INTERSECTION PAIRING IN HYPERBOLIC MANIFOLDS, VECTOR BUNDLES AND CHARACTERISTIC CLASSES

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Abstract. We present some constraints on the intersection pairing in hyperbolic manifolds. We also show that, given a closed negatively curved manifold M of dimension at least three, only finitely many rank k oriented vector bundles over M can admit complete hyperbolic metrics.

§0. Introduction

Hyperbolic manifolds are locally symmetric Riemannian manifolds of negative sectional curvature. Any complete hyperbolic manifold is a quotient of a rank one symmetric space by a discrete groups of isometries that acts freely. The main result is the following

0.1. Theorem. *Let Y be a finite connected CW-complex such that $\pi_1(Y)$ is torsion free group which is not virtually nilpotent. Assume that $\pi_1(Y)$ has no nontrivial decomposition as an amalgamated product or an HNN extension over a virtually nilpotent group. Then, given a nonnegative integer n and homology classes $[\alpha] \in H_m(Y)$ and $[\beta] \in H_{n-m}(Y)$, there exists $K > 0$ such that,*

(1) for any continuous map $f : Y \to N$ of Y into an oriented complete hyperbolic n-manifold N that induces an isomorphism of fundamental groups, and

(2) for any embedding $f : Y \to N$ of Y into an oriented complete hyperbolic n-manifold N that induces a monomorphism of fundamental groups.

the intersection number of the cycles $f_*\alpha$ and $f_*\beta$ in N satisfies $|\langle f_*\alpha, f_*\beta\rangle| \leq K$.

The assumptions on $\pi_1(Y)$ are used to invoke a compactness theorem due to Rips that the space of conjugacy classes of faithful discrete representations of $\pi_1(Y)$ into the group of isometries of a rank one symmetric space is compact. For example, these assumptions hold if Y is a closed aspherical manifold of dimension at least three where $\pi_1(Y)$ is word-hyperbolic. In particular, we get the following

0.2. Corrolary. *Let M be a closed oriented $K(\Gamma, 1)$ manifold of dimension $n \geq 3$ such that Γ is word-hyperbolic. Then there is a finite subset $F \subset H^k(M)$ that contains the Euler class of any π_1-incompressible topological embedding of M into a complete oriented hyperbolic $(n+k)$-manifold.*

Thus, F contains the Euler class of any oriented vector bundle $\mathbb{R}^k \to E \to M$ if E admits a complete $\mathbb{F}$-hyperbolic structure.

The corollary generalizes a result of Kapovich [Kap2] who proved that, given a closed oriented negatively curved manifold M of dimension $n \geq 3$, there is a finite subset $F \subset H^n(M)$ that contains the Euler class of any rank n vector bundle whose total space admits a complete *real* hyperbolic structure.

Kapovich [Kap2] also proves a similar result in case M is a closed orientable surface. Namely, the Euler number of any oriented $\mathbb{R}^2$-bundle over a closed oriented surface of genus g is bounded by $\exp(\exp(10^8(g-1)))$ provided the total space of the bundle is a complete real hyperbolic manifold.

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More generally, Kapovich [Kap2] proved that there is a universal function $C(-,-)$ such that for any incompressible singular surfaces Σ_{g_1}, Σ_{g_2} in a complete oriented real hyperbolic 4-manifold we have $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \leq C(g_1,g_2)$.

Reznikov [Rez] showed that for any singular surfaces Σ_{g_1}, Σ_{g_2} in a closed oriented negatively curved 4-manifold M with the sectional curvature pinched between $-k^2$ and $-K^2$ there is a bound $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \leq C(g_1,g_2,k,K,\chi(M))$ for some universal function C.

Some vector bundles appear naturally as hyperbolic manifolds. For example, given a real hyperbolic manifold M the manifold $M\times\mathbb{R}^k$ admits a real hyperbolic structure while the tangent bundle TM admits a complex hyperbolic structure. However, aside from these trivial cases, it is not easy to construct hyperbolic structures on vector bundles.

All known examples are based on some subtle geometric constructions. An orientable $\mathbb{R}^2$-bundle over a closed oriented surface of genus g is known to admit a real hyperbolic structure provided the Euler class e of the bundle satisfies $|e|<g$ [Luo1,2] (cf. [GLT], [Kap1], [Kui1,2]). For nonorientable bundles over nonorientable surfaces the condition $|e|\leq[\frac{g}{8}]$ implies the existance of a real hyperbolic structure [Bel1]. Complex hyperbolic structures exist on orientable $\mathbb{R}^2$-bundles over closed oriented surfaces provided $|e+2g-2|<g$ [GKL]. There are examples of real hyperbolic structures on nontrivial orientable $\mathbb{R}^2$-bundles over closed real hyperbolic 3-manifolds [Bel2].

For any stable characteristic class c (e.g. the total Pontrjagin class) there is a uniform bound on the number of elements of the ring $H^*(M,\mathbb{Z})$ that can be represented as the class c of the normal bundle of an immersion of M into a (G,X)-manifold where G is real algebraic or semisimple with finite center. In fact, the bound is the number of connected components of the space $\mathrm{Hom}(\pi_1(M),G)$.

Note that a vector bundle is determined up to a finite number of possibilities by its rational Pontrjagin classes and the rational Euler class. In particular, we get the following.

0.3. Theorem. *Let M be a closed smooth manifold and k be a positive integer. Assume that either k is odd or $k>dim(M)$.*

Then the set of isomorphism classes of the normal bundles of smooth immersions of M into (possibly incomplete) hyperbolic $(n+k)$-manifolds is finite.

Combining 0.2 and 0.3 we get

0.4. Theorem. *Let M be a closed oriented negatively curved n-manifold with $n\geq 2$ and k be a positive integer with $n+k\geq 5$. Then the set of isomorphism classes of the normal bundles of smooth π_1-incompressible embeddings of M into a complete hyperbolic $(n+k)$-manifold is finite.*

In particular, the total spaces of only finitely many smooth oriented rank k vector bundles over M can admit complete hyperbolic structures.

Note that according to a result of Anderson [A] the total space of any vector bundle over a closed negatively curved manifold admits a complete metric with the sectional curvature pinched between two negative constants.

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§1. Varieties of representations and vector bundles

Let B be a finite connected CW-complex with the universal cover $\tilde{B}$. Let G be a Lie group that acts effectively and analytically on a contractible n-manifold X. Given a homomorphism $\rho:\pi_1(B)\to G$, the group $\pi_1(B)$ acts on $\tilde{B}\times X$ via $\gamma(b,x)=(\gamma b,\rho(\gamma)x)$. The quotient map $(\tilde{B}\times X)/\pi_1(B)=E_\rho\to\tilde{B}/\pi_1(B)=B$ is a flat (G,X)-bundle with fiber X and holonomy ρ. Isomorphism classes of flat (G,X)-bundles are in one-to-one correspondence with the space of conjugacy classes of representations $\mathrm{Hom}(\pi_1(B),G)/G$.

The vector bundle $\tilde{B}\times TX\to\tilde{B}\times X$ decsendes to a vector bundle ν over E_ρ. The bundle ν is "vertical" in the sense that its fibers are tangent to the fibers of E_ρ. Clearly, isomorphic (G,X)-bundles (disregarding the flat structure) have isomorphic vertical bundles.

Since X is contractible, the bundle E_ρ has a section s which is unique up to homotopy. For homotopic sections s_2 and s_1, the bundles $s_1^*\nu$ and $s_2^*\nu$ are isomorphic, hence, the isomorphism class

of $s^*\nu$ is determined by ρ. In fact, the isomorphism class of $s^*\nu$ only depends on the conjugacy class of ρ in $\mathrm{Hom}(\pi_1(B), G)$ since isomorphic (G, X)-bundles have isomorphic vertical bundles.

Moreover, by the covering homotopy theorem, if representations ρ_1 and ρ_2 lie in the same connected component of the representation variety $\mathrm{Hom}(\pi_1(B), G)$, the bundles E_{ρ_1} and E_{ρ_2} are isomorphic as (G, X)-bundles. Therefore, the isomorphism class of an (G, X)-bundle E_ρ (and, hence, the isomorphism class of its vertical vector bundle) depends only on the connected component of ρ in $\mathrm{Hom}(\pi_1(B), G)$. Thus we have constructed a locally-constant map $V : \mathrm{Hom}(\pi_1(B), G) \to Vect^n(B)$ that takes a representation from $\mathrm{Hom}(\pi_1(B), G)$ to the isomorphism class of the corresponding vertical bundle. Furthermore, given a characteristic class $c : Vect^n(B) \to H^*(B)$ we get a locally-constant map $c : \mathrm{Hom}(\pi_1(B), G) \to H^*(B)$.

