PROJECTIVES ARE FREE FOR NILPOTENT VARIETIES

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Dedicated to the memory of Professor Tengiz Gegelia

0. Introduction. Projective objects in algebraic categories are retracts of free objects. We will show that finitely generated projective objects in the category of nilpotent groups of class n are free. The similar result is true for the category of nilpotent objects in other kind of algebraic categories, like Lie k-algebras, associative k-algebras etc. Here k is any commutative ring with property, that finitely generated projective k-modules are free. Moreover we will show that a retract of free finitely generated object is again free in complete case. We will derive such results from a very general statement on the lifting of idempotents in linear extensions of algebraic theories which generalize the well-known principle for modules (see for example Proposition III.2.10 of [1]).

1. Preliminaries on algebraic theories. We recall the notion of algebraic theory in the sense of Lawvere. Good reference for algebraic theories is [4]. The algebraic theory, or simply theory, formalizes the concept of an algebraic object as a set together with n-ary operations for various $n \in \mathbf{N}$ and equational relations.

1.1. Definition. *A theory is a category T with the following properties: T has a set of objects equal to the set of natural numbers $\mathbf{N} = \{0, 1, 2, \dots\}$. The object n is the n-fold product of the object 1: there are morphisms $p_i^n \in \mathrm{Hom}_T(n, 1)$ $i = 1, \dots, n$, such that the induced map*

$$\mathrm{Hom}_T(k, n) \ \to \ \mathrm{Hom}_T(k, 1)^n$$

is an isomorphism of sets. A morphism of theories is a product preserving functor which is the identity on objects. A theory T is called pointed, if 0 is a zero object in T.

One should think of the set $\mathrm{Hom}_T(n, 1)$ as the space of n-ary operations in the theory T.

1.2. Definition. *If T is a theory, a T-algebra X is a product preserving functor $X : T \to Sets$ to the category of sets. $X(1)$ is called the underlying set of the T-algebra X. A morphism of T-algebras is a natural transformation of functors, it is called surjective, if it yields surjection on the underlying sets.*

For a T-algebra X, each morphism $\phi \in \mathrm{Hom}_T(n, 1)$ gives rise to a map $X(1)^n \cong X(n) \xrightarrow{X(\phi)} X(1)$. This justifies thinking of a T-algebra as the underlying set together with n-ary operations parametrized by the set $\mathrm{Hom}_T(n,1)$.

1.3. Free T-algebras. The forgetful functor $T\text{-}alg \to Sets, X \mapsto X(1)$ has a left adjoint, the free T-algebra functor $F^T : Sets \to T\text{-}alg$. We let $< n >_T$ (or simply $< n >$) denote the free T-algebra generated by $\{1, \cdots, n\}$. The category T^{op} is equivalent to the category of finitely generated free T-algebras.

1.4. Algebraic functors. A morphism of theories $\phi : T_1 \to T$ induces an adjoint functor pair analogous to scalar restrictions and extensions: if X is a T-algebra, i.e., a product preserving functor $X : T \to Sets$, then the composite $X\phi : T_1 \to Sets$ is again product preserving, so it defines an T_1-algebra denoted by ϕ^*X. Note that the underlying set of ϕ^*X is the same as for X. ϕ^* has a left adjoint ϕ_*. ([4, Thm. 3.7.7]). We remark that the functor ϕ_* sends free objects to free ones, and its restriction to finitely generated free objects is the dual of $\phi : T_1 \to T$.

1.5. Examples. i) Let R be a ring and R^{mod} be the category, whose objects are the natural numbers, while the morphisms are given by

$$\mathrm{Hom}_{R^{\mathrm{mod}}}(n, m) := \mathrm{Hom}_R(R^m, R^n).$$

Here Hom_R denotes the Hom in the category of left R-modules. Then R^{mod} is a theory called the theory of left R-modules. One can proved, that R^{mod}-algebras are the same as the left R-modules. In this way we get a functor from the category of rings to the category of theories, which is full embedding. We see that theories are generalization of rings and we will try to generalize some well-known notion of rings to theories (see 1.6 and section 4).

