A NEW ASYMPTOTIC EXPANSION AND ASYMPTOTICALLY BEST CONSTANTS IN LYAPUNOV'S THEOREM

G. P. CHISTYAKOV[1]

B.Verkin Institute for Low Temperature Physics and Engineering,
National Academy of Sciences of Ukraine

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ABSTRACT. A new asymptotic expansion is obtained in Lyapunov's Central Limit Theorem for distribution functions of centered and normed sums of independent random variables which are not necessary identically distributed. It is applied to determine the asymptotically best constants in the Berry–Esseen inequality, thus solving problems about their optimal values raised by Kolmogorov and Zolotarev.

1. Introduction

The main result of this paper is a new asymptotic expansion in the Central Limit Theorem (CLT) for distribution functions of sums of independent non-identically distributed random variables having finite moments of the third order (see Theorem 2.1). This result differs from previous results since we do not require any additional restrictions on the distributions of summands (such as, for instance, Cramér's condition (C), see Petrov (1975, 1987)). Theorem 2.1 is new even in the particular case of identically distributed summands, since the remainder term is estimated uniformly over the class of distributions of summands with fixed value of Lyapunov's fraction L. As an application of Theorem 2.1, we compute the asymptotically best constants in the Berry–Esseen inequality (see Theorem 2.2 and Corollary 2.3)

Denote by $\mathfrak{X}_3$ the set of all non-zero random variables (r.v.'s) with mean zero and finite absolute moments of order 3. We denote the set of distribution functions (d.f.'s) of these r.v.'s by $\mathfrak{F}_3$. In the case of symmetric r.v.'s we denote the corresponding sets by $\mathfrak{X}_{3s}$ and $\mathfrak{F}_{3s}$ respectively. Consider a finite sequence of independent r.v.'s $X_1, \dots, X_n$ of the class $\mathfrak{X}_3$ such that X_j have d.f.'s F_j, characteristic

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functions (c.f.'s) f_j, variances σ_j^2, third moments α_j, third absolute moments β_j respectively. Write

$$s = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}, \quad B = \beta_1 + \cdots + \beta_n, \quad L = \frac{B}{s^3}, \quad Z = \frac{X_1 + X_2 + \cdots + X_n}{s}.$$

We denote the d.f. of Z by F, and the corresponding c.f. by f.

According to Lyapunov's theorem, the d.f. F converges to the standard normal d.f. Φ in the Kolmogorov metric

$$\rho(F, \Phi) = \sup_x \bigl| F(x) - \Phi(x) \bigr|$$

if Lyapunov's fraction $L \to 0$. Moreover (see Berry (1941) and Esseen (1942)), there exists an absolute constant c such that

$$\rho(F, \Phi) \leqslant cL. \tag{1.1}$$

Kolmogorov (1953) raised the question of finding the smallest value, say c_e, of the constant c in (1.1). He conjectured that $c_e = 1/\sqrt{2\pi}$. Esseen (1956) showed that $c_e \geqslant (\sqrt{10} + 3)/6\sqrt{2\pi} = 1/\sqrt{2\pi} + 0.0107899\ldots$

Theorem A (Esseen (1956)). *Assume that $X_1, \ldots, X_n$ are independent identically distributed (i.i.d.) and $F_1 \in \mathfrak{F}_3$. Then*

$$\frac{\sigma_1^3}{\beta_1} \limsup_{n\to\infty} \sqrt{n}\, \rho(F, \Phi) \leqslant \frac{\sqrt{10} + 3}{6\sqrt{2\pi}}.$$

This inequality turns into equality if and only if $F_1(x)$ is the lattice d.f., having jumps $(\sqrt{10} - 2)/2$ and $(4 - \sqrt{10})/2$ at the points $x_1 = -h(4 - \sqrt{10})/2$ and $x_2 = h(\sqrt{10} - 2)/2$ respectively, for some $h > 0$.

Rogozin (1960) refined Esseen's result. Denote by $\mathcal{N}$ the set of all normal d.f.'s and $\rho(F, \mathcal{N}) = \inf\limits_{Q \in \mathcal{N}} \rho(F, Q)$.

Theorem B (Rogozin (1960)). *Assume that $X_1, \ldots, X_n$ are i.i.d. and $F_1 \in \mathfrak{F}_3$. Then*

$$\frac{\sigma_1^3}{\beta_1} \limsup_{n\to\infty} \sqrt{n}\, \rho(F, \mathcal{N}) \leqslant \frac{1}{\sqrt{2\pi}}.$$

This inequality turns into equality if $F_1(x)$ is the symmetric Bernoulli d.f., having jumps $\frac{1}{2}$ at $x = \pm 1$.

The exact value of the constant c_e is not known until now. This problem is discussed in the monographs of Ibragimov and Linnik (1971), Petrov (1975, 1987) and Zolotarev (1986). In these books, the reader can find a discussion of the constant problem in other metrics (in particular, in the metric of the space L_p, $1 \leqslant p < \infty$) as well.

Starting with Esseen, many authors used asymptotic expansions for the d.f. F. To describe them we need the following notation. We denote by $\mathrm{Fr}\{y\}$ the fractional part of an arbitrary real number y and $\psi_n(x) = \frac{1}{2} - \mathrm{Fr}\left\{\left(x\sqrt{n} - \frac{a\,n}{\sigma_1}\right)\frac{\sigma_1}{h}\right\}$. If $F_1(x)$ is a lattice d.f., the parameters a and $h > 0$ characterize the maximal lattice $x_m = a + mh$ $(m = 0, \pm 1, \pm 2, \dots)$ containing the jump points of the d.f. F_1. If F_1 is not a lattice d.f., one assumes $h = \psi_n(x) = 0$.

Theorem C (Esseen (1945)). *Assume that $X_1, \dots, X_n$ are i.i.d. and $F_1 \in \mathfrak{F}_3$. Then the expansion*

$$F(x) = \Phi(x) + \frac{\alpha_1}{6\sigma_1^3\sqrt{2\pi n}}\,(1 - x^2)\,e^{-x^2/2} + \frac{h}{\sigma_1\sqrt{2\pi n}}\,e^{-x^2/2}\,\psi_n(x) + o\left(\frac{1}{\sqrt{n}}\right) \tag{1.2}$$

holds uniformly in $x \in \mathbb{R}^1$ as $n \to \infty$.

Unfortunately, known estimates of the remainder term in (1.2) are not uniform in $F_1 \in \mathfrak{F}_3$. In the non-i.i.d. case expansions of type (1.2) are obtained under certain additional restrictions only, see, for example, Petrov (1975).

Introduce the quantities

$$r(l, x) = \sup\bigl|\,F(x) - \Phi(x)\,\bigr|, \qquad r(l) = \sup \rho(F, \Phi), \tag{1.3}$$

where the supremums are taken over all $F_1, \dots, F_n \in \mathfrak{F}_3$ with Lyapunov's fraction $L = l$, and write

$$c^* = \limsup_{l \to 0} \frac{r(l)}{l}, \qquad c^*(x) = \limsup_{l \to 0} \frac{r(l, x)}{l}. \tag{1.4}$$

In the symmetric case we define $r_s(l, x)$, $r_s(l)$ and c_s^*, $c_s^*(x)$ similarly to (1.3) and (1.4) just replacing $\mathfrak{F}_3$ by $\mathfrak{F}_{3s}$. Kolmogorov (1953) raised the question of computing the quantities defined above (see Ibragimov and Linnik (1971) and Zolotarev (1986) as well). Consider also the function $r_*(l) = \sup \inf_{Q \in \mathcal{N}} \rho(F, Q)$, where the supremum is taken over all $F_1, \dots, F_n \in \mathfrak{F}_3$ with $L = l$ and write

$$c_* = \limsup_{l \to 0} \frac{r_*(l)}{l}$$

The question of computing c_* was proposed by Zolotarev (1986, p. 231). To solve these problems it does not suffice to prove the asymptotic expansion (1.2). In fact, it is necessary to prove an analog of (1.2) in the non-i.i.d. case with a uniform remainder term of type $O(L^{1+\delta})$, $\delta > 0$.

2. Statements of main results

The notation $c, c_1, c_2, \dots$ will be used throughout for absolute positive constants. By c we denote different constants in different (or even in the same) formulas. The symbols $c_1, c_2, \dots$ are applied for concrete constants, for example, we assume that $c_1 = 1/39$. We denote by Θ different quantities with $|\Theta| \leqslant 1$. We suppose without loss of generality that Lyapunov's fraction L satisfies $L \leqslant e^{-1}$. For a subset $J \subset \{1, 2, \dots, n\}$, let F_J denote the d.f. of the r.v. $\frac{1}{s}\sum_{j\in J} X_j$. Then its c.f. f_J has the form $f_J(t) = \prod_{j\in J} f_j(t/s)$. We denote by $\overline{J}$ the set $\{1, 2, \dots, n\}\setminus J$. If $J = \varnothing$, then we assume that the d.f. $F_\varnothing(x)$ is degenerate at zero and its c.f. $f_\varnothing(t) \equiv 1$. Introduce the measure m of subsets of $\{1, 2, \dots, n\}$ by

$$m(J) = \frac{1}{s^2}\sum_{j\in J}\sigma_j^2, \qquad J \subset \{1, 2, \dots, n\}.$$

Introduce, as an analog of the second term on the right-hand side of (1.2), the quantity

$$\alpha(x) = \sum_{j=1}^{n}\int_{-\infty}^{\infty}\Big(\Phi\big(x - u/s\big) - \Phi(x) + \Phi'(x)\,\frac{u}{s} - \Phi^{(2)}(x)\,\frac{u^2}{2s^2}\Big)\,dF_j(u).$$

In order to define an analog of the third term, consider the closed set

$$S = \Big\{t \in \Gamma : \big|f(t)\big| \geqslant L^2,\ \frac{d}{dt}\big|f(t)\big| = 0\Big\},$$

where $\Gamma = \big[1/4L,\, L^{-1-c_1}\big]$. Assume that $S \neq \varnothing$. In this case we define a finite sequence $t_1, \dots, t_r$ of elements of the set S by induction as follows. Denote by t_1 the minimal point of the set S. Suppose that the points $t_1, \dots, t_k$ are already constructed. Let $\Gamma^{(m)}$ be the $(10^{-3}L^{-1+c_1})$-neighborhood of t_m, $m = 1, \dots, k$. We consider throughout open neighborhoods only. If the set $\Lambda_k = S \setminus (\Gamma^{(1)} \cup \dots \cup \Gamma^{(k)})$ is not empty, we choose t_{k+1} as the minimal point of this set. We iterate this procedure as long as the sets Λ_k, $k = 1, \dots, r-1$, are not empty.

We have the estimate

$$\big|f(t_q)\big| \geqslant L^2, \qquad q = 1, \dots, r. \tag{2.1}$$

Fix now some of $t_1, \dots, t_r$, say t_q, and consider those c.f.'s f_j which satisfy the relation

$$\big|f_j(t_q/s)\big| \geqslant \exp\Big\{\frac{2\sigma_j^2\log L}{L^{9c_1}s^2}\Big\}. \tag{2.2}$$

The set of indices j which corresponds to these c.f.'s we denote by J_q. In view of (2.1), we have

$$m(J_q) \geqslant 1 - L^{9c_1}, \qquad q = 1, \dots, r. \tag{2.3}$$

Consider the set J^* of indices j such that

$$\frac{\beta_j}{B} \leqslant L^{-8c_1} \frac{\sigma_j^2}{s^2}. \tag{2.4}$$

It is easily seen that

$$m(J^*) \geqslant 1 - L^{8c_1}. \tag{2.5}$$

Indeed, we have for the measure of the set of indices $\overline{J}^*$

$$m(\overline{J}^*) \leqslant L^{8c_1} \sum_{j \in \overline{J}^*} \frac{\beta_j}{B} \leqslant L^{8c_1}.$$

Thus, the measures of the sets J_q $(q = 1, \dots, r)$ and J^* are close to 1 if L is small enough. Note that

$$J_1 = J_2 = \dots = J_r = J^* = \big\{1, 2, \dots, n\big\}$$

if all r.v.'s X_j are i.i.d.

Introduce the sets of indices

$$J_* = J^* \cap \bigcap_{q=1}^{r} J_q, \qquad \overline{J}_* = \big\{1, 2, \dots, n\big\} \setminus J_* \tag{2.6}$$

and define

$$S_* = \big\{t_1, \dots, t_r\big\} \cap \big\{t \in \Gamma : \big|f_{\overline{J}_*}(t)\big| \geqslant L^{c_1}\big\}.$$

Let $S_* \neq \varnothing$. In this case we define t_* as the smallest of the points $t \in S_*$. Throughout $\mathbb{N}$ denotes the set of all positive integers. Select the minimal $d_* \in \mathbb{N}$ such that every of the points $t_1, \dots, t_r$ is contained in the $(-10^6 L^{-2c_1} \log L)$-neighborhood of some point of the form $m t_*/d_*$ where $m \in \mathbb{N}$. Using this notation, we define the parameter

$$h \stackrel{\text{def}}{=} \frac{2\pi d_* s}{t_*}. \tag{2.7}$$

Introduce the d.f. $H_\varepsilon(x; h)$ with points of increase in the interval $\big[-h/2s,\, h/2s\big]$ by

$$H_\varepsilon(x; h) = \sum_{m \in \mathbb{Z}} \Big(\big(F * \Phi_\varepsilon\big)\Big(x + \frac{mh}{s}\Big) - \big(F * \Phi_\varepsilon\big)\Big(\frac{(m-1/2)\,h}{s}\Big)\Big)$$
$$\text{for} \quad x \in \big[-h/2s,\, h/2s\big].$$

Here and below $\mathbb{Z}$ is the set of all integers, $\varepsilon = 8\,L^{1+c_1}\sqrt{|\log L|}$ and $\Phi_\sigma(x) = \Phi(x/\sigma)$ for $\sigma > 0$. Obviously, $H_\varepsilon\bigl(-h/2s;\,h\bigr) = 0$ and $H_\varepsilon\bigl(h/2s;\,h\bigr) = 1$. Define, as an analog of the third summand of the sum (1.2), the quantity

$$\frac{h}{s\sqrt{2\pi}}\,e^{-x^2/2}\,\Psi(x;h),$$

where

$$\Psi(x;h) \stackrel{\mathrm{def}}{=} \int_{-h/2s}^{h/2s} \Bigl(\frac{1}{2} - \mathrm{Fr}\Bigl\{\frac{s}{h}\,(x-u)\Bigr\}\Bigr)\,dH_\varepsilon(u;h).$$

Define $M_f = \max\limits_{t\in\Gamma}\bigl|f(t)\bigr|$.

The main result of this paper is the following theorem.

Theorem 2.1. *Let $M_f > L^2$. If $S_* \neq \varnothing$, then the representation*

$$(F * \Phi_\varepsilon)(x) = \Phi(x) + \alpha(x) + \frac{h}{s\sqrt{2\pi}}\,e^{-x^2/2}\,\Psi(x;h) + c\,\Theta\,L^{1+c_1}|\log L|^{7/6} \qquad (2.8)$$

holds for all real x. Moreover, the parameter h admits the estimate

$$h \leqslant 2\,(1 + cL^{2c_1})\,L\,s, \qquad (2.9)$$

the measure of the set J_* has the lower bound

$$m(J_*) \geqslant 1 - cL^{8c_1} \qquad (2.10)$$

and, for $j \in J_*$, the following estimates hold

$$\bigl|f_j\bigl(2\pi/h\bigr)\bigr| \geqslant 1 + c\,\frac{\log L}{L^{12c_1}}\,\frac{\sigma_j^2}{s^2}\,. \qquad (2.11)$$

The representation (2.8) holds for all real x without the third summand of the sum (2.8) in the cases where $S_* = \varnothing$ or $M_f \leqslant L^2$.

The representation (2.8) permits to study the behavior of the functions $c^*(x)$ and $c_s^*(x)$. The first who studied these functions was Linnik (1947). His investigation was continued by Petrov (1955), Chistyakov (1990) and Bentkus (1994).

The following result is a simple consequence of Theorem 2.1.

Corollary 2.1. *For any fixed real x the following equality holds*

$$\limsup_{l\to 0} \sup \frac{1}{L}\left| F(x) - \Phi(x) - \alpha(x) \right| = \Phi'(x), \tag{2.12}$$

where the supremum is taken over all $F_1, \ldots, F_n$ such that $L = l$.

This equality is due to Chistyakov (1990). Furthermore, in the paper just mentioned, the asymptotic behavior of the functions $c^*(x)$ and $c_s^*(x)$ was derived from the equality (2.12). It was proved that

$$c^*(x)\,|x|^3 \to 1 \quad \text{and} \quad c_s^*(x)\,|x|^3 \to 1/2 \qquad \text{as} \quad |x| \to \infty.$$

Denote by $\mathfrak{F}_{3s}^*$ a subset of the d.f.'s of the set $\mathfrak{F}_{3s}$ such that

$$\limsup_{l\to 0} \sup \frac{1}{B} \sum_{j=1}^{n} \int\limits_{|u|>\gamma s} |u|^3 \, dF_j(u) = 0 \tag{2.13}$$

for any $\gamma > 0$. Here the supremum is taken over all $F_1, \ldots, F_n$ from the set $\mathfrak{F}_{3s}^*$ with $L = l$. As an example of $\mathfrak{F}_{3s}^*$ one can consider a set of symmetric d.f.'s corresponding to r.v.'s bounded in absolute value by the same constant and having the same variance.

It was shown by Chistyakov (1990), that (2.13) implies

$$c^*\left(x, \mathfrak{F}_{3s}^*\right) = \Phi'(x).$$

The quantity $c^*\left(x, \mathfrak{F}_{3s}^*\right)$ is defined in the same way as $c_s^*(x)$ with only difference that in its definition the supremum is taken over all $F_1, \ldots, F_n$ such that $L = l$ in the set $\mathfrak{F}_{3s}^*$ instead of d.f.'s $F_1, \ldots, F_n$ in the set $\mathfrak{F}_{3s}$. It was also shown that the condition (2.13) is in some sense essential.

Corollary 2.2. *Assume that the r.v.'s $X_1, \ldots, X_n$ are i.i.d. Then the representation* (1.2) *follows from* (2.8).

Thus, Theorem 2.1 generalizes Theorem C. Moreover, the remainder term is uniformly estimated by Lyapunov's fraction with power greater then one.

Remark. In the case when the r.v.'s X_j are i.i.d. the statement of Theorem 2.1 is valid for $J_* = \{1, 2, \ldots, n\}$.

Using Theorem 2.1 and some ideas of Esseen (1956) and Rogozin (1960), we compute the constants c^* and c_*. Moreover, we prove the following theorem.

Theorem 2.2. *The following estimates hold*

$$r(l) \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\, l + c\, l^{1+c_1}\, |\log l\,|^{7/6}, \tag{2.14}$$

$$r_*(l) \leqslant \frac{1}{\sqrt{2\pi}}\, l + c\, l^{1+c_1}\, |\log l\,|^{7/6}. \tag{2.15}$$

The answer to the question about constants is contained in

Corollary 2.3. *The constants c^* and c_* are given by*

$$c^* = \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\,, \qquad c_* = \frac{1}{\sqrt{2\pi}}\,.$$

Theorem 2.2 and Corollary 2.3 sharpen Theorem A and B and solve Kolmogorov's and Zolotarev's problems.

Bentkus (1994) proved earlier that

$$r_s(l) \leqslant \frac{1}{\sqrt{2\pi}}\, l + cl^{4/3} \qquad \text{and} \qquad c_s^* = \frac{1}{\sqrt{2\pi}}\,. \tag{2.16}$$

In this case Theorem 2.1 implies the estimate

$$r_s(l) \leqslant \frac{1}{\sqrt{2\pi}}\, l + cl^{1+c_1}\, |\log l\,|^{7/6}. \tag{2.17}$$

The inequality (2.17) is essentially weaker than (2.16). We shall present the proof of (2.17) in Section 4 in order to provide another way for computing the constant c_s^*.

