Sum-free sets and related sets

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Abstract

A set A of integers is *sum-free* if $A\cap(A+A)=\emptyset$. Cameron conjectured that the number of sum-free sets $A\subseteq\{1,\ldots,N\}$ is $O(2^{N/2})$. As a step towards this conjecture, we prove that the number of sets $A\subseteq\{1,\ldots,N\}$ satisfying

$$(A+A+A)\cap(A+A+A+A)=\emptyset$$

is $2^{[(N+1)/2]}(1+o(1))$.

1 Introduction

We use the notation

$$\begin{aligned} A+B &= \{a+b : a\in A,\ b\in B\},\\ A-B &= \{a-b : a\in A,\ b\in B\},\\ A+x &= \{a+x : a\in A\}, \end{aligned}$$

etc., where $A,B\subseteq\mathbf{Z}$ and $x\in Z$.

A symbol "$O(\ldots)$" or "$\ll$" provided with an index implies a constant depending on the parameter(s) in the index. When there is no index, the implied constant is absolute.

A set $A\subset\mathbf{Z}$ is a *sum-free* if

$$A\cap(A+A)=\emptyset. \tag{SF}$$

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Cameron (see [4]) conjectured that the number $\mathrm{SF}(N)$ of sum-free sets $A \subseteq \{1, \ldots, N\}$ satisfies $\mathrm{SF}(N) \ll 2^{N/2}$. Note that the exponent $N/2$ cannot be improved because any set of odd numbers is sum-free, as well as any subset of $\{[N/2]+1, \ldots, N\}$.

The conjecture of Cameron is still open, in spite of a number of partial or similar results. For instance, Erdős and Granville (personal communication), Alon [1, Proposition 4.1], and Calkin [2] had independently proved that

$$\mathrm{SF}(N) = 2^{N/2+o(N)} \tag{1}$$

as $N \to \infty$. Cameron and Erdős [4] proved that there are at most $O\left(2^{N/2}\right)$ sum-free subsets of $\{[N/3], \ldots, N\}$.

Deshouillers, Freiman, Sós, and Temkin [5] investigated the structure of sum-free sets $A \subseteq \{1, \ldots, N\}$ with the additional condition

$$|A| \geq 2N/5 - x, \tag{2}$$

where x is a fixed positive integer. They obtained quite an explicit characterization, which, together with the result of Cameron and Erdős quoted above, yields that the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ subject to (2) is $O_x\left(2^{N/2}\right)$.

On the other hand, as pointed to me J.-M. Deshouillers, Calkin's proof of (1) can be adapted to show that the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ with at most $(1/4 - \delta)\, N$ elements is $O_\delta\left(2^{N/2}\right)$. Implementing this idea, we obtain the following result.

Theorem 1.1 *For any $\delta > 0$ there are at most $O_\delta\left(2^{\left(1/2-\delta^2/16\right)N}\right)$ sum-free sets $A \subseteq \{1, \ldots, N\}$ such that*

$$|A| \leq (1/4 - \delta)\, N. \tag{3}$$

(The optimal choice of parameters in the proof allows one to replace $\delta^2/16$ by $c\delta^2 + O(\delta^4)$, where $c = 1/\log 2$.)

Very recently Calkin and Taylor [3] proved the following: given an integer $k \geq 3$, there are at most $O_k\left(2^{\frac{k-1}{k}N}\right)$ sets $A \subseteq \{1, \ldots, N\}$ subject to

$$A \cap (\underbrace{A + \cdots + A}_{k}) = \emptyset. \tag{4}$$

Unfortunately, their method does not extend to $k = 2$.

In this paper we modify the definition of sum-free sets in a different, than Calkin and Taylor, manner. Given a positive integer k, a set $A \subset \mathbf{Z}$ is an *SF_k-set* if

$$(\underbrace{A + \cdots + A}_{k}) \cap (\underbrace{A + \cdots + A}_{k+1}) = \emptyset. \tag{SF$_k$}$$

In particular, SF_1-sets are just sum-free set.