ii) We recall that the lower cental series of a group G is defined by the induction:

$$\Gamma_1 G = G,\ \Gamma_{i+1}G \ = \ [G, \Gamma_i G],\ i \geq 0.$$

By definition G is a nilpotent group of class n if $\Gamma_{n+1}G = 0$. For any group G, the group $G/\Gamma_{n+1}G$ is a group of nilpotency class n. If G is a free group generated by the set $\{1, \cdots, m\}$, then $< m >_n := G/\Gamma_{n+1}G$ is called free group of nilpotency class n generated by the set $\{1, \cdots, m\}$. One can consider the full subcategory of the category of nilpotent groups of class n whose objects are $< m >_n,\ m \geq 0$. We let $\mathbf{Nil_n(Gr)}$ denote the dual of this category. Then $\mathbf{Nil_n(Gr)}$ is a theory, called theory of nilpotent groups of class n. Obviously the category of $\mathbf{Nil_n(Gr)}$-algebras is isomorphic to the category of nilpotent groups of class n. The functor $G \mapsto G/\Gamma_{n+1}G$ yields the morphism of algebraic theories:

$$-/\Gamma_{n+1} : \mathbf{Nil_{n+1}(Gr)} \to \mathbf{Nil_n(Gr)}.$$

iii) We recall that *a Leibniz algebra* (see [8]) is k-module $\mathbf{g}$ equipped with a bracket $\mathbf{g} \otimes \mathbf{g} \to \mathbf{g}$, satisfying the identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

Any Lie algebra is a Lebiniz algebra. For a Leibniz algebra $\mathbf{g}$ we let $\mathbf{g}_{Lie}$ denote the Lie algebra obtained by quotient $\mathbf{g}/[x, x] \sim 0$. A Leibniz algebra $\mathbf{g}$ is called two-sided Leibniz algebra if it satisfies the following identity:

$$[[x, y], z] = [x, [y.z]] - [y, [x, z]].$$

The theory of Lie (resp, Leibniz, two-sided Leibniz) algebras are denoted by **Lie**, **Leibniz**, $\mathbf{Leibniz_2}$. Then $(-)_{Lie}$ yields morphisms of algebraic theories:

$$\mathbf{Leibniz} \to \mathbf{Lie} \text{ and } \mathbf{Leibniz_2} \to \mathbf{Lie}.$$

iv) Let $\mathcal{P}$ be an operad. We recall, that $\mathcal{P}$ is a collection of S_n-modules $\mathcal{P}(n)$, $n \geq 1$, together with multiplications

$$\mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \cdots \mathcal{P}(i_n) \to \mathcal{P}(i_1 + \cdots + i_n)$$

and chosen unit $1 \in \mathcal{P}(1)$, satisfying certain associative and unital conditions (see [6]). Here S_n denotes the symmetric group. An algebra over $\mathcal{P}$ is a vector space A equipped with a sequence of S_n-equivariant maps

$$\mathcal{P}(n) \otimes A^{\otimes n} \to A,$$

which are associative with respect to the operad product and operad unit act as identity. The free $\mathcal{P}$-algebra generated by a vector space V, is defined by

$$F(V, \mathcal{P}) := \bigoplus_{i \geq 1} \mathcal{P}(i) \otimes_{S_i} V^{\otimes i}.$$

Then the dual of the category of finitely generated free $\mathcal{P}$-algebras form a theory. In this way we get a functor from the category of operads to the category of theories. For simplicity, we let $\mathcal{P}$ denote also the theory corresponding to the operad $\mathcal{P}$. A $\mathcal{P}$-algebra A is called *nilpotent* of class n, if for any $m > n$ the product $\mathcal{P}(m) \otimes A^{\otimes m} \to A$ vanishes. It is easy to remark, that the category of nilpotent $\mathcal{P}$-algebras of class n is the same as the category of $\mathcal{P}_n$-algebras. Here $\mathcal{P}_n$ is the operad given by $\mathcal{P}_n(m) := \mathcal{P}(m)$ if $m \leq n$ and $\mathcal{P}_n(m) := 0$ if $m > n$. The multiplication in $\mathcal{P}_n$ is defined in such way, that the natural projection $\mathcal{P} \to \mathcal{P}_n$ is a morphism of operads. Moreover the projections yield the morphisms of operads:

$$\mathcal{P} \to \cdots \to \mathcal{P}_{n+1} \to \mathcal{P}_n \to \cdots \to \mathcal{P}_1 \, .$$

It is well-known that many classical algebras are in fact algebras over an operad. For example. There are operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$, whose algebras are respectively the associative algebras without unit, the associative and commutative algebras without unit and the Lie algebras. Moreover corresponding notion of nilpotent object is exactly the classical one.

