Local to Global Structure in Twin Buildings

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1 Introduction

Twin buildings are a generalization of spherical buildings, and the present paper shows that the local structure of a twin building with no "tree residues" determines its global structure (subject to a mild restriction on four cases of finite residues). This is a generalization of the same result for spherical buildings given by J. Tits in [Ti74](Theorem 4.1.2). That theorem is crucial in the classification of spherical buildings, and the present paper is a step towards an eventual classification of twin buildings.

The sense in which twin buildings generalize spherical buildings is the following. If W denotes the Coxeter group of a building, then spherical buildings are those for which W is finite. In such cases there exists a unique element w_0 of longest length in W, and when the W-distance between two chambers is w_0 one says they are *opposite* (they are then antipodal to one another in the unique spherical apartment containing both). When W is infinite there is no element of longest length, hence no opposites. However in a twin building there is the idea of a chamber in one of the two buildings being opposite a chamber in the other. This feature is essential to our work.

We mentioned above that our twin buildings have no "tree residues". This means that if s_i and s_j are distinguished generators of the Coxeter group W, then the order m_{ij} of $s_i s_j$ is finite. Without that restriction each building would contain rank 2 residues which are trees, and we would lose control of the global structure because there are non-isomorphic twinnings of a given tree. Twin trees have special features of their own, and a start on the study of such objects is made in [RT94].

Main Theorems

As mentioned above one of the main results in [Ti74] asserts that a building of spherical type is determined by its local structure. The techniques leading to a proof of this result can be applied also to twin buildings whose diagrams have no entry equal to infinity. This fact motivated Conjecture 1 in [Ti90], which we state here in a slightly different version. For the notation the reader is referred to Sections 2 and 3.

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Conjecture: *Let* $(\Delta = (\Delta_+, \Delta_-, \delta_*)$, $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ *be thick twin buildings of type* M, *where* $m_{ij} \neq \infty$ *for any* $i, j \in I$. *Given two pairs of opposite chambers* $(c_+, c_-) \in \Delta_+ \times \Delta_-$, $(c'_+, c'_-) \in \Delta'_+ \times \Delta'_-$ *and an isometry from* $E_2(c_+) \cup \{c_-\}$ *onto* $E_2(c'_+) \cup \{c'_-\}$, *then there is a unique extension to an isometry from* Δ *onto* Δ'.

A step towards a proof of the conjecture above is made in [Ti90], where the following result is stated.

Theorem 1.1: *The assumptions being as in the conjecture above, each isometry from* $E_2(c_+) \cup \{c_-\}$ *onto* $E_2(c'_+) \cup \{c'_-\}$ *extends to an isometry from* $\Delta_+ \cup \{c_-\}$ *onto* $\Delta'_+ \cup \{c'_-\}$.

The theorem above is proved in [Ti90] under the additional hypothesis, that each panel contains a sufficiently large number of chambers. A slightly different approach [Ro] shows that this hypothesis can be dropped. See also the remarks in [Ti90] 5.7.

Theorem 1.1 says that the local structure determines Δ_+ and - with some additional arguments - also Δ_-; however, it is not clear at this point, whether the twinning δ_* is determined as well. It is not even clear whether the extension in Theorem 1.1 is unique (see 5.9 (a) in [Ti90]). In this paper we prove that the extension is indeed unique, if the set of chambers opposite to a given chamber is connected, which is our condition (co). This is then used to prove the conjecture when Δ satisfies (co).

All buildings considered in this paper are assumed to have no "tree residues", i.e. there is no entry equal to infinity in the diagram. Our results are the following:

Theorem 1.2: *If* Δ *satisfies (co), then the extension in Theorem 1.1 is unique.*

The proof will be given at the end of Section 4.

Theorem 1.3: *If* Δ *satisfies (co), then any isometry from* $\Delta_+ \cup \{c_-\}$ *onto* $\Delta'_+ \cup \{c'_-\}$ *extends uniquely to an isometry from* Δ *onto* Δ'.

Theorem 1.3 will be proved in Section 5.

Corollary 1.4: *If* Δ *satisfies (co), then the local structure determines the global structure.*

Our last result is a local criterion for the validity of condition (co), which is probably well known for spherical buildings. By assumption all rank 2 residues are spherical, and since (co) is satisfied for most rank 2 buildings (see below), the following theorem gives a useful criterion for Δ to satisfy (co).