In case the group G is real algebraic, complex algebraic, or semisimple with finite center, $\mathrm{Hom}(\pi_1(B), G)$ has finitely many connected components [G1], in particular, the map V has finite image.

1.1. Proposition. *Let f be a continuous map of B into an (G, X)-manifold N with holonomy $h : \pi_1(N) \to G$. Then the pullback f^*TN of the tangent bundle to N via f is isomorphic to the pullback of the vertical bundle of the (G, X)-bundle E_ρ via a section, where $\rho = h \circ f_* : \pi_1(B) \to G$.*

Proof. Let f be a continuous map of B into an (G, X)-manifold N with holonomy homomorphism $h : \pi_1(N) \to G$. Lift f to an equivariant map of universal coverings $\tilde{f} : \tilde{B} \to \tilde{N}$ and compose with the developing map $d : \tilde{N} \to X$. Thus, we get an equivariant map $d \circ \tilde{f} : \tilde{B} \to X$.

The group $\pi_1(B)$ acts via f_* by bundle isomorphisms on $T\tilde{N}$, and $\pi_1(B)$ acts via h by bundle isomorphisms on TX. Since d is an equivariant local diffeomorhism, we have an equivariant isomorphism of pullback bundles $\tilde{f}^*(T\tilde{N}) \cong \tilde{f}^*d^*(TX)$.

Note that the bundle $\tilde{f}^*d^*(TX)$ is equivariantly isomorphic to the pullback of the bundle $\tilde{B} \times TX \to \tilde{B} \times X$ via the map $\tilde{F} = (\mathrm{id}, d \circ \tilde{f}) : \tilde{B} \to \tilde{B} \times X$.

The group $\pi_1(B)$ acts on $\tilde{B} \times TX$ by bundle isomorphisms, so $\tilde{B} \times TX$ descendes to the vertical bundle ν over E_ρ. Since $\tilde{F}$ is equivariant, $\pi_1(B)$ also acts on $\tilde{F}^*(\tilde{B} \times X)$ by bundle isomorphisms and, therefore, the bundle $\tilde{F}^*(\tilde{B} \times X)$ descendes to $F^*\nu$, the pullback of the vertical bundle via the section F.

Similarly, the bundle $\tilde{f}^*(T\tilde{N})$ descendes to f^*TN. Thus the equivariant isomorphism $\tilde{f}^*(T\tilde{N}) \cong \tilde{f}^*d^*(TX) \cong \tilde{F}^*(\tilde{B} \times X)$ induces the bundle isomorphism $f^*TN \cong F^*\nu$. $\square$

1.2. Corollary. *Assume the space $Hom(\pi_1(B), G)$ has $d < \infty$ connected components. Then there exist d real vector bundles over B such that, for any continuous map f of B into an (G, X)-manifold N, the pullback bundle f^*TN is isomorphic to one of the bundles.*

We denote $[E]$ the stable equivalence class of a vector bundle E in the real K-theory $\widetilde{KO}(B)$. Notice that, if B is a closed smooth manifold and f is a smooth immersion, the normal bundle of the immersion ν_f has the property $\nu_f \oplus TB \cong f^*TN$; therefore, in this case $[\nu_f] = [f^*TN] - [TB]$. Characteristic classes are natural transformations of contravariant functors $\mathrm{Vect}_{\mathbb{R}}(-) \to H^*(-, \Lambda)$. A characteristic class is called stable if descendes to a natural transformation $\widetilde{KO}(-) \to H^*(-, \Lambda)$. For example, the total Pontrjagin class is stable.

1.3. Corollary. *Let B is a closed smooth manifold and the space $Hom(\pi_1(B), G)$ has $d < \infty$ connected components. Then there exists a finite subset of $\widetilde{KO}(B)$ of size at most d that contains the equivalence class of the normal bundle of any immersion of B into an (G, X)-manifold N. In particular, for any stable characteristic class $c : Vect(B) \to H^*(B, \mathbb{Z})$, there is a finite subset of $H^*(B, \mathbb{Z})$ of size at most d that contains the c-image of the normal bundle of any immersion of B into an (G, X)-manifold N.*

Proof. Let $i : B \to N$ be an immerion with the normal bundle ν_i. Then $\nu_i \oplus TB = i^*TN$. Passing to real K-theory we get $[\nu_i] = [i^*TN] - [TB]$. By 1.2, there are at most d possibilities for the isomorphism class of i^*TN while TB is a constant. $\square$

1.4. Remark. In case the G-action preserves an almost complex structure on X, any (G,X)-manifold is almost complex. In paricular, the vertical and tangent bundles are complex and 1.3 holds for almost complex immersions (of course KO must be replaced with KU).

I am most grateful to J. M. Rosenberg for explaining to me the following nice fact.

1.5. Proposition. *Let Y be a finite CW-complex and k be a positive integer. Then the set of isomorphism classes of oriented real (complex, respectively) rank k vector bundles over Y with the same rational Pontrjagin classes and the rational Euler class (rational Chern classes, respectively) is finite.*

Proof. To simplify notations we only give a proof for complex vector bundles and then indicate necessary modifications for real vector bundles. We need to show that the "Chern classes map" $(c_1,\dots,c_k):[Y,BU(k)]\to H^*(Y,\mathbb{Q})$ is finite-to-one. First, notice that c_1 classifies the line bundles (cf. 2.8), so we can assume $k>1$.

The integral ith Chern class $c_i\in H^*(BU(k),\mathbb{Z})\cong[BU(k),K(\mathbb{Z},4i)]$ can be represented by a continuous map $f_i:BU(k)\to K(\mathbb{Z},2i)$ such that $c_i=f_i^*(\alpha_{2i})$ where α_{2i} is the fundamental class of $K(\mathbb{Z},2i)$. (Recall that, by Hurewicz theorem $H_{n-1}(K(\mathbb{Z},n),\mathbb{Z})=0$ and $H_n(K(\mathbb{Z},n),\mathbb{Z})\cong\mathbb{Z}$; the class in $H^n(K(\mathbb{Z},n),\mathbb{Z})\cong\mathrm{Hom}(H_n(K(\mathbb{Z},n),\mathbb{Z});\mathbb{Z})$ corresponding to the identity homomorphism is called the fundamental class and is denoted α_n.)

It defines "Chern classes map" $c=(f_1,\dots,f_k):BU(k)\to K(\mathbb{Z},2)\times\cdots\times K(\mathbb{Z},2k)=\Pi$. We now check that the map induces an isomorphism on rational cohomology. According to [GM, 7.5, 7.6] for even n, $H^n(K(\mathbb{Z},n),\mathbb{Q})\cong\mathbb{Q}[\alpha_n]$. By Künnet formula

$$H^*(\times_{i=1}^k K(\mathbb{Z},2i),\mathbb{Q})\cong\otimes_{i=1}^k H^*(K(\mathbb{Z},2i),\mathbb{Q})\cong\otimes_{i=1}^k\mathbb{Q}[\alpha_{2i}]\cong\mathbb{Q}[\alpha_2,\dots\alpha_{2k}]$$

and under this ring isomorphism $\alpha_2^{s_1}\times\cdots\times\alpha_{2k}^{s_k}$ corresponds to $\alpha_2^{s_1}\dots\alpha_{2k}^{s_k}$. It is well known that $H^*(BU(k),\mathbb{Q})\cong\mathbb{Q}[c_1,\dots,c_k]$ [MS, 14.5]. Thus the homomorphism $c^*:H^*(\times_{i=1}^k K(\mathbb{Z},2i),\mathbb{Q}))\to H^*(BU(k),\mathbb{Q})$ defines a homomorphism $\mathbb{Q}[\alpha_2,\dots\alpha_{2k}]\cong\mathbb{Q}[c_1,\dots,c_k]$ that takes α_{2i} to c_i. Since this is an isomorphism, so is c^* as promised.

Therefore c induces an isomorphism on rational homology. Then, since $BU(k)$ is simply connected, the map c must be a rational homotopy equivalence [GM, 7.7]. In other words the homotopy theoretic fiber F_c of the map c has finite homotopy groups.