1.6. Projective T-algebras. An object $P \in T\text{-}alg$ is called *projective* if for any surjective homomorphism f of T-algebras, the map $Hom_{T-alg}(P, f)$ is also surjective. Any free object in T-alg is projective and any projective object is a retract of a free object. A projective object is called *finitely generated* if it is a retract of a finitely generated free T-algebra.

2. The idempotent completion. Let **C** be a category. An idempotent in $\mathcal{C}$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(c, c)$, such that $f^2 = f$. We will say that an idempotent f is split if there exist a diagram in $\mathcal{C}$:

$$c \xrightarrow{q} x \xrightarrow{j} c$$

such that $f = j \circ q$ and $q \circ j = 1_x$. Easily shows that if such a splitting exists, then it is unique up to isomorphism. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ splits idempotents if for any idempotent f in $\mathcal{C}$, the idempotent $F(f)$ splits in $\mathcal{D}$. The idempotent completion of the category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which splits idempotents and which is universal with respect of this property. The idempotent completion exists. We recall the construction due to Karubi.

2.1. The category $\mathrm{Idem}(\mathcal{C})$. For a category $\mathcal{C}$ we let $\mathrm{Idem}(\mathcal{C})$ denote, the category of idempotents of $\mathcal{C}$. Objects of $\mathrm{Idem}(\mathcal{C})$ are idempotents in $\mathcal{C}$. In order to specify that f is considered as an object of $\mathrm{Idem}(\mathcal{C})$, we let $[f]$ denote the object f in $\mathrm{Idem}(\mathcal{C})$. If $f \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(d, d)$ are idempotents, then we put

$$\mathrm{Hom}_{\mathrm{Idem}(\mathcal{C})}([f], [g]) \;:=\; \{h : c \to d \mid g \circ h \circ f = h\}.$$

The composition $h' * h$ in $\mathrm{Idem}(\mathcal{C})$ is given by

$$h' * h := h' \circ g \circ h,$$

while the identity morphism $1_{[f]}$ is f. We remark that the functor

$$u_{\mathcal{C}} : \mathcal{C} \to \mathrm{Idem}(\mathcal{C}),$$

given by $u(c) = [1_c]$ is full, faithful and splits idempotents. Moreover $\mathrm{Idem}(\mathcal{C})$ is the idempotent completion of $\mathcal{C}$. Clearly, the idempotent completion of the category of finitely generated free objects is the category of finitely generated projective objects. Since $\mathrm{Idem}(\mathcal{C}^{op})$ is the dual of $\mathrm{Idem}(\mathcal{C})$, we get the following

2.2. Lemma. *If T is a theory, then the category* $\mathrm{Idem}(T)$ *is equivalent to the dual of the category of finitely generated projective objects in T-alg.*

The next result is straightforward.

2.3. Lemma. *The functor* Idem *from the category of small categories to itself preserves inverse limits.*

3. Linear Extensions of Categories. We recall the definition of linear extension of categories in the sense of Baues and Wirsching (see [3] and Chapter 4 of [2]).