One immediately observes that, on the one hand,

$$(\mathrm{SF}_k) \Longrightarrow (\mathrm{SF}_{k'}) \quad \text{for} \quad k \geq k',$$

and on the other hand, any set of odd numbers is an SF_k-set for any k. Therefore

$$\mathrm{SF}(N) = \mathrm{SF}_1(N) \geq \mathrm{SF}_2(N) \geq \mathrm{SF}_3(N) \geq \ldots \geq 2^{[(N+1)/2]},$$

where we denote by $\mathrm{SF}_k(N)$ the number of SF_k-subsets of $\{1, \ldots, N\}$.

In addition to the conjecture of Cameron

$$\mathrm{SF}_1(N) \ll 2^{N/2},$$

we conjecture that

$$\mathrm{SF}_2(N) = 2^{[(N+1)/2]}(1 + o(1)). \tag{5}$$

The main result of this note is the following relaxed version of (5):

$$\mathrm{SF}_3(N) = 2^{[(N+1)/2]}(1 + o(1)). \tag{6}$$

Actually, we give an estimate for the error term.

Theorem 1.2 *There exists an absolute constant $C > 0$ such that*

$$SF_3(N) = 2^{[(N+1)/2]} + O\left(2^{(1/2-C)N}\right). \tag{7}$$

The constant C can be easily computed from the argument: for instance, $C = 10^{-5}$ will do, and this value can be improved without difficulty. We did not try to optimize C because the present method is unlikely to give the best possible value for it.

It is also worth mentioning that the constant implied by $O(\ldots)$ cannot be explicitly computed from the proof, because the theorem of Szemerédi is involved (via Calkin's argument, see Section 2).

The method of proof of Theorem 1.2 goes back to Freiman [7] and was developed in [5] in a somewhat different direction. This method seems to be not strong enough to attack the conjecture of Cameron, but (I believe) a suitably sharpened version of it would allow one to prove (5). At present, we have the following partial result.

Theorem 1.3 *The number of SF_2-sets $A \subseteq \{1, \ldots, N\}$ satisfying $|A| > 2N/7 + 9/7$ is $2^{[(N+1)/2]} + O\left(N2^{8N/21}\right)$.*

Thus, to establish (5) in its full strength, one has to obtain the corresponding estimate for the number of SF_2-sets $A \subseteq \{1, \ldots, N\}$ with

$$(1/4 - \delta)\, N \leq |A| \leq 2N/7 + 9/7,$$

where δ is any positive number.

Theorems 1.1, 1.2 and 1.3 are proved in Sections 2, 3 and 4, respectively.

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I would like to thank Gregory Freiman and Jean-Marc Deshouillers for valuable discussions, and Seva Lev for having made his papers [8, 9] available to me prior to publication. Originally, I was merely able to prove that $\mathrm{SF}_4(N) = 2^{[(N+1)/2]}(1+o(1))$ and $\mathrm{SF}_3(N) \ll 2^{N/2}$. It was Lev's remarkable result (see Lemma 3.3) that allowed me to supersede both these estimate by (6).

2 Sum-free sets and the argument of Calkin

In this section we prove Theorem 1.1, following Calkin's argument [2] with some changes.

Lemma 2.1 *Let U be a set of n elements and $0 < \varepsilon \le 1/4$. Then there are at most $O_\varepsilon\left(2^{(1-2\varepsilon^2)n}\right)$ subsets $A \subseteq U$ with $|A| \le (1/2-\varepsilon)n$.*

Proof The required quantity is equal to $\sum_{k=0}^{\kappa} \binom{n}{k} \le n\binom{n}{\kappa}$, where $\kappa = [(1/2-\varepsilon)n]$. We can assume that $n \ge 4$, whence $\kappa > 0$. Using Stirling's formula, one estimates:

$$\begin{aligned} n\binom{n}{\kappa} &\ll n^{1/2}\frac{n^n}{\kappa^\kappa (n-\kappa)^{n-\kappa}} \\ &\ll n^{1/2}\frac{(\kappa+1)^\kappa}{\kappa^\kappa}\frac{n^{n+1}}{(\kappa+1)^{\kappa+1}(n-\kappa)^{n-\kappa}} \end{aligned}$$

