Grundy Sets of Partial Orders.

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Abstract

Two players I and II are playing the Hackendot game, a slight variant of Hackenbush (see Berlekamp, Conway and Guy [1]) played on a partially ordered set P : First I chooses $x \in P$ and deletes the final section generated by x: he leaves the order P^x, then II chooses $y \in P^x$ and deletes the final section generated by y... and so on alternately. The player who cannot move loses the game. This kind of game was introduced by Von-Neumann when the order P is a reverse ordered tree (the root is maximal). He concluded in this case that player I wins by a strategy stealing argument. In [5], Úlehla described how player I could win. We give an easy argument to calculate such a strategy for a wider class of partially ordered sets.

In the following definitions, A and B are finite sets of integers. The *Nim sum* of two integers n and m is denoted by $n +_2 m$, this sum consists of adding, digit to digit without carry, the binary expression of n and m : for example $9 + 5 = 1001 + 101 = 1100 = 12$. We define also the following sum $n +_2 A = \{n +_2 i \mid i \in A\}$. The *Grundy value* of A, $g(A)$, is the least integer which is not in A. For any $i \in \mathbb{N}$, the $i^{t=h}$ *Grundy value of* A, $g^i(A)$, is the integer n_i where $\{n_0, n_1, \ldots\}$ is the natural order on $\mathbb{N} - A$. In particular $g^0(A) = g(A)$, we will always use g instead of g^0. The *Grundy sum* and the *Grundy product* of A and B are respectively :

$$A \oplus B = \{g(A) +_2 b \mid b \in B\} \cup \{g(B) +_2 a \mid a \in A\}$$

$$A \odot B = A \cup \{g^i(A) \mid i \in B\}$$

Partial orders will be called *posets*, a poset which is linearly ordered is a *chain*. Let P be a finite poset, we denote by P^x the poset we obtain from P by deleting x and all the elements greater than x in P. Let P and Q be two disjoint posets, $P \cup Q$ denotes the *disjoint union* of P and Q. The *sum* of P and Q is obtained by letting all elements of Q greater than all elements of P, we denote it by $P \to Q$. The Hackendot game described in the abstract when played on disjoint unions of chains is the usual Nim game introduced by C. Bouton [2] in his famous paper. The class of *reverse trees* is the smallest class of posets containing the singleton which is closed by disjoint unions $P \cup Q$ and sums $P \to Q$ provided for the sum that Q is a singleton. In [5], Úlehla analyzed the Hackendot game on reverse trees and described how to find a winning strategy. We extend his result to the class of *N-free orders*, the smallest class of orders which contains the singleton, and which is stable by disjoint unions and sums. To do so, we will show how to calculate the Grundy value of N-free orders, this provides a way to find a winning strategy for I or for II.

Define inductively the *Grundy set* of P, $G_s(P)$, for convenience we will write $g(P)$ instead of $g(G_s(P))$, this is the *Grundy value* of P.
i) $G_s(\emptyset) = \emptyset$
ii) $G_s(P) = \{g(P^x) \mid x \in P\}$

It is well known that I has a winning strategy for the Hackendot game on P if and only if $g(P) \neq 0$. The classical result of Bouton is that $g(P \cup P') = g(P) +_2 g(P')$ so that the disjoint union operation is easy to handle. But of course in the general case the length of the calculation of the Grundy value of a poset is exponential. However the Grundy value of some posets which belong to certain classes can be calculated, see for example the paper of Fraenkel and Scheinerman [3]. This is also the case for N-free orders :

Theorem 1 *The Grundy value of an N-free poset P of order n can be computed in $O(n^4)$.*

Proof. - Following a result of Valdes, Tarjan and Lawler [6], one can check if a poset P is N-free in $O(n^2)$. Moreover, if the answer is YES, then the algorithm gives a decomposition of P into a binary tree. The nodes of this decomposition tree are operations union and sum and the leaves are the elements of P.