3.1. Definition. *Let $\mathcal{C}$ be a small category and let $D : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ be a bifunctor. We say that*

$$0 \to D \to \mathcal{E} \xrightarrow{p} \mathcal{C} \to 0 \tag{3.1.1}$$

is a linear extension of the category $\mathcal{C}$ by D if (a), (b) *and* (c) *hold.*

a) *$\mathcal{E}$ and $\mathcal{C}$ have the same objects and p is a full functor which is the identity on objects.*

b) *The abelian group $D(c, d)$ acts on the set* $\mathrm{Hom}_{\mathcal{E}}(c, d)$. *Moreover $p(f_0) = p(g_0)$ iff there is unique $\alpha \in D(A, B)$ such that $g_0 = f_0 + \alpha$. Here for each $f_0 : c \to d$ in $\mathcal{E}$ and $\alpha \in D(A, B)$ we write $f_0 + \alpha$ for the action of α on f_0. Any $f_0 \in p^{-1}(f)$ is called a lift of f.*

c) *The action satisfies the linear distributivity law:*

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_*\beta + g^*\alpha \ .$$

The morphism between two linear extensions

$$\begin{array}{ccccccccc} 0 & \to & D & \to & \mathcal{E} & \xrightarrow{p} & \mathcal{C} & \to & 0 \\ & & \downarrow \phi_1 & & \downarrow \phi_0 & & \downarrow \phi & & \\ 0 & \to & D' & \to & \mathcal{E}' & \xrightarrow{p'} & \mathcal{C}' & \to & 0 \end{array} \tag{3.1.2}$$

consists of functors ϕ and ϕ_0, such that $\phi p = p' \phi_0$. Moreover the natural transformation $\phi_1(c, d) : D(c, d) \to D'(\phi c, \phi d)$ is given, such that

$$\phi_0(f_0 + \alpha) = \phi_0(f_0) + \phi_1(\alpha).$$

One has the following additional properties :

d) There is a canonical bijection

$$M(\mathcal{C}, D) \cong H^2(\mathcal{C}, D).$$

Here $M(\mathcal{C}, D)$ is the set of equivalence classes of linear extensions and $H^*(\mathcal{C}, D)$ is the cohomology of $\mathcal{C}$ with coefficients in D (see [3]).

e) For any $c \in Ob\ (\mathcal{C}) = Ob\ (\mathcal{E})$ one has a short exact sequence of groups:

$$0 \to D(c,c) \to Aut_{\mathcal{E}}(c) \xrightarrow{p} Aut_{\mathcal{C}}(c) \to 1.$$

f) The functor p reflects isomorphisms and yields bijection on the sets of isomorphism classes $\mathrm{Iso}(\mathcal{E}) \cong \mathrm{Iso}(\mathcal{C})$.

g) Assuming $\phi_1(c,d)$ in (3.1.2) is an isomorphism for any $c, d \in \mathcal{C}$, then ϕ is an equivalence of categories iff ϕ_0 is an equivalence of categories.

We wish to have the linear extension of the form:

$$0 \to D \to \mathrm{Idem}(\mathcal{E}) \xrightarrow{p} \mathrm{Idem}(\mathcal{C}) \to 0.$$

Unfortunatly this is not the case, because $\mathrm{Idem}(\mathcal{E})$ and $\mathrm{Idem}(\mathcal{C})$ have different objects. As we will show this is the only obstruction.

3.2. Proposition. i) *The bifunctor D has the unique extension*

$$D : (\mathrm{Idem}\ \mathcal{C})^{op} \times (\mathrm{Idem}\ \mathcal{C}) \to Ab.$$

ii) *The action*

$$\mathrm{Hom}_{\mathcal{E}}(c,d) \times D(c,d) \to \mathrm{Hom}_{\mathcal{E}}(c,d)$$

$c, d \in Ob\ (\mathcal{C})$ has the unique extension

$$\mathrm{Hom}_{\mathrm{Idem}(\mathcal{E})}([f],[g]) \times D([f],[g]) \to \mathrm{Hom}_{\mathrm{Idem}(\mathcal{E})}([f],[g]).$$

$[f],[g] \in Ob\ \mathrm{Idem}(\mathcal{E})$.

iii) *There is a linear extension of categories*

$$0 \to D \to \mathrm{Idem}(\mathcal{E}) \xrightarrow{p} \mathcal{X} \to 0,$$

where $\mathcal{X}$ is a category with the same objects as $\mathrm{Idem}(\mathcal{E})$, but with following morphisms:

$$\mathrm{Hom}_{\mathcal{X}}([f_0],[g_0]) = \mathrm{Hom}_{\mathrm{Idem}(\mathcal{C})}([f],[g]),$$

where as usual $f = p(f_0)$ and $g = p(g_0)$.