Theorem 1.5: *If every rank 2 residue satisfies (co), then* Δ *satisfies (co).*

This will be proved in Section 6. As a consequence of Theorems 1.1,1.3 and 1.5 we obtain the following.

Corollary 1.6: *If every rank 2 residue satisfies (co), then the local structure* $E_2(c_+)$ *determines the global structure.*

We now ask about the validity of (co) for rank 2 buildings. Here are some facts.

(1) All rank 2 buildings of types $A_1 \times A_1$ and A_2 satisfy (co). This is a straightforward exercise.

(2) All rank 2 buildings of type B_2 except the unique example with three chambers per panel (i.e. that for the group $Sp_4(2)$) satisfy (co) (see [Br93]).

(3) All finite rank 2 buildings except those for the groups $Sp_4(2), G_2(2), G_2(3)$ and ${}^2F_4(2)$ satisfy (co) (see [Br93]).

(4) Theorem 1.1 implies that all rank 2 residues must be Moufang (see [Ti90] Corollary 5.6).

(5) All Moufang rank 2 buildings except the four listed in (3) satisfy (co). By our first and second remark, it remains to consider the Moufang hexagons and octagons which are not contained in the list of Remark (3). For such buildings condition (co) has been checked by P. Abramenko using a group theoretic argument (private communication).

Using Corollary 1.6 and the previous remarks we obtain the following.

Corollary 1.7: *If Δ has no residues of type $Sp_4(2), G_2(2), G_2(3)$ or ${}^2F_4(2)$, then an isometry from $E_2(c_+) \cup \{c_-\}$ onto $E_2(c'_+) \cup \{c'_-\}$ extends uniquely to an isometry from Δ onto Δ'. In particular the local structure $E_2(c_+)$ determines the global structure.*

It should be mentioned that there are twin buildings for which condition (co) is not satisfied. We will give a series of examples at the end of section 6. Our proof of the conjecture above relies heavily on the condition (co) and it seems that our arguments cannot be modified to settle the general case. However, assuming a weaker form of condition (co), which is satisfied for all affine twin buildings, we can give a similar but more elaborated proof for the conjecture.

In Section 2 we recall the basic definitions about buildings in order to fix notation and to state some preliminary results. We introduce twin buildings in Section 3 along the lines of [Ti90]. The reader may also find additional information about twin buildings in [Ti88/89] and [Ti89/90]. In Section 4 we consider isometries between twin buildings and prepare the proof of Theorem 1.3, which will be completed in section 5. Section 6 concerns condition (co).

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2 Preliminaries

Throughout this paper I denotes a finite set and $M = (m_{ij})_{i,j\in I}$ a Coxeter matrix over I. For a set $J \subseteq I$ we set $M_J = (m_{ij})_{i,j\in J}$.

The pair $(W, (s_i)_{i\in I})$ denotes the Coxeter system of type M. For $J \subseteq I$ we set $W_J = \langle s_j | j \in J\rangle \leq W$. The pair $(W_J, (s_j)_{j\in J})$ is the Coxeter system of type M_J (cf. [Bo68]).

The length of $w \in W$ relative to the generating set $S = \{s_i | i \in I\}$ is denoted by $l(w)$. A set $J \subseteq I$ is called *spherical* if W_J is finite. Given a spherical subset of I there exists a unique element of maximal length in W_J, which we denote by r_J.

A *building* of type M is a set Δ endowed with a *distance function* $\delta : \Delta \times \Delta \longrightarrow W$ satisfying the following axioms where $x, y \in \Delta$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in \Delta$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The elements of Δ are called *chambers*. For any two chambers x and y we set $l(x, y) = l(\delta(x, y))$. Two chambers x, y are called *i-adjacent* if $\delta(x, y) = s_i$; they are called *adjacent* if they are i-adjacent for some $i \in I$. A *gallery* joining x and y is a sequence of chambers $x = x_0, x_1, \ldots, x_k = y$ such that x_l and x_{l+1} are adjacent for any $0 \leq l < k$; the number k is called the *length* of the gallery.

A subset X of Δ is called *connected* if for any two $x, y \in X$ there exists a gallery $x = x_0, x_1, \ldots, x_k = y$ such that $x_l \in X$ for all $0 \leq l \leq k$. A connected component of $X \subseteq \Delta$ is defined in the usual way.