$$
\begin{aligned}
&\leq \quad n^{1/2}\left(\frac{1}{(1/2-\varepsilon)^{1/2-\varepsilon}(1/2+\varepsilon)^{1/2+\varepsilon}}\right)^{n} \\
&= \quad n^{1/2}2^{n}\exp\left(-n\sum_{k=1}^{\infty}\frac{(2\varepsilon)^{2k}}{2k(2k-1)}\right) \\
&\leq \quad n^{1/2}2^{n}e^{-2n\varepsilon^{2}} \\
&\ll_{\varepsilon} \quad 2^{(1-2\varepsilon^{2})n},
\end{aligned}
$$

as desired. ■

Lemma 2.2 *Let U be a set of n elements, partitioned into $2t$ disjoint subsets:*

$$U = U_1 \cup \ldots \cup U_t \cup V_1 \cup \ldots \cup V_t,$$

where

$$U_i \cap U_j = \emptyset, \quad V_i \cap V_j = \emptyset \quad (1 \leq i < j \leq t), \quad U_i \cap V_j = \emptyset \quad (1 \leq i, j \leq t).$$

Assume that $|U_i| = |V_i|$ for $1 \leq i \leq t$. Then at most $O_\varepsilon\left(2^{(1/2-4\varepsilon^2)n+t}\right)$ subsets $A \subseteq U$ satisfy the conditions

$$|A| \leq (1/4 - \varepsilon)n \tag{8}$$

and

$$\text{either} \quad A \cap U_i = \emptyset \quad \text{or} \quad A \cap V_i = \emptyset \quad (1 \leq i \leq t). \tag{9}$$

Proof There are 2^t sets of the form $W_1 \cup \ldots \cup W_t$, where each W_i is either U_i or V_i. Each of the sets $W_1 \cup \ldots \cup W_t$ has $n/2$ elements, and by Lemma 2.1 it has at most $O_\varepsilon\left(2^{(1-8\varepsilon^2)n/2}\right)$ subsets A satisfying $|A| \leq (1/2 - 2\varepsilon)n/2$. Therefore there are at most $2^t O_\varepsilon\left(2^{(1/2-4\varepsilon^2)n}\right)$ sets $A \subseteq U$ satisfying (8) and (9), which proves the lemma. ■

The following lemma is the heart of (our version of) Calkin's argument.

Lemma 2.3 *Let n, d, and k be positive integers and $0 < \varepsilon \leq 1/4$. Put*

$$\mathcal{P} = \mathcal{P}_{k,d} = \{-kd, -(k-1)d, \ldots, -d, 0, d, \ldots, kd\}. \tag{10}$$

Then there are at most $O_\varepsilon\left(2^{(1/2-4\varepsilon^2+(2k)^{-1})(n+2kd)}\right)$ sets $A \subseteq \{1, \ldots, n-1\}$ satisfying (8) and

$$(A+A) \cap (\mathcal{P}+n) = \emptyset \tag{11}$$

Proof Assume first that $n \equiv 1 \bmod 2kd$. Write $n = 2kdm + 1$ and partition the set $\{1, \ldots, n-1\}$ as follows:

$$\{1, \ldots, n-1\} = \left(\bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d}} U_{ij}\right) \bigcup \left(\bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d}} V_{ij}\right)$$

where $U_{ij} = \{(i-1)kd + j, (i-1)kd + d + j, \ldots, ikd - d + j\}$ and $V_{ij} = n - U_{ij}$. Then $U_{ij} + V_{ij} \subseteq \mathcal{P} + n$. Therefore any A satisfying (11) has either $A \cap U_{ij} = \emptyset$ or $A \cap V_{ij} = \emptyset$. By Lemma 2.2, there are at most $O_\varepsilon\left(2^{(1/2-4\varepsilon^2)(n-1)+md}\right) = O_\varepsilon\left(2^{(1/2-4\varepsilon^2+(2k)^{-1})n}\right)$ possible A.

Similarly, when $n \equiv 0 \bmod 2kd$, write $n = 2kdm$ and partition the set $\{0, \ldots, n-1\}$ as

$$\{0, \ldots, n-1\} = \left(\bigcup_{\substack{1 \leq i \leq m \\ 0 \leq j \leq d-1}} U_{ij}\right) \bigcup \left(\bigcup_{\substack{1 \leq i \leq m \\ 0 \leq j \leq d-1}} V_{ij}\right)$$

where U_{ij} and V_{ij} are defined as above. Again, by Lemma 2.2, there are at most $O_\varepsilon\left(2^{(1/2-4\varepsilon^2+(2k)^{-1})n}\right)$ possible A.