iv) *The fuctor $\bar{p} : \mathcal{X} \to \mathrm{Idem}(\mathcal{C})$ which sends $[f_0]$ to $[p(f_0)] = [f]$ and is identity on morphisms, is an equivalence of categories.*

v) *The functor $u_{\mathcal{C}} : \mathcal{C} \to \mathrm{Idem}(\mathcal{C})$ is an equivalence of categories iff $u_{\mathcal{E}} : \mathcal{E} \to \mathrm{Idem}(\mathcal{E})$ is an equivalence of categories.*

Proof. i) and ii) are consequence of the universality of the idempotent completion. Straightforward but a bit tedious chacking shows that iii) is true, while iv) is a consequence of Lemma 3.4. Finaly v) follows from iii), iv) and g) of 3.1.

3.3. Lift of idempotents in the linear extensions. Here we prove the following lemma, which is the nonlinear generalization of Proposition III.2.10 of [1]. Let $D:\mathcal{C}^{op}\times\mathcal{C}\to Ab$ be a bifunctor and

$$0\to D\to\mathcal{E}\xrightarrow{p}\mathcal{C}\to 0$$

be a linear extension of $\mathcal{C}$ by D, whose characteristic element is $ch(\mathcal{E})\in H^2(\mathcal{C},D)$.

3.4. Lemma. *Any idempotent element has an idempotent lift. Thus for any $f\in \mathrm{Hom}_{\mathcal{C}}(c,c)$, such that $f^2=f$, there exists $f_0\in \mathrm{Hom}_{\mathcal{E}}(c,c)$, such that $p(f_0)=f$ and $f_0^2=f_0$.*

Proof. We consider the category $\mathcal{I}$, which has only one object and only one non-identity morphism τ with $\tau^2=\tau$. Then we have a functor $F:\mathcal{I}\to\mathcal{C}$ which assigns f to τ. We wish to lift F as a functor $F_0:\mathcal{I}\to\mathcal{E}$. But existence of such lifting is equivalent to show that

$$0=(F)^*(ch(\mathcal{E}))\in \mathrm{H}^2(\mathcal{I},D\circ(F^{op}\times F)).$$

One easily notices that the category of bifunctors

$$\mathcal{I}^{op}\times\mathcal{I}\to Ab$$

is isomorphic to the category of bimodules over the ring $\mathbf{Z}[x]/(x^2-x)$ and moreover the cochains of the category $\mathcal{I}$ with coefficients in a bifunctor are the same as Hochschild cochains of the ring $\mathbf{Z}[x]/(x^2-x)$, hence there exists an isomorphism

$$\mathrm{H}^*(\mathcal{I},-)\cong \mathrm{Hoch}^*(\mathbf{Z}[x]/(x^2-x),-)$$

and the statement follows from Lemma 3.5.

3.5. Lemma. *The Hochschild cohomology for the ring $\mathbf{Z}[x]/(x^2-x)$ vanishes in positive dimensions.*

Proof. Let M be a $\mathbf{Z}[x]/(x^2-x)$-bimodule. Then reduced Hochschild cochain complex has the form:

$$M\xrightarrow{b}M\xrightarrow{a}M\xrightarrow{b}M\xrightarrow{a}\cdots\ ,$$

where $b(m)=xm-mx$ and $a(x)=xm-m+mx$. Let $c:M\to M$ be a homomorphism given by $c(m)=2xm-m$. Then $bc+ca=1_M$ as well as $ac+cb=1_M$ and the proof is finished.

3.7. Corollary. *The functor p yields a bijection on the set of isomorphic clases of objects*:

$$\mathrm{Iso}(\mathrm{Idem}(\mathcal{E}))\cong \mathrm{Iso}(\mathrm{Idem}(\mathcal{C})).$$

Proof. This a consequence of The property f) of 3.1 and Proposition 3.2.