In Section 3 we prove auxiliary results, in Section 4 Theorem 2.2, Corollary 2.3 and (2.17), and in Section 5 Theorem 2.1 and Corollaries 2.1 and 2.2. One should emphasize that in the proof of Theorem 2.1 we used some ideas of the seminal papers of Esseen (1945) and Linnik (1947).

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3. Auxiliary results

Let P_1 and P_2 be probability distributions (pr.d.'s). By $\rho(P_1, P_2)$ we denote the Kolmogorov distance between their d.f.'s.

Lemma 3.1 (Le Cam (1965), see Arak and Zaitsev (1988, p. 61)). *Let P_1, P_2 and H be pr.d.'s. Then, for any $\tau \geqslant 0$,*

$$\big(2H\big(\{x : |x| \leqslant \tau/2\}\big) - 1\big)\, \rho(P_1, P_2) \leqslant \rho(P_1 * H, P_2 * H) + \gamma(\tau)\, H\big(\{x : |x| \leqslant \tau/2\}\big),$$

where $\gamma(\tau) = \min\{\, Q(P_1, \tau),\, Q(P_2, \tau)\,\}$ and $Q(P_j, \tau) = \sup_x P_j\big([x,\, x + \tau]\big)$.

We now formulate two auxiliary lemmas which sharpen the corresponding Esseen results (1956). Let $\varkappa \in \big(0,\, 1/10\big]$ and introduce the set $\mathfrak{F}_{3,\varkappa}$ of d.f.'s in the following way. We shall say that a d.f. G with a finite third absolute moment belongs to $\mathfrak{F}_{3,\varkappa}$ if there exists a real number $\alpha = \alpha(G)$ such that

a) the spectrum of G is contained in the set of the form $\bigcup_{m \in \mathbb{Z}} \big(\alpha + m - \varkappa,\, \alpha + m + \varkappa\big)$

b) at least two intervals $\big(\alpha + m - \varkappa,\, \alpha + m + \varkappa\big)$, $m \in \mathbb{Z}$, contain points of increase of the d.f. G.

Denote by d_G the minimal distance between the points of the form $\alpha + m$, $m \in \mathbb{Z}$, which contain points of increase of the d.f. G in their $\varkappa$-neighborhoods.

Lemma 3.2. *Let X be a random variable with a d.f. G from the set $\mathfrak{F}_{3,\varkappa}$. Then*

$$\frac{1}{2}\, (d_G - 8\,\varkappa)\ \mathrm{Var}\ X \leqslant \inf_b\ \mathbf{E}\, |X - b|^3.$$

Lemma 3.3. *Let X be a random variable with a d.f. from the set $\mathfrak{F}_{3,\varkappa}$. Then*

$$\frac{1}{2}\, (1 - 8\,\varkappa)\ \mathrm{Var}\ X + \frac{1}{6}\, \big|\, \mathbf{E}\, (X - \mathbf{E}\, X)^3 \,\big| \leqslant \frac{\sqrt{10} + 3}{6}\, \mathbf{E}\, |\, X - \mathbf{E}\, X\, |^3.$$

The proof of these lemmas may be obtained by repeating the reasoning which has been used to prove the corresponding Esseen results (1956) (see, for example, Ibragimov and Linnik (1971)). Therefore we omit the proof of Lemmas 3.2 and 3.3.

Lemma 3.4. *Let X be a random variable and let its c.f. $g(t)$ satisfy the relation*

$$\big|\, g(2\,\pi) \big| \geqslant 1 - \nu \tag{3.1}$$

for some $\nu \in \big[0,\, 1\big)$. Then there exists a number θ such that $\theta \in \big[-1/2,\, 1/2\big)$ and for any $b \in \big(0,\, 1/2\big]$ the following estimate holds

$$\mathbf{P}\, \Big\{\, X \in \bigcup_{m \in \mathbb{Z}} \big\{\, x : b \leqslant |\, x - m - \theta\, | \leqslant 1/2 \,\big\} \setminus \big\{ -1/2 \big\} \,\Big\} \leqslant \frac{1}{8\, b^2}\, \nu.$$

If $g(2\,\pi)$ is positive, then the last inequality is valid with $\theta = 0$.

Proof. Write $g(2\pi) = \bigl|g(2\pi)\bigr|\, e^{ia}$ where $a \in \bigl[-\pi,\, \pi\bigr)$. In view of (3.1) one obtains the estimate

$$2\int_{-\infty}^{\infty} \sin^2 \pi x\, dG(x + a/2\pi) \leqslant \nu,$$

where G is the d.f. of X. Using the inequality $\sin^2 \pi x \geqslant 4x^2$ which is valid for $x \in \bigl[-1/2,\, 1/2\bigr]$, we get from the last estimate that

$$\sum_{m\in\mathbb{Z}} \int\limits_{-1/2<x\leqslant 1/2} x^2\, dG(m + x + \theta) \leqslant \frac{1}{8}\,\nu, \qquad \text{where} \quad \theta = a/2\pi. \tag{3.2}$$

The statement of Lemma 3.4 follows immediately from this inequality. □

Lemma 3.5. *Let X be a random variable with a d.f. G from the class $\mathfrak{F}_3$ such that*

$$\operatorname{Var} X \leqslant \eta^{-2K}, \tag{3.3}$$

$$\mathbf{E}\,|X|^3 \leqslant \eta^{-K} \operatorname{Var} X \tag{3.4}$$

for some $K \in \bigl(0,\, 1/6\bigr]$ and $\eta \in \bigl(0,\, 10^{-3/K}\bigr)$. Let its c.f. $g(t)$ satisfy the relation

$$\bigl|g(2\pi)\bigr| \geqslant 1 - \eta \operatorname{Var} X. \tag{3.5}$$

Then there exists a random variable X^* with a d.f. from the class $\mathfrak{F}_{3,\varkappa}$, where $\varkappa = \eta^{K_1/8}$ and $K_1 = 13\,K/8$, such that

$$\bigl|\operatorname{Var} X - \operatorname{Var} X^*\bigr| \leqslant 13\,\eta^{5K/8} \operatorname{Var} X, \tag{3.6}$$

$$\Biggl|\ \int\limits_{|x|\leqslant\eta^{-K_1}} x^3\, dG(x) - \mathbf{E}\,(X^* - \mathbf{E}\,X^*)^3\Biggr| \leqslant 73\,\eta^{K/4} \operatorname{Var} X, \tag{3.7}$$

$$\mathbf{E}\,\bigl|X^* - \mathbf{E}\,X^*\bigr|^3 \leqslant \int\limits_{|x|\leqslant\eta^{-K_1}} |x|^3\, dG(x) + 100\,\eta^{K/4} \operatorname{Var} X. \tag{3.8}$$

Proof. We first note that, by (3.5), the r.v. X satisfies the conditions of Lemma 3.4 with $\nu = \eta \operatorname{Var} X$. By this lemma, there exists a number $\theta \in \bigl[-1/2,\, 1/2\bigr)$ such that

$$\mathbf{P}\bigl\{X \in S_{\theta,\eta}\bigr\} \leqslant \frac{1}{8}\,\eta^{1-K_1/4} \operatorname{Var} X, \tag{3.9}$$

where $S_{\theta,\eta}$ is the set of the form

$$\bigl\{x\in\mathbb{R}^1 : |x| \leqslant \eta^{-K_1}\bigr\} \bigcap \bigcup_{m\in\mathbb{Z}} \Bigl(\bigl\{x \in \mathbb{R}^1 : \eta^{K_1/8} \leqslant |x - m - \theta| \leqslant 1/2\bigr\} \setminus \bigl\{-1/2\bigr\}\Bigr).$$

Introduce the r.v. X^* in the following way

$$X^* = \begin{cases} X, & \text{if } X \notin S_{\theta,\eta}, \\ \theta, & \text{if } X \in S_{\theta,\eta}. \end{cases}$$

Let us now verify the validity of the relations (3.6)–(3.8). As the first step we prove (3.6). Using (3.9), one easily sees that

$$\left| \int\limits_{|x| \leqslant \eta^{-K_1}} x\, dG(x) - \mathbf{E}\, X^* \right| \leqslant \frac{1}{8} \left(\eta^{1-5K_1/4} + \eta^{1-K_1/4} \right) \operatorname{Var} X.$$

Taking into account the assumption $\mathbf{E}\, X = 0$, we rewrite the last inequality in the form

$$\left| \int\limits_{|x| > \eta^{-K_1}} x\, dG(x) + \mathbf{E}\, X^* \right| \leqslant \frac{1}{8} \left(\eta^{1-5K_1/4} + \eta^{1-K_1/4} \right) \operatorname{Var} X. \tag{3.10}$$

To estimate the integral on the left-hand side, we need the following relation which can be obtained with the help of Chebyshev's inequality and (3.4)

$$W_G(x) = G(-x) + 1 - G(x) \leqslant \frac{\mathbf{E}\, |X|^3}{x^3} \leqslant \eta^{-K} \frac{\operatorname{Var} X}{x^3}, \qquad x > 0.$$

Then, integrating by parts, we get

$$\begin{aligned} \int\limits_{|x| > \eta^{-K_1}} |x|\, dG(x) &\leqslant \eta^{-K_1} W_G(\eta^{-K_1}) + \int\limits_{x > \eta^{-K_1}} W_G(x)\, dx \\ &\leqslant \eta^{2K_1 - K} \operatorname{Var} X + \eta^{-K} \operatorname{Var} X \int\limits_{x > \eta^{-K_1}} \frac{1}{x^3}\, dx \\ &\leqslant 2\,\eta^{9K/4} \operatorname{Var} X. \end{aligned}$$

Applying the last estimate to (3.10), we derive the required inequality

$$|\,\mathbf{E}\, X^*| \leqslant \left(\frac{1}{4}\, \eta^{1-5K_1/4} + 2\,\eta^{9K/4} \right) \operatorname{Var} X \leqslant 3\eta^{9K/4} \operatorname{Var} X. \tag{3.11}$$

In the same way we obtain the inequalities

$$\left| \int\limits_{|x| \leqslant \eta^{-K_1}} x^2\, dG(x) - \mathbf{E}\, (X^*)^2 \right| \leqslant \frac{1}{8} \left(\eta^{1-9K_1/4} + \eta^{1-K_1/4} \right) \operatorname{Var} X$$

and

$$\int\limits_{|x| > \eta^{-K_1}} x^2\, dG(x) \leqslant 3\eta^{5K/8} \operatorname{Var} X.$$

Thus, we have

$$\big|\,\mathbf{E}\,X^2-\,\mathbf{E}\,(X^*)^2\big|\leqslant\Big(\frac{1}{4}\,\eta^{1-9K_1/4}+3\eta^{5K/8}\Big)\ \mathrm{Var}\ X\leqslant 4\eta^{5K/8}\ \mathrm{Var}\ X. \tag{3.12}$$

The inequalities (3.11), (3.3) and (3.12) imply the estimate

$$\big|\,\mathrm{Var}\ X-\mathrm{Var}\ X^*\big|\leqslant\big(4\eta^{5K/8}+9\,\eta^{5K/2}\big)\ \mathrm{Var}\ X\leqslant 13\,\eta^{5K/8}\ \mathrm{Var}\ X \tag{3.13}$$

that proves (3.6).

As the second step we verify (3.7). Write

$$\mathbf{E}\,(X^*-\,\mathbf{E}\,X^*)^3=\,\mathbf{E}\,(X^*)^3-3\,\mathbf{E}\,(X^*)^2\,\mathbf{E}\,X^*+2\,(\,\mathbf{E}\,X^*)^3$$

and estimate the second and the third terms on the right-hand side. By (3.12) we have

$$\big(\,\mathbf{E}\,|X^*|\big)^{\,2}\,\leqslant\,\,\mathbf{E}\,(X^*)^2\leqslant(1+4\eta^{5K/8}\,)\ \mathrm{Var}\ X. \tag{3.14}$$

According to (3.14), (3.11) and (3.3), one has

$$\begin{aligned}3\,\mathbf{E}\,(X^*)^2\,|\,\mathbf{E}\,X^*|+2\,|\,\mathbf{E}\,X^*|^3&\leqslant 18\,\eta^{9K/4}\,(\mathrm{Var}\ X)^2+54\,\eta^{27K/4}\,(\mathrm{Var}\ X)^3\\&\leqslant 72\,\eta^{K/4}\ \mathrm{Var}\ X.\end{aligned}$$

Note, finally, that

$$\Bigg|\int\limits_{|x|\leqslant\eta^{-K_1}}x^3\,dG(x)-\mathbf{E}\,(X^*)^3\Bigg|\leqslant\frac{1}{8}\Big(\eta^{1-13K_1/4}+\eta^{1-K_1/4}\Big)\ \mathrm{Var}\ X\leqslant\frac{1}{4}\,\eta^{K/4}\ \mathrm{Var}\ X.$$

As before, we used here the estimate (3.9). Comparing the two last estimates, we get (3.7).

It remains to verify (3.8). It is easy to see that

$$\mathbf{E}\,\big|\,X^*-\,\mathbf{E}\,X^*\big|^3\leqslant\,\mathbf{E}\,|X^*|^3+3\,\mathbf{E}\,(X^*)^2\,|\,\mathbf{E}\,X^*|+3\,\mathbf{E}\,|X^*|\,(\,\mathbf{E}\,X^*)^2+|\,\mathbf{E}\,X^*|^3.$$

Let us show that this inequality leads to (3.8). Note that

$$\mathbf{E}\,|X^*|^3\leqslant\Bigg|\int\limits_{|x|\leqslant\eta^{-K_1}}|x|^3\,dG(x)\Bigg|+\eta^{1-K_1/4}\ \mathrm{Var}\ X.$$

Therefore, using (3.14), (3.11) and (3.3), we obtain (3.8).

Let us finally prove that the d.f. of X^* belongs to the set $\mathfrak{F}_{3,\varkappa}$ where $\varkappa=\eta^{K_1/8}$. Assume that this is not true. Then there exists an integer m such that all points of increase of the d.f. of X^* lay in the $\varkappa$-neighborhood of the point $m+\theta$. In addition, by (3.2), we have

$$\mathbf{E}\left(X^*-m-\theta\right)^2 \leqslant \frac{1}{8}\,\eta \text{ Var } X.$$

On the other hand, the inequality (3.13) may be rewritten in the form

$$\Big|\text{ Var } X - \text{Var}\left(X^*-m-\theta\right)\Big| \leqslant 13\,\eta^{5K/8} \text{ Var } X.$$

The last two inequalities yield

$$\text{Var } X \leqslant \left(\frac{1}{8}\,\eta + 13\,\eta^{5K/8}\right) \text{ Var } X < \text{Var } X,$$

that is impossible. □

Let c_2 be an absolute positive constant which will be chosen in Section 5. The constants below will depend on c_2 but they turn into absolute constants when the value of c_2 is fixed.

Lemma 3.6. *Let X be a r.v. with a d.f. $H(x)$ of the class $\mathfrak{F}_3$, a c.f. $\varphi(t)$, a variance σ^2 and a third absolute moment β. Let a point $a \geqslant c_2$ satisfy the relation*

$$\left|\varphi(a)\right| \geqslant 1-\eta\,\sigma^2, \tag{3.15}$$

where $\eta \in \left(0,\, (4+2\,\pi/c_2)^{-3}\right)$. Suppose also that

$$\beta \leqslant \eta^{-1/3}\sigma^2 \leqslant \eta^{-2/3}. \tag{3.16}$$

Then the c.f. $\varphi(t)$ admits the representation

$$\varphi(t) = \varphi(q\,a)\exp\left\{\text{ Re }\frac{\varphi'(q\,a)}{\varphi(q\,a)}\,(t-q\,a) - \frac{1}{2}\,\sigma^2\,(t-q\,a)^2 + r_q(t)\right\} \tag{3.17}$$

in the $\dfrac{1}{8\,\sigma}$-neighborhood of every point $q\,a$ $\left(q = 1,\ldots,[\eta^{-1/9}]\right)$, where the functions $r_q(t)$ have the form

$$r_q(t) = i\,C_{1q}\,(t-q\,a) + C_{2q}\,(t-q\,a)^2 + C_{3q}\,(t-q\,a)^3.$$

Here the real-valued quantities C_{1q} and the complex-valued quantities C_{2q}, C_{3q} satisfy the estimates

$$|\,C_{1q}\,| \leqslant c\,(q^2\,\sigma^2+\beta)\,\eta^{2/3}, \tag{3.18}$$

$$|\,C_{2q}\,| \leqslant c\,(q^2\,\sigma^2+\beta)\,\eta^{1/3}, \tag{3.19}$$

$$|\,C_{3q}\,| \leqslant c\beta. \tag{3.20}$$

Furthermore, the following relations hold

$$\Big| \operatorname{Re} \frac{\varphi'(qa)}{\varphi(qa)} - q \frac{\varphi'(a)}{\varphi(a)} \Big| \leqslant c\,(q^3\sigma^2 + q\beta)\,\eta^{2/3} \tag{3.21}$$

and

$$\Big| \operatorname{Re} \frac{\varphi'(qa)}{\varphi(qa)} \Big| \leqslant cq\eta^{1/2}\sigma^2. \tag{3.22}$$

Remark. The symbol $[\eta^{-1/9}]$ denote here the largest integer which is not greater than $\eta^{-1/9}$.

Proof. Write $\varphi(a) = e^{i\theta}\big|\varphi(a)\big|$ where $\theta \in \big[-\pi,\, \pi\big)$. Introduce the d.f. G with the c.f. of the form

$$g(t) = e^{-i\theta t/a}\varphi(t).$$

Obviously, $G(x) = H\big(x + \theta/a\big)$. We study next the behavior of the c.f. $g(t)$ and its derivatives.

Lemma 3.7. *The estimates*

$$1 - \operatorname{Re} g(qa) \leqslant q^2\eta\sigma^2, \tag{3.23}$$

$$\big|\operatorname{Im} g(qa)\big| \leqslant 2q^3\eta\sigma^2 \tag{3.24}$$

hold for all $q = 1, 2, \dots$.

Proof. By the well-known inequality

$$\sin^2 \frac{qax}{2} \leqslant q^2 \sin^2 \frac{ax}{2}$$

which is true for all real x, we easily obtain the estimate

$$1 - \operatorname{Re} g(qa) \leqslant q^2\big(1 - g(a)\big).$$

Recalling (3.15) we arrive at the relation (3.23).

The equality

$$\begin{aligned}\sin(qax) &= \sin(ax)\,\cos\big((q-1)\,ax\big) + \cos(ax)\,\sin\big((q-1)\,ax\big)\\ &= \sin(ax) - 2\sin(ax)\,\sin^2\frac{(q-1)\,ax}{2} + \sin\big((q-1)\,ax\big)\\ &\qquad - 2\sin\big((q-1)\,ax\big)\,\sin^2\frac{ax}{2}\end{aligned}$$

permits us to establish the relation

$$\operatorname{Im} g(qa) = \operatorname{Im} g(a) + \operatorname{Im} g\big((q-1)\,a\big) + A_1 + A_{q-1},$$

where the quantities A_q admit the estimate $|A_q| \leqslant 1 - \operatorname{Re} g(qa)$. From the last relation we finally get

$$\operatorname{Im} g(qa) = q\, \operatorname{Im} g(a) + q\,A_1 + \sum_{m=2}^{q-1} A_m. \tag{3.25}$$

Since $\operatorname{Im} g(a) = 0$ we have with the help of (3.23)

$$\big|\operatorname{Im} g(qa)\big| \leqslant q\eta\sigma^2 + \eta\sigma^2 \sum_{m=2}^{q-1} m^2 \leqslant 2q^3\eta\sigma^2$$

that proves (3.24). □

Lemma 3.8. *The estimates*

$$\big|\operatorname{Re} g'(qa) - q\, \operatorname{Re} g'(a)\big| \leqslant c\,(q^3\sigma^2 + q\beta)\,\eta^{2/3}, \tag{3.26}$$

$$\big|\operatorname{Im} g'(qa) + \theta/a\big| \leqslant c\,(q^2\sigma^2 + \beta)\,\eta^{2/3} \tag{3.27}$$

hold for all $q = 1, 2, \ldots$.