In the general case there exists a non-negative integer $\lambda < kd$ such that $n + 2\lambda \equiv 0 \bmod 2kd$ or $n + 2\lambda \equiv 1 \bmod 2kd$. As already proved, there are at most $O_\varepsilon\left(2^{(1/2-4\varepsilon^2+(2k)^{-1})(n+2\lambda)}\right)$ sets $A \subset \{1, \ldots, n-1\}$ satisfying (8) and

$$\Big((A+\lambda) + (A+\lambda)\Big) \cap \Big(\mathcal{P} + (n+2\lambda)\Big) = \emptyset.$$

This proves the lemma. ■

Lemma 2.4 *Let $f(n)$ be a function satisfying $f(n) = o(n)$ as $n \to \infty$. Then for any $\varepsilon > 0$ there are at most $O_{\varepsilon,f}\left(2^{\varepsilon N}\right)$ sets $A \subseteq \{1, \ldots, N\}$ such that $|A| \leq f(N)$.*

Proof See, for instance, [2, Lemma 2]. ■

Proof of Theorem 1.1 Fix a positive integer k, to be specified later. For any positive integer n denote by $s_k(n)$ the smallest integer s with the following property: any s-element set $A \subseteq \{1, \ldots, n\}$ contains a $(2k+1)$-term arithmetic progression. Szemerédi [13] proved that

$$s_k(n) = o(n) \tag{12}$$

as $n \to \infty$.

We may assume that

$$0 < \delta \leq 1/4. \tag{13}$$

Put $N' = [(1 - \delta^2/3)N]$. We say that a set $A \subseteq \{1, \ldots, N\}$ is *of the first type* if $|A \cap \{N'+1, \ldots, N\}| \geq s_k(N)$, and *of the second type* otherwise. We estimate separately the number of sum-free sets of each type.

We begin with the second type. As follows from (1) with N' instead of N, there are at most $O_\delta\left(2^{(1/2+\delta^2/12)(1-\delta^2/3)N}\right) = O_\delta\left(2^{(1/2-\delta^2/12)N}\right)$ sum-free subsets of $\{1, \ldots, N'\}$. Further, by (12) and Lemma 2.4, there are at most $O_k\left(2^{\delta^2 N/48}\right)$ subsets of $\{N' + 1, \ldots, N\}$ with less than $s_k(N)$ elements. Therefore there are at most $O_{\delta,k}\left(2^{(1/2-\delta^2/16)N}\right)$ sum-free sets of the second type.

Now estimate the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ of the first type, subject to (3). For any such A, put $A' = A \cap \{1, \ldots, N'\}$ and $A'' = A \cap \{N' + 1, \ldots, N\}$.

Since the set A'' has at least $s_k(N)$ elements, it contains a $(2k+1)$-term progression. Write it as $\mathcal{P}_{k,d}+n$ (see (10)), where $n \geq N'+1$ and $d > 0$ are integers. We have

$$(A'+A') \cap (\mathcal{P}_{k,d}+n) = \emptyset, \tag{14}$$

because A is sum-free. Further, it follows from (3) and (13) that

$$|A'| \leq (1/4-\delta/2)n. \tag{15}$$

Also, since $n+kd \leq N$ and $2kd \leq N-N'-1 \leq \delta^2 N/3$, we have $n+2kd \leq (1+\delta^2/6)N$.

For fixed n and d, we have, by Lemma 2.3, at most $O_\delta\left(2^{(1/2-\delta^2+(2k)^{-1})(n+2kd)}\right) = O_\delta\left(2^{(1/2-\delta^2/2+k^{-1})N}\right)$ sets A' subject to (14) and (15).

Since there are at most N choices for n and at most N choices for d, there are at most $N^2 O_\delta\left(2^{(1/2-\delta^2/2+k^{-1})N}\right)$ possibilities for A'. The number of possible A'' can be estimated trivially as $2^{N-N'} = O\left(2^{\delta^2 N/3}\right)$. Therefore there are at most $N^2 O_\delta\left(2^{(1/2-\delta^2/6+k^{-1})N}\right) = O_\delta\left(2^{(1/2-\delta^2/7+k^{-1})N}\right)$ sum-free sets of the first type, subject to (3).