4. Linear and Nilpotent Extensions of Theories. We start with following

4.1 Lemma. *Let T be a theory and*

$$0 \to D \to T_1 \stackrel{\phi}{\to} T \to 0$$

be a linear extension of T by D, where $D : T^{op} \times T \to Ab$ is a bifunctor preservind finite product with respect of the second variable. Then T_1 is a theory too.

Proof. See Lemma 3.4 of [5].

In the case of Lemma 4.1, we will say that $\phi : T_1 \to T$ is a *linear extension of theories.*

4.2. Examples. i) Let

$$0 \to M \to R \to S \to 0$$

be a square zero ideal extension of the ring S by the S-bimodule M. Then

$$0 \to D \to R^{\mathrm{mod}} \to S^{\mathrm{mod}} \to 0$$

is a linear extension of theories, where $D(n, m) := Hom_S(S^m, S^n \otimes M)$ (see 4.14 of [5]). Hence the notion of the linear extension of theories is a generalization of the notion of square zero ideal extension for rings.

ii) The morphism of theories

$$-/\Gamma_{n+1} : \mathbf{Nil_{n+1}(Gr)} \to \mathbf{Nil_n(Gr)}$$

from example ii) of 1.5 is a linear extension. Indeed, by theorem of Witt [9] we know that for any free group G of nilpotency class $n+1$ one has a central extension of groups

$$0 \to L_{n+1}A \to G \to G/\Gamma_{n+1}G \to 0,$$

where $A = G_{ab} = (G/\Gamma_{n+1}G)_{ab}$ and $L_*(A)$ is the free Lie ring generated by A. Now if G_1 is another free group of nilpotency class $n+1$, then it follows from the properties of central extensions that

$$\mathrm{Hom}(G/\Gamma_{n+1}G, L_{n+1}A_1) = \mathrm{Hom}(A, L_{n+1}A_1)$$

acts on $\mathrm{Hom}(G, G_1)$, where $A_1 = (G_1)_{ab} = (G_1/\Gamma_{n+1}G_1)_{ab}$. Assuming f and g are homomorphisms from $\mathrm{Hom}(G, G_1)$, then

$$(-/\Gamma_{n+1})(f) = (-/\Gamma_{n+1})(g)$$

iff there exist unique

$$h \in \mathrm{Hom}(G/\Gamma_{n+1}G, L_{n+1}A_1) = \mathrm{Hom}(A, L_{n+1}A_1)$$

such that $g = f + h$. Hence we have the linear extension (see also 4.16 of [5]):

$$0 \to D \to \mathbf{Nil_{n+1}}(Gr) \to \mathbf{Nil_n(Gr)} \to 0,$$

where D is given by

$$D(< m >_n, < m' >_n) = \mathrm{Hom}((< m' >_n)_{ab}\ , L_{n+1}((< m >_n)_{ab})).$$

iii) The morphism of theories $(-)_{Lie} : \mathbf{Leibniz}_2 \to \mathbf{Lie}$ is a linear extension. Indeed in two-sided Leibniz algebras the following identities holds:

$$[x, [y, y]] = 0;\ \ [[x, x], y] = 0.$$

Thus for any two-sided Leibniz algebra $\mathbf{g}$ one has a central extension

$$0 \to \mathbf{g}^{ann} \to \mathbf{g} \to \mathbf{g}_{Lie} \to 0.$$

Based on Lemma 1.3 of [8], one easily shows that, if $\mathbf{g}$ is a free two-sided Leibniz algebra, then $\mathbf{g}^{ann} \cong \Gamma(\mathbf{g}_{ab})$, where

$$\Gamma(V) := Ker(V \otimes V \to \Lambda^2 V).$$

Then the properties of central extensions shows that

$$0 \to D \to \mathbf{Leibniz}_2 \to \mathbf{Lie} \to 0$$

is a linear extension, where $D(n, m) = \mathrm{Hom}((< m >)_{ab}\ , \Gamma(< n >_{ab}))$.

iv) For an operad $\mathcal{P}$, the morphisms of operads $\mathcal{P}_{n+1} \to \mathcal{P}_n$ from iv) of Examples 1.5, yields a linear extension of theories.