Consider an oriented rank k vector bundle $f:Y\to BU(k)$ with characteristic classes $c\circ f:Y\to\Pi$. Our goal is to show that the map $c\circ f$ has at most finitely many nonhomotopic liftings to $BU(k)$. Look at the set of liftings of $c\circ f$ to $BU(k)$ and try to construct homotopies skeleton by skeleton using the obstruction theory. The obstructions lie in the groups of cellular cochains of Y with coefficients in the homotopy groups of F_c. (Note that the fibration $F_c\to BU(k)\to\Pi$ has simply connected base and fiber (since $k>1$), so the coefficients are not twisted.) Since the homotopy groups of F_c are finite, there are at most finitely many nonhomotopic liftings. This completes the proof for complex vector bundles.

For oriented real vector bundles of odd rank the same argument works with $c=(p_1,\dots,p_{[k/2]})$, where p_i is the ith Pontrjagin class. Similarly, for oriented real vector bundles of even rank we set $c=(e,p_1,\dots,p_{k/2-1})$, where e is the Euler class.

We point out as a warning that the argument given above fails for $BO(k)$, due to the fact $BO(k)$ is not simply connected (i.e. the map $C=(p_1,\dots,p_{[k/2]})$ of $BO(k)$ to the product of Eilenberg-MacLane spaces is *not* a rational homotopy equivalence even though it induces an isomorphism on rational cohomology). Probably the conclusion of the theorem is true if we use characteristic classes with twisted coefficients. The need for something like "twisted Euler class" can be seen from the computation $\pi_i(BSO(k))\cong\pi_i(BO(k))$ for $i\geq 2$; thus the image of the usual "Euler class" survives in $\pi_k(BO(k))$. We choose to avoid dealing with twisted classes by considering oriented bundles. □

1.6. Remark. Note that, in case either k is odd or $k>\dim(Y)$, the rational Euler class is zero, and, hence rational Pontrjagin classes determine the vector bundle up to a finite number of possibilities.

1.7. Corollary. *Let B is a closed smooth oriented manifold and the space $Hom(\pi_1(B), G)$ has $d < \infty$ connected components. Then the following holds:*

(1) Set $k = dim(X) - dim(B)$. If either k is odd or $k > dim(B)$, then the set o f isomorphism classes of the normal bundles of smooth immersion of B into an (G, X)-manifold N is finite.

(2) If the G-action preserves an almost complex structure on X, then the set of isomorphism classes of the (complex) normal bundles of almost complex immersion of B into an (G, X)-manifold N is finite.

§2. Invariants of continuous maps

Throughout the paper we assume that all manifolds are Hausdorff, paracompact and connected.

2.1. Definition. Let B be a topological space and X_B be a set that may depend on B. Let ι be a map that, given an oriented manifold N and a continuous map from B into N, produces an element of X_B. We call ι an *invariant* if the two following conditions hold:

(1) Homotopic maps $f_1 : B \to N$ and $f_2 : B \to N$ have the same invariant.

(2) Let $h : N \to L$ be an orientation-preserving homeomorphism of N onto an open subset of L. Then, for any continuous map $f : B \to N$, the maps $f : B \to N$ and $h \circ f : B \to L$ have the same invariant.

2.2. Example: tangent bundle. Assume B is paracompact and X_B is the semigroup $\mathrm{Vect}_{\mathbb{R}}(B)$ of real vector bundles over B. Given a continuous map $f : B \to N$, set $\iota(f : B \to N) = f^*TN$, the pullback of the tangent bundle to N under f. Then ι is an invariant.

More generally, let τ be a functor from the category of manifolds and open embeddings to the category of vector bundles over the manifolds. Then $\iota(f : B \to N) = f^*\tau N$ is an invariant.

For instance, one can take τN to be an iterated tangent bundle, or their Whitney sums, tensor, or exterior products. Furthermore, stable equivalence class of $f^*\tau N$ and the characteristic classes of $f^*\tau N$ are invariants.

2.3. Example: normal bundles. Assume B is a closed smooth manifold and $X_B = KO(B)$. Given a continuous map $f : B \to N$, set $\iota(f : B \to N) = [f^*TN] - [TB]$. Notice that, if f is a smooth immersion, the normal bundle of the immersion ν_f has the property $\nu_f \oplus TB \cong f^*TN$; therefore, in this case $[\nu_f] = [f^*TN] - [TB]$. Clearly, any stable characteristic class of $[f^*TN] - [TB]$ is an invariant with values in $X_B = H^*(B, \Lambda)$.

In case each N is almost complex all bundles are complex and KO should be replaced with KU.

2.4. Euler class. Since the Euler class is non-stable we need a different approach to define the Euler class of a continuous map. Assume B is a compact topological space and N is an oriented n-manifold. All (co)homology groups in this section have integer coefficients. Set $X_B = \mathrm{Hom}(H_*(B), H^*(B))$.

Let $f : B \to N$ be a continuous map. For every positive integer m we define a group homomorphism $\varepsilon_m(f)$ of $H_m(B)$ into $H^{n-m}(B)$ as the composition of the following homomorthisms

$$H_m(B) \xrightarrow{f_*} H_m(N, N \backslash f(B)) \cong H^{n-m}(f(B)) \xrightarrow{f^*} H^{n-m}(B)$$

where the middle isomorphism is the Poincaré Duality (see [D, VIII.7.2]). Notice that we do not have to use Čech cohomology because the natural transformation $\check{H}^{n-m}(f(B)) \to H^{n-m}(f(B))$ is an isomorphism since $f(B)$ is ENR [D, VIII.6.12] (recall that in a Hausdorff paracompact manifold any compact subset is ENR [D, VIII.1.3, IV.8.10, IV.8.11]).

We denote $\varepsilon(f)$ the collection of homomorphism $\{\varepsilon_m(f)\}$. We next show that $\varepsilon(f)$ is an invariant.

2.5 Intersections. There is an obvious connection between $\varepsilon(f)$ and the intersection pairing in N. Let $\alpha \in H_m(B)$ and $\beta \in H_{n-m}(B)$ where $n = \dim(N)$. The intersection of $f_*\alpha$ and $f_*\beta$ in N can be computed as follows. Since B is compact one can find a compact K with $f(B) \subset K \subset N$.

Thus we get the following commutative diagram.

$$\begin{array}{ccccccc} H_m(B) & \xrightarrow{f_*} & H_m(N, N\backslash f(B)) & \xrightarrow{D^{-1}} & H^{n-m}(f(B)) & \xrightarrow{f^*} & H^{n-m}(B) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i^*} & & \| \\ H_m(K) & \xrightarrow[D^{-1}]{f_*} & H_m(N, N\backslash K) & \longrightarrow & H^{n-m}(K) & \xrightarrow{f^*} & H^{n-m}(B) \end{array}$$

Let $A \in H^*(K)$ and $B \in H^*(K)$ be the Poincaré duals of $f_*\alpha, f_*\beta \in H_*(K)$ respectively. Then the intersection number is given by $I(f_*\alpha, f_*\beta) = (-1)^{m(n-m)} < A, f_*\beta >= (-1)^{m(n-m)} < B, f_*\alpha >$. It is easy to check that $I(f_*\alpha, f_*\beta)$ does not depend on K. Furthermore,

$$(-1)^{m(n-m)} I(f_*\alpha, f_*\beta) =< f^*A, \beta >=< \varepsilon(\alpha), \beta >=< f^*B, \alpha >=< \varepsilon(\beta), \alpha > .$$

2.6. Definition. Let B be a closed oriented m-manifold. We define the *Euler class* $e(f)$ of a continuous map $f : B \to N$ to be $\varepsilon_m(f)$-image of the fundamental class $[B]$ of B. Thus, $e(f)$ depends on the choice of orientation on B.

Recall that the Euler class of a smooth embedding $f : M \to N$ is the Euler class of the normal bundle of $f(M)$ in N. If f is a topological embedding, one can still define its Euler class [D, VIII.11.10].

Notice that our definition generalizes the one in [D]. Indeed, according to [D, VIII.11.18] the Euler class χ_M^N defined in [D] has the property $\chi_M^N \cap [N, N\backslash f(M)] = f_*[M]$ where $[N, N\backslash f(M)]$ is the fundamental class of the pair $(N, N\backslash f(M))$.

Our definition of the Euler class has been of some use in topology (see e.g. [CS p137]). However no careful treatment is available.