- To calculate the Grundy set of P (and thus the Grundy value of P), we associate to any leaf of the tree its Grundy set, that is $\{0\}$. To calculate the Grundy set of the branches of the tree, and then finally the Grundy set of P, one just have to be able to compute the Grundy set of sums and unions :

i) The Grundy set of $P \cup Q$ is $G_s(P) +_2 G_s(Q)$:

$$G_s(P \cup Q) = \{g((P \cup Q)^x) \mid x \in P\}$$
$$= \{g(P^x \cup Q) \mid x \in P\} \cup \{g(P \cup Q^x) \mid x \in Q\}$$
$$= \{g(P^x) +_2 g(Q)) \mid x \in P\} \cup \{g(P) +_2 g(Q^x)) \mid x \in Q\}$$
$$= G_s(P) +_2 G_s(Q).$$

ii) The Grundy set of of $P \rightarrow Q$ is $G_s(P) \odot G_s(Q)$:

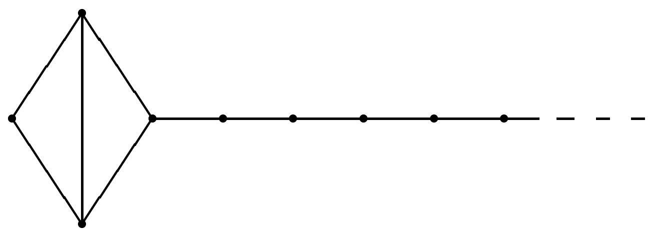
We prove the result by induction on $|P| + |Q|$, this is true if both P and Q are empty. Note that if A and B are finite sets of integers then $g(A \odot B) = g^{g(B)}(A)$.

$$G_s(P \rightarrow Q) = \{g((P \rightarrow Q)^x) \mid x \in (P \rightarrow Q)\}$$
$$= \{g((P \rightarrow Q)^x) \mid x \in P\} \cup \{g((P \rightarrow Q)^x) \mid x \in Q\}$$
$$= \{g(P^x) \mid x \in P\} \cup \{g(P \rightarrow Q^x) \mid x \in Q\}$$
$$= G_s(P) \cup \{g(G_s(P \rightarrow Q^x)) \mid x \in Q\}$$

$$
\begin{aligned}
&= G_s(P) \cup \{g(G_s(P) \odot G_s(Q^x)) \mid x \in Q\} \\
&= G_s(P) \cup \{g^{g(G_s(Q^x))}(G_s(P)) \mid x \in Q\} \\
&= G_s(P) \cup \{g^{g(Q^x)}(G_s(P)) \mid x \in Q\} \\
&= G_s(P) \cup \{g^{i}(G_s(P)) \mid i \in G_s(Q)\} \\
&= G_s(P) \odot G_s(Q).
\end{aligned}
$$

It is routine then to check that the calculation along the tree can be done in $O(n^2)$. □

It is likely, by a more careful analysis, that one could obtain a faster algorithm, for example by computing the Grundy set in the same time of the decomposition. But some other questions could be asked. The Grundy value of a poset can be any positive integer, just take the chains. But is there a characterization of Grundy sets of posets? An easy observation shows that the number of elements of $G_s(P)$ is greater or equal to the height of P. Since an order of height 1 has only possible Grundy value 0 or 1, this shows that if a Grundy set is a singleton, then it is $\{0\}$ or $\{1\}$. However, the Grundy value of an order of height 2 can be arbitrarily large. The graphs depicted below (the path can be arbitrarily long) represent posets of height 2 in the following way : the vertices of the graphs are minimal elements and the edges are maximal elements, an edge being greater than its endvertices only. The proof that the so-defined orders can have arbitrarily large Grundy values is left to the reader. Those examples answer a question of Fraenkel and Scheinerman (see [3], p.160) and one can notice that removing the vertical edge of any of those graph gives a bipartite graph, which associated order is known to have a Grundy value bounded by 3 (see [3], corollary 2.2).



To conclude this paper, we recall a conjecture of D. Gale and A. Neymann (see [4], first question) which have motivated some work in this area:

Problem 1 Is the Grundy value of Boolean lattices without min and max equal to 0?

A more general problem would be to find a way of calculating the Grundy value of the cartesian product of two orders. As for the sum, the Grundy value does not behave well with the cartesian product. But even worse, one can find some posets P, P', Q and Q' such that $G_s(P) = G_s(P')$, $G_s(Q) = G_s(Q')$ and $G_s(P \times Q) \neq G_s(P' \times Q')$ (one can choose $Q = Q' = C_2$, the chain with 2 elements). Thus the approach via Grundy sets is not elaborated enough to serve as the tool. A challenging problem would be to find an appropriate notion, invariant with the cartesian product.

References

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