y_{11} + x_2 x_4 y_{12} y_{14} y_{33}) \\
2(y_{12} y_{14} y_{24} y_{33} + y_{13} y_{14} y_{34} y_{22} + y_{12} y_{13} y_{23} y_{44} + y_{23} y_{24} y_{34} y_{11}) \\
-6(y_{13} y_{14} y_{23} y_{24} + y_{12} y_{13} y_{24} y_{34}) \\
+3(y_{14}^2 y_{23}^2 + y_{12}^2 y_{34}^2 + y_{13}^2 y_{24}^2) \\
+2 y_{12} y_{14} y_{23} y_{34} \\
-(y_{14}^2 y_{22} y_{33} + y_{12}^2 y_{33} y_{44} + y_{13}^2 y_{22} y_{44} + y_{34}^2 y_{11} y_{22} + y_{23}^2 y_{11} y_{44} + y_{24}^2 y_{11} y_{33}).
\end{gathered}
$$

Observe the absence of any z-coordinate; this is to be expected since inverting all factors in λ conjugates it, so this will have no effect on $s(\lambda)$.

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