2.7. Remark. In case $n - m$ is odd and $f : M \to N$ is homotopic to an embedding one can prove that $e(f)$ has order two in $H^{n-m}(M)$. Indeed, it is enough to check the assumptions of VIII.11.24 in [D]. As remarked after VIII.11.20 of [D], the assumptions are always satisfied if M and N are ENR, which is the case since M and N are paracompact, Hausdorff manifolds.

Moreover if $n - m = 1$, then $e(f) = 0$ because $H^1(M)$ is torsion free by the universal coefficients theorem.

2.8. Remark. For vector bundles, the Euler class is the primary obstruction to existence of an everywhere nonzero section. Moreover, for 2-plane bundles, the Euler class turns out to be the only invariant, i.e. orientable 2-plane bundles over a Hausdorff paracompact space X are isomorphic if and only if their Euler classes are equal. In addition, every element of $H^2(X)$ can be realized as the Euler class of some orientable 2-plane bundle [H, I.3.8, I.4.3.1].

2.9. Euler class is an invariant. Let L be a compact subset of N such that $f(K) \subset L$ and let U be an open subset of N with $L \subset U$. The following diagram commutes by naturality of the cap product,

$$\begin{array}{ccccccc} H_m(K) & \xrightarrow{f_*} & H_m(N, N\backslash f(K)) & \xrightarrow{D^{-1}} & H^{n-m}(f(K)) & \xrightarrow{f^*} & H^{n-m}(K) \\ \| & & {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i^*} & & \| \\ H_m(K) & \xrightarrow{f_*} & H_m(U, U\backslash L) & \xrightarrow{D^{-1}} & H^{n-m}(L) & \xrightarrow{f^*} & H^{n-m}(K) \end{array}$$

thus, the homomorphism $\varepsilon_m(f)$ can be computed using the lower raw of the diagram. In other words we can replace K by a larger compact set and N by a smaller open subset.

The property (1) in the definition of an invariant says that, for homotopic maps $f_1, f_2 : K \to N$, we have $\varepsilon_m(f_1) = \varepsilon_m(f_2)$. Indeed, choose a homotopy $F : [0, 1] \times K \to N$ and put $L = \mathrm{Im}(F)$. Then both $\varepsilon_m(f_1)$ and $\varepsilon_m(f_2)$ can be computed as the compositions of the following homomorphisms.

$$H_m(K) \longrightarrow H_m(N, N\backslash L) \cong H^{n-m}(L)) \longrightarrow H^{n-m}(K)$$

Now since the maps $f_1, f_2 : K \to L$ are homotopic, $\varepsilon_m(f_1) = \varepsilon_m(f_2)$.

The property (2) in the definition of an invariant follows from the observation below. Let $f_1, f_2 : K \to U$ be continuous maps and let g be a self-homeomorphism of U such that $g \circ f_1 = f_2$. Then $\varepsilon_m(f_1) = \varepsilon_m(f_2)deg(g)$ as the diagram below shows.

$$\begin{array}{ccccccc}
H_m(K) & \xrightarrow{f_{1*}} & H_m(N, N\backslash f_1(K)) & \xrightarrow{D^{-1}} & H^{n-m}(f_1(K)) & \xrightarrow{f_1^*} & H^{n-m}(K) \\
\| & & \downarrow{\scriptstyle g_*} & & \uparrow{\scriptstyle g^*} & & \| \\
H_m(K) & \xrightarrow{f_{2*}} & H_m(U, U\backslash f_2(K)) & \xrightarrow{D^{-1}} & H^{n-m}(f_2(K)) & \xrightarrow{f_2^*} & H^{n-m}(K)
\end{array}$$

2.10. Remark. Invariants of a continuous map $f : B \to N$ can be computed in any open neighborhood of $f(B)$ thanks to the property (2).

Let B be a finite connected CW-complex and $f : B \to N$ be an embedding (i.e. homeomorphism onto its image). Let $p : \tilde{N} \to N$ be a covering map with $p_*\pi_1(\tilde{N}) = f_*\pi_1(B)$. One can lift f to an embedding $\tilde{f} : B \to \tilde{N}$. Thus the map $p|_{\tilde{f}(B)} \to f(B)$ is a homeomorhism. Using compactness of B, one can find an open neighborhood $\tilde{U}$ of $\tilde{f}(B)$ such that $p|_{\tilde{U}} \to U$ is a homeomorphism.

Since invariants $\iota(\tilde{f})$ and $\iota(f)$ can be computed in $\tilde{U}$ and U repectively and $p : \tilde{U} \to U$ is an orientation-preserving homeomorphism (with respect to the induced orientation on $\tilde{N}$) we conclude $\iota(\tilde{f}) = \iota(f)$.

§3. Invariants of discrete representations.

3.1. Any invariant of continuous maps gives rise to an invariant of discrete representations as follows. Let Y be a finite connected CW-complex and let X be a contractible n-manifold without boundary (thus, Y is compact and X is orientable).

Start with a discrete representation $\rho : \pi_1(Y, y) \to \mathrm{Homeo}_+(X)$ such that the group $\Gamma = \rho(\pi_1(Y, y))$ acts freely and properly on X. Thus $p : X \to N = X/\Gamma$ is a covering. By choosing a point $x \in X$ we fix an isomorphism $\phi : \Gamma \cong \pi_1(N, p(x))$. The homomorphism $\phi \circ \rho : \pi_1(Y, y) \to \pi_1(X, p(x))$ defines a homotopy class of maps $[f] \in [(Y, y); (N, p(x))]$ [Sp, 8.11, Thm 11]. It is easy to see the free homotopy class $[f] \in [Y; N]$ is independent of the choice of x (see Lemma 3.2 for the proof).

Given an invariant ι of continuous maps, we define the *associated invariant of discrete representations* ι as $\iota(\rho) = \iota(f)$. In particular, the *Euler class of a discrete representation* ρ is defined as $e(\rho) = e(f)$. Since $\iota(f)$ depends only on the free homotopy class of f, $\iota(\rho)$ and $e(\rho)$ are well defined.

3.2. Lemma. *The free homotopy class $[f] \in [Y; N]$ is independent of the choice of a point $x \in X$*

Proof. Choosing a point $x_k \in X, k \in \{1, 2\}$ we fix an isomorphism $\phi_k : \Gamma \cong \pi_1(N, p(x_k))$. The homomorphism $\phi_k \circ \rho : \pi_1(Y, y) \to \pi_1(N, p(x_k))$ defines a homotopy class of maps $[f_k] \in [(Y, y); (N, p(x_k))]$ [Sp, 8.11, Thm 11]. Similarly, the isomorphism $\phi_2 \circ \phi_1^{-1} : \pi_1(N, p(x_1)) \to \pi_1(N, p(x_2))$ yields a unique homotopy class of maps $[h] \in [N, p(x_1); N, p(x_2]$. Notice that $(\phi_2 \circ \phi_1^{-1}) \circ (\phi_1 \circ \rho) = \phi_2 \circ \rho$, hence $[h \circ f_1] = [f_2]$.

It remains to show that h is freely homotopic to the identity. Let $\tilde{\omega}$ be a path in X joining $\tilde{\omega}(0) = x_1$ and $\tilde{\omega}(1) = x_2$; denote $\omega = p(\tilde{\omega})$. According to [Sp 7.3.2], there is a map $\alpha : (N, p(x_1)) \to (N, p(x_2))$ ω-homotopic to the identity (that is, there is a homotopy F_t of the identity and α such that $F_t(p(x_1)) = \omega(t)$). We are to show that α and h are homotopic.

Let σ be an arbitrary loop at $p(x_1) = \omega(0)$, then the loop $\alpha(\sigma)$ is homotopic to the loop $\omega \circ \sigma \circ \omega^{-1}$. (Indeed, put $\omega_t = \omega|_{[0,t]}$; then $\omega_t^{-1} \circ F_t(\sigma) \circ \omega_t$ is an $\omega(0)$-preserving homotopy of σ and $\omega^{-1} \circ \alpha(\sigma) \circ \omega$). In fact, since α_* is an isomorphism, any element of $\pi_1(N, \omega(1))$ is of the form $[\alpha(\sigma)] = [\omega \circ \sigma \circ \omega^{-1}]$.