4.3. Definition. *A morphism of algebraic theories $\phi : T_1 \to T$ said to be nilpotent extension if ϕ can be decomposed as a composition of linear extensions. A theory is called linear if it is additive category.*

If T is a linear theory then it is isomorphic to the theory R^{mod}, where $R = \mathrm{Hom}_T(1, 1)$. Moreover for any theory T there is a universal morphism into linear theory. This linear theory can be described as the Kroneker product (see [4]) of T and the theory of abelian groups and will be denoted by $T \otimes \mathbf{Z}$. So one can think on $T \otimes \mathbf{Z}$ as a ring. By the properties of Kroneker product, the category of modules over $T \otimes \mathbf{Z}$ is isomorphic to the category of abelian group object in T-alg. We will say that T is *a nilpotent* theory if $T \to T \otimes \mathbf{Z}$ is a nilpotent extension. By Examples 4.2 the theory of nilpotent groups of class n as well as nilpotent $\mathcal{P}$-algebras are nilpotent theories. We leave as an exercise to the interesting reader to show that other kind of nilpotent algebras (say restricted Lie p-algebras or Moufang loops) gives rise to nilpotent theories in our sense.

We say that a theory T has the *PF-property* if any finitely generated projective object in T-alg is a free object. Thus T has PF property iff

$u_T : T \to \mathrm{Idem}(T)$ is an equivalence of categories. If T is a pointed theory, then a summand of a free object is a projective object in T-*alg*. We say that a theory T has the *PS-property* if any finitely generated projective object in T-*alg* is a summand of a finitely generated free object.

4.4. Proposition. *Let $\phi : T_1 \to T$ be a nilpotent extension of theories. Then*

i) *ϕ yields a bijection on isomorphic clases of finitely generated projective objectes.*

ii) *Moreover T has PF-property iff T_1 has PF-property.*

iii) *A finitely generated projective T_1-algebra P_1 is free iff $\phi_*(P_1)$ is a free T-algebra.*

iv) *If additionally T and T_1 are pointed theories, then T has PS-property iff T_1 has PS-property.*

Proof. i) follows from Lemma 2.2 and Corollary 3.7, while ii) is a consequence of v) of Proposition 3.2. Since ϕ_* sends free objects to free objects, we see that "if" part of iii) holds. In order to prove "only if" part of iii) we choose an isomorphism into free object: $\alpha : \phi_*(P_1) \to < n >_T$. One can consider α as a morphism in $\mathcal{X}^{op}$, where $\mathcal{X}$ is the category from iii) of Proposition 3.2. Then a) and f) of 3.1) show that there is an isomorphism $\alpha_0 : P_1 \to < n >_{T_1}$ such that $\phi_*(\alpha_0) = \alpha$ and iii) is proved. Assume now that T_1 has PS-property, and take a projective T-algebra P. Then there exists a projective T_1-algebra P_1 and an isomorphism $\phi_*(P_1) \cong P$. We may choose a projective T_1-algebra P_1' such that $P_1 \vee P_1'$ is free, where $\vee$ denotes the sum. Then $\phi_*(P_1 \vee P_1') \cong P \vee \phi_*(P_1')$ is also free. Hence T has PS-property. Conversely, if T has PS-property and P_1 is a projective T_1-algebra, then there exists a projective T-algebra P' such that $\phi_*(P_1) \vee P'$ is free. By i) we know, that there exists projective T_1-algebra P_1', such that $\phi_* P_1' \cong P'$. Since ϕ_* preserves sums $\phi_*(P_1 \vee P_1')$ is free and the result follows from iii).

4.5. Corollary. *Any nilpotent theory T has PS-proprty. Moreover it has PF-proprerty iff the ring $T \otimes \mathbf{Z}$ has PF-property.*

4.6. Examples. i) For $T = \mathbf{Nil_n(Gr)}$ the corresponding ring is $\mathbf{Z}$, hence it has PF-proprerty.

ii) Let k be a field. Then it is well known that a subalgebra of a free Lie k-algebra is free. Hence **Lie** has PF-proprty. Now by Proposition 4.4 the theory $\mathbf{Leibniz_2}$ has the same property.

iii) Let $\mathcal{P}$ be an operad. By iii) of 4.2, $\mathcal{P}_n$ is a nilpotent theory. Hence it has PS property. Moreover if $\mathcal{P}_1$ has PF property, then $\mathcal{P}_n$ has the same property. As particular case we obtain that projective objects in the category of nilpotent (commutative, associative, or Lie) k-algebras are free. Here k is any commutative ring with property, that finitely generated projective k-modules are free.