Given a set $J \subseteq I$ and $x \in \Delta$, the *J-residue* of x is the set $R_J(x) = \{y \in \Delta | \delta(x, y) \in W_J\}$. Each J-residue is a building of type M_J with the distance function induced by δ (cf [Ro89];p.30).

A *residue* is a subset R of Δ such that there exist $J \subseteq I$ and $x \in \Delta$ with $R = R_J(x)$; the set J is called the *type* of R; the rank of R is defined to be the cardinality of J. A *panel* is a residue of rank 1.

The building Δ is called *thick* if each panel contains at least three chambers.

Let J be a spherical subset of I and let R be a J-residue of Δ. Two chambers $x, y \in R$ are called *opposite* in R if $\delta(x, y) = r_J$.

For $x \in \Delta$ and $k \in N$ we set $E_k(x) = \bigcup_{|J|\leq k} R_J(x)$.

Given $x \in \Delta$ and a J-residue $R \subseteq \Delta$, then there exists a unique chamber $z \in R$ such that $l(x, y) = l(x, z) + l(z, y)$ for any $y \in R$ (cf [Ti74] 3.19.6). The chamber z is called the *projection* of x onto R and is denoted by $proj_R x$.

Let $(\Delta, \delta), (\Delta', \delta')$ be two buildings of type M and let $X \subseteq \Delta, X' \subseteq \Delta'$. A mapping $\varphi : X \longrightarrow X'$ is called an *isometry* if it is bijective and if $\delta'(\varphi(x), \varphi(y)) = \delta(x, y)$ for all $x, y \in X$.

3 Twin Buildings

A *twin building* of type M is a pair of buildings (Δ_+, δ_+) and (Δ_-, δ_-) of type M endowed with a *codistance function* $\delta_* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \longrightarrow W$ satisfying the following axioms, where $\epsilon \in \{+, -\}, x \in \Delta_\epsilon, y \in \Delta_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw 1) $\delta_*(y, x) = w^{-1}$;

(Tw 2) if $z \in \Delta_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw 3) if $s \in S$, there exists $z \in \Delta_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

Throughout this section let $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta_*)$ be a twin building.

We set $\Delta = \Delta_+ \cup \Delta_-$. Let $\epsilon : \Delta \longrightarrow \{+, -\}$ be the map which indicates whether $x \in \Delta$ belongs to Δ_+ or Δ_-; we call $\epsilon(x)$ the *sign* of x. If $x, y \in \Delta$ have different signs we set $l_*(x, y) = l(\delta_*(x, y))$. We define the mapping $\delta : \Delta \times \Delta \longrightarrow W$ by setting $\delta(x, y) = \delta_*(x, y)$ if $\epsilon(x) \neq \epsilon(y)$ and $\delta(x, y) = \delta_{\epsilon(x)}(x, y)$ otherwise.

A J-residue in Δ is a J-residue in Δ_+ or Δ_-; the sign of a residue is $+$ or $-$ according to whether it lies in Δ_+ or Δ_-.

Two chambers x, y are called *opposite* if they have different signs and if $\delta_*(x, y) = 1$; two residues R, T are called *opposite* if they have the same type and if there exist two chambers $x \in R$ and $y \in T$ which are opposite in Δ. If x is a chamber, then x^{op} denotes the set of chambers which are opposite to x.

The following lemma is an immediate consequence of the axioms for a twinning:

Lemma 3.1: *Let $w \in W$ and let $x, y \in \Delta$ have different signs, then the following are equivalent:*

(1) $\delta_*(x, y) = w$.

(2) There exists $z_0 \in x^{op}$ such that $\delta(z_0, y) = w$ and $l(w) = l(z_0, y) \leq l(z, y)$ for each $z \in x^{op}$.

The following observation is an easy consequence of Proposition 3 (iii) in [Ti90]:

Lemma 3.2: *Let $x, y \in \Delta$ be such that $x^{op} = y^{op}$. Then $x = y$.*

Let J be a spherical subset of I, let R be a J-residue in Δ_ϵ and let $x \in \Delta_{-\epsilon}$. Then there exists a unique chamber $z \in R$ such that $l_*(x, y) = l_*(x, z) - l_\epsilon(z, y)$ for each $y \in R$. The chamber z is called the projection of x onto R; it will be denoted by $proj_R x$. Note that $proj_R x$ is defined for each pair (R, x) consisting of a spherical residue $R \subseteq \Delta$ and a chamber $x \in \Delta$.