Now specialize $k = [14/\delta^2]+1$. Then there are at most $O_\delta\left(2^{(1/2-\delta^2/16)N}\right)$ sum-free sets of the second type, and at most $O_\delta\left(2^{(1/2-\delta^2/14)N}\right)$ sum-free sets of the first type, subject to (3). This proves the theorem. ■

3 SF_3-sets

In this section we prove Theorem 1.2. We need some additional notation. For a finite set $A \subseteq \mathbf{Z}$ denote by $\min A$, $\max A$, $\gcd(A)$, and A_+ the smallest element of A, the largest element of A, the greatest common divisor of the elements of A, and the set of non-negative elements of A, respectively. Also, put

$$\ell(A) = \max A - \min A, \quad \gcd'(A) = \gcd(A - A).$$

Note that if $\gcd'(A) = d > 1$ then all elements of A belong to the same residue class $\bmod d$.

A trivial estimate shows that there are at most $\sum_{d=3}^{N} d2^{N/d+1} \ll 2^{N/3}$ sets $A \subseteq \{1, \ldots, N\}$ with $\gcd'(A) \geq 3$. The number of sum-free sets $A \subseteq \{1, \ldots, N\}$ with $\gcd(A) = 2$ is $\mathrm{SF}(N/2)$, which is $O\left(2^{N/3}\right)$ by (1). Thus, there are at most $O\left(2^{N/3}\right)$ sum-free sets $A \subseteq \{1, \ldots, N\}$ with $\gcd'(A) \geq 3$ or $\gcd(A) = 2$. Since any SF_3-set is sum-free, the same holds for SF_3-sets.

The SF_3-sets $A \subseteq \{1, \ldots, N\}$ with $\gcd(A) = 1$ and $\gcd'(A) = 2$ are just the sets of odd numbers. There are exactly $2^{[(N+1)/2]}$ such sets.

It remains to prove that the number of SF_3-sets $A \subseteq \{1, \ldots, N\}$ with $\gcd'(A) = 1$ is $O\left(2^{(1/2-C)N}\right)$. This is an immediate consequence (with $C = 10^5$) of Theorem 1.1 and the following proposition.

Proposition 3.1 *Let $A \subseteq \{1, \ldots, N\}$ be an SF_3-set with $\gcd'(A) = 1$. Then either $|A| \leq 4N/17 + O(1)$ or $\ell(A) \leq 13N/34 + O(1)$.*

Indeed, the number of sets $A \subseteq \{1, \ldots, N\}$ with $\ell(A) \leq 13N/34 + O(1)$ is trivially estimated as $O\left(N2^{13N/34}\right)$.

The proof of Proposition 3.1 is based on the following two lemmas.

Lemma 3.2 (Lev and Smeliansky [10, Th. 2]) *Let A and B be finite sets of integers with $\ell(A) \geq \ell(B)$ and $\gcd'(A) = 1$. Then*

$$|A + B| \geq |B| + \min(\ell(A), |A| + |B| - 3). \qquad \blacksquare$$

Lemma 3.3 (Lev [9]) *Let A be a finite set of integers with* $\gcd'(A) = 1$. *Put*

$$n = |A|, \quad l = \ell(A), \quad k = [(l-1)/(n-2)]\,. \tag{16}$$

Then for any non-negative integers h_1 and h_2 we have

$$\Big|(\underbrace{A+\dots+A}_{h_1}) - (\underbrace{A+\dots+A}_{h_2})\Big| \ge \begin{cases} \dfrac{h(h+1)}{2}(n-2)+h+1, & h \le k, \\ \dfrac{k(k+1)}{2}(n-2)+k+1+ & \\ (h-k)l, & h > k, \end{cases}$$

where $h = h_1 + h_2$. ■

Mention that cases $h_1 = h_2 = 1$ and $h_1 = 2, h_2 = 0$ follow from Lemma 3.2, as well as from an old result of Freiman [6] (see also [12, 11]), and the case $h_2 = 0$ (and h_1 arbitrary) was done in the previous paper of Lev [8].