Proof. Using the argument which we employed in the proof of formula (3.25), we get the relation

$$\operatorname{Re} g'(qa) = q\, \operatorname{Re} g'(a) + q\,B_1 + \sum_{m=2}^{q-1} B_m, \tag{3.28}$$

where the quantities B_m admit the estimates

$$|B_m| \leqslant 2 \int_{-\infty}^{\infty} |x|\, \sin^2 \frac{max}{2}\, dG(x) \tag{3.29}$$

for $m = 1, 2, \ldots, q-1$. Let us estimate the integrals on the right–hand sides of these inequalities. We write them as sums of two integrals I_{m1} and I_{m2}, where

$$I_{m1} = \int\limits_{|x| \leqslant \eta^{-1/3}} |x|\, \sin^2 \frac{max}{2}\, dG(x) \quad \text{and} \quad I_{m2} = \int\limits_{|x| > \eta^{-1/3}} |x|\, \sin^2 \frac{max}{2}\, dG(x).$$

By (3.23), we have for the integrals I_{m1} $(m = 1, 2, \dots, q-1)$ the bound

$$I_{m1} \leqslant \eta^{-1/3}\big(1 - \operatorname{Re} g(ma)\big) \leqslant m^2\eta^{2/3}\sigma^2. \tag{3.30}$$

We estimate the integrals I_{m2} $(m = 1, 2, \dots, q-1)$ as follows

$$I_{m2} \leqslant \int\limits_{|x|>\eta^{-1/3}} |x|\, dG(x) \leqslant \int\limits_{|x|>\eta^{-1/3}-|\theta/a|} \big(|x| + |\theta/a|\big)\, dH(x).$$

In view of Chebyshev's inequality we see that

$$W_H(x) = H(-x) + 1 - H(x) \leqslant \frac{\beta}{x^3}$$

for all $x > 0$. Then, integrating by parts, one easily gets

$$\int\limits_{|x|>\eta^{-1/3}-\pi/c_2} \Big(|x| + \frac{\pi}{c_2}\Big)\, dH(x) \leqslant \frac{2\beta}{(\eta^{-1/3}-\pi/c_2)^2} \leqslant c\eta^{2/3}\beta. \tag{3.31}$$

From estimates (3.30) and (3.31) we obtain the upper bound

$$\int\limits_{-\infty}^{\infty} |x| \sin^2 \frac{max}{2}\, dG(x) \leqslant m^2\eta^{2/3}\sigma^2 + c\beta\eta^{2/3} \leqslant c(m^2\sigma^2 + \beta)\eta^{2/3} \tag{3.32}$$

for $m = 1, 2, \dots, q-1$, which permit us to derive from (3.28) and (3.29) the estimate (3.26).

Write the formula

$$\begin{aligned}\operatorname{Im} g'(qa) = \int\limits_{-\infty}^{\infty} x \cos(qax)\, dG(x) &= \int\limits_{-\infty}^{\infty} x\, dG(x) - 2\int\limits_{-\infty}^{\infty} x \sin^2 \frac{qax}{2}\, dG(x)\\ &= -\frac{\theta}{a} - 2\int\limits_{-\infty}^{\infty} x \sin^2 \frac{qax}{2}\, dG(x).\end{aligned}$$

According to this equality, the inequality (3.27) follows immediately from (3.32). □

Lemma 3.9. *The estimates*

$$\Big| \int_{-\infty}^{\infty} x^2\, dG(x) + \operatorname{Re} g^{(2)}(q\,a) \Big| \leqslant c\,(q^2\,\sigma^2 + \beta)\,\eta^{1/3}, \tag{3.33}$$

$$\big| \operatorname{Im} g^{(2)}(q\,a) \big| \leqslant c\,(q^2\,\sigma^2 + \beta)\,\eta^{1/3} \tag{3.34}$$

hold for all $q = 1, 2, \ldots$.

Proof. We write the formula

$$\int_{-\infty}^{\infty} x^2\, dG(x) + \operatorname{Re} g^{(2)}(q\,a) = 2 \int_{-\infty}^{\infty} x^2 \sin^2 \frac{q\,a\,x}{2}\, dG(x)$$

and estimate the integral on the right-hand side of this equality similarly to the integral from (3.29). Using upper bounds similar to (3.30) and (3.31), we get (3.33).

In order to derive (3.34) we represent $-\operatorname{Im} g^{(2)}(q\,a)$ in the form

$$\begin{aligned} -\operatorname{Im} g^{(2)}(q\,a) &= \int_{-\infty}^{\infty} x^2 \sin(q\,a\,x)\, dG(x) \\ &= \int_{|x|\leqslant 3\pi/2a} x^2 \sin(q\,a\,x)\, dG(x) + \int_{|x|> 3\pi/2a} x^2 \sin(q\,a\,x)\, dG(x) \\ &\overset{\text{def}}{=} I_{q1}^* + I_{q2}^*. \end{aligned}$$

Since $\big|\sin(a\,x/2)\big| \geqslant \sqrt{2}\,a\,|x|/3\,\pi$ for $x \in \big[-3\,\pi/2\,a,\ 3\,\pi/2\,a\big]$, we have the estimate

$$\begin{aligned} \big|I_{q1}^*\big| &\leqslant \int_{|x|\leqslant 3\pi/2a} x^2\, dG(x) = \Big(\frac{3\,\pi}{\sqrt{2}\,a}\Big)^2 \int_{|x|\leqslant 3\pi/2a} \Big(\frac{\sqrt{2}\,a\,x}{3\,\pi}\Big)^2 dG(x) \\ &\leqslant \Big(\frac{3\,\pi}{\sqrt{2}\,a}\Big)^2 \int_{-\infty}^{\infty} \sin^2 \frac{a\,x}{2}\, dG(x) \leqslant \Big(\frac{3\,\pi}{\sqrt{2}c_2}\Big)^2 \big(1 - g(a)\big). \end{aligned}$$

Together with (3.23), this inequality implies

$$\big|I_{q1}^*\big| \leqslant c\eta\,\sigma^2. \tag{3.35}$$

The integral I_{q2}^* we estimate with the help of the Cauchy–Bunyakovskii inequality. It is easy to see that

$$\begin{aligned}\big|I_{q2}^*\big| &\leqslant \int\limits_{|x|>3\pi/2a} |x|^{3/2}\cdot|x|^{1/2}\,\big|\sin(q\,a\,x)\big|\,dG(x)\\ &\leqslant \Bigg(\int\limits_{|x|>3\pi/2a}|x|^3\,dG(x)\;\cdot\int\limits_{|x|>3\pi/2a}|x|\,\sin^2(q\,a\,x)\,dG(x)\Bigg)^{1/2}.\end{aligned}$$

Noting that, for $|x|>3\pi/2\,a$, the estimate $\big|x-\theta/a\big|\leqslant 2\,|x|$ is true, we obtain the upper bound

$$\int\limits_{|x|>3\pi/2a}|x|^3\,dG(x)=\int\limits_{|x|>3\pi/2a}\big|x-\theta/a\big|^3\,dH(x)\leqslant 8\,\beta.$$

The estimate (3.32) implies therefore the relation

$$\big|I_{q2}^*\big|\leqslant c\beta^{1/2}(q^2\sigma^2+\beta)^{1/2}\eta^{1/3}\leqslant c\big(q\sigma\sqrt{\beta}+\beta\big)\,\eta^{1/3}\leqslant c(q^2\sigma^2+\beta)\,\eta^{1/3}.$$

Combining it with (3.35), we get (3.34). $\square$

Lemma 3.10. *The representation*

$$g(t)=g(q\,a)\,\exp\left\{\frac{g'(q\,a)}{g(q\,a)}\,(t-q\,a)+\frac{1}{2}\left(\frac{g^{(2)}(q\,a)}{g(q\,a)}-\left(\frac{g'(q\,a)}{g(q\,a)}\right)^2\right)(t-q\,a)^2+r_q\,(t-q\,a)^3\right\} \tag{3.36}$$

holds in the $\dfrac{1}{8\sigma}$*-neighborhood of every point* $q\,a$ $\big(q=1,\dots,[\eta^{-1/9}]\big)$, *where the quantity* r_q *admits the estimate*

$$|r_q|\leqslant 10\,\beta. \tag{3.37}$$

Proof. We first note that the modulus of the c.f. $g(t)$ satisfies the lower bound

$$\big|\,g(t)\big|\geqslant\frac{1}{2} \tag{3.38}$$

in the $\dfrac{1}{8\sigma}$-neighborhood of the points $q\,a$ $\big(q=1,\dots,[\eta^{-1/9}]\big)$. Indeed, by the Taylor formula, we have

$$\varphi(t)=\varphi(q\,a)+\Big\{\,\mathrm{Re}\,\varphi'\big(q\,a+\varkappa_1\,(t-q\,a)\big)+i\,\,\mathrm{Im}\,\varphi'\big(q\,a+\varkappa_2\,(t-q\,a)\big)\Big\}\,(t-q\,a), \tag{3.39}$$

where $0<\varkappa_p<1$, $p=1,2$. In view of the inequality $\mathbf{E}\,|X|\leqslant\sqrt{\mathbf{E}\,X^2}$ we get the estimate

$$\Big|\varphi'\big(2\,\pi q+\varkappa_p\,(t-q\,a)\big)\Big|\cdot|\,t-q\,a\,|\leqslant\mathbf{E}\,|X|\cdot|\,t-q\,a\,|\leqslant\sigma\,\frac{1}{8\,\sigma}=\frac{1}{8}$$

which permits us to obtain the required lower bound (3.38) from (3.39), (3.23) and (3.16).

Consider the functions $\log g(t)$ in the $\frac{1}{8\sigma}$-neighborhood of the points qa with $q = 1, \dots, [\eta^{-1/9}]$. We denote by $\log g(t)$ the principal branch of the logarithm of the c.f. $g(t)$. It is easy to see that if we exponentiate the Taylor formula for $\log g(t)$, we get the relation which coincides with the formula (3.36) with

$$r_q = \frac{1}{6}\Big(\frac{d^3}{dt^3}\,\mathrm{Re}\,\log g(t)\Big|_{t=qa+\varkappa_3\,(t-qa)} + i\,\frac{d^3}{dt^3}\,\mathrm{Im}\,\log g(t)\Big|_{t=qa+\varkappa_4\,(t-qa)}\Big)\,(t-qa)^3. \tag{3.40}$$

Here $0 < \varkappa_p < 1$, $p = 3, 4$. Now we need the simple formula

$$\frac{d^3}{dt^3}\,\log g(t) = \frac{d^3}{dt^3}\,\log\varphi(t) = \frac{\varphi^{(3)}(t)}{\varphi(t)} - 3\,\frac{\varphi^{(2)}(t)}{\varphi(t)}\cdot\frac{\varphi^{(1)}(t)}{\varphi(t)} + 2\Big(\frac{\varphi^{(1)}(t)}{\varphi(t)}\Big)^3.$$

Using the estimates $\left|\varphi^{(m)}(t)\right| \leqslant \mathbf{E}\,|X|^m$ which are true for all $m = 1, 2, \dots$ and $t \in \mathbb{R}^1$ and as well the inequality

$$\mathbf{E}\,|X| \leqslant \sqrt{\mathbf{E}\,X^2} \leqslant \sqrt[3]{\mathbf{E}\,|X|^3},$$

we see, keeping in mind (3.38), that

$$\Big|\frac{d^3}{dt^3}\,\log g(t)\Big| = \Big|\frac{d^3}{dt^3}\,\log\varphi(t)\Big| \leqslant 30\,\beta$$

for all t from the $\frac{1}{8\sigma}$-neighborhood of every point qa where $q = 1, \dots, [\eta^{-1/9}]$. Applying this estimate to the formula (3.40), we obtain the inequality (3.37). □

We also need some assertions concerning the coefficients corresponding to $t - qa$ and $(t - qa)^2$ in the formula (3.36). To obtain them, we need the formulas

$$\mathrm{Re}\,\frac{g^{(p)}(qa)}{g(qa)} = \mathrm{Re}\,g^{(p)}(qa)\,\frac{\mathrm{Re}\,g(qa)}{|g(qa)|^2} + \mathrm{Im}\,g^{(p)}(qa)\,\frac{\mathrm{Im}\,g(qa)}{|g(qa)|^2}\,, \tag{3.41}$$

$$\mathrm{Im}\,\frac{g^{(p)}(qa)}{g(qa)} = \mathrm{Im}\,g^{(p)}(qa)\,\frac{\mathrm{Re}\,g(qa)}{|g(qa)|^2} - \mathrm{Re}\,g^{(p)}(qa)\,\frac{\mathrm{Im}\,g(qa)}{|g(qa)|^2} \tag{3.42}$$

which are valid for $p = 1, 2$.

Lemma 3.11. *The estimates*

$$\Big|\,\mathrm{Re}\,\frac{g'(qa)}{g(qa)} - q\,\mathrm{Re}\,\frac{g'(a)}{g(a)}\,\Big| \leqslant c\,(q^3\sigma^2 + q\,\beta)\,\eta^{2/3}, \tag{3.43}$$

$$\Big|\,\mathrm{Re}\,\frac{g'(qa)}{g(qa)}\,\Big| \leqslant c\,q\,\eta^{1/2}\sigma^2, \tag{3.44}$$

$$\Big|\,\mathrm{Im}\,\frac{g'(qa)}{g(qa)} + \frac{\theta}{a}\,\Big| \leqslant c\,(q^2\sigma^2 + \beta)\,\eta^{2/3} \tag{3.45}$$

hold for $q = 1, \dots, [\eta^{-1/9}]$.

Proof. Let us first prove the inequality (3.43). According to (3.23) and (3.38), we have

$$\Big|\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-1\Big| \leqslant \frac{1-\operatorname{Re} g(qa)}{|g(qa)|^2}+\frac{1}{|g(qa)|^2}-1 \leqslant 12\,\big(1-\operatorname{Re} g(qa)\big) \leqslant 12\,q^2\eta\sigma^2. \quad (3.46)$$

In view of (3.24) and (3.38), one gets the estimate

$$\frac{|\operatorname{Im} g(qa)|}{|g(qa)|^2} \leqslant 8q^3\eta\sigma^2. \quad (3.47)$$

From the equality (3.41) with $p=1$ we obtain the bound

$$\Big|\operatorname{Re}\frac{g'(qa)}{g(qa)}-q\operatorname{Re}\frac{g'(a)}{g(a)}\Big| \leqslant \Big|\operatorname{Re} g'(qa)\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-q\operatorname{Re}\frac{g'(a)}{g(a)}\Big| + \big|\operatorname{Im} g'(qa)\big|\frac{|\operatorname{Im} g(qa)|}{|g(qa)|^2}. \quad (3.48)$$

Using (3.27), (3.47) and (3.16), one has

$$\big|\operatorname{Im} g'(qa)\big|\frac{|\operatorname{Im} g(qa)|}{|g(qa)|^2} \leqslant \Big(c(q^2\sigma^2+\beta)\eta^{2/3}+\frac{\pi}{c_2}\Big)\,8q^3\eta\sigma^2 \leqslant cq^3\eta\sigma^2. \quad (3.49)$$

On the other hand, we have, by (3.26), (3.38), (3.46) and (3.49),

$$\begin{aligned}\Big|\operatorname{Re} g'(qa)\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-q\operatorname{Re}\frac{g'(a)}{g(a)}\Big| &\leqslant \Big|\big(\operatorname{Re} g'(qa)-q\operatorname{Re} g'(a)\big)\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}\Big| \\ &+ q\,|\operatorname{Re} g'(a)|\left(\Big|\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-1\Big|+\Big|\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-1\Big|\right)+q\,|\operatorname{Im} g'(a)|\,\frac{|\operatorname{Im} g(a)|}{|g(a)|^2} \\ &\leqslant c(q^3\sigma^2+q\beta)\eta^{2/3}+q\int_{-\infty}^{\infty}|x\sin(ax)|\,dG(x)\cdot 12\,(q^2+1)\eta\sigma^2+cq\eta\sigma^2.\end{aligned}$$

Let us estimate the integral on the right-hand side-hand side of this inequality in the way which we used for estimating $\operatorname{Im} g^{(2)}(qa)$ (see the proof of Lemma 3.9). By (3.23), we have

$$\begin{aligned}\int_{-\infty}^{\infty}|x\sin(ax)|\,dG(x) &\leqslant \frac{6\pi}{\sqrt{2}\,a}\int_{-\infty}^{\infty}\sin^2\frac{ax}{2}\,dG(x) \\ &\quad + 2\Big(\int_{|x|>3\pi/2a} x^2\,dG(x)\cdot\int_{|x|>3\pi/2a}\sin^2\frac{ax}{2}\,dG(x)\Big)^{1/2} \\ &\leqslant c\eta\sigma^2+4\sigma\cdot\eta^{1/2}\sigma \leqslant c\eta^{1/2}\sigma^2. \end{aligned} \quad (3.50)$$

Applying this estimate to the preceding inequality and taking into account (3.49) and (3.16), we get from (3.48) the inequality (3.43).

We next prove the inequality (3.45). From the equality (3.42) with $p = 1$ one easily obtains

$$\Big|\operatorname{Im}\frac{g'(qa)}{g(qa)}+\frac{\theta}{a}\Big| \leqslant \Big|\operatorname{Im} g'(qa)+\frac{\theta}{a}\Big|\,\frac{|\operatorname{Re} g(qa)|}{|g(qa)|^2} + \Big|\frac{\theta}{a}\Big|\,\Big|\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}-1\Big| + \big|\operatorname{Re} g'(qa)\big|\,\frac{|\operatorname{Im} g(qa)|}{|g(qa)|^2}.$$

Since, by (3.50), the estimate

$$\big|\operatorname{Re} g'(qa)\big| \leqslant q\int_{-\infty}^{\infty}\big|x\,\sin(ax)\big|\,dG(x) \leqslant cq\eta^{1/2}\sigma^2$$

is true, we see in view of (3.27), (3.38), (3.46), (3.47) and (3.16) that the right-hand side of the preceding inequality does not exceed

$$c(q^2\sigma^2+\beta)\eta^{2/3} + cq^2\eta\sigma^2 + q\eta^{1/2}\sigma^2\cdot q^3\eta\sigma^2 \leqslant c(q^2\sigma^2+\beta)\eta^{2/3}$$

that proves (3.45).

To prove the inequality (3.44) we repeat the argument above with necessary changes. ☐

Lemma 3.12. *The estimate*

$$\Big|\frac{g^{(2)}(qa)}{g(qa)} - \Big(\frac{g'(qa)}{g(qa)}\Big)^2 + \sigma^2\Big| \leqslant c(q^2\sigma^2+\beta)\,\eta^{1/3}$$

holds for all $q = 1, \dots, [\eta^{-1/9}]$.

Proof. Consider the equality (3.41) with $p = 2$ and estimate the second summand on the right-hand side. Using (3.34), (3.47) and (3.16), we have

$$\Big|\operatorname{Im} g^{(2)}(qa)\,\frac{\operatorname{Im} g(qa)}{|g(qa)|^2}\Big| \leqslant c(q^2\sigma^2+\beta)\eta^{1/3}\cdot q^3\eta\sigma^2 \leqslant c(q^2\sigma^2+\beta)\eta^{2/3}. \tag{3.51}$$

We study below the first summand on the right-hand side of (3.41), using estimates (3.33), (3.46) and (3.16). One easily obtains

$$-\operatorname{Re} g^{(2)}(qa)\,\frac{\operatorname{Re} g(qa)}{|g(qa)|^2} = \bigg(\int_{-\infty}^{\infty} x^2\,dG(x) - \Big(\int_{-\infty}^{\infty} x^2\,dG(x) + \operatorname{Re} g^{(2)}(qa)\Big)\bigg)\frac{\operatorname{Re} g(qa)}{|g(qa)|^2}$$
$$= \int_{-\infty}^{\infty} x^2\,dG(x) + r_q^*, \tag{3.52}$$

where the quantity r_q^* admits the estimate

$$|r_q^*| \leqslant cq^2\eta\sigma^2 \cdot \int\limits_{-\infty}^{\infty} x^2\,dG(x) + c(q^2\sigma^2+\beta)\eta^{1/3}$$
$$\leqslant cq^2\eta\sigma^2\cdot(\sigma^2+1) + c(q^2\sigma^2+\beta)\eta^{1/3} \leqslant c(q^2\sigma^2+\beta)\eta^{1/3}. \qquad (3.53)$$

We now consider the equality (3.42) with $p=2$. Repeating arguments above, we see that

$$\Big|\operatorname{Im}\frac{g^{(2)}(qa)}{g(qa)}\Big| \leqslant c(q^2\sigma^2+\beta)\eta^{1/3} + c(\sigma^2+1)\cdot q^3\eta\sigma^2 \leqslant c(q^2\sigma^2+\beta)\eta^{1/3}. \qquad (3.54)$$

Using the simple formula

$$\Big(\frac{g'(qa)}{g(qa)}\Big)^2 = \Big(\operatorname{Re}\frac{g'(qa)}{g(qa)}\Big)^2 - \Big(\operatorname{Im}\frac{g'(qa)}{g(qa)}\Big)^2 + 2i\Big(\operatorname{Re}\frac{g'(qa)}{g(qa)}\Big)\Big(\operatorname{Im}\frac{g'(qa)}{g(qa)}\Big)$$

and the relations (3.44), (3.45) and (3.16), we get the estimate

$$\Big|\Big(\frac{g'(qa)}{g(qa)}\Big)^2 + \Big(\frac{\theta}{a}\Big)^2\Big| \leqslant cq^2\eta\sigma^4 + c(q^2\sigma^2+\beta)\eta^{2/3}\cdot\big((q^2\sigma^2+\beta)\eta^{2/3}+1\big)$$
$$+ cq\eta^{1/2}\sigma^2\cdot\big((q^2\sigma^2+\beta)\eta^{2/3}+1\big) \leqslant c(q^2\sigma^2+\beta)\eta^{1/3}. \qquad (3.55)$$

It is easy to see that the statement of Lemma 3.12 can be easily derived from relations (3.51)–(3.55). □

We can now complete the proof of Lemma 3.6. Its statement follows immediately from the representation (3.36) and from Lemmas 3.11 and 3.12. □

4. Proof of the results about asymptotically best constants

Proof of Theorem 2.2. We shall derive Theorem 2.2 from Theorem 2.1. In Sections 4 and 5 we shall use the notation of Sections 1 and 2. Without loss of generality we assume that Lyapunov's fraction $L \leqslant c$, where c is so small as it is necessary for the arguments below.

We first prove the theorem in the case where $M_f > L^2$, $S_* = \varnothing$ and in the case where $M_f \leqslant L^2$. In order to use the second part of Theorem 2.1 we should estimate the quantity $\alpha(x)$. By the Taylor formula, $\alpha(x)$ can be written in the form

$$\alpha(x) = -\frac{1}{6\,s^3}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\Phi^{(3)}\big(x-u_0/s\big)\,u^3\,dF_j(u) \tag{4.1}$$

where $u_0 = u_0(u)$ is real and $|u_0| < |u|$. It is easy to see that

$$\Big|\Phi^{(3)}\big(x-u_0/s\big)\Big| \leqslant \frac{1}{\sqrt{2\pi}}\,, \tag{4.2}$$

and

$$\Big|\Phi^{(3)}\big(x-u_0/s\big)-\Phi^{(3)}(x)\Big| \leqslant \sup_{v\in\mathbb{R}^1}\Big|\Phi^{(4)}\big(x-v/s\big)\Big|\,\frac{|u|}{s} \leqslant \frac{|u|}{s}\,. \tag{4.3}$$

In view of (4.1) and (4.2) we have the estimate

$$\big|\alpha(x)\big| \leqslant \frac{1}{6\sqrt{2\pi}}\,\frac{1}{s^3}\sum_{j=1}^{n}\int_{-\infty}^{\infty}|u|^3\,dF_j(u) = \frac{1}{6\sqrt{2\pi}}\,L$$

which together with the second part of Theorem 2.1 implies the inequality

$$\rho(F*\Phi_\varepsilon,\Phi) \leqslant \frac{1}{6\sqrt{2\pi}}\,L + c\,L^{1+c_1}|\log L|^{7/6}.$$

It is obvious that $\rho(\Phi*\Phi_\varepsilon,\Phi) \leqslant cL^2$. Then, applying Lemma 3.1 to the pr.d.'s P_1, P_2 and H with the d.f.'s $F(x), \Phi(x)$ and $\Phi_\varepsilon(x)$ respectively and $\tau = 10\,\varepsilon\sqrt{|\log\varepsilon|}$, one immediately obtains the estimate

$$\rho(F,\Phi) \leqslant \frac{1}{6\sqrt{2\pi}}\,L + c\,L^{1+c_1}|\log L|^{7/6} \tag{4.4}$$

which proves the theorem in the considered case.

Let now $M_f > L^2$ and $S_* \neq \varnothing$. Denote $\delta = L^{-5/12}h/s$ and write formula (4.1) in the form

$$\begin{aligned}\alpha(x) = &-\frac{1}{6\,s^3}\sum_{j=1}^{n}\int_{|u|\leqslant\delta s}\Phi^{(3)}\big(x-u_0/s\big)u^3\,dF_j(u)\\ &-\frac{1}{6\,s^3}\sum_{j=1}^{n}\int_{|u|>\delta s}\Phi^{(3)}\big(x-u_0/s\big)u^3\,dF_j(u) \stackrel{\text{def}}{=} \alpha_1(x)+\alpha_2(x).\end{aligned}$$

According to (4.2), the term $\alpha_2(x)$ admits the estimate

$$\big|\alpha_2(x)\big| \leqslant \frac{1}{6\sqrt{2\pi}\,s^3}\sum_{j=1}^{n}\int\limits_{|u|>\delta s}|u|^3\,dF_j(u). \tag{4.5}$$

Using the estimate (4.3), we rewrite the term $\alpha_1(x)$ in the form

$$\alpha_1(x) = \frac{1}{6\sqrt{2\pi}}(1-x^2)\,e^{-x^2/2}\,\frac{1}{s^3}\sum_{j=1}^{n}\int\limits_{|u|\leqslant\delta s}u^3\,dF_j(u) + \varDelta(x) \overset{\text{def}}{=} \alpha_{1,1}(x) + \varDelta(x),$$

where

$$\big|\varDelta(x)\big| \leqslant c\,\frac{\delta}{s^3}\sum_{j=1}^{n}\int\limits_{|u|\leqslant\delta s}|u|^3\,dF_j(u) \leqslant c\delta L.$$

By virtue of (2.9), we conclude that $\delta \leqslant cL^{7/12}$ and get finally

$$\big|\varDelta(x)\big| \leqslant cL^{19/12}. \tag{4.6}$$

The quantity $\alpha_{1,1}(x)$ has the obvious estimate

$$\big|\alpha_{1,1}(x)\big| \leqslant \frac{1}{6\sqrt{2\pi}}\,\frac{1}{s^3}\sum_{j=1}^{n}\Bigg|\int\limits_{|u|\leqslant\delta s}u^3\,dF_j(u)\Bigg|. \tag{4.7}$$

We now note that the modulus of the third term on the right-hand side of (2.8) does not exceed

$$\frac{h}{2\sqrt{2\pi}\,s}\,e^{-x^2/2}. \tag{4.8}$$

Introduce the quantity

$$\gamma_{J_*} = \frac{1}{6\sqrt{2\pi}}\,\frac{1}{s^3}\sum_{j\in J_*}\Bigg|\int\limits_{|u|\leqslant\delta s}u^3\,dF_j(u)\Bigg| + \frac{h}{2\sqrt{2\pi}\,s},$$

where J_* is the set of indices from Theorem 2.1. Using (4.5)–(4.8), we easily derive from the representation (2.8) the inequality

$$\rho(F*\Phi_\varepsilon,\Phi) \leqslant \gamma_{J_*} + \frac{1}{6\sqrt{2\pi}\,s^3}\sum_{j\in J_*}\int\limits_{|u|>\delta s}|u|^3\,dF_j(u)$$
$$+\frac{1}{6\sqrt{2\pi}\,s^3}\sum_{j\notin J_*}\beta_j + cL^{1+c_1}|\log L|^{7/6}. \tag{4.9}$$

Introduce random variables $Y_j = X_j/h$ with the d.f.'s $G_j(x) = F_j(hx)$ and c.f.'s $g_j(t) = f_j(t/h)$ respectively. It can be easily seen that

$$\mathbf{E}\, Y_j = 0, \quad \mathrm{Var}\ Y_j = \frac{\sigma_j^2}{h^2}, \quad \mathbf{E}\,|Y_j|^3 = \frac{\beta_j}{h^3}.$$

We shall show now that the r.v.'s Y_j $(j \in J_*)$ satisfy the assumptions of Lemma 3.5 with $\eta = L^{2-13c_1}$ and $K = 6c_1$. To this end we shall use (2.11) and the estimate $h \leqslant cLs$ which is in some sense weaker than (2.9). Applying these bounds, we see that

$$\begin{aligned} \big|g_j(2\pi)\big| = \big|f_j(2\pi/h)\big| &\geqslant 1 + c\,\frac{\log L}{L^{12c_1}}\,\frac{\sigma_j^2}{s^2} = 1 + c\,\frac{\log L}{L^{12c_1}}\Big(\frac{h}{s}\Big)^2\,\mathrm{Var}\ Y_j \\ &\geqslant 1 - L^{2-13c_1}\,\mathrm{Var}\ Y_j \end{aligned}$$

so that the condition (3.5) is satisfied. Put $T = s^2/B$. In view of (2.4), we have

$$\sigma_j^2\,T^2 = \sigma_j^2 \cdot \frac{s^4}{B^2} \leqslant \sigma_j^2 \cdot L^{-16c_1}\,\frac{\sigma_j^4}{\beta_j^2} \leqslant L^{-16c_1}$$

and therefore, taking into account the simple estimate $hT \geqslant L^{c_1}$ which follows from the definition of the parameter h (see (2.7)),

$$\mathrm{Var}\ Y_j \leqslant L^{-16c_1}(hT)^{-2} \leqslant L^{-18c_1}$$

so that the condition (3.3) is also valid. Using again (2.4), we obtain

$$\mathbf{E}\,|Y_j|^3 = \frac{\beta_j}{B} \cdot \frac{B}{h^3} \leqslant L^{-8c_1}\,\frac{\sigma_j^2}{s^2} \cdot \frac{B}{h^3} = L^{-8c_1}(hT)^{-1}\,\mathrm{Var}\ Y_j \leqslant L^{-9c_1}\,\mathrm{Var}\ Y_j,$$

i.e., the condition (3.4) holds. Thus, all assumptions of Lemma 3.5 are satisfied and, by this theorem, there exist r.v.'s Y_j^* $(j \in J_*)$ with d.f.'s from the class $\mathfrak{F}_{3,L^{2c_1}}$ such that the relations (3.6)–(3.8) are true for them with the indicated above values of the parameters η and K. From (3.6) we get the relation

$$(1 - L^{6c_1})\,\frac{1}{h^2}\sum_{j \in J_*}\sigma_j^2 \leqslant \sum_{j \in J_*}\mathrm{Var}\ Y_j^* \leqslant (1 + L^{6c_1})\,\frac{1}{h^2}\sum_{j \in J_*}\sigma_j^2$$

which, by (2.10), gives

$$(1 - 2L^{6c_1})\,\frac{s^2}{h^2} \leqslant \sum_{j \in J_*}\mathrm{Var}\ Y_j^* \leqslant (1 + L^{6c_1})\,\frac{s^2}{h^2}\,. \tag{4.10}$$

In addition, we have from (3.7) and (3.8)

$$\sum_{j\in J_*}\Big|\int\limits_{|x|\leqslant L^{-5/12}} x^3\,dG_j(x)\Big| \leqslant \sum_{j\in J_*}\Big(\big|\,\mathbf{E}\,(Y_j^*-\mathbf{E}\,Y_j^*)^3\big|+L^{2c_1}\operatorname{Var}Y_j\Big) \tag{4.11}$$

and

$$\sum_{j\in J_*}\mathbf{E}\,\big|Y_j^*-\mathbf{E}\,Y_j^*\big|^3 \leqslant \sum_{j\in J_*}\Big(\int\limits_{|x|\leqslant L^{-5/12}}|x|^3\,dG_j(x)+L^{2c_1}\operatorname{Var}Y_j\Big) \tag{4.12}$$

respectively.

Rewrite the expression for the quantity γ_{J_*} in the form

$$\begin{aligned}\gamma_{J_*} &= \frac{1}{6\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\Big|\int\limits_{|u|\leqslant\delta s/h} u^3\,dG_j(u)\Big|+\frac{h}{2\sqrt{2\pi}\,s}\\ &= \frac{1}{6\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\Big|\int\limits_{|u|\leqslant L^{-5/12}} u^3\,dG_j(u)\Big|+\frac{h}{2\sqrt{2\pi}\,s}\end{aligned}$$

and estimate it with the help of (4.11) and (4.10). We have

$$\begin{aligned}\gamma_{J_*} &\leqslant \frac{1}{6\sqrt{2\pi}}\,\frac{h^3}{s^3}\Big(\sum_{j\in J_*}\big|\,\mathbf{E}\,(Y_j^*-\mathbf{E}\,Y_j^*)^3\big|+L^{2c_1}\,\frac{s^2}{h^2}\Big)+\frac{h}{2\sqrt{2\pi}\,s}\\ &\leqslant \frac{1+L^{2c_1}}{\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\Big(\frac{1}{6}\,\big|\,\mathbf{E}\,(Y_j^*-\mathbf{E}\,Y_j^*)^3\big|+\frac{1}{2}\operatorname{Var}Y_j^*\Big).\end{aligned}$$

Using Lemma 3.3 and the inequality (4.12), we see that

$$\begin{aligned}\sum_{j\in J_*}\Big(\frac{1}{6}\,\big|\,\mathbf{E}\,(Y_j^*-\mathbf{E}\,Y_j^*)^3\big|+\frac{\operatorname{Var}Y_j^*}{2}\Big) &\leqslant \frac{\sqrt{10}+3}{6}\,(1+9L^{2c_1})\sum_{j\in J_*}\mathbf{E}\,|Y_j^*-\mathbf{E}\,Y_j^*|^3\\ &\leqslant \frac{\sqrt{10}+3}{6}\,(1+9L^{2c_1})\sum_{j\in J_*}\Big(\int\limits_{|x|\leqslant L^{-5/12}}|x|^3\,dG_j(x)+L^{2c_1}\operatorname{Var}Y_j\Big)\end{aligned}$$

and therefore

$$\begin{aligned}\gamma_{J_*} &\leqslant (1+11\,L^{2c_1})\,\frac{\sqrt{10}+3}{6\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\Big(\int\limits_{|x|\leqslant L^{-5/12}}|x|^3\,dG_j(x)+L^{2c_1}\operatorname{Var}Y_j\Big)\\ &\leqslant (1+11\,L^{2c_1})\,\frac{\sqrt{10}+3}{6\sqrt{2\pi}}\,\frac{1}{s^3}\sum_{j\in J_*}\int\limits_{|x|\leqslant\delta s}|x|^3\,dF_j(x)+c\,L^{2c_1}\,\frac{h}{s}.\end{aligned}$$

By (2.9), we have $h/s \leqslant 3L$. Applying the last two inequalities to (4.9), we obtain the estimate

$$\rho(F * \Phi_\varepsilon, \Phi) \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\, L + L^{1+c_1} |\log L|^{7/6}.$$

We now get (2.14), using Lemma 3.1 in the same way as in the proof of (4.4).

Let us prove the estimate (2.15). By virtue of (4.4), it is sufficient to establish (2.15) in the case where $M_f > L^2$ and $S_* \neq \varnothing$. Introduce the two subsets $J_*^{(1)}$ and $J_*^{(2)} = J_* \setminus J_*^{(1)}$ of the set J_* as follows. We shall say that an index j belongs to $J_*^{(1)}$ if

$$h\sigma_j^2 > \frac{2}{3} \Bigg| \int\limits_{|u| \leqslant \delta s} u^3 \, dF_j(u) \Bigg|. \tag{4.13}$$

Put

$$\zeta = -\frac{1}{6 s^3} \sum_{j=1}^{n} \int\limits_{|u| \leqslant \delta s} u^3 \, dF_j(u).$$

It is easy to see that

$$\Phi(x) - \Phi(x-\zeta) = \zeta\, \Phi'(x - \zeta_1 \zeta),$$

where $\zeta_1 \in \big(0,\, 1\big)$, and there exists some $\zeta_2 \in \big(0,\, 1\big)$ such that

$$\big| \Phi'(x - \zeta_1 \zeta) - \Phi'(x) \big| \leqslant \frac{1}{\sqrt{2\pi}}\, e^{-(x-\zeta_2\zeta)^2/2} |x - \zeta_2 \zeta| \cdot |\zeta|.$$

Since $|\zeta| \leqslant L$, the right-hand side of the last inequality does not exceed cL so that we have

$$\Phi(x) - \Phi(x-\zeta) = \zeta \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} + c\,\Theta L^2.$$

Recall that by Θ we denote quantities with $|\Theta| \leqslant 1$. According to the last formula and (4.6), we get

$$\big| \Phi(x) - \Phi(x-\zeta) + \alpha_1(x) \big| \leqslant \frac{1}{6\sqrt{2\pi}}\, x^2 e^{-x^2/2} \frac{1}{s^3} \sum_{j=1}^{n} \Bigg| \int\limits_{|u| \leqslant \delta\, s} u^3 \, dF_j(u) \Bigg| + cL^{19/12}.$$

Therefore, using (4.5) and (4.8), we obtain from the representation (2.8) that

$$\begin{aligned} \big| F * \Phi_\varepsilon(x) - \Phi(x-\zeta) \big| &\leqslant \gamma_{J_*^{(1)}}(x) + \gamma_{J_*^{(2)}} + \frac{h}{2\sqrt{2\pi}\, s}\, m(\bar{J}_*) \\ &+ \frac{1}{6\sqrt{2\pi}\, s^3} \sum_{j \in J_*} \int\limits_{|u| > \delta s} |u|^3 \, dF_j(u) + \frac{1}{6\sqrt{2\pi}\, s^3} \sum_{j \notin J_*} \beta_j + cL^{1+c_1} |\log L|^{7/6}, \end{aligned} \tag{4.14}$$

where

$$\gamma_{J_*^{(1)}}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, \frac{1}{s^3} \sum_{j\in J_*^{(1)}} \biggl(\frac{x^2}{6} \biggl| \int\limits_{|u|\leqslant \delta s} u^3\, dF_j(u) \biggr| + \frac{h}{2}\,\sigma_j^2 \biggr)$$

and

$$\gamma_{J_*^{(2)}} = \frac{1}{\sqrt{2\pi}}\, \frac{1}{s^3} \sum_{j\in J_*^{(2)}} \biggl(\frac{1}{6} \biggl| \int\limits_{|u|\leqslant \delta s} u^3\, dF_j(u) \biggr| + \frac{h}{2}\,\sigma_j^2 \biggr).$$

Under the assumption (4.13) we have

$$\sup_{x\in\mathbb{R}^1} e^{-x^2/2} \biggl(\frac{x^2}{6} \biggl| \int\limits_{|u|\leqslant \delta s} u^3\, dF_j(u) \biggr| + \frac{h}{2}\,\sigma_j^2 \biggr) \leqslant \frac{h}{2}\,\sigma_j^2.$$

Hence, in view of (3.6) for $X = Y_j$ and $X^* = Y_j^*$ $(j \in J_*)$, we obtain that

$$\begin{aligned} \gamma_{J_*^{(1)}}(x) &\leqslant \frac{1}{\sqrt{2\pi}}\, \frac{1}{s^3} \sum_{j\in J_*^{(1)}} \frac{h}{2}\,\sigma_j^2 = \frac{1}{\sqrt{2\pi}}\, \frac{h^3}{s^3} \sum_{j\in J_*^{(1)}} \frac{1}{2} \operatorname{Var} Y_j \\ &\leqslant \frac{1+L^{6c_1}}{\sqrt{2\pi}}\, \frac{h^3}{s^3} \sum_{j\in J*^{(1)}} \frac{1}{2} \operatorname{Var} Y_j^*. \end{aligned}$$

According to Lemma 3.2, we have for the r.v.'s Y_j^* $(j \in J_*)$:

$$(1 - 8L^{2c_1})\, \frac{1}{2} \operatorname{Var} Y_j^* \leqslant \mathbf{E}\, | Y_j^* - \mathbf{E}\, Y_j^* |^3. \tag{4.15}$$

Using (4.15) and (4.12), we get finally

$$\begin{aligned} \gamma_{J_*^{(1)}}(x) &\leqslant \frac{1+L^{c_1}}{\sqrt{2\pi}}\, \frac{h^3}{s^3} \sum_{j\in J_*^{(1)}} \mathbf{E}\, | Y_J^* - \mathbf{E}\, Y_J^* |^3 \\ &\leqslant \frac{1+L^{c_1}}{\sqrt{2\pi}} \cdot \frac{1}{s^3} \biggl(\sum_{j\in J_*^{(1)}} \int\limits_{|u|\leqslant \delta s} |u^3|\, dF_j(u) + L^{2c_1} h s^2 \biggr). \end{aligned} \tag{4.16}$$

According to the definition of the set $J_*^{(2)}$, the quantity $\gamma_{J_*^{(2)}}$ does not exceed

$$\frac{1}{\sqrt{2\pi}}\, \frac{1}{s^3} \sum_{j\in J_*^{(2)}} \int\limits_{|u|\leqslant \delta s} |u^3|\, dF_j(u). \tag{4.17}$$

Applying (4.16) and (4.17) to (4.14) and taking into account (2.9) and (2.10), we obtain the inequality

$$\bigl|\,F * \Phi_\varepsilon(x) - \Phi(x-\zeta)\,\bigr| \leqslant \frac{1}{\sqrt{2\pi}}\,L + c\,L^{1+c_1}|\,\log L\,|^{7/6}.$$

It is easy to see that this inequality implies

$$\bigl|\,F * \Phi_\varepsilon(x) - (\Phi * \Phi_\varepsilon)(x-\zeta)\,\bigr| \leqslant \frac{1}{\sqrt{2\pi}}\,L + c\,L^{1+c_1}|\,\log L\,|^{7/6}.$$

To prove (2.15), we apply Lemma 3.1 to the last inequality similarly to the proof of (4.4). □

Proof of (2.17). We shall argue similarly to the proof of (2.14). Therefore we use the same notation. By virtue of (4.4), it suffices to establish (2.17) in the case $M_f > L^2$ and $S_* \neq \varnothing$. We use again the relation (4.9) in which the expression for γ_F has now the form

$$\gamma_F = \frac{h}{2\sqrt{2\pi}\,s}$$

and, by (2.10) and (3.6) for r.v.'s $X = Y_j$, $X^* = Y_j^*$, admits the estimate

$$\gamma_F \leqslant (1 + L^{7c_1})\,\frac{1}{s^3}\sum_{j\in J_*}\frac{h}{2}\,\sigma_j^2 \leqslant \frac{1+L^{6c_1}}{\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\frac{1}{2}\,\mathrm{Var}\,Y_j^*.$$

According to this inequality and estimates (4.15) and (4.12), we get

$$\begin{aligned}
\gamma_F &\leqslant \frac{1+L^{c_1}}{\sqrt{2\pi}}\,\frac{h^3}{s^3}\sum_{j\in J_*}\mathbf{E}\,|\,Y_j^* - \mathbf{E}\,Y_j^*|^3 \\
&\leqslant \frac{1+L^{c_1}}{\sqrt{2\pi}}\,\frac{1}{s^3}\sum_{j\in J_*}\biggl(\int\limits_{|u|\leqslant\delta s}|u|^3\,dF_j(u) + L^{2c_1}h\,\sigma_j^2\biggr) \\
&= \frac{1+L^{c_1}}{\sqrt{2\pi}}\,\biggl(\frac{1}{s^3}\sum_{j\in J_*}\int\limits_{|u|\leqslant\delta s}|u|^3\,dF_j(u) + L^{2c_1}\,\frac{h}{s}\biggr).
\end{aligned}$$

Applying this estimate to (4.9) and using that $h/s \leqslant 3L$, we easily obtain

$$\rho(F * \Phi_\varepsilon, \Phi) \leqslant \frac{1}{\sqrt{2\pi}}\,L + c\,L^{1+c_1}|\,\log L\,|^{7/6}.$$

Now (2.17) follows from Lemma 3.1. □

It remains to note that Corollary 2.3 follows directly from this theorem if one takes into account the second parts of Theorems A and B.

5. A new asymptotic expansion in Lyapunov's theorem

In this Section we shall use all the notation of Sections 1 and 2. Our main aim is to prove Theorem 2.1 with its corollaries. Their proofs are based on the well-known inversion formula

$$(F*\Phi_\varepsilon)(x)-(\Phi*\Phi_\varepsilon)(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-itx}}{-it}\big(f(t)-e^{-t^2/2}\big)\,e^{-\varepsilon^2t^2/2}\,dt. \tag{5.1}$$

We write the integral in (5.1) as the sum of the two integrals I_1 and I_2 taken over the sets $\Gamma_1=\big[-1/4L,\,1/4L\big]$ and $\Gamma_2=\big(-\infty,\,-1/4L\big)\cup\big(1/4L,\,\infty\big)$ respectively, and estimate these integrals separately.

For this we need two auxiliary Theorems 5.1 and 5.2. Below $\alpha(x)$ is the quantity introduced in Section 2.

Theorem 5.1. *The following relation holds*

$$I_1=\alpha(x)+c\,\Theta L^{4/3}.$$

A similar result was proved in Chistyakov (1990, pp. 298–300), see also Bentkus (1994). We shall provide a proof for the sake of completeness.

Proof. We shall use the well-known inequality

$$\big|f_j(t/s)-1\big|\leqslant\frac{\sigma_j^2}{2s^2}\,t^2\leqslant\frac{\beta_j^{2/3}}{2\,s^2}\,t^2\leqslant\frac{1}{2}\,L^{2/3}t^2,\qquad t\in\mathbb{R}^1. \tag{5.2}$$

Taking into account (5.2), we get the expansion

$$\log f_j(t/s)=-\sum_{k=1}^{\infty}\frac{1}{k}\big(1-f_j(t/s)\big)^k$$

for $|t|\leqslant L^{-1/3}$ and $j=1,2,\dots,n$. This expansion and the estimate (5.2) permit to obtain that, for $|t|\leqslant L^{-1/3}$,

$$\begin{aligned}\sum_{j=1}^{n}\big|\log f_j(t/s)+1-f_j(t/s)\big|&\leqslant\sum_{j=1}^{n}\sum_{k=2}^{\infty}\frac{1}{k}\Big(\frac{\sigma_j^2}{2\,s^2}\,t^2\Big)^k\leqslant\frac{1}{4\,s^4}\sum_{j=1}^{n}\sigma_j^4t^4\\&\leqslant\frac{t^4}{4}\big(\max_m\sigma_m/s\big)\sum_{j=1}^{n}\frac{\beta_j}{s^3}\leqslant\frac{1}{4}\,L^{4/3}t^4.\end{aligned} \tag{5.3}$$

For the same t we have

$$f(t) - e^{-t^2/2} = e^{-t^2/2} R(t) \prod_{j=1}^{n} \exp\Big\{ f_j(t/s) - 1 + \frac{\sigma_j^2}{2\,s^2}\, t^2 \Big\} + e^{-t^2/2} \bigg(\prod_{j=1}^{n} \exp\Big\{ f_j(t/s) - 1 + \frac{\sigma_j^2}{2\,s^2}\, t^2 \Big\} - 1 \bigg), \tag{5.4}$$

where

$$R(t) = f(t) \prod_{j=1}^{n} e^{-f_j(t/s)+1} - 1.$$

According to (5.3),

$$\big|R(t)\big| \leqslant \frac{1}{4}\, e^{1/4} L^{4/3} t^4, \qquad |t| \leqslant L^{-1/3}. \tag{5.5}$$

We need also the following well-known bound

$$\sum_{j=1}^{n} \Big| \, f_j(t/s) - 1 + \frac{\sigma_j^2}{2\,s^2} \Big| \leqslant \frac{|t|^3}{6\,s^3} \sum_{j=1}^{n} \beta_j = \frac{1}{6}\, L\, |t|^3, \qquad t \in \mathbb{R}^1. \tag{5.6}$$

Applying (5.5) and (5.6) to (5.4), we easily get the inequality

$$\Big| \, f(t) - e^{-t^2/2} - e^{-t^2/2} \Big(\frac{1}{2}\, t^2 + \sum_{j=1}^{n} \big(f_j(t/s) - 1 \big) \Big) \Big| \leqslant c L^{4/3} t^4 \, \exp\Big\{ -\frac{1}{2}\, t^2 + \frac{1}{6}\, L\, |t|^3 \Big\} + c L^2\, |t|^6\, e^{-t^2/2} \tag{5.7}$$

for $|t| \leqslant L^{-1/3}$. We need also the following well-known result

$$\big| f(t) - e^{-t^2/2} \big| \leqslant 16\, L\, |t|^3 e^{-t^2/3}, \qquad |t| \leqslant 1/4L \tag{5.8}$$

(see Petrov (1975, p. 109)). Using (5.8), we obtain

$$\Big| \big(e^{-\varepsilon^2 t^2/2} - 1 \big) \big(f(t) - e^{-t^2/2} \big) \Big| \leqslant c\, \varepsilon^2 L\, |t|^5\, e^{-t^2/3}, \qquad |t| \leqslant 1/4\,L. \tag{5.9}$$

It is easy to see that

$$\alpha(x) = -\frac{1}{2\,\pi} \int_{-\infty}^{\infty} \frac{e^{-itx}}{it}\, e^{-t^2/2} \Big(\frac{1}{2}\, t^2 + \sum_{j=1}^{n} \big(f_j(t/s) - 1 \big) \Big)\, dt.$$

Therefore, by the definition of the integral $I_1(x)$, we have

$$\begin{aligned}\big|I_1(x)-\alpha(x)\big| &\leqslant c\int\limits_{|t|\leqslant L^{-1/3}}\frac{1}{|t|}\Big|f(t)-e^{-t^2/2}-e^{-t^2/2}\Big(\frac{1}{2}t^2+\sum_{j=1}^n\big(f_j(t/s)-1\big)\Big)\Big|\,dt\\ &+c\int\limits_{|t|\leqslant L^{-1/3}}\frac{1}{|t|}\big|e^{-\varepsilon^2t^2/2}-1\big|\cdot\big|f(t)-e^{-t^2/2}\big|\,dt\\ &+c\int\limits_{L^{-1/3}<|t|\leqslant 1/4L}\frac{1}{|t|}\,e^{-\varepsilon^2t^2/2}\big|f(t)-e^{-t^2/2}\big|\,dt\\ &+c\int\limits_{|t|>L^{-1/3}}\frac{1}{|t|}\,e^{-t^2/2}\Big(\frac{1}{2}t^2+\sum_{j=1}^n\big|f_j(t/s)-1\big|\Big)\,dt.\end{aligned}$$

The first, the second, the third and the fourth integrals on the right-hand side of this inequality can be estimated with the help of (5.7), (5.9), (5.8) and (5.2) respectively. Each of these integrals does not exceed $cL^{4/3}$ that it was to be proved. $\square$

Theorem 5.2. *Let $M_f > L^2$ and let τ be a point of the interval Γ such that*

$$\big|f(\tau)\big| \geqslant L^2. \tag{5.10}$$

Then there exists a positive integer d, satisfying the inequality $d \leqslant 80\,\tau L$, such that the estimate

$$\big|f(t)\big| < L^2 \tag{5.11}$$

holds for $t \in \Gamma$ outside the union of the $\varkappa_q$-neighborhoods of the points $q\tau/d$ $\big(q = 1,\dots,[L^{-1-c_1}d/\tau]+1\big)$, where

$$\varkappa_q = 4\cdot 10^5\,\frac{\tau}{d}\,\Big(\frac{q+1}{d}+\tau L\Big)\,L\,|\log L|^{1/2}.$$

Proof. Introduce the r.v.'s $Z_j = TX_j$ $(j = 1,2,\dots,n)$ where, as in the proof of Theorem 2.2, $T = s^2/B$. The r.v.'s Z_j have the d.f.'s $H_j(x) = F_j(x/T)$ and the c.f.'s $\varphi_j(t) = f_j(Tt)$. Consider the $\varphi_j(t)$ satisfying the inequality

$$\big|\varphi_j(\tau^*)\big| \geqslant \exp\Big\{4\,\frac{\sigma_j^2}{s^2}\,\log L\Big\}, \tag{5.12}$$

where $\tau^* = \tau L$. The set of indices j which corresponds to these c.f.'s we denote by $J^{(1)}$. Using (5.10), we see that $m(J^{(1)}) \geqslant 1/2$.

Introduce the set $J^{(2)}$ of indices j for which

$$\frac{\beta_j}{B} \leqslant 4\,\frac{\sigma_j^2}{s^2}\,. \tag{5.13}$$

It is easy to see that $m(J^{(2)}) \geqslant 3/4$. Note that in the case when all X_j are i.i.d. we have $J^{(1)} = J^{(2)} = \big\{1, 2, \dots, n\big\}$.

Define the set $J^{(3)} = J^{(1)} \cap J^{(2)}$. It is obvious that

$$m(J^{(3)}) \geqslant 1/4. \tag{5.14}$$

Denote by Z_{js} a symmetrization of the r.v. Z_j, i.e., $Z_{js} = Z_j - Z_j'$, where Z_j' is an independent copy of Z_j. Let $H_{js}(x)$ be the d.f. of Z_{js}.

Lemma 5.1. *For all $j \in J^{(1)}$, we have the inequality*

$$\int\limits_{|x| \leqslant \pi/\tau^*} x^2\, dH_{js}(x) \leqslant -2\,\pi^2 \Big(\frac{\sigma_j T L}{\tau^*}\Big)^2 \log L.$$

Proof. Assume that there exists an index j such that

$$\int\limits_{|x| \leqslant \pi/\tau^*} x^2\, dH_{js}(x) > -2\,\pi^2 \Big(\frac{\sigma_j T L}{\tau^*}\Big)^2 \log L.$$

With the help of the inequality $\sin x \geqslant 2\,x/\pi$ $\big(0 \leqslant x \leqslant \pi/2\big)$ we conclude that

$$\begin{aligned}
1 - \big|\varphi_j(\tau^*)\big|^2 &= 2 \int\limits_{-\infty}^{\infty} \sin^2 \frac{\tau^* x}{2}\, dH_{js}(x) \geqslant 2 \int\limits_{|x| \leqslant \pi/\tau^*} \sin^2 \frac{\tau^* x}{2}\, dH_{js}(x) \\
&\geqslant 2\,\Big(\frac{\tau^*}{\pi}\Big)^2 \int\limits_{|x| \leqslant \pi/\tau^*} x^2\, dH_{js}(x) > -2\,\Big(\frac{\tau^*}{\pi}\Big)^2\, 2\,\pi^2 \Big(\frac{\sigma_j T L}{\tau^*}\Big)^2 \log L \\
&= -4\,\frac{\sigma_j^2}{s^2}\, \log L.
\end{aligned}$$

Therefore

$$\big|\varphi_j(\tau^*)\big|^2 < 1 + 4\,\frac{\sigma_j^2}{s^2}\, \log L \leqslant \exp\Big\{4\,\frac{\sigma_j^2}{s^2}\, \log L\Big\}.$$

This contradicts the estimate (5.12). □

Given an integer k, let us introduce the set

$$V_k = \Big\{x : \big|x - 2\,\pi\, k/\tau^*\big| < c_3\, \frac{L\,|\log L|^{1/2}}{\tau^*}\Big\},$$

where $c_3 = 400\,\pi$.

Lemma 5.2. *For every $j \in J^{(3)}$ there exists an integer k_j satisfying the inequalities $0 < |k_j| \leqslant 40\,\tau^*/\pi$ and*

$$\int\limits_{V_{k_j}} dH_{js}(x) \geqslant c_4 \,\frac{\sigma_j^2 T^2}{\tau^*}\,, \tag{5.15}$$

where $c_4 = 250^{-3}$.

Proof. Denote by $\widetilde{\beta}_{js}$ the third absolute moment of the r.v. Z_{js}. Using the simple estimate $\widetilde{\beta}_{js} \leqslant 8\beta_j T^3$, we obtain with the help of Chebyshev's inequality the estimate

$$W_{js}(x) = H_{js}(-x) + 1 - H_{js}(x) \leqslant \frac{8\beta_j T^3}{x^3} \tag{5.16}$$

for all $x > 0$. Then, integrating by parts and taking into account (5.13), we obtain, for any $N > 0$,

$$\begin{aligned} \int\limits_{|x|>N} x^2\, dH_{js}(x) &\leqslant N^2 W_{js}(N) + 2 \int\limits_{x>N} x\, W_{js}(x)\, dx \\ &\leqslant \frac{8}{N}\,\beta_j T^3 + 16\,\beta_j\, T^3 \int\limits_{x>N} \frac{1}{x^2}\, dx \\ &\leqslant \frac{24}{N}\,\beta_j T^3 \leqslant \frac{96}{N}\,\sigma_j^2\, T^2. \end{aligned} \tag{5.17}$$

Note now that for $j \in J^{(1)}$ the r.v. $X = \tau^* Z_{js}/2\,\pi$ satisfy the conditions of Lemma 3.4 (with $\nu = -8\,(\sigma_j/s)^2 \log L$). By this lemma, we obtain the inequality

$$\sum_{k\in\mathbb{Z}} \int\limits_{\{a/\tau^* \leqslant |x| \leqslant \pi/\tau^*\}\setminus\{-\pi/\tau^*\}} dH_{js}\left(x + \frac{2\,\pi\, k}{\tau^*}\right) \leqslant -\left(\frac{2\,\pi\,\sigma_j}{a\, s}\right)^2 \log L \tag{5.18}$$

which is valid for all $a \in \big(0,\,\pi\big)$. Choosing $a = a_0 = 400\,\pi\, L\,|\log L\,|^{1/2}$, we see that the right-hand side of the inequality (5.18) is then equal to $(\sigma_j\, T/200)^2$. Denote by V the set

$$\bigcup_{k\in\mathbb{Z}} \left(\left\{x:\, \frac{a_0}{\tau^*} \leqslant \left|x - \frac{2\,\pi\, k}{\tau^*}\right| \leqslant \frac{\pi}{\tau^*}\right\} \setminus \left\{-\frac{\pi}{\tau^*}\right\}\right).$$

Using Lemma 5.1 and the inequality (5.17) with $N = 80$, we obtain the estimate

$$\int\limits_{\pi/\tau^* \leqslant |x| \leqslant 80} x^2\, dH_{js}(x) \geqslant 2\,\sigma_j^2\, T^2 + 2\,\pi^2 \left(\frac{\sigma_j\, T\, L}{\tau^*}\right)^2 \log L - \frac{6}{5}\,\sigma_j^2\, T^2 \geqslant \frac{3}{4}\,\sigma_j^2\, T^2.$$

Therefore, from (5.18) with $a = a_0$ it follows the inequality

$$\int\limits_{\{\pi/\tau^*\leqslant|x|\leqslant 80\}\setminus V} x^2\, dH_{js}(x) \geqslant \frac{3}{4}\,\sigma_j^2\, T^2 - \frac{1}{5}\,\sigma_j^2\, T^2 > \frac{1}{2}\,\sigma_j^2\, T^2 \tag{5.19}$$

which implies the statement of Lemma 5.2. Indeed, if Lemma 5.2 were not true, then the inequality opposite to (5.15) would be valid for some $j \in J^{(3)}$ and for all integer k_j satisfying the inequalities $0 < |k_j| \leqslant 40\,\tau^*/\pi$. This implies an estimate which contradicts (5.19). $\square$

Denote by $P_{1s}, \dots, P_{ns}$ the pr.d.'s of the r.v.'s $Z_{1s}, \dots, Z_{ns}$ respectively. By Lemma 5.2, for every measure P_{js} with $j \in J^{(3)}$ there exist at least two integers k such that

$$P_{js}(V_k) \geqslant c_4\, \frac{\sigma_j^2\, T^2}{\tau^*}\,. \tag{5.20}$$

Denote by d_j the minimal distance between different integers k satisfying (5.20).

Lemma 5.3. *For all $j \in J^{(3)}$, the parameters d_j admit the estimate*

$$d_j \leqslant 80\,\tau^*.$$

Proof. For all $j \in J^{(3)}$, denote by A_j the set union of the neighborhoods V_k, satisfying the condition (5.20), and introduce the pr.d.'s P_{js}^* by

$$P_{js}^*(S') = P_{js}(S' \cap A_j)\big/P_{js}(A_j)$$

for all Borel's sets S'. In order to apply Lemma 3.2 to the pr.d. $P_{js}^*(S')$ we need some auxiliary inequalities.

In view of (5.17) we have

$$\int\limits_{|x|>192} x^2\, dP_{js} \leqslant \frac{1}{2}\,\sigma_j^2\, T^2. \tag{5.21}$$

In addition, taking into account (5.18) and the definition of the set A_j, one obtains

$$\int\limits_{\overline{A}_j\cap\,[-192,192]} x^2\, dP_{js} \leqslant \frac{192^2}{200^2}\,\sigma_j^2\, T^2 + 192^2 \cdot \frac{192\,\tau_*}{\pi}\, c_4\, \frac{\sigma_j^2\, T^2}{\tau^*} \leqslant \frac{4}{3}\,\sigma_j^2\, T^2 \tag{5.22}$$

where $\overline{A}_j = \mathbb{R}^1 \setminus A_j$. Since

$$\int\limits_{-\infty}^{\infty} x^2\, dP_{js} = 2\,\sigma_j^2\, T^2,$$

we see, by (5.21) and (5.22), that

$$\int\limits_{-\infty}^{\infty} x^2\, dP_{js}^* \geqslant \frac{1}{6}\,\big(P_{js}(A_j)\big)^{-1} \sigma_j^2\, T^2. \tag{5.23}$$

On the other hand, it can be easily seen that

$$\int\limits_{-\infty}^{\infty} |x|^3 \, dP^*_{js} \leqslant \big(P_{js}(A_j)\big)^{-1} \int\limits_{-\infty}^{\infty} |x|^3 \, dP_{js}. \tag{5.24}$$

For $j \in J^{(3)}$ introduce the r.v.'s W_j with pr.d. P^*_{js} respectively. The d.f.'s of the r.v.'s $\tau^* W_j/2\pi$ belong to the set $\mathfrak{F}_{3,\varkappa}$, where $\varkappa = \frac{c_3}{2\pi} L\,|\log L\,|^{1/2}$. Then, applying Lemma 3.2 and taking into account the estimates (5.23), (5.24) and (5.13), we obtain the inequalities

$$\frac{1}{2}\Big(d_j - \frac{1}{2}\Big)\frac{2\pi}{\tau^*}\,\frac{1}{6}\,\frac{\sigma_j^2 T^2}{P_{js}(A_j)} \leqslant \frac{1}{2}\Big(d_j - \frac{1}{2}\Big)\frac{2\pi}{\tau^*}\int\limits_{-\infty}^{\infty} x^2 \, dP^*_{js}$$

$$\leqslant \int\limits_{-\infty}^{\infty} |x|^3 \, dP^*_{js} \leqslant \frac{8\beta_j}{P_{js}(A_j)} \leqslant \frac{32\,\sigma_j^2 T^2}{P_{js}(A_j)}$$

which imply the statement of Lemma 5.3. □

Lemma 5.4. *For all $j \in J^{(3)}$, the c.f. $\varphi_j(t)$ admits the estimate*

$$\big|\,\varphi_j(t)\big| < \exp\Big\{640\,\tau^* \frac{\sigma_j^2}{s^2}\,\log L\Big\} \tag{5.25}$$

for $t \in \big[1/4,\, L^{-c_1}\big]$ outside the union of δ_q-neighborhoods of the points $q\tau^*/d_j$ $\big(q = 1, \dots, [L^{-c_1} d_j/\tau^*] + 1\big)$, where

$$\delta_q = 4 \cdot 10^5 \,\frac{\tau^*}{d_j}\Big(\frac{q+1}{d_j} + \tau^*\Big) L\,|\log L\,|^{1/2}.$$

Proof. Assume that there exist an index $j \in J^{(3)}$ and a real point $w^*_j \in \big[1/4,\, L^{-c_1}\big]$ of the form $w^*_j = q_j\,\tau^*/d_j + \varDelta_j$, where $q_j \in \mathbb{Z}$ and $|\varDelta_j| \in \big[\delta_{q_j},\, \tau^*/2\,d_j\big]$, such that

$$\big|\,\varphi_j(w^*_j)\big| \geqslant \exp\Big\{640\,\tau^* \frac{\sigma_j^2}{s^2}\,\log L\Big\}. \tag{5.26}$$

In order to simplify the notation we write below q, w, $\varDelta$ instead of q_j, w^*_j, $\varDelta_j$ respectively. We obtain from (5.26) the inequality

$$\sum_{p=1}^{2} \int\limits_{V_{\nu_p}} \sin^2 \frac{w\,x}{2}\, dH_{js}(x) \leqslant -1280\,\tau^* \frac{\sigma_j^2}{s^2}\,\log L, \tag{5.27}$$

where ν_1 and ν_2 are integers such that $\nu_2 - \nu_1 = d_j$ and

$$P_{js}(V_{\nu_p}) \geqslant c_4\,\frac{\sigma_j^2 T^2}{\tau^*} \tag{5.28}$$

for $p = 1, 2$.

Consider the integral on the left-hand side of (5.27) with $p = 1$. Each point x from V_{ν_1} can be represented in the form $x = w_1 + u$, where $w_1 = 2\pi\nu_1/\tau^*$ and $|u| \leqslant c_3 L\,|\log L\,|^{1/2}/\tau^*$. Hence, we have the formula

$$\sin\left(\frac{w\,w_1}{2} + \frac{w\,u}{2}\right) = \sin\frac{w\,w_1}{2}\,\cos\frac{w\,u}{2} + \cos\frac{w\,w_1}{2}\,\sin\frac{w\,u}{2}.$$

Since $\bigl|\sin(w\,u/2)\bigr| \leqslant |w\,u|/2$ and $1 - \cos^2(w\,u/2) \leqslant (w\,u)^2/2$, we obtain the inequality

$$\left|\sin^2\left(\frac{w\,w_1}{2} + \frac{w\,u}{2}\right) - \sin^2\frac{w\,w_1}{2} - \frac{1}{2}\sin(w\,w_1)\,\sin(w\,u)\right| \leqslant (w\,u)^2. \tag{5.29}$$

Consider two possible cases:

$$\left|\sin\frac{w\,w_1}{2}\right| \leqslant 4\,w\,\frac{c_3 L\,|\log L\,|^{1/2}}{\tau^*} \tag{5.30}$$

and

$$\left|\sin\frac{w\,w_1}{2}\right| > 4\,w\,\frac{c_3 L\,|\log L\,|^{1/2}}{\tau^*}.$$

Consider the second case. According to the estimate

$$\left|\sin\frac{w\,w_1}{2}\right| > 4\,|w\,u| \geqslant 4\,\bigl|\sin(w\,u)\bigr|,$$

we get from (5.29) that

$$\sin^2\left(\frac{w\,w_1}{2} + \frac{w\,u}{2}\right) \geqslant \frac{3}{4}\sin^2\frac{w\,w_1}{2} - (w\,u)^2 \geqslant \frac{1}{2}\sin^2\frac{w\,w_1}{2}.$$

Then, by (5.27), one concludes that

$$\frac{1}{2}\sin^2\frac{w\,w_1}{2}\,P_{js}(V_{\nu_1}) \leqslant -1280\,\tau^*\,\frac{\sigma_j^2}{s^2}\,\log L.$$

Therefore, taking into account (5.28), we obtain that

$$\sin^2\frac{w\,w_1}{2} \leqslant -\frac{2560}{c_4}\,(\tau^*)^2\,L^2\log L.$$

Combining this inequality with (5.30) we see that

$$\left|\sin\frac{w\,w_1}{2}\right| \leqslant \varepsilon_q = \max\left\{4c_3\,w\,\frac{L\,|\log L\,|^{1/2}}{\tau^*},\ \frac{51}{\sqrt{c_4}}\,\tau^* L\,|\log L\,|^{1/2}\right\}.$$

In a similar way one can verify the same inequality for the quantity $\left|\sin(w\,w_2/2)\right|$ where $w_2 = 2\,\pi\,\nu_2/\tau^*$. From the last two inequalities it follows that there exist integers n_1 and n_2 such that

$$\Big(\frac{q\tau^*}{d_j}+\Delta\Big)\frac{\pi\,\nu_1}{\tau^*}=n_1\pi+\theta_1,\qquad \Big(\frac{q\tau^*}{d_j}+\Delta\Big)\frac{\pi\,\nu_2}{\tau^*}=n_2\pi+\theta_2,$$

where $|\theta_p|\leqslant\frac{\pi}{2}\varepsilon_q$ $(p=1,2)$. Deducting from the second equality the first one, we conclude that

$$\Delta\,\frac{\pi\,d_j}{\tau^*}=(n_2-n_1-q)\,\pi+\theta_2-\theta_1.$$

Since $|\Delta|\leqslant\tau^*/2\,d_j$, then $n_2-n_1-q=0$ and we have $\Delta=\tau^*(\theta_2-\theta_1)/\pi\,d_j$. This relation contradicts the inequality $|\Delta|\geqslant\delta_q$ because of the chain of the estimates

$$|\,\theta_2-\theta_1|\,\frac{\tau^*}{\pi\,d_j}\leqslant\pi\varepsilon_q\,\frac{\tau^*}{\pi\,d_j}<\delta_q.\qquad\square$$

Lemma 5.5. *There exists a set of indices* $J^{(4)}$ $\big(J^{(4)}\subset J^{(3)}\big)$ *such that its measure satisfies the inequality* $m\big(J^{(4)}\big)\geqslant 1/320\,\tau^*$ *and* d_j *have only one value* d *for all* $j\in J^{(4)}$.

Proof. We split the set $J^{(3)}$ into disjoint sets such that d_j is constant on each of them. From Lemma 5.4 it follows that the number of such sets does not exceed $80\,\tau^*$. Therefore we obtain the statement of Lemma 5.5 from the estimate (5.14). $\square$

We now complete the proof of Theorem 5.2. Its statement follows from Lemmas 5.4 and 5.5. Indeed, by Lemma 5.4, the inequality

$$\begin{aligned}\big|f(t/L)\big|=\prod_{j=1}^{n}|\,\varphi_j(t)|\leqslant\prod_{j\in J^{(4)}}|\,\varphi_j(t)|<\prod_{j\in J^{(4)}}\exp\Big\{640\,\tau^*\,\frac{\sigma_j^2}{s^2}\,\log L\Big\}\\ =\exp\Big\{640\,\tau^*\,m\big(J^{(4)}\big)\,\log L\Big\}\end{aligned}$$

is valid for $t\in\big[1/4,\,L^{-c_1}\big]$ outside the union of the δ_q-neighborhoods of the points $\big\{q\tau^*/d\,:\,q=1,\ldots,\,[L^{-c_1}d/\tau^*]+1\big\}$. Then the statement of Theorem 5.2 easily follows from Lemma 5.5. $\square$

Proof of Theorem 2.1. In order to obtain the representation (2.8) we return to the formula (5.1) and estimate the integrals I_1 and I_2. By Theorem 5.1, we have the required relation for the integral I_1. We therefore need to estimate the integral I_2 only.

We first consider the case where $M_f>L^2$ and $S_*\neq\varnothing$. In Section 2 we have defined the set of points $t_1,\ldots,t_r$ and the point t_*. By Theorem 5.2, in which we assume $\tau = t_*$, there exists a positive integer d, satisfying the inequality $d \leqslant 80\, t_* L$, and such that the estimate (5.11) holds for $t \in \Gamma$ outside the union of the $\varkappa_q^*$-neighborhoods of the points $q t_*/d$ $\bigl(q = 1, \ldots, [L^{-1-c_1} d/t_*] + 1\bigr)$, where

$$\varkappa_q^* = 4 \cdot 10^5\, \frac{\tau_*}{d} \left(\frac{q+1}{d} + \tau_* \right) |\log L|^{1/2}. \tag{5.31}$$

Here and below $\tau_* = t_* L$. Write

$$a_{q,k} = \left(q + \frac{k}{d} \right) \tau_*, \tag{5.32}$$

where q and k are integers such that $0 \leqslant k < d$. Denote by Q the set of the points $a_{q,k}$ possessing the property: there exists at least one point t_m in the $\varkappa_{qd+k}^*$-neighborhood of the point $a_{q,k}/L$.

Introduce now a subset Q_* of the set Q in the following way. Let

$$\mathbb{K} = \bigl\{ k : 0 \leqslant k < d,\ \exists\, q : a_{q,k} \in Q \bigr\}.$$

Then we define

$$Q_* = \Bigl\{ \min\bigl\{ a_{q,k} : q = 1, \ldots, [L^{-1-c_1} d/t_*] + 1 \bigr\} :\ k \in \mathbb{K} \Bigr\}.$$

It is obvious that the point τ_* belongs to Q_*.

Denote by D the greatest common divisor of the integers from the set $\mathbb{K} \cup d$. It is easy to see that the parameter d_* introduced in Section 2 is equal to d/D and the number of points of the set Q_* does not exceed d_*.

From the definition of the set $\bigl\{t_1, \ldots, t_r\bigr\}$ and Theorem 5.2 it follows that the number of these points r does not exceed $L^{-c_1} d/\tau_* \leqslant 80\, L^{-c_1} + 1$. Hence, the estimates (2.3) and (2.5) immediately imply the lower bound (2.10).

Our next step is to show that for $j \in J_*$ the c.f.'s $f_j(t)$ satisfy the estimate (2.11). For this it is convenient to return to the r.v.'s $Z_j = T X_j$ with the d.f. $H_j(x)$ and c.f. $\varphi_j(t)$ which were introduced at the beginning of the proof of Theorem 5.2. We shall prove a lemma which will give us the required result.

Lemma 5.6. *The estimate*

$$\bigl|\varphi_j\bigl(\tau_*/d_*\bigr)\bigr| \geqslant 1 - c\eta_*\, \sigma_j^2\, T^2 = 1 - c\,(\eta_*\, L^{-2})\, \frac{\sigma_j^2}{s^2} \tag{5.33}$$

is valid for all indices $j \in J_*$, where

$$\eta_* = d_*^3 \left(1 + \frac{L^{-2c_1}}{\tau_*^2} \right) L^{2-9c_1} |\log L|. \tag{5.34}$$

Proof. According to Theorem 5.2, the $(L\varkappa^*_{qd+k})$-neighborhood of a point $a_{q,k}$ of the set Q_* contains a unique point of the set $\{Lt_1, Lt_2, \dots, Lt_r\}$. Write this point in the form $a_{q,k}+w_{q,k}$, where, by (5.31), the numbers $w_{q,k}$ satisfy the inequality

$$|w_{q,k}| \leqslant \delta \stackrel{\text{def}}{=} \frac{c}{d}\left(L^{-c_1}+\tau_*^2\right) L\,|\log L|^{1/2} \tag{5.35}$$

with some $c \geqslant 10^3$. Taking into account (2.2), it is easy to see that the estimate (5.33) follows from the inequality

$$1-\left|\varphi_j(\tau_*/d_*)\right|^2 \leqslant d_*^2\left(1+\pi^2\,\frac{L^{-2c_1}}{\tau_*^2}\right)\sum_{a_{q,k}\in Q_*}\left(1-\left|\varphi_j(a_{q,k}+w_{q,k})\right|^2\right) \\ + c(L^{-c_1}+\tau_*^2)^3\,L^{3-8c_1}|\log L|^{3/2}\,\sigma_j^2\,T^2 \tag{5.36}$$

and from the simple estimate

$$\left(L^{-c_1}+\tau_*^2\right)^3 L^{3-8c_1}|\log L|^{3/2} \leqslant L^{2-9c_1}.$$

Let us prove the relation (5.36). By (5.16) and (2.4), we have the estimate

$$1-H_{js}(1/d_*\,\delta)+H_{js}(-1/d_*\,\delta) \leqslant 8\,d_*^3\,\delta^3\,\beta_j T^3 \leqslant\; 8\,d_*^3\,\delta^3\,L^{-8c_1}\sigma_j^2\,T^2 \\ \leqslant c(L^{-c_1}+\tau_*^2)^3 L^{3-8c_1}|\log L|^{3/2}\sigma_j^2\,T^2.$$

Therefore,

$$1-\left|\varphi_j(\tau_*/d_*)\right|^2 \leqslant 2\int\limits_{-1/d_*\delta}^{1/d_*\delta} \sin^2\frac{\tau_*\,x}{2\,d_*}\;dH_{js}(x)+c(L^{-c_1}+\tau_*^2)^3L^{3-8c_1}|\log L|^{3/2}\sigma_j^2\,T^2.$$

Thus, for the proof of (5.36) it suffices to verify that

$$\sin^2\frac{\tau_*\,x}{2\,d_*} \;\leqslant d_*^2\left(1+\pi^2\,\frac{L^{-2c_1}}{\tau_*^2}\right)\sum_{a_{q,k}\in Q_*}\sin^2\frac{(a_{q,k}+w_{q,k})\,x}{2}, \tag{5.37}$$

where $x\in\left[-1/d_*\,\delta,\,1/d_*\,\delta\right]$.

We rewrite x in the form $x = w+2\pi p/\tau_*$, where p is an integer and $|w| \leqslant \pi/\tau_*$. Suppose that $|w| \leqslant L^{c_1}/d_*$. If $p \neq md_*$ where $m\in\mathbb{Z}$, then $p = m_1 d_* + p_1$ where m_1 and p_1 are integers and $0<|p_1|<d_*$. In this case every term of the sum on the right-hand side of (5.37) may be represented in the form

$$\sin^2\left(\frac{a_{q,k}x}{2}+\frac{w_{q,k}\,x}{2}\right) = \sin^2\left(\frac{\pi\,k\,p}{d}+\frac{\tau_*\,w}{2}\left(q+\frac{k}{d}\right)+\frac{w_{q,k}\,x}{2}\right) \\ = \sin^2\left(\frac{\pi\,(k/D)\,p_1}{d_*}+\frac{\tau_*\,w}{2}\left(q+\frac{k}{d}\right)+\frac{w_{q,k}\,x}{2}\right).$$

Since the integers from the set $\mathbb{K} \cup d$ are relatively prime, then for every p_1 there exists a k from the representation (5.32) of the points $a_{q,k} \in Q_*$ such that $(k/D)\, p_1 = m_2\, d_* + p_2$, where m_2 and p_2 are integers and $0 < |p_2| < d_*$. Therefore we finally have, for this k,

$$\sin^2\Big(\frac{\pi\,(k/D)\,p_1}{d_*} + \frac{\tau_*\,w}{2}\,\Big(q + \frac{k}{d}\Big) + \frac{w_{q,k}\,x}{2}\Big) \geqslant \sin^2\frac{\pi}{2\,d_*} \geqslant \frac{1}{d_*^2}\,.$$

This estimate implies inequality (5.37) for the w being considered.

Let now $|w| > L^{c_1}/d_*$. Then the inequality (5.37) follows from the bound

$$\sin^2\frac{\tau_*\,x}{2} \geqslant \Big(\frac{\tau_*\,L^{c_1}}{\pi\,d_*}\Big)^2\,. \qquad \square$$

In order to deduce (2.11) from Lemma 5.6 we shall estimate the quantity η_*. Taking into account that $d_* \leqslant d \leqslant 80\,\tau_* \leqslant 80\,L^{-c_1}$, one has

$$\eta_* \leqslant -c L^{2-12c_1}\log L = c L^2 \cdot L^{-12c_1}\,|\log L|.$$

Recall that $\varphi_j(t) = f_j(T\,t)$ and that, by the definition, $h = 2\pi\,d_*\,s/t_*$. Since

$$\big|f_j\big(2\pi/h\big)\big| = \big|f_j\big(t_*/d_*\,s\big)\big| = \big|\varphi_j\big(\tau_*/d_*\big)\big|,$$

the assertion (2.11) follows immediately from Lemma 5.6 and the preceding estimate.

Let us show that the assertion (2.9) is also true. Consider the r.v.'s Y_j $(j \in J_*)$ introduced in the proof of Theorem 2.2. We have seen that every r.v. Y_j with $j \in J_*$ satisfies the assumptions of Lemma 3.5 with $\eta = L^{2-13c_1}$ and $K = 6c_1$. Note that, verifying this, we used instead of the assertion (2.9) of Theorem 2.1 the estimate $h \leqslant c L s$. The last inequality follows now from the relation $d_* \leqslant 80\,\tau_* = 80 t_* L$ and from the definition of h. By Lemma 3.5, there exists a r.v. Y_j^* with a d.f. from the set $\mathfrak{F}_{3,L^{2c_1}}$ such that

$$\operatorname{Var} Y_j^* \geqslant (1 - L^{6c_1}) \operatorname{Var} Y_j$$

and

$$\mathbf{E}\,|Y_j^* - \mathbf{E}\,Y_j^*|^3 \leqslant \mathbf{E}\,|Y_j|^3 + L^{2c_1} \operatorname{Var} Y_j.$$

In addition, by Lemma 3.2, we have the estimate

$$\Big(\frac{1}{2} - 8\,L^{2c_1}\Big) \operatorname{Var} Y_j^* \leqslant \mathbf{E}\,|Y_j^* - \mathbf{E}\,Y_j^*|^3$$

which together with the two preceding inequalities yields

$$\frac{1}{2}\,(1 - 20\,L^{2c_1}) \operatorname{Var} Y_j \leqslant \mathbf{E}\,|Y_j|^3 + L^{2c_1} \operatorname{Var} Y_j.$$

In order to deduce the required estimate (2.9) from this inequality, it suffices to show that there exists an index j from the set J_* such that

$$\mathbf{E}\,|Y_j|^3 \leqslant L^{2c_1}\,\frac{1}{h}\,L\,s\ \mathrm{Var}\ Y_j.$$

To prove this, we note that the set $J^{(5)}$ of indices, for which

$$\frac{\beta_j}{B} \leqslant L^{2c_1}\,\frac{\sigma_j^2}{s^2}\,,$$

has the measure $m\big(J^{(5)}\big) \geqslant L^{2c_1}$. Together with (2.10) this implies the inequality $m\big(J^{(5)} \cap J_*\big) > 0$. Therefore we have $J^{(5)} \cap J_* \neq \varnothing$ and, choosing an index j from $J^{(5)} \cap J_*$,

$$\mathbf{E}\,|Y_j|^3 = \frac{\beta_j}{B}\cdot\frac{B}{h^3} \leqslant L^{2c_1}\,\frac{\sigma_j^2}{s^2}\cdot\frac{B}{h^3} = L^{2c_1}(hT)^{-1}\,\mathrm{Var}\ Y_j = L^{2c_1}\,\frac{1}{h}\,L\,s\ \mathrm{Var}\ Y_j.$$

Thus, the assertion (2.9) is proved.

Our following step is to establish the representation (2.8). It is easy to see that every r.v. Z_j, for $j \in J_*$, satisfies the conditions of Lemma 3.6 with $H(x) = H_j(x)$, $\varphi(t) = \varphi_j(t)$, $\sigma^2 = \sigma_j^2 T^2$, $\beta = \beta_j T^3$, $a = \tau_*/d_*$, $\eta = c\eta_*$. Indeed, (3.15) follows from (5.33). The condition $a \geqslant c_2 = 1/80$ is satisfied since $d_* \leqslant 80\,\tau_*$. In order to verify (3.16) it suffices to note that the inequality (2.4) can be written in the form

$$\beta_j T^3 \leqslant L^{-8c_1}\sigma_j^2 T^2.$$

Moreover, by (2.4), we have

$$\sigma_j^2 T^2 = \sigma_j^2\cdot\frac{s^4}{B^2} \leqslant \sigma_j^2\cdot L^{-16c_1}\,\frac{\sigma_j^4}{\beta_j^2} = L^{-16c_1}\,\frac{\sigma_j^6}{\beta_j^2} \leqslant L^{-16c_1}$$

and, in view of (5.34), $L^{-16c_1} \leqslant \eta_*^{-1/3}$ for sufficiently small L.

By Lemma 3.6, every c.f. $\varphi_j(t)$ $(j \in J_*)$ admits the representation (3.17) in the $\dfrac{1}{8\sigma_j T}$-neighborhood of every point $q\,\tau_*/d_*$ with $q = 1, \ldots, \big[L^{-c_1}d_*/\tau_*\big] + 1$. In this representation the function $r_q(t)$ has the form

$$r_q(t) = r_{qj}(t) = i\,C_{1qj}\Big(t - q\,\frac{\tau_*}{d_*}\Big) + C_{2qj}\Big(t - q\,\frac{\tau_*}{d_*}\Big)^2 + C_{3qj}\Big(t - q\,\frac{\tau_*}{d_*}\Big)^3, \qquad (5.38)$$

where the real-valued quantity C_{1qj} and the complex-valued quantities C_{2qj} and C_{3qj} satisfy the inequalities (3.18)–(3.20) respectively with the parameters indicated above. Moreover, the estimates (3.21) and (3.22) hold.

In the sequel for the sake of brevity we shall denote $a_q = q\tau_*/d_*$ for $q \in \mathbb{Z}$. Let A_* be the set of the points $a_q = q\tau_*/d_*$ with $q = 1, \dots, \left[L^{-c_1} d_*/\tau_*\right] + 1$. Below we assume that the parameter δ is defined in (5.35).

Denote by $\varphi_{J_*}(t)$ and $\varphi_{\overline{J}_*}(t)$ the c.f.'s

$$\prod_{j\in J_*} \varphi_j(t) \quad \text{and} \quad \prod_{j\in \overline{J}_*} \varphi_j(t)$$

and by $H_{J_*}(x)$ and $H_{\overline{J}_*}(x)$ their d.f.'s respectively. In order to study the behavior of these c.f.'s we need Lemmas 5.7 and 5.8 below.

Lemma 5.7. *Let a_q be an arbitrary point from the set A_*. Then the estimate*

$$\delta \sum_{j\in J_*} \Bigl| \operatorname{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} \Bigr| \leqslant c L^{c_1} |\log L|^{7/6} \tag{5.39}$$

holds.

Proof. As a first step we prove the estimate

$$\Bigl| \sum_{j\in J_*} \operatorname{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} \Bigr| \leqslant c q d_*^2 L^{-2} \eta_*^{2/3} + c \frac{q}{d_*} \bigl(m(\overline{J}_*)\bigr)^{1/2} L^{-1-c_1}. \tag{5.40}$$

In view of (3.21) we have

$$\begin{aligned} \Bigl| \sum_{j\in J_*} \operatorname{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} - q \operatorname{Re} \frac{\varphi_j'(a_1)}{\varphi_j(a_1)} \Bigr| &\leqslant c\Bigl(q^3 T^2 \sum_{j=1}^{n} \sigma_j^2 + q T^3 \sum_{j=1}^{n} \beta_j \Bigr) \eta_*^{2/3} \\ &= c q (q^2+1) L^{-2} \eta_*^{2/3}. \end{aligned} \tag{5.41}$$

The point τ_* is chosen so that

$$\operatorname{Re} \sum_{j\in J_*} \frac{\varphi_j'(\tau_*)}{\varphi_j(\tau_*)} = -\operatorname{Re} \sum_{j\in \overline{J}_*} \frac{\varphi_j'(\tau_*)}{\varphi_j(\tau_*)} = -\operatorname{Re} \frac{\varphi_{\overline{J}_*}'(\tau_*)}{\varphi_{\overline{J}_*}(\tau_*)}.$$

Note that the relation

$$|\varphi_{\overline{J}_*}'(t)| \leqslant \int_{-\infty}^{\infty} |x| \, dH_{\overline{J}_*}(x) \leqslant \Bigl(\int_{-\infty}^{\infty} x^2 \, dH_{\overline{J}_*}(x) \Bigr)^{1/2} = \bigl(m(\overline{J}_*)\bigr)^{1/2} \frac{1}{L} \tag{5.42}$$

is valid for all real t. Therefore, we obtain

$$\Bigl| \operatorname{Re} \frac{\varphi_{\overline{J}_*}'(\tau_*)}{\varphi_{\overline{J}_*}(\tau_*)} \Bigr| \leqslant \bigl(m(\overline{J}_*)\bigr)^{1/2} L^{-1-c_1}. \tag{5.43}$$

The last estimate permits to derive from (5.41) with $q = d_*$ and (5.43) the inequality

$$\Big| \mathrm{Re} \sum_{j \in J_*} \frac{\varphi_j'(a_1)}{\varphi_j(a_1)} \Big| \leqslant c d_*^2 L^{-2} \eta_*^{2/3} + \frac{1}{d_*} \big(m(\bar{J}_*)\big)^{1/2} L^{-1-c_1}$$

which together with (5.41) implies (5.40).

Let us show that (5.39) follows from (5.40). Indeed, by (5.35), (5.34) and (2.10), one has

$$\delta \sum_{j \in J_*} \Big| \mathrm{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} \Big| \leqslant c \frac{\tau_*}{d_*} L^{1-c_1} |\log L|^{1/2} \cdot L^{-c_1} \frac{d_*^3}{\tau_*} \cdot L^{-2} \cdot d_*^{2/3} L^{2(2-11c_1)/3} |\log L|^{2/3}$$
$$+ c \frac{\tau_*}{d_*} L^{1-c_1} |\log L|^{1/2} \cdot \frac{L^{-c_1}}{\tau_*} \cdot L^{4c_1} \cdot L^{-1-c_1}$$
$$\leqslant c L^{c_1} |\log L|^{7/6}$$

for all points $a_q \in A_*$. □

Lemma 5.8. *Let a_q be an arbitrary point from the set A_*. Then the estimate*

$$\sum_{j \in J_*} \big| r_{qj}(t) \big| \leqslant c L^{2c_1} |\log L|^{7/6}$$

holds for real t satisfying the inequality $|t - a_q| < \delta$.

Proof. We see with the help of (5.35), (3.18) and (5.34) that

$$|t - a_q| \sum_{j \in J_*} |C_{1qj}| \leqslant c\delta \cdot \Big(q^2 T^2 \sum_{j \in J_*} \sigma_j^2 + T^3 \sum_{j \in J_*} \beta_j \Big) \cdot \eta_*^{2/3}$$
$$\leqslant c \frac{\tau_*}{d_*} L^{1-c_1} |\log L|^{1/2} \cdot L^{-2c_1} \Big(\frac{d_*}{\tau_*}\Big)^2 L^{-2} \cdot d_*^{2/3} L^{2(2-11c_1)/3} |\log L|^{2/3}$$
$$\leqslant c L^{2c_1} |\log L|^{7/6}$$

for t satisfying $|t - a_q| < \delta$. In a similar way one gets in view of (3.19)

$$(t - a_q)^2 \sum_{j \in J_*} |C_{2qj}| \leqslant c\delta^2 \cdot \Big(q^2 T^2 \sum_{j \in J_*} \sigma_j^2 + T^3 \sum_{j \in J_*} \beta_j \Big) \cdot \eta_*^{1/3}$$
$$\leqslant c \Big(\frac{\tau_*}{d_*}\Big)^2 L^{2-2c_1} |\log L| \cdot L^{-2c_1} \Big(\frac{d_*}{\tau_*}\Big)^2 L^{-2} \cdot d_*^{1/3} L^{(2-11c_1)/3} |\log L|^{2/3}$$
$$\leqslant c L |\log L^{18c_1}|^{5/3} \tag{5.44}$$

and, by (3.20),

$$|t-a_q|^3 \sum_{j\in J_*} |C_{3qj}| \leqslant c\delta^3 \cdot T^3 \sum_{j\in J_*} \beta_j \leqslant c\Big(\frac{\tau_*}{d_*}\Big)^3 L^{3-3c_1} |\log L|^{3/2} \cdot L^{-2}$$

$$\leqslant c L^{1-6c_1} |\log L|^{3/2}. \tag{5.45}$$

Comparing the three last estimates, we obtain the assertion of Lemma 5.8. □

The two preceding lemmas permit to formulate and prove a lemma on a behavior of the c.f. $\varphi_{J_*}(t)\cdot\varphi_{\overline{J}_*}(t)$.

Lemma 5.9. *Let a_q be an arbitrary point from the set A_*. Then the representation*

$$\prod_{j=1}^n \varphi_j(t) = \prod_{j=1}^n \varphi_j(a_q) \cdot \exp\Big\{ -\frac{(t-a_q)^2}{2L^2}\Big\} \big(1 + c\,\Theta L^{c_1} |\log L|^{7/6}\big) \tag{5.46}$$

is true for real t satisfying the inequality $|t-a_q| < \delta$.

Proof. Let t satisfy the conditions of the Lemma 5.9. By Lemmas 5.7 and 5.8, the relation

$$\varphi_{J_*}(t) = \varphi_{J_*}(a_q)\,\exp\Big\{ -\frac{m(J_*)\,(t-a_q)^2}{2L^2}\Big\} \big(1 + c\,\Theta L^{c_1} |\log L|^{7/6}\big)$$

can be easily obtained from the representation (3.17) for the c.f.'s $\varphi_j(t)$ with $j\in J_*$. Note that the representation (3.17) is applicable in a wider neighborhood of the point a_q. Taking into account (2.10), we finally get

$$\varphi_{J_*}(t) = \varphi_{J_*}(a_q)\,\exp\Big\{ -\frac{(t-a_q)^2}{2L^2}\Big\} \big(1 + c\,\Theta L^{c_1} |\log L|^{7/6}\big). \tag{5.47}$$

On the other hand, using the Taylor formula and the estimate (5.42), we derive the formula

$$\varphi_{\overline{J}_*}(t) = \varphi_{\overline{J}_*}(a_q) + c\,\Theta L^{2c_1} |\log L|^{1/2}. \tag{5.48}$$

The relations (5.47) and (5.48) imply the assertion of Lemma 5.9. □

Denote $\widetilde{f}(t) = f(t)\,\exp\{-\varepsilon^2 t^2/2\}$ and introduce the integrals

$$\widetilde{I}_q(x) = -\frac{1}{2\pi i} \int_{a_q-\delta}^{a_q+\delta} \frac{1}{t} \exp\Big\{ -\frac{ixt}{L}\Big\} \widetilde{f}(t/L)\,dt, \qquad x\in\mathbb{R}^1, \tag{5.49}$$

for $q=\pm1,\pm2,\dots$. Note that $\widetilde{I}_q(x) = -\overline{\widetilde{I}_{-q}(x)}$.

Lemma 5.10. *The formula*

$$\widetilde{I}_q(x) = i\,L\,\Phi'(x)\,\frac{1}{a_q}\,\widetilde{f}\big(a_q/L\big)\,\exp\big\{-i\,a_q x/L\big\}\,\big(1 + c\,\Theta L^{c_1}|\log L|^{7/6}\big), \qquad x \in \mathbb{R}^1,$$

is true for all points $a_q \in A_*$.

Proof. We get from Lemma 5.9 with the help of a simple computation the formula

$$\widetilde{I}_q(x) = i\,\widetilde{f}\big(a_q/L\big)\,\frac{1}{a_q}\,\exp\Big\{-\frac{i\,a_q\,x}{L}\Big\}\,\frac{1}{2\pi}\int\limits_{a_q-\delta}^{a_q+\delta}\exp\Big\{-\frac{i\,x\,(t-a_q)}{L} - \frac{(t-a_q)^2}{2\,L^2}\Big\}\,dt$$
$$+ c\,\Theta\,L^{1+c_1}|\log L|^{7/6}.$$

By the definition of the parameter δ, the integral on the right-hand side of this equality has the form $\sqrt{2\pi}\,e^{-x^2/2}L + c\,\Theta\,L^{1+c_1}$. This completes the proof of Lemma 5.10. □

For the integral $I_2(x)$ defined at the beginning of Section 5 we have the following important representation.

Lemma 5.11. *The equality*

$$I_2(x) = i\,L\,\Phi'(x)\,\frac{1}{a_1}\sum\frac{1}{q}\,\widetilde{f}(a_q/L)\,\exp\big\{-i\,a_q x/L\big\} + c\,\Theta L^{1+c_1}|\log L|^{7/6} \tag{5.50}$$

holds for all real x, where the sum is taken over all integers $q \neq 0$.

Remark. As before we denote here by a_q the points of the form $q\tau_*/d_*$ where $q \in \mathbb{Z}$.

Proof. Introduce the sets $\Gamma_{2,1}, \Gamma_{2,2}$ and $\Gamma_{2,3}$ as follows. Denote by U the union of the δ-neighborhoods of the points $a_q \in A_*$ and assume

$$\Gamma_{2,1} = (-U \cup U) \cap \Gamma_3, \quad \Gamma_{2,2} = (\Gamma_2 \setminus \Gamma_{2,1}) \cap \Gamma_3, \quad \Gamma_{2,3} = \mathbb{R}^1 \setminus \Gamma_3,$$

where $\Gamma_3 = \big[-L^{-1-c_1},\, L^{-1-c_1}\big]$. We now write the integral $I_2(x)$ as the sum of three integrals $I_{2,1}(x), I_{2,2}(x)$ and $I_{2,3}(x)$ over the sets $\Gamma_{2,1}, \Gamma_{2,2}$ and $\Gamma_{2,3}$ respectively. We see from Theorem 5.2 that $\big|I_{2,1}(x)\big| \leqslant -cL^2\log L$. Note that the same estimate is true for the modulus of the integral $I_{2,3}(x)$. It remains to study the integral $I_{2,2}(x)$. According to Lemma 5.10, we have

$$I_{2,2}(x) = i\,L\,\Phi'(x)\sum\frac{1}{a_q}\,\widetilde{f}(a_q/L)\,\exp\{-i\,a_q x/L\} + c\,\Theta L^{1+c_1}|\log L|^{7/6}$$

for all real x, where the sum is taken over all integer $q \neq 0$ such that $|q| \leqslant L^{-c_1}d_*/\tau_*$. Comparing this relation with the estimates of the integrals $I_{2,1}(x)$ and $I_{2,3}(x)$, we obtain the statement of Lemma 5.11. □

Let us complete the proof of Theorem 2.1 in the case where $M_f > L^2$ and $S_* \neq \varnothing$. We note that

$$\widetilde{f}(a_q/L) = \int\limits_{-\infty}^{\infty} \exp\Big\{\frac{iq\tau_* u}{d_* L}\Big\}\, d(F * \Phi_\varepsilon)(u) = \int\limits_{-h/2s}^{h/2s} \exp\Big\{\frac{2\pi iqsu}{h}\Big\}\, dH_\varepsilon(u;h),$$

where the d.f. $H_\varepsilon(x;h)$ is defined in Section 2. Then the sum in the formula (5.50) can be rewritten in the form

$$\begin{aligned}\sum_{q\neq 0} \frac{1}{q} \exp\Big\{-\frac{2\pi iqsx}{h}\Big\} &\int\limits_{-h/2s}^{h/2s} \exp\Big\{\frac{2\pi isqu}{h}\Big\}\, dH_\varepsilon(u;h) \\ &= -i \int\limits_{-h/2s}^{h/2s} \sum_{q=1}^{\infty} \frac{2}{q} \sin\Big(\frac{2\pi s}{h}(x-u)\Big)\, dH_\varepsilon(u;h) \\ &= -2\pi i \int\limits_{-h/2s}^{h/2s} \Big(\frac{1}{2} - \mathrm{Fr}\Big\{\frac{s}{h}(x-u)\Big\}\Big)\, dH_\varepsilon(u,h).\end{aligned}$$

Substituting this expression in the relation (5.50), we obtain the representation (2.8) in the case where $M_f > L^2$ and $S_* \neq \varnothing$.

Consider now the case where $M_f > L^2$ and $S_* = \varnothing$. Our aim is to establish that in this case $\big|I_2(x)\big| \leqslant cL^{c_1}$ for all real x. Then Theorem 2.1 will be proved.

Suppose first that $M_f > L^2$, $S_* = \varnothing$ and $S \neq \varnothing$. We repeat the arguments which was used to prove Theorem 2.1 in the case where $M_f > L^2$ and $S_* \neq \varnothing$. These arguments remain in force with one change only. We consider the set of the points $\big\{t_1,\dots,t_r\big\}$ and the set of indices J_* defined in Section 2 and assume now that $t_* = t_1$. Then we define d_* in just the same way as in Section 2. We denote as before $\tau_* = t_* L$. The rest of notation will be used below as well.

We see that c.f.'s $\varphi_j(t)$ $(j \in J_*)$ satisfy the conditions of Lemma 3.6 with $a = \tau_*/d_*$ and $\eta = c\eta_*$. Therefore they admit an analog of the representation (3.17) with the remainder term $r_q(t) = r_{qj}(t)$ (see (5.38)) and satisfy the corresponding analogs of the estimates (3.18)–(3.22). We now consider this representation in the $\widetilde{\delta}$-neighborhoods of the points $a_q = q\tau_*/d_*$ $\big(q = 1,\dots,\big[L^{-c_1}d_*/\tau_*\big]+1\big)$, where $\widetilde{\delta} = L^{1-9c_1}$. To estimate the absolute value of the c.f. $\varphi_{J_*}(t)$ in these neighborhoods we need the following lemma.

Lemma 5.12. *Let* a_q *be an arbitrary point of the form* $a_q = q\tau_*/d_*$, *where* $q = 1, \dots, \left[L^{-c_1} d_*/\tau_*\right] + 1$. *Then*

$$\sum_{j\in J_*} \Big| \operatorname{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} \Big| \leqslant c L^{-1-7c_1} |\log L|^{1/2}. \tag{5.51}$$

Moreover, the inequality

$$\Big| \prod_{j\in J_*} e^{r_{qj}(t)} \Big| \leqslant 1 + cL^{2c_1} |\log L|^{1/3} \tag{5.52}$$

is true for real t *satisfying the inequality* $|t - a_q| < \widetilde{\delta}$.

Proof. We shall argue similarly to the proof of (5.44) and (5.45). Indeed, the inequality (3.22) implies

$$\begin{aligned}\sum_{j\in J_*} \Big| \operatorname{Re} \frac{\varphi_j'(a_q)}{\varphi_j(a_q)} \Big| &\leqslant cq \cdot T^2 \sum_{j\in J_*} \sigma_j^2 \cdot \eta_*^{1/2} \\ &\leqslant cL^{-c_1} \cdot L^{-2} \cdot L^{1-6c_1} |\log L|^{1/2} = cL^{-1-7c_1} |\log L|^{1/2}\end{aligned}$$

that proves (5.51). Taking into account (3.19) and (3.20), we see that

$$\begin{aligned}(t-a_q)^2 \sum_{j\in J_*} |C_{2qj}| &\leqslant c\widetilde{\delta}^2 \cdot \Big(q^2 T^2 \sum_{j\in J_*} \sigma_j^2 + T^3 \sum_{j\in J_*} \beta_j \Big) \cdot \eta_*^{1/3} \\ &\leqslant cL^{2(1-9c_1)} \cdot L^{-2c_1} L^{-2} \cdot L^{(2-12c_1)/3} |\log L|^{1/3} \\ &= cL^{2c_1} |\log L|^{1/3}\end{aligned}$$

and

$$|t-a_q|^3 \sum_{j\in J_*} |C_{3qj}| \leqslant c\widetilde{\delta}^3 \cdot T^3 \sum_{j\in J_*} \beta_j \leqslant cL^{3(1-9c_1)} \cdot L^{-2} = cL^{1-9c_1}$$

for $|t - a_q| < \widetilde{\delta}$. Applying these estimates to (5.38), we get the assertion (5.52) of Lemma 5.12. □

We can now prove the required bound for $\left|\varphi_{J_*}(t)\right|$.

Lemma 5.13. *For every $q = 1, \dots, \left[L^{-c_1} d_*/\tau_*\right] + 1$ there exists a real quantity b_q which does not depend on t and satisfies the inequality*

$$\left|\varphi_{J_*}(t)\right| \leqslant c \exp\Big\{ -\frac{1}{2L^2}\left(t - a_q + b_q\right)^2 \Big\}$$

for real t satisfying the inequality $|t - a_q| < \widetilde{\delta}$.

Proof. Applying Lemma 5.12 and (3.17) to the c.f.'s $\varphi_j(t)$ $(j \in J_*)$ and taking into account (2.10), we have

$$\left|\varphi_{J_*}\left(t + a_q\right)\right| = \exp\Big\{ D_q t - \frac{1}{2L^2} t^2 + c\,\Theta L^{c_1} |\log L|^{1/3} \Big\} \qquad (-\widetilde{\delta} < t < \widetilde{\delta}) \quad (5.53)$$

for $q = 1, \dots, \left[L^{-c_1} d_*/\tau_*\right] + 1$. In this formula D_q is a real quantity which does not depend on t and satisfies the inequality $|D_q| \leqslant L^{-1-7c_1} |\log L|^{1/2}$. Since $\left|\varphi_{J_*}\left(t + a_q\right)\right| \leqslant 1$, we get from (5.53) the inequality

$$D_q t - t^2/2L^2 - cL^{c_1} |\log L|^{1/3} \leqslant 0 \qquad (-\widetilde{\delta} < t < \widetilde{\delta})$$

which may be rewritten in the form

$$-\frac{1}{2}\left(D_q L - t/L\right)^2 + \frac{1}{2}\left(D_q L\right)^2 - cL^{c_1} |\log L|^{1/3} \leqslant 0.$$

Since $|D_q| L^2 \leqslant cL^{1-7c_1} |\log L|^{1/2} < \widetilde{\delta}$, then, putting in the preceding inequality $t = D_q L^2$, we conclude that $\frac{1}{2}\left(D_q L\right)^2 - cL^{c_1} |\log L|^{1/3} \leqslant 0$. Using this estimate, we easily obtain from the formula (5.53) the statement of Lemma 5.13. □

For the integrals $\widetilde{I}_q(x)$ $(q = 1, 2, \dots)$, which were defined by the formula (5.49), we now have the following bound.

Lemma 5.14. *The estimate*

$$\left|\widetilde{I}_q(x)\right| \leqslant \frac{c}{q} L^{1+c_1}$$

holds for all real x and $q = 1, \dots, \left[L^{-c_1} d_*/\tau_*\right]$.

Proof. We first note that

$$\left|\widetilde{I}_q(x)\right| \leqslant \frac{1}{2\pi} \int\limits_{-\delta}^{\delta} \frac{1}{|t + a_q|} \left|\varphi_{J_*}(t + a_q)\right| \cdot \left|\varphi_{\overline{J}_*}(t + a_q)\right| dt.$$

Using the Taylor formula and the relations $S_* = \varnothing$, (5.42) and (2.10), we have the inequality $\big|\varphi_{\overline{J}_*}(t+a_q)\big| \leqslant cL^{c_1}$ for $t \in \big(-\delta,\, \delta\big)$ and $q = 1, \dots, \big[L^{-c_1} d_*/\tau_*\big]$. In addition, in view of the estimate $a_1 \geqslant 1/80$, one concludes that

$$|t + a_q| \geqslant \frac{1}{2}\, a_q \geqslant \frac{q}{160} \qquad \text{for} \quad t \in \big(-\delta,\, \delta\big).$$

Therefore, one gets the estimate

$$\big|\widetilde{I}_q(x)\big| \leqslant c\,\frac{1}{q}\, L^{c_1} \int\limits_{-\delta}^{\delta} \big|\varphi_{J_*}(t+a_q)\big|\, dt.$$

By Lemma 5.13, we obtain

$$\int\limits_{-\delta}^{\delta} \big|\varphi_{J_*}(t+a_q)\big|\, dt \leqslant c \int\limits_{-\infty}^{\infty} \exp\Big\{ -\frac{t^2}{2\,L^2} \Big\}\, dt \leqslant cL.$$

The last two inequalities prove Lemma 5.14. □

In order to complete the proof of Theorem 2.1 in the case where $M_f > L^2$, $S_* = \varnothing$ and $S \neq \varnothing$ we note that

$$\big|I_2(x)\big| \leqslant \big|I_{2,1}(x)\big| + \big|I_{2,2}(x)\big| + \big|I_{2,3}(x)\big|,$$

where the integrals $I_{2,1}(x), I_{2,2}(x)$ and $I_{2,3}(x)$ are defined in the same way that in the proof of Lemma 5.11 with only difference that the points a_q have now the form $a_q = qt_1/d_*$. It is easy to see that the previous bounds for $\big|I_{2,2}(x)\big|$ and $\big|I_{2,3}(x)\big|$ remains true. By Lemma 5.14, the modulus of the integral $I_{2,1}(x)$ does not exceed $cL^{1+c_1}\,|\log L|$. Thus, $\big|I_2(x)\big| \leqslant cL^{1+c_1}|\log L|$ for all real x that it was to be proved.

It remains to prove Theorem 2.1 in the case where $M_f > L^2$ and $S = \varnothing$. In order to estimate the integral $I_2(x)$ in this case we note that the function $\big|f(t)\big|$ is either non-increasing or non-decreasing on the interval Γ.

The first case is impossible. Indeed, by (5.8), we have at the point $t = 1/4L$ the inequality $\big|f(t)\big| < L^2$ and this inequality is valid for all points of Γ that contradicts the condition $M_f > L^2$.

Let $\big|f(t)\big|$ be a non-decreasing function on Γ. Let us show that every point $\tau \in \Gamma$, at which the estimate (5.10) holds, satisfies the inequality $\tau > 1/2\,L^{1+c_1}$. Assume that there exists a point τ_1 such that $\tau_1 < 1/2\,L^{1+c_1}$ and at which the estimate (5.10) is true. Then, by Theorem 5.2, there exists a point $\tau_2 \in \big[1/2\,L^{1+c_1},\, 1/L^{1+c_1}\big]$ such that $\left|f(\tau_2)\right| \leqslant L^2$. Since $\left|f(t)\right|$ is a non-decreasing function, we obtain a contradiction. If $\left|f(t)\right| \leqslant L^2$ for all $t \in \left[1/4L,\, 1/2L^{1+c_1}\right]$, then, taking into account the choice of the parameter ε, we can easily compute that

$$\left|\, I_2(x)\right| \leqslant c L^2 \log L. \tag{5.54}$$

Now the statement of Theorem 2.1 follows immediately from Theorem 5.1. Thus, Theorem 2.1 in the case where $M_f > L^2$ is completely proved.

It remains to prove it in the case where $M_f \leqslant L^2$. It can be easily seen that the estimate (5.54) remains true and the statement of Theorem 2.1 follows from Theorem 5.1. □

Proof of Corollary 2.1. Assume that the parameters A, $\mathit{\Delta}$ and M satisfy the inequalities $A \geqslant 1,\ 0 < \mathit{\Delta} \leqslant 1,\ M \geqslant \mathit{\Delta}$. Introduce the normal d.f. $\Phi_\gamma(u) = \Phi(u/\gamma)$ with γ satisfying

$$\frac{1}{\gamma} \geqslant \frac{8MA}{\mathit{\Delta}} \log\Big(\frac{8MA}{\mathit{\Delta}}\Big).$$

We shall need the following auxiliary result.

Lemma 5.15 (Chistyakov (1990)). *Let G be a d.f. and let V_1 and V_2 be absolutely continuous functions of bounded variation such that, for all real u,*

$$\left|V_j(u)\right| \leqslant A \qquad \text{and} \qquad \left|V_j'(u)\right| \leqslant A.$$

Denote

$$\widetilde{G} = G * \Phi_\gamma, \qquad \widetilde{V}_j = V_j * \Phi_\gamma, \quad j = 1, 2,$$

and assume that $\left|\widetilde{V}_2(u) - V_2(u)\right| \leqslant \mathit{\Delta}/8M$ for all real u.

If the inequality $\left|\widetilde{G}(u) - \widetilde{V}_1(u) - V_2(u)\right| \leqslant \mathit{\Delta}$ is true in the $\mathit{\Delta}/MA$-neighborhood of a point ζ, then the following estimate holds

$$\left|\,G(\zeta) - V_1(\zeta) - V_2(\zeta)\right| \leqslant \Big(1 + \frac{1}{M}\Big)\mathit{\Delta}.$$

We note that from the representation (2.8) it follows that

$$\left|\,(F * \Phi_\varepsilon)(u) - \Phi(u) - \alpha(u)\right| \leqslant \Phi'(u)L + cL^{1+c_1}\left|\,\log L\,\right|^{7/6}$$

for all real u. We now apply Lemma 5.15 to the functions $G = F$, $V_1 = \Phi$ and $V_2 = \alpha$. It is not difficult to verify that $\left|\alpha(u) - \widetilde{\alpha}(u)\right| \leqslant c L^2$ for all real u and the assumptions of Lemma 5.15 are satisfied at the point $\zeta = x$, for the parameters $\gamma = \varepsilon$, $A = 1$, $\mathit{\Delta} = \Phi'(x)\, L + L^{1+c_1/2}\left|\,\log L\,\right|^2$ and $M = L^{-c_1/2}$. By Lemma 5.15, we have

$$\left|\,F(x) - \Phi(x) - \alpha(x)\right| \leqslant \Phi'(x)\, L + cL^{1+c_1/2}\left|\,\log L\,\right|^2.$$

From this estimate it follows that

$$\limsup_{l\to 0}\ \sup\ \frac{1}{L}\bigl|F(x)-\Phi(x)-\alpha(x)\bigr|\leqslant\Phi'(x),$$

where the supremum is taken over all $F_1,\dots,F_n\in\mathfrak{F}_3$ with $L=l$. The corresponding lower bound

$$\limsup_{l\to 0}\ \sup\ \frac{1}{L}\bigl|F(x)-\Phi(x)-\alpha(x)\bigr|\geqslant\Phi'(x)$$

is a simple fact which was proved in Chistyakov (1990). Comparing the two last estimates, we obtain the formula (2.12). □

Proof of Corollary 2.2. Without loss of generality we assume below that $n>c(F_1)$, where $c(F_1)$ is a sufficiently large quantity depending on F_1 only.

In order to deduce Esseen's representation (2.1) from the representation (2.8) we first consider the case where the d.f. F_1 is not lattice. We shall show then that in the case where $M_f>L^2$ and $S_*\neq\varnothing$ the third term in the representation (2.8) tends to zero as $n\to\infty$. For this we note that the modulus of the c.f. $f_1(t)$ is strictly less than 1 for all non-zero real t. Therefore for every $\lambda>0$ the inequality

$$\bigl|f_1(t)\bigr|\leqslant 1-\lambda \tag{5.55}$$

holds for real t such that $\sigma_1^2/4\beta_1\leqslant|t|\leqslant K_\lambda$ where $K_\lambda\to\infty$ as $\lambda\to 0$. In addition, by (5.8), we see that the inequality

$$\bigl|f_1(t)\bigr|^n\leqslant L^3 \tag{5.56}$$

is valid for $t\in\bigl[10\sqrt{|\log L|}/\sigma_1\sqrt{n},\,\sigma_1^2/4\beta_1\bigr]$. Recall that the parameter h in the representation (2.8) has the form (2.7). Taking into account (5.55), we see that $t_*L\to\infty$ as $L\to 0$. Moreover, from the relations (2.11), (5.55), (5.56) and from Theorem 5.2 it follows that $d_*=1$. Therefore we can conclude that $h/Ls\to 0$ $(L\to 0)$ that implies the required result for the third term on the right-hand side of (2.8).

Repeating the arguments used in the proof of Theorem 2.2 for studying the quantity $\alpha(x)$, we easily obtain that

$$\alpha(x)=\frac{\alpha_1}{6\sigma_1^3\sqrt{2\pi n}}\,(1-x^2)\,e^{-x^2/2}+o\Bigl(\frac{1}{\sqrt{n}}\Bigr)$$

uniformly in $x\in\mathbb{R}^1$. Therefore Esseen's representation follows from (2.8) in the case where the d.f. F_1 is not lattice.

It remains to consider the case where the d.f. F_1 is lattice. The jump points of F_1 have then the form $a+kd$, where a,d are constant, $d>0$, $k\in\mathbb{Z}$. Assume that

d is the maximal span of the d.f. F_1. It is easy to see that in this case $M_f = 1$, $S_* \neq \varnothing$ and the span d coincides with the parameter h from the representation (2.8) for sufficiently large n. The simple computation shows that

$$\left|(F * \Phi_\varepsilon)(x) - F(x)\right| \leqslant L^{1+c_1/2}$$

and

$$\left|\psi_n(x) - \Psi(x;h)\right| \leqslant L^{1+c_1/2}$$

for x outside the $(L^{1+c_1/2})$-neighborhood of the union of the points of the form $kh/\sigma_1\sqrt{n} + a\sqrt{n}/\sigma_1$, $k \in \mathbb{Z}$. According to these two estimates, Esseen's representation (1.2) follows from (2.8) for x indicated above. The absolute values of the increments of the functions $F(x)$, $\Phi(x)$ and $\psi_n(x)$ on the left and right open $(L^{1+c_1/2})$-half-neighborhoods of each of the points $kh/\sigma_1\sqrt{n} + a\sqrt{n}/\sigma_1$ are not greater than $cL^{1+c_1/2}$. Therefore the representation (2.1) follows from (2.8) for all real x. $\square$

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Gennadii Chistyakov
Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave.
310164 Kharkov
Ukraine
E-mail address: chistyakov@ilt.kharkov.ua