We close this section with two lemmas which are easy generalizations of the corresponding results in [Ti74](Proposition 3.29 and Theorem 3.28).

Lemma 3.3: *Let x, y be two opposite chambers in Δ and let R be a spherical residue containing x. Then $proj_R y$ is opposite to x in R.*

Lemma 3.4: *Let R, T be opposite spherical residues. Then $proj_R |_T$ and $proj_T |_R$ are inverse bijections.*

4 Isometries

Throughout this section M is a Coxeter diagram over a finite set I satisfying $m_{ij} \neq \infty$ and Δ, Δ' are thick twin buildings of type M.

Definition 4.1: Let $X \subseteq \Delta$ and $X' \subseteq \Delta'$. A mapping $\varphi : X \longrightarrow X'$ is called an *isometry* if the following conditions are satisfied:

(Is 1) φ preserves signs;

(Is 2) φ is bijective;

(Is 3) $\delta'(\varphi(x), \varphi(y)) = \delta(x, y)$ for any two chambers $x, y \in X$.

Lemma 4.2: *Let $R \subseteq \Delta, R' \subseteq \Delta'$ be spherical residues of the same type and $\varphi : R \longrightarrow R'$ be an isometry. Let $(x, x') \in \Delta \times \Delta'$ be such that the map $x \to x'$ extends φ to an isometry from $\{x\} \cup R$ onto $\{x'\} \cup R'$. Then $\varphi(proj_R x) = proj_{R'} x'$.*

Proof: Clearly, φ preserves the numerical distance l and the numerical codistance l_*. Thus the lemma follows from the characterization of projections given in the previous sections. $\square$

Proposition 4.3: *Let J be a spherical subset of I and let $(c_+, c_-) \in \Delta_+ \times \Delta_-, (c'_+, c'_-) \in \Delta'_+ \times \Delta'_-$ be pairs of opposite chambers. Let R, T, R' and T' denote the J-residues of c_+, c_-, c'_+ and c'_- respectively. Given an isometry*

$$\varphi : R \cup T \longrightarrow R' \cup T',$$

then $\varphi \mid_T = proj_{T'} \circ \varphi \circ proj_R$.

Proof: It follows from Lemma 4.2 that $\varphi \circ proj_T = proj_{T'} \circ \varphi$ and from Lemma 3.4 we infer that $\varphi \mid_T = \varphi \circ proj_T \circ proj_R$ and hence the claim follows. $\square$

Corollary 4.4: *Let $J, c_+, c_-, c'_+, c'_-, R, T, R', T'$ and φ be as in the previous proposition. Given an isometry $\psi : R \cup T \longrightarrow R' \cup T'$ such that $\varphi \mid_R = \psi \mid_R$, then $\varphi = \psi$.*

Proposition 4.5: *Given two pairs of opposite chambers $(c_+, c_-) \in \Delta_+ \times \Delta_-, (c'_+, c'_-) \in \Delta'_+ \times \Delta'_-$ and an isometry*

$$\varphi : \{c_-\} \cup E_2(c_+) \longrightarrow \{c'_-\} \cup E_2(c'_+),$$

then φ has a unique extension to an isometry

$$\varphi^* : E_2(c_-) \cup E_2(c_+) \longrightarrow E_1(c'_-) \cup E_2(c'_+).$$

The proposition above is stated in [Ti90] in section 5.8.1 and a proof is given in [Ro]. It is the generalization to twin buildings of Lemma 4.13 in [Ti74]. The arguments yielding Lemma 4.13 in [Ti74] generalize to the situation of twin buildings but are somewhat involved.

Lemma 4.6: *Given c_+, c_-, c'_+, c'_- and φ be as in 4.5, the map φ has a unique extension to an isometry*

$$\varphi^* : E_1(c_-) \cup E_2(c_+) \longrightarrow E_1(c'_-) \cup E_2(c'_+).$$

Proof: The existence of φ^* is immediate from 4.5; the uniqueness follows from Corollary 4.4. □

Given an isometry $\varphi : \{c_-\} \cup \Delta_+ \longrightarrow \{c'_-\} \cup \Delta'_x$ and $x \in c_-^{op}$, we define $\varphi_x = \varphi \mid_{\{c_-\} \cup E_2(x)}$. Lemma 4.6 gives us an isometry

$$\varphi^x : E_1(c_-) \cup E_2(x) \longrightarrow E_1(c'_-) \cup E_2(\varphi(x))$$

extending φ_x. We wish to consider whether $\varphi^x \mid_{E_1(c_-)} = \varphi^y \mid_{E_1(c_-)}$ for $x, y \in c_-^{op}$.