4.7. Remark. The proposition 4.4 is a nonlinear generalization of Proposition III.2.12 of [1].

5. Completion. In the same line as was done in Proposition III.2.12 of [1] one can prove the corresponding statement in complete case. Let

$$\mathbf{T} = (\cdots \to T_n \to T_{n-1} \to \cdots \to T_1)$$

be an inverse system of theories. We assume, that

$$T_n \to T_{n-1}$$

is a linear extension of theories for each $n \geq 2$. We recall that the category of theories has inverse limits and these are calculated pointwise (see Prop. 3.11.1 of [4]).

5.1 Proposition. Let T be the inverse limit of $\mathbf{T}$. If T_1 has the property PF, then T has the same property.

Proof. In Lemma 5.4 we put $\mathcal{C}_n = T_n$, $\mathcal{E}_n = \mathrm{Idem}(T_n)$ and $f_n = u_{T_n}$. By Lemma 2.3 and v) of Proposition 3.2 we see that the conditions of Lemma 5.4 hold. Hence u_T is an equivalence of categories and proposition is done.

5.2. Nil-complete groups. A group G is called *nil-complete* if the natural map $G \to lim\ G/\Gamma_n G$ is an isomorphism. Any group G has nil-completion $G^{nil} := lim\ G/\Gamma_n G$. The nil-completion of (finitely generated) free group is called (finitely generated) free nil-complete group. Let $\mathbf{Gr^{Nil}}$ be the inverse limit of the inverse system of algebraic theories

$$\cdots \to \mathbf{Nil_{n+1}(Gr)} \to \mathbf{Nil_n(Gr)} \to \cdots \to \mathbf{Nil_2(Gr)} \to \mathbf{Nil_1(Gr)} = \mathbf{Ab}$$

One can show that the dual of the category $\mathbf{Gr^{Nil}}$ is the category of finitely generated free nil-complete groups and continuous homomorphisms. So as a consequence we can conclude that, *a retract of a finitely generated free nil-complete group is still free nil-complete group.*

5.3. Nil-complete algebras. Let $\mathcal{P}$ be an operad. A *free complete* $\mathcal{P}$-algebra generated by a vector space V is given by

$$\hat{\mathcal{P}}(V) := \prod_{i \geq 1} \mathcal{P}(i) \otimes_{S_i} V^{\otimes i}.$$

The inverse system of theories introduced in iv) of Examples 1.5 shows that *a retract of a finitely generated free complete $\mathcal{P}$-algebra is still free complete $\mathcal{P}$-algebra.* In the case $\mathcal{P} = \mathcal{C}om$ this was proved by Gubeladze [7].

5.4. Lemma. Let $(\mathcal{C}_n)_{n\geq 1}$ and $(\mathcal{E}_n)_{n\geq 1}$ be inverse systems of small categories and let $(f_n)_{n\geq 1}$ be a morphism from the first inverse system to the second one. Suppose that the following assumptions hold

i) The functor $f_n : \mathcal{C}_n \to \mathcal{E}_n$ is an equivalence of categories for each $n \geq 1$.

ii) Each $\mathcal{C}_{n+1} \to \mathcal{C}_n$ is full functor, which is identity on objects and reflects isomorphisms.

Then induced functor between limits $f : lim\ \mathcal{C}_n \to lim\ \mathcal{E}_n$ is an equivalence of categories.

Proof is obvious.

Acknowledgements

The author is very grateful to Mamuka Jibladze for very useful discussions on the subject. I am very grateful to Friedhelm Waldhausen for his hospitality at the University of Bielefeld, while this paper was being written. Author partially supported by Grant No GM1-115 of the U.S. CRDF.

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