Lift σ to a (unique) path $\tilde{\sigma} : [0, 1] \to X$ with $\sigma(0) = x_1$, so $\sigma(1) = \gamma(x_1)$ where $\gamma \in \Gamma$ such that $\phi_1(\gamma) = [\sigma]$. Consider the path $\gamma(\tilde{\omega}) \circ \tilde{\sigma} \circ \tilde{\omega}^{-1}$ joining $\tilde{\omega}(1) = x_2$ to $\gamma(x_2)$; the path is a lift of $\omega \circ \sigma \circ \omega^{-1}$. Thus $\phi_2(\gamma) = [\omega \circ \sigma \circ \omega^{-1}] = \alpha_*([\sigma]) = \alpha_*\phi_1(\gamma)$, that is, $\alpha_* = \phi_2 \circ \phi_1^{-1}$. Therefore, $[\alpha] = [h]$ in $[N, p(x_1); N, p(x_2)]$ (see [Sp, 8.11, Thm 11]). Since α is freely homotopic to the identity, we are done. $\square$

3.3. Claim. *Given $k \in \{1,2\}$, let $\rho_k : \pi_1(Y,y) \to Homeo_+(X)$ be a discrete representation such that the group $\rho_k(\pi_1(Y,y))$ acts freely and properly on X and. Assume ρ_1 and ρ_2 are conjugate by an orientation-preserving homeomorphism $\tilde{g}$ of X. Then $\iota(\rho_1) = \iota(\rho_2)$*

Proof. Denote $\Gamma_k = \rho_k(\pi_1(Y,y))$. Let $\iota : \Gamma_1 \to \Gamma_2$ be the isomorphism induced by $\tilde{g}$, so that $\iota \circ \rho_1 = \rho_2$, and $p_k : X \to X/\Gamma_k = N_k$ be the natural projection.

Choose a point $x_1 \in X$ and put $x_2 = \tilde{g}(x_1)$ This fixes isomorphisms $\phi_k : \Gamma_k \cong \pi_1(N_k, p_k(x_k))$. The homomorphism $\phi_k \circ \rho_k : \pi_1(Y,y) \to \pi_1(N_k, p_k(x_k))$ defines a homotopy class of maps $[f_k] \in [(Y,y);(N_k,p_k(x_k))]$ [Sp, 8.11, Thm 11]. The isomorphism

$$\phi_2 \circ \iota \circ \phi_1^{-1} : \pi_1(N, p(x_1)) \to \pi_1(N, p(x_2))$$

is induced by a homeomorphism $[g] \in [N, p(x_1); N, p(x_2)]$, where $\tilde{g}$ is the lift of g. Notice that

$$(\phi_2 \circ \iota \circ \phi_1^{-1}) \circ (\phi_1 \circ \rho_1) = \phi_2 \circ \iota \circ \rho_1 = \phi_2 \circ \rho_2$$

hence $[g \circ f_1] = [f_2]$. Sinve ι is an invariant, $\iota(f_1) = \iota(f_2)$ as desired. □

3.4. Remark. Let G be a Lie group that acts smoothly on a contractible manifold X. In § 1 we constructed a locally-constant map $V : \mathrm{Hom}(\pi_1(Y), G) \to Vect^n(Y)$ that takes a representation from $\mathrm{Hom}(\pi_1(Y), G)$ to the isomorphism class of the corresponding vertical bundle. Set

$$D(\pi_1(Y)) = \{\rho \in \mathrm{Hom}(\pi_1(Y), G) : \rho(\pi_1(Y)) \text{ acts freely and properly discontinuously on } X\}.$$

Let $\iota : D(\pi_1(Y)) \to Vect^n(Y)$ be an invariant of discrete representations associated with the invariant of maps ι defined in 2.2. Recall that ι is defined as follows. Take a continuous map $f : Y \to N = X/\rho(\pi_1(Y))$ that induces ρ; then $\iota(\rho)$ is the isomorphism class of the bundle f^*TN. Then, according to the proposition 1.1, ι is the restriction of V. Therefore, the stable equivalence class of f^*TN and characterisic classes of f^*TN are also restrictions of the corresponding invariants defined on the whole space $\mathrm{Hom}(\pi_1(Y), G)$.

However, in general, the Euler class is not a restriction of an invariant coming from the map V.

§4. Geometric convergence

In this section we discuss some basic facts on geometric convergence. The notion of geometric convergence was introduced by Chaubaty (see [Bou]). More details relevant to our exposition can be found in [CEG], [Lok] and [BP]. In this section we let G be a Lie group equiped with some left invariant Riemmanian metric.

4.1. Definition. Let $C(G)$ be the set of all closed subgroups of G. Define a topology on $C(G)$ as follows. We say that a sequence $\{\Gamma_n\} \in C(G)$ converges to $\Gamma_{\mathrm{geo}} \in C(G)$ geometrically if the following two conditions hold:

(1) If $\gamma_{n_k} \in \Gamma_{n_k}$ converges to $\gamma \in G$, then $\gamma \in \Gamma_{\mathrm{geo}}$;
(2) If $\gamma \in \Gamma_{\mathrm{geo}}$, then there is a sequence $\gamma_n \in \Gamma_n$ with $\gamma_n \to \gamma$ in G.

4.2. Fact. [CEG] The space $C(G)$ is compact and metrizable.

4.3. Fact. [Lok] $\Gamma_n \to \Gamma_{\mathrm{geo}}$ iff for every compact subset $K \subset G$ the sequence $\Gamma_n \cap K \to \Gamma_{\mathrm{geo}} \cap K$ in the Hausdorff topology (i.e. for any $\varepsilon > 0$, there is N such that, if $n > N$, then $\Gamma_n \cap K$ lies in the ε-neihgborhood of $\Gamma_{\mathrm{geo}} \cap K$ and $\Gamma_{\mathrm{geo}} \cap K$ lies in the ε-neihgborhood of $\Gamma_n \cap K$).

4.4. Fact. [Lok] Let $\Gamma_{\mathrm{geo}} \subset G$ is a discrete subgroup. Then there is $\varepsilon > 0$ such that, for any $\Gamma_n \to \Gamma_{\mathrm{geo}}$ in $C(G)$ and any compact $K \subset G$, there is N such that, if $n > N$ and $\gamma \in \Gamma_{\mathrm{geo}} \cap K$, then there is a unique $\gamma_n \in \Gamma_n \cap K$ that is ε-close to γ.

In particular, Γ_n is discrete for $n > N$, since $e \in \Gamma_n$ is the only element of Γ_n, that is in the ε-neighbourhood of $e \in G$.

4.5. Lemma. *Let G be a Lie group and Γ_n be a sequence of discrete groups that converges geometrically to $\Gamma = \Gamma_{\text{geo}}$. If the identity component Γ_0 of Γ_{geo} is compact, the sequence $\{\Gamma_n\}$ has unbounded torsion (i.e. for any positive integer N, there is $\gamma \in \Gamma_{n(N)}$ of finite order which is greater than N).*

Proof. Choose $\epsilon > 0$ so small that the closed 4ϵ-neighborhood U of Γ_0 is disjoint from $\Gamma \setminus \Gamma_0$ (this is possible since Γ_0 is compact). Given $M > 1$, for large n, Γ_n and Γ are e/M-Hausdorff close on the compact set U. Take an arbitrary element $g \in \Gamma_0$ in ϵ-neighborhood of the identity e and choose $g_n \in \Gamma_n$ so that it is e/M-close to g.

First, show that g_n has finite order. Suppose not. Since U is compact and $< g_n >$ is discrete and infinite, there is a smallest $k > 1$ with $(g_n)^k \notin U$. So, $(g_n)^{k-1} \in U$. The metric is left invariant, hence,

$$d((g_n)^{k-1}, (g_n)^k) = d(g_n, e) < d(g, e) + d(g_n, g) < \epsilon + \epsilon/M < 2\epsilon.$$

Since Γ_n and Γ are e/M-Hausdorff close on U and since $\Gamma \cap U = \Gamma_0$, we get $d((g_n)^{k-1}, \Gamma_0) < \epsilon/M < \epsilon$. This implies $d((g_n)^k, \Gamma_0) < 3\epsilon$ and, thus, $(g_n)^k \in U$, a contradiction. Thus g_n has finite order.

We have just showed that any neighborhood of g contains an element of finite order $g_n \in \Gamma_n$ (for n large enough). So we get a sequence g_n converging to g. Assume the torsion of Γ_n is uniformly bounded by N. Then, for any finite order element $h \in \Gamma_n$, we have $h^{N!} = e$. In particular $e = (g_n)^{N!}$ converges to $g^{N!}$. We, thus, proved that any element g in Γ_0 that is ϵ-close to the identity satisfies $g^{N!} = e$. This is absurd. (Indeed, it would mean that the map $\Gamma_0 \to \Gamma_0$ that takes g to $g^{N!}$ maps the ϵ-neighborhood of the identity to the identity. But the map is clearly a diffeomorphism on some neighborhood of the identity). Thus, Γ_n has unbounded torsion as desired. □

4.6. Remark. Let Γ_n be a sequence of torsion free groups converging to a discrete group Γ_{geo} in $C(G)$. Then Γ_{geo} is torsion free.

Indeed, choose $\gamma \in \Gamma_{\text{geo}}$ with $\gamma^k = e$. Find a sequence $\gamma_n \to \gamma$. Then $\gamma_n^k \to e$. By 4.4 we have $\gamma_n^k = e$ for large n. Since Γ_n are torsion free, $k = 1$ as desired.

4.7. Lemma. *Let G be a Lie group with the identity component G_0 and Γ_n be a sequence of discrete groups that converges geometrically to $\Gamma = \Gamma_{\text{geo}}$. Then the identity component Γ_0 of Γ_{geo} is nilpotent.*

Proof. The group Γ_{geo} is closed, so it is a Lie group. So the identity component Γ_0 of Γ_{geo} is a Lie group; we are to show that $\Gamma_0 \leq G_0$ is nilpotent. Let U be a neighborhood of the identity that lies in a Zassenhaus neighbourhood in G_0 (see [Rag, 8.16]). Being a Lie group Γ_0 is generated by any neigborhood of the identity, in particular by $V = U \cap \Gamma_0$. To show Γ_0 is nilpotent, it suffices to check that, for some k, $V^{(k)} = \{e\}$, that is, any iterated commutator of weight k with entries in V is trivial [Rag 8.17]. Fix an iterated commutator $[v_1 \dots [v_{k-2}[v_{k-1}, v_k]] \dots]$ with $v_k \in V$ and choose a sequences $g_k^n \to v_k$ where $g_k^n \in \rho_n(\pi)$. For n large enough, the elements $g_k^n, \dots g_k^n$ lie in a Zassenhaus neighbourhood. Then by Zassenhaus-Kazhdan-Margulis lemma [Rag 8.16] the group $< g_k^n, \dots g_k^n >$ lies in a connected nilpotent Lie subgroup of G_0. The class of any connected nilpotent Lie subgroup of G_0 is bounded by $\dim(G_0)$. Thus, for $k > \dim(G_0)$, the k-iterated commutator $[g_1^n \dots [g_{k-2}^n[g_{k-1}^n, g_k^n]] \dots]$ is trivial, for large n. This implies $[v_1 \dots [v_{k-2}[v_{k-1}, v_k]] \dots]$ is trivial and, therefore, Γ_0 is nilpotent. □

4.8. Theorem. *Let G be a Lie group with the identity component G_0 and π be a finitely generated group without nontrivial normal nilpotent subgroups. Let $\rho_k \in Hom(\pi, G)$ be a sequence of discrete faithful representations that converges algebraically to a representation ρ. Then ρ is faithful.*

Proof. It suffices to prove that the group $K = \text{Ker}(\rho)$ is nilpotent. Let V be a set of generators for K (maybe infinite). To show K is nilpotent, it suffices to check that, for some k, $V^{(k)} = \{e\}$, that is, any iterated commutator of weight k with entries in V is trivial [Rag 8.17]. Fix an iterated commutator $[v_1 \dots [v_{k-2}[v_{k-1}, v_k]] \dots]$ with $v_k \in V$.

For n large enough, the elements $\rho_n(v_1), \dots, \rho_n(v_k)$ lie in a Zassenhaus neighbourhood of G_0. Then by Zassenhaus-Kazhdan-Margulis lemma [Rag 8.16] the group $< \rho_n(v_1), \dots, \rho_n(v_k) >$ lies in

a connected nilpotent Lie subgroup of G_0. The class of any nilpotent subgroup of G_0 is bounded by $\dim(G_0)$. Thus, for $k > \dim(G_0)$, the k-iterated commutator

$$[\rho_n(v_1)\dots[\rho_n(v_{k-2})[\rho_n(v_{k-1}),\rho_n(v_k)]]\dots]$$

is trivial, for large n. Since ρ_n is faithful $[v_1\dots[v_{k-2}[v_{k-1},v_k]]\dots]$ is trivial and we are done. □

4.9. Theorem. [Lok] Let X be a simply connected homogeneous Riemannian manifold and G be a transitive group of isometries. Let Γ_n be a sequence of torsion free subgroups of G converging geometrically to a discrete subgroup $\Gamma_{\rm geo}$. Let $U \subset X$ be a relatively compact open set.

Then, if n is large enough, there exists a relatively compact open set V with $U \subset V$ and smooth embeddings $\tilde\varphi_n : U \to V$ such that $\tilde\varphi_n$ cover embeddings $\varphi_n : U/\Gamma_{\rm geo} \to V/\Gamma_n$

$$\begin{array}{ccc} U & \xrightarrow{\tilde\varphi_n} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_n} \\ U/\Gamma_{\rm geo} & \xrightarrow[\varphi_n]{} & V/\Gamma_n \end{array}$$

The embeddings $\tilde\varphi_n$ converges C^r-uniformly to the inclusion.

Sketch of the Proof. Let $K = \{g \in G : g(\overline{V}) \cap \overline{V} \neq \emptyset\}$, so K is compact. By 4.4, if we take n sufficiently large, then for any $\gamma \in \Gamma_{\rm geo} \cap K$ there exists a unique $\gamma_n \in \Gamma_n \cap K$ that is close to γ. It defines a bijective map r_n from $\Gamma_{\rm geo} \cap K$ onto $\Gamma_n \cap K$.

Our goal is to construct embeddings $\varphi_n : U \to V$ that are close to the inclusion and r_n-equivariant (in the sense that $x = \gamma(y)$ for $\gamma \in \Gamma_{\rm geo}$ iff $\varphi_n(x) = r_n(\gamma)\varphi_n(x)$) The construction is non-trivial and we refer to [Lok] for a complete proof. □

4.10. Let π be a finitely generated group and $\mathbf{H}^n_{\mathbb{F}}$ be the rank one negatively curved symmetric space over $\mathbb{F}$ (i.e. $\mathbf{H}^n_{\mathbb{F}}$ is a real, complex or quaternion hyperbolic n-space or the Cayley plane). Let $G = \mathrm{Isom}(\mathbf{H}^n_{\mathbb{F}})$; we equip $\mathrm{Hom}(\pi, G)$ with algebraic (i.e. pointwise convergence) topology.

4.11. Theorem. *Let $\rho_k \in Hom(\pi, G)$ be a sequence of representations converging algebraically to a representation ρ where, for every k, the group $\rho_k(\pi)$ is discrete and the sequence $\{\rho_k(\pi)\}$ has uniformly bounded torsion. Suppose that $\rho(\pi)$ is an infinite group without nontrivial normal nilpotent subgroups. Let $\rho_{n_k}(\pi)$ be a subsequence of $\rho_k(\pi)$ that converges geometrically to $\Gamma_{\rm geo}$ (such a subsequence always exists since $C(G)$ is compact).*

Then $\Gamma_{\rm geo}$ is discrete (and so is $\rho(\pi) \subset \Gamma_{\rm geo}$).

Proof. Being a closed subgroup $\Gamma_{\rm geo}$ is a Lie group. By the lemma 4.7 the identity component Γ_0 of $\Gamma_{\rm geo}$ is nilpotent Lie group. Since $\rho(\pi)$ does not have a nontrivial normal nilpotent subgroup, $\rho(\pi) \cap \Gamma_0 = \{e\}$, so $\rho(\pi)$ is discrete. Using the Selberg lemma we find a torsion free subgroup $\Pi \leq \rho(\pi)$ of finite index. The group Π is infinite since $\rho(\pi)$ is. Notice that the group Π may have at most two fixed points at infinity (any isometry that fixes at least three points is elliptic [BGS, p84]).

Suppose the identity component Γ_0 of $\Gamma_{\rm geo}$ is nontrivial.

Lemma. *The group Π is virtually nilpotent.*

Proof. According to [Bow1, 3.3.1], the nilpotence of Γ_0 implies one the following mutually exclusive conclusions:

Case 1. Γ_0 has a fixed point in $\mathbf{H}^n_{\mathbb{F}}$ (hence Γ_0 is compact which is is impossible by 4.5).

Case 2. The fixed point set of Γ_0 is a point $p \in \partial\mathbf{H}^n_{\mathbb{F}}$.

Since Π normalizes Γ_0 (in fact, Γ_0 is a normal subgroup of $\Gamma_{\rm geo}$) it has to fix p too. If p is the only fixed point of Π, then every nontrivial element of Π is parabolic [EO, 8.9P] Any parabolic preserves horospheres centered at p [BGS, p84], therefore, according to [Bow2] Π is virtually nilpotent.

If Π fixes two points at infinity, then it acts freely and properly discontinuously on the geodesic joining the points. Hence $\Pi \cong \mathbb{Z}$, the fundamental group of a circle.

Case 3. Γ_0 has no fixed points in $\mathbf{H}^n_{\mathbb{F}}$ and preserves setwise some bi-infinite geodesic. In this case the fixed point set of Γ_0 is the endpoints p and q of the geodesic. Indeed, Γ_0 fixes each of the endpoints, since Γ_0 is connected. Assume Γ_0 fixes some other point of $\partial\mathbf{H}^n_{\mathbb{F}}$. Then any element of Γ_0 is elliptic ([BGS, p85]) and, as such, it fixes the bi-infinite geodesic pointwise. Thus Γ_0 has a fixed point in $\mathbf{H}^n_{\mathbb{F}}$ which contradicts the assumptions of the Case 3.

Since Π normalizes Γ_0, it leaves the set $\{p, q\}$ invariant. Moreover, Π preserves $\{p, q\}$ pointwise because Π contains no elliptics (any isometry that flips p and q has a fixed point on the geodesic joining p and q). Therefore, $\Pi \cong \mathbb{Z}$ as before. Hence, Π is virtually nilpotent as desired. $\square$

Thus, $\rho(\pi)$ has a nilpotent subgroup of finite index and, therefore, has a normal nilpotent subgroup of finite index. This contradicts the assumption that $\rho(\pi)$ is an infinite group without nontrivial normal nilpotent subgroups. $\square$

4.12. Remark. Let Γ be a finitely generated *discrete* subgroup of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{F}})$ that has a normal nilpotent subgroup. Then Γ is, in fact, virtually nilpotent. Indeed, by Selberg lemma we can assume Γ is torsion free. Then, repeating the arguments above, we see that Γ fix a point at infinity and, hence, is virtually nilpotent. Conversely, any virtually nilpotent group clearly has a normal nilpotent subgroup of finite index.

4.13. Corrolary. *Let π be a finitely generated discrete subgroup of $G = Isom(\mathbf{H}^n_{\mathbb{F}})$ that is not virtually nilpotent. Then the set of faithful discrete representations of π into G is a closed subset of $Hom(\pi, G)$.*

Proof. Let a sequence ρ_n of faithful discrete representations converge to $\rho \in \mathrm{Hom}(\pi, G)$. According to 4.12 the group π has no normal nilpotent subgroup. Then 4.8 implies ρ is faithful. By compactness $\rho_n(\pi)$ has a subsequence that converges geometrically to Γ_{geo} which is a discrete group by 4.11. Therefore, $\rho(\pi) \subset \Gamma_{\mathrm{geo}}$ is also discrete as wanted. $\square$

§5. A compactness theorem

5.1. In this section we state a compactness theorem due to Rips (see [BF] and the review of the paper by Paulin in MR96h:20056). Earlier versions of the theorem have been proved by Thurston, Morgan and Shalen [Mor]. We are grateful to M. Bestvina for useful communications on the Rips' theorem.

Assume that $G = \mathrm{Isom}_+(\mathbf{H}^n_{\mathbb{F}})$, the group of all orientation-preserving isometries of $\mathbf{H}^n_{\mathbb{F}}$. Let $FD(\pi, G) \subset \mathrm{Hom}(\pi, G)$ be a subspace of faithful representations such that the groups $\rho_n(\pi(Y, y))$ are discrete.

5.2. Definition. We say that a group π splits over a virtually nilpotent group if π can be decomposed as a nontrivial amalgamated product over a virtually nilpotent group or as an HNN-extension over a virtually nilpotent group.

5.3. Theorem (Rips). *Let π be a finitely presented group which is not virtually nilpotent and does not split over virtually nilpotent. Then $FD(\pi, G)/G$ is compact (notice that the empty set is compact).*

5.4. Proposition. *Let M be a closed (possibly nonorientable) $K(\Gamma, 1)$ topological n-manifold with $n \geq 3$ such that Γ is word-hyperbolic. Then Γ is not virtually nilpotent and does not split over virtually nilpotent.*

Proof. Before starting the proof we recall several useful algebraic facts. First, note that Γ is torsion free since it is the fundamental group of an aspherical manifold. Second, any torsion free virtually cyclic group is, in fact, infinite cyclic because it has two ends [St, 4.A.6.2, 5.B.1]. Thus, any virtually cyclic subgroup of Γ is infinite cyclic.

Third, show that any virtually nilpotent subgroup of a torsion free word-hyperbolic group Γ is infinite cyclic. By the remark above, it suffices to prove that any nilpotent subgroup $N \leq \Gamma$ is cyclic.

Recall that in a word hyperbolic group Γ the centralizer $C_x(\Gamma)$ of a nontorsion element x is a finite extension of an infinite cyclic group generated by x [CDP, 10.7.3]. We have $< x > \leq C_x(N) \leq C_x(\Gamma)$ for any $x \in N$. Therefore, $< x >$ has finite index in $C_x(N)$; so $C_x(N)$ is virtually cyclic, hence $C_x(N) \cong \mathbb{Z}$.

Being a nilpotent group N must have a nontrivial center $Z(N)$; clearly $Z(N) \leq C_x(N)$, so $Z(N) \cong \mathbb{Z}$. This impies that, for any $x \in N$, there exist an integer m such that $x^m \in Z(N)$. Therefore, for any $x, y \in N$ there is an integer m with the property $x^m y = yx^m$, or $x^m = yx^m y^{-1} = (yxy^{-1})^m$. Any element of a torsion free nilpotent group has at most one m-th root, hence $x = yxy^{-1}$. Thus N must be abelian, that is $N = Z(N) \cong \mathbb{Z}$ as desired.

In particular, the group Γ itself is not virtually nilpotent because $\mathrm{cd}\Gamma = \dim(M) \geq 3$ while $\mathbb{Z}$ has cohomological dimension one.

We now start the proof. Argue by contradiction. Thus, either $\Gamma = \Gamma_1 *_C \Gamma_2$ or $\Gamma = \Gamma_0 *_C$ where C is infinite cyclic and Γ_k is a proper subgroup of Γ, $k \in \{0, 1, 2\}$.

First, we assume that either $\Gamma = \Gamma_1 *_C \Gamma_2$ and $|\Gamma : \Gamma_k| = \infty$ for $k \in \{1, 2\}$ or $\Gamma = \Gamma_0 *_C$. (notice that by the definition of HNN-extension Γ_0 has to have an infinite index in Γ).

Then it follows from Mayer-Vietoris sequence [Bro VII.9.1] that $\mathrm{cd}\Gamma \leq \max(\mathrm{cd}\Gamma_1, \mathrm{cd}\Gamma_2, \mathrm{cd}C + 1)$ (cf. [Bro, VIII.2.2.4(c)]). Since $|\Gamma : \Gamma_k| = \infty$, Γ_k is a fundamental group of a noncompact n-manifold that is a cover of M, therefore, $\mathrm{cd}\Gamma_k < \mathrm{cd}\Gamma$ [Bro, VIII.8.1(a)]. As $\mathrm{cd}C = 1$, we get $\mathrm{cd}\Gamma \leq 2$; on the other hand $\mathrm{cd}\Gamma = \dim(M) = n \geq 3$, a contradiction.

Second, suppose that $\Gamma = \Gamma_1 *_C \Gamma_2$ and for some k, $|\Gamma : \Gamma_k| < \infty$. Say Γ_1 has finite index in Γ. This implies $|\Gamma_2 : C| < \infty$ because $C = \Gamma_1 \cap \Gamma_2$. Then Γ_2 is virtually cyclic and, hence, infinite cyclic.

We now use the Mayer-Vietoris sequence [Bro VII.9.1] for homology with twisted coefficients where $n = \dim(M)$

$$\cdots \longrightarrow H_n C \longrightarrow H_n\Gamma_1 \oplus H_n\Gamma_2 \longrightarrow H_n\Gamma \longrightarrow H_{n-1}C \longrightarrow \cdots.$$

Since $n \geq 3$ we get $0 = H_nC = H_{n-1}C = H_n\Gamma_2$. In particular, the inclusion $\Gamma_1 \hookrightarrow \Gamma$ induces an isomorphism in homology.

We next show that the map $H_n(\Gamma_1) \to H_n(\Gamma)$ induced by the inclusion $\Gamma_1 \hookrightarrow \Gamma$ is the multiplication by $|\Gamma : \Gamma_1|$. Thus $|\Gamma : \Gamma_1| = 1$ which contradicts the assumption that Γ_1 is a proper subgroup of Γ. Indeed, being a compact manifold M is homotopy equivalent to a finite CW-complex Y [We]. The universal cover $\tilde{Y}$ of Y is a free Γ-space (and, hence, $\tilde{Y}$ is a free Γ_1-space). The identity map of $\tilde{Y}$ is equivariant with respect to the inclusion $\Gamma_1 \hookrightarrow \Gamma$, therefore it induces an equivariant map of cell complexes $C_*(\tilde{Y}) \to C_*(\tilde{Y})$. Passing to invariants we have a map $C_*(\tilde{Y})_{\Gamma_1} \to C_*(\tilde{Y})_\Gamma$. We know that $C_*(\tilde{Y})_{\Gamma_1} \cong C_*(\tilde{Y}/\Gamma_1)$ and $C_*(\tilde{Y})_\Gamma \cong C_*(\tilde{Y}/\Gamma) = C_*(Y)$. Moreover, the corresponding map $C_*(\tilde{Y}/\Gamma_1) \to C_*(Y)$ is induced by a finite cover $\tilde{Y}/\Gamma_1 \to Y = \tilde{Y}/\Gamma$. Thus the induced homomorphism in homology with twisted coefficients $H_n(\tilde{Y}/\Gamma_1) \to H_n(Y)$ is the multiplication by $|\Gamma : \Gamma_1| = \deg(\tilde{Y}/\Gamma_1 \to Y)$ (we recall that Y and $\tilde{Y}/\Gamma_1$ are homotopy equivalent to closed n-manifolds, therefore, $H_n(Y) \cong \mathbb{Z} \cong H_n(\tilde{Y}/\Gamma_1)$). $\square$

§6. The Main theorem.

Let Y be a finite CW-complex and $y \in Y$. Let X be a contractible homogeneous Riemannian manifold and G be a transitive group of orientation-preserving isometries. We equip $\mathrm{Hom}(\pi_1(Y, y), G)$ with algebraic (i.e. pointwise convergence) topology. Let ι be an invariant of continuous maps. We also use the letter ι for the corresponding invariant of discrete representations.

6.1. Theorem. *Let $\rho_n \in Hom(\pi_1(Y, y), G)$ be a sequence of representations such that the groups $\rho_n(\pi_1(Y, y))$ are discrete and torsion free. Assume that ρ_n converges to ρ algebraically and the groups $\rho_n(\pi_1(Y, y))$ converge geometrically to a discrete group Γ_{geo}. Let $f : Y \to X/\Gamma_{\mathrm{geo}}$ be a continuous*

map that induces the homomorphism $\rho : \pi_1(Y,y) \to \rho(\pi_1(Y,y)) \subset \Gamma_{\text{geo}}$. *Then there exists a positive integer* N *such that*

(1) for every $n > N$, $\iota(\rho_n) = \iota(f)$,

(2) moreover, if either f *is homotopic to an embedding or* $\rho(\pi_1(Y,y)) = \Gamma_{\text{geo}}$, *then* $\iota(\rho_n) = \iota(\rho)$ *for* $n > N$.

Proof. According to 4.11, the group Γ_{geo} is discrete. Being a limit of torsion free groups the discrete group Γ_{geo} is itself torsion free by 4.6. Thus Γ_{geo} acts freely and properly discontinuously on X, so X/Γ_{geo} is a manifold.

We consider the universal covering $q : \tilde{Y} \to Y$; clearly $\tilde{Y}$ has a natural cell structure so that the covering group π acts by permuting cells. We fix an isomorphism $\psi : \pi \to \pi(Y,y)$ by choosing a point $\tilde{y} \in q^{-1}(y)$. Since Y is a finite complex, one can choose a finite connected subcomplex $D \subset \tilde{Y}$ with $q(D) = Y$. (Pick a representing cell in every orbit, the union of the cells is a finite subcomplex that is mapped onto Y by q. Adding finitely many cells, one can assume the complex is connected.)

The representation ρ defines a continuous map $\bar{f} : Y \to X/\rho(\pi)$ which is defined up to homotopy by 3.2. The map $\bar{f}$ can be lifted to a $\rho \circ \psi$-equivariant map $\tilde{f} : \tilde{Y} \to X$ of universal coverings. The identity map $\mathrm{id} : X \to X$ is equivariant with respect to the inclusion $\rho(\pi) \hookrightarrow \Gamma_{\text{geo}}$, therefore, $\tilde{f}$ is equivariant with respect to the homomorphism $\rho \circ \psi : \pi \to \Gamma_{\text{geo}}$. We denote f the composition of $\bar{f}$ and the covering $X/\rho(\pi) \to X/\Gamma_{\text{geo}} = N$ induced by the inclusion $\rho(\pi) \subset \Gamma_{\text{geo}}$.

Let $U \subset X$ be an open relatively compact neighborhood of $\tilde{f}(D)$. We are in position to apply the Theorem 4.9. Thus, if n is large enough, there is a sequence of embeddings $\tilde{\varphi}_n : U \to V \subset X$ that is converging to the inclusion and is r_n-equivariant. So the map $\tilde{\varphi}_n \circ \tilde{f} : D \to V$ is $r_n \circ \rho \circ \psi$-equivariant. By the very definition of r_n we have $r_n \circ \rho = \rho_n$ whenever left hand side is defined. We now extend the map $\tilde{\varphi}_n \circ \tilde{f} : D \to V$ by equivariance to a $\rho_n \circ \psi$-equivariant map $\tilde{\xi}_n : \tilde{Y} \to X$. The map $\tilde{\xi}_n$ descendes to $\xi_n : Y \to V/\rho_n(\pi) \subset X/\rho_n(\pi)$. Notice that by construction $\xi_n = \varphi_n \circ f$. Since φ_n converge to the inclusion, φ_n is orientation-preserving for large n. Thus $\iota(\xi_n) = \iota(f)$ for large n (see 2.9).

Notice that $\iota(\xi_n) = \iota(\rho_n)$. Denote $\tilde{w} = \tilde{\xi}_n(\tilde{y})$ and $w = p_n(\tilde{w})$ where p_n is the covering $p_n : X \to X/\rho_n(\pi_1(Y,y)) = N_n$ and consider the corresponding isomorphism $\phi_n : \rho_n(\pi_1(Y,y)) \to \pi_1(N_n, w)$. We need to verify that the homomorphism $\xi_{n*} : \pi_1(Y,y) \to \pi_1(N_n, w)$ coincide with $\phi_n \circ \rho_n$.

Start with a loop $[\omega] \in \pi_1(Y,y)$; it lifts to a path $\tilde{\omega}$ with endpoints $\tilde{y}$ and $\psi^{-1}([\omega])(\tilde{y})$. Since $\tilde{\xi}_n$ is $\rho_n \circ \psi$-equivariant, $\tilde{\xi}_n(\tilde{\omega})$ has endpoints $\tilde{w}$ and $\tilde{\xi}_n(\psi^{-1}([\omega])(\tilde{y})) = \rho_n \circ \psi(\psi^{-1}([\omega])(\tilde{\xi}(\tilde{y})) = \rho_n([\omega])(\tilde{w})$. Thus $\xi_{n*}([\omega]) = [\xi_n(\omega)] = \phi_n \circ \rho_n([\omega])$, hence, $\iota(\xi_n) = \iota(\rho_n)$ as desired.

Therefore, for large n, $\iota(\rho_n) = \iota(f)$ which proves (1). We are to prove (2) now. Clearly, if $\rho(\pi_1(Y,y)) = \Gamma_{\text