Proposition 4.7: *With the notation above, if x and y are adjacent, then*

$$\varphi^x \mid_{E_1(c_-)} = \varphi^y \mid_{E_1(c_-)} .$$

Proof: We set $x' = \varphi^x(x) = \varphi(x) = \varphi^y(x)$. Let $i \in I$ and let P, Q, P' and Q' be the $\{i\}$-residue of x, c_-, x' and c'_-, respectively. It follows that $\varphi(P) = P', \varphi^x(Q) = Q' = \varphi^y(Q)$ and $\varphi^x \mid_P = \varphi \mid_P = \varphi^y \mid_P$. Thus $\varphi^x \mid_{P \cup Q}$ and $\varphi^y \mid_{P \cup Q}$ are isometries from $P \cup Q$ onto $P' \cup Q'$ which agree on P. By Corollary 4.4 φ^x and φ^y also agree on Q and hence on $E_1(c_-)$. □

Corollary 4.8: *If x, y lie in the same connected component of c_-^{op}, then $\varphi^x \mid_{E_1(c_-)} = \varphi^y \mid_{E_1(c_-)}$.*

Lemma 4.9: *Given a map $\varphi : \{c_-\} \cup \Delta_+ \longrightarrow \{c'_-\} \cap \Delta'_+$ such that*

(i) $\varphi \mid_{\Delta_+}$ is an isometry of Δ_+ onto Δ'_+, and

(ii) φ maps c_-^{op} to ${c'_-}^{op}$,

then φ is an isometry.

Proof: This follows from the characterization of the codistance given in Lemma 3.1. □

Lemma 4.10: *Given chambers $x_- \in \Delta_-$ and $x'_-, y'_- \in \Delta'_-$, and two isometries $\varphi : \{x_-\} \cup \Delta_+ \longrightarrow \{x'_-\} \cup \Delta'_+$ and $\psi : \{x_-\} \cup \Delta_+ \longrightarrow \{y'_-\} \cup \Delta'_+$ such that $\varphi \mid_{x_-^{op}} = \psi \mid_{x_-^{op}}$ then $x'_- = y'_-$ and $\varphi = \psi$.*

Proof: This is an immediate consequence of Lemma 3.2. □

Lemma 4.11: *Let J be a spherical subset of I, let $(c_+, c_-) \in \Delta_+ \times \Delta_-, (c'_+, c'_-) \in \Delta'_+ \times \Delta'_-$ be pairs of opposite chambers and let R and R' denote the J-residues of c_+ and c'_+ respectively. If two isometries φ, ψ from $\{c_-\} \cup R$ onto $\{c'_-\} \cup R'$ agree on $E_1(c_+) \cap R$, then they are equal.*

Proof: By Lemma 3.3 $proj_R c_-$ is opposite to c_+ in R and by Lemma 4.2 we have $\varphi(proj_R c_-) = proj_{R'} \varphi(c_-) = proj_{R'} c'_- = proj_{R'} \psi(c_-) = \psi(proj_R c_-)$. Hence $\varphi \mid_R$ and $\psi \mid_R$ are isometries from R onto R' which agree on $E_1(c_+) \cap R$ and on a chamber which is opposite to c_+ in R. The assertion follows now from Theorem 4.1.1. in [Ti74]. $\square$

Corollary 4.12: *If two isometries $\varphi, \psi : E_2(c_+) \cup \{c_-\} \longrightarrow E_2(c'_+) \cup \{c'_-\}$ agree on $E_1(c_+)$, then they are equal.*

Proposition 4.13: *Given isometries $\varphi, \psi : \Delta_+ \cup \{c_-\} \longrightarrow \Delta'_+ \cup \{c'_-\}$ and two chambers x and y in the same connected component of c_-^{op}, then if φ and ψ agree on $E_2(x)$, they also agree on $E_2(y)$.*