Proof of Proposition 3.1 Let $A \subseteq \{1, \dots, N\}$ be an SF_3-set with $\gcd'(A) = 1$. We use the notation (16). Also, put

$$A_2 = (A-A)_+, \quad A_3 = A_2 + A, \quad A_4 = (A+A-A-A)_+.$$

Then

$$A_2 \subseteq \{0, \dots, l\}, \quad A_3 \subseteq \{1, \dots, N+l\}, \quad A_4 \subseteq \{0, \dots, 2l\}.$$

Since $(N-1)/5 + 2 \le 4N/17 + O(1)$, we can assume that $n > (N-1)/5 + 2 \ge (l-1)/5 + 2$, in particular $k \le 4$.

By Lemma 3.3,

$$|A_4| = \frac{|A+A-A-A|+1}{2} \ge \frac{k(k+1)}{4}n + \frac{(4-k)}{2}l + O(1)$$

and

$$|A_2| = \frac{|A-A|+1}{2} \ge \begin{cases} 3n/2 - 1, & k \ge 2, \\ n/2 + l/2 + 1/2, & k = 1. \end{cases}$$

By Lemma 3.2

$$|A_3| \geq |A_2| + \min(l, |A| + |A_2| - 3) \geq \begin{cases} 7n/2 - 4, & k \geq 2, \\ n/2 + 3l/2 + 1/2, & k = 1, \end{cases}$$

because $\min(l, |A| + |A_2| - 3) \geq 2n - 3$ when $k \geq 2$.

Since A is an SF_3-set, the sets A_3 and A_4 are disjoint. Since both are subsets of $\{0, \ldots, N + l\}$, we have

$$l+N+1 \geq |A_3|+|A_4| \geq \begin{cases} (k^2 + k + 14)n/4 + (4 - k)l/2 + O(1), & k \geq 2, \\ n + 3l - 7, & k = 1. \end{cases}$$

Now if $k = 1$ then either $n \leq 4N/17$ or $l \leq (N - n)/2 + O(1) \leq 13N/34 + O(1)$, as desired. If $k \geq 2$ then

$$n \leq \frac{2(k-2)l + 4N}{k^2 + k + 14} + O(1) \leq \frac{2k}{k^2 + k + 14} N + O(1) \leq 4N/17 + O(1),$$

because $2k/(k^2 + k + 14) \leq 4/17$ when $k \in \{2, 3, 4\}$. The proof is complete. ■

4 SF_2-sets

In this section we prove Theorem 1.3. The argument goes along the same lines as that of the previous section. Again, the number of SF_2-sets $A \subseteq \{1, \ldots, N\}$ with $\gcd'(A) \geq 2$ is $2^{[(N+1)/2]} + O\left(2^{N/3}\right)$, and it remains to prove the following.

Proposition 4.1 *Let $A \subseteq \{1, \ldots, N\}$ be an SF_2-set with* $\gcd'(A) = 1$. *Then either $|A| \leq 2N/7 + 9/7$ or $\ell(A) \leq 8N/21$.*

Proof We use the same notation as in the proof of Proposition 3.1. The argument splits into three cases.

Case 1: $l \leq 2n - 4$. By Lemma 3.2

$$\begin{aligned} |A - A| &\geq n + l, \quad |A_2| \geq n/2 + l/2 + 1/2, \\ |A_2 - A| &\geq |A_2| + l \geq n/2 + 3l/2 + 1/2. \end{aligned}$$

Since A is an SF_2-set, the sets $A-A$ and A_2-A are disjoint. Since they both are subsets of $\{-N,-N+1,\ldots,l\}$, we have $N+l+1\geq 3n/2+5l/2+1$, whence either $n\leq 2N/7$ or $l\leq 8N/21$, as desired.

Case 2: $2n-3\leq l<5n/2-4$. Now

$$\begin{aligned} |A-A| &\geq 3n-3, \quad |A_2| \geq 3n/2-1, && (17)\\ |A_2-A| &\geq |A_2|+l \geq 3n/2+l-1, && (18)\end{aligned}$$

whence $N+l+1\geq 9n/2+l-4$, whence $n\leq 2N/9+10/9\leq 2N/7+9/7$, as desired.

Case 3: $l\geq 5n/2-4$. We again have (17), and instead of (18) we obtain

$$|A_2-A|\geq 2|A_2|+|A|-3\geq 4n-5.$$

Now $2N+1\geq N+l+1\geq 7n-8$, whence $n\leq 2N/7+9/7$, as desired. ■

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