Proof: It suffices to consider the case where x and y are adjacent, the general statement then following by induction on the length of a gallery joining x and y in c_-^{op}. Since $E_1(y) \subseteq E_2(x)$ it follows that φ and ψ agree on $E_1(y) \cup \{c_-\}$ and hence on $E_2(y)$ by the previous corollary. $\square$

Corollary 4.14: *Suppose c_-^{op} is connected. If two isometries $\varphi, \psi : \{c_-\} \cup \Delta_+ \longrightarrow \{c'_-\} \cup \Delta'_+$ agree on $E_2(x)$ for a chamber $x \in c_-^{op}$, then they are equal.*

Proof: Since c_-^{op} is connected, the mappings φ and ψ agree on c_-^{op} by the previous proposition. The assertion follows now from Lemma 4.10. $\square$

Proof of Theorem 1.2: Theorem 1.2 is immediate from Corollary 4.14.

5 Proof of Theorem 1.3

Throughout this section M is a Coxeter diagram over a finite set I satisfying $m_{ij} \neq \infty$ and Δ, Δ' are thick twin buildings of type M. Moreover we assume the following:

(co) the set of chambers opposite to a given chamber in Δ (resp. Δ') is connected.

Proposition 5.1: *Every isometry $\varphi : \{c_-\} \cup \Delta_+ \longrightarrow \{c'_-\} \cup \Delta'_+$ extends to an isometry $\bar{\varphi} : E_1(c_-) \cup \Delta_+ \longrightarrow E_1(c'_-) \cup \Delta'_+$.*

Proof: To simplify notation, write $x' = \varphi(x)$ for all $x \in \Delta_+$. For $x \in c_-^{op}$ let $\varphi^x : E_1(c_-) \cup E_2(x) \longrightarrow E_1(c'_-) \cup E_2(x')$ be the unique isometry extending $\varphi \mid_{\{c_-\} \cup E_2(x)}$ (Lemma 4.6). Since we assume that c_-^{op} is connected, Corollary 4.8 shows that φ^x and φ^y agree on $E_1(c_-)$ for any two $x, y \in c_-^{op}$. This common restriction gives a map from $E_1(c_-)$ to $E_1(c'_-)$ which together with φ yields a map $\bar{\varphi}$ as in the statement of the proposition.

To show that $\bar{\varphi}$ is an isometry, let $d_- \in E_1(c_-)$. By Lemma 4.9 it suffices to show that if x is opposite to d_-, then $\varphi(x)$ is opposite to $\bar{\varphi}(d_-)$. Since $d_- \in E_1(c_-)$ we can find a chamber y opposite to c_- such that $x \in E_1(y)$. Now $\varphi(x) = \varphi^y(x)$ and $\bar{\varphi}(d_-) = \varphi^y(d_-)$, and since φ^y is an isometry, we are done. $\square$

Proposition 5.2: *Given an isometry $\varphi_+ : \Delta_+ \longrightarrow \Delta'_+$ and a pair $(c_-, c'_-) \in \Delta_- \times \Delta'_-$ such that φ_+ extends to an isometry $\varphi_c : \{c_-\} \cup \Delta_+ \longrightarrow \{c'_-\} \cup \Delta'_+$, then for any chamber $d_- \in \Delta_-$ there exists a unique chamber $d'_- \in \Delta'_-$ such that φ_+ extends to an isometry $\varphi_d : \{d_-\} \cup \Delta_+ \longrightarrow \{d'_-\} \cup \Delta'_+$. Moreover, if d_- is i-adjacent to c_-, then d'_- is i-adjacent to c'_-.*

Proof: If d_- is i-adjacent to c_-, then the existence of d'_- and φ_d is immediate from Proposition 5.1, and d'_- is i-adjacent to c'_-.

If d_- is any chamber of Δ_- we may therefore work by induction along a gallery from c_- to d_-, obtaining the existence of some d'_- and φ_d as required. The uniqueness of d'_- is immediate from Lemma 4.10. □

Proof of Theorem 1.3: Let φ be an isometry from $\Delta_+ \cup \{c_-\}$ onto $\Delta'_+ \cup \{c'_-\}$. By the previous Proposition, φ extends uniquely to a map ψ from Δ to Δ' preserving codistance and i-adjacency. Similarly φ^{-1} extends to a map ψ_1 from Δ' to Δ, and $\psi_1\psi$ extends the identity map on $\Delta_+ \cup \{c_-\}$. By the uniqueness part of 5.2 $\psi_1\psi$ must be the identity, and therefore $\psi_1 = \psi^{-1}$. Thus $\psi\,|_{\Delta_-}$ must be a bijection from Δ_- to Δ'_-, and hence an isometry, since it preserves i-adjacency.

6 On condition (co)

Lemma 6.1: *Let Δ be a spherical building of rank 2 satisfying (co). Let c, d_0, d_1 and d_2 be chambers such that d_i is adjacent to d_{i+1} for $i = 0, 1$ and such that $l(c, d_0) \geq l(c, d_1) < l(c, d_2)$.*

Then there exists a gallery $d_0 = f_0, f_1, \ldots, f_k = d_2$ satisfying $l(c, d_1) < l(c, f_i)$ for $1 \leq i \leq k$.

Proof: Take galleries $\gamma_0 = (d_0, \ldots, x)$ and $\gamma_2 = (d_2, \ldots, y)$ with $x, y \in c^{op}$ and for which $l(c, \ )$ increases monotonically. Since c^{op} is connected, there is a gallery γ_1 in c^{op} from x to y. Now $\gamma_0\gamma_1\gamma_2$ is a gallery as required. □

Proof of Theorem 1.5: Let $c \in \Delta$ and $x, y \in c^{op}$. For each gallery $G : x = x_0, x_1, \ldots, x_k$ joining x and y set

$$\alpha(G) = max\{l_*(c, x_i) | 0 \leq i \leq k\}$$

and

$$\beta(G) = |\{x_i | l_*(c, x_i) = \alpha(G)\}|.$$

We will show that if a gallery G satisfies $\beta(G) > 1$, then there exists a gallery G_1 such that $\alpha(G_1) = \alpha(G)$ and $\beta(G_1) = \beta(G) - 1$; if it satisfies $\beta(G) = 1$ we will construct a gallery G_1 with $\alpha(G_1) = \alpha(G) - 1$. Applying this procedure successively, we will obtain a gallery joining x and y which is contained in c^{op} and therefore the claim follows.

Let G be as above and choose $0 \leq j \leq k$ such that $l_*(c, x_j) = \alpha(G)$ and $l_*(c, x_{j+1}) = \alpha(G) - 1$. Now choose a rank 2 residue R contaning x_{j-1}, x_j and x_{j+1}, and let c_1 denote the projection of c onto R. We apply Lemma 6.1 to obtain a gallery $x_{j-1} = y_0, y_1, \ldots, y_l = x_{j+1}$ satisfying $l(c_1, y_i) > l(c_1, x_j)$ (and hence $l_*(c, y_i) < l_*(c, x_j)$) for each $1 \leq i \leq l$. The gallery $G_1 : x = x_0, x_1, \ldots, x_{j-1} = y_0, y_1, \ldots, y_l = x_{j+1}, \ldots, x_k = y$ is as required above and we are done.

A series of buildings which do not satisfy (co):

Proposition 6.2: *Let M be an irreducible Coxeter diagram over I satisfying $m_{ij} \in \{2,4\}$ for $i \neq j \in I$. Given a twin building Δ of type M whose panels contain exactly 3 chambers, then c^{op} is not connected for any chamber $c \in \Delta$.*

The proof of the proposition above is based on the following observations:

(1) Given a building of type $A_1 \times A_1$ whose panels contain exactly three chambers, then c^{op} is a thin building of type $A_1 \times A_1$.

(2) Given a building of type B_2 whose panels contain exactly three chambers. Then c^{op} has two connected components; each of them is a thin building of type B_2.

The observations above and the results of [Ti81] imply:

(3) Given a chamber c in a twin building Δ (as above) and x, y lying in the same connected component of c^{op} then each minimal gallery joining x and y is contained in c^{op}; in particular, a connected component of c^{op} is a thin building of type M.

By observation (2) we may choose two chambers $x, y \in c^{op}$ in a common rank 2 residue R of type B_2, but in distinct connected components of $c^{op} \cap R$. Suppose they lie in a common connected component Δ_0 of c^{op}. Since Δ_0 is a (thin) building, a gallery from x to y that is minimal in Δ_0 must be of reduced type, and hence also minimal in Δ. Since residues are convex, this gallery lies in R, contradicting our choice of x and y in distinct components of $c^{op} \cap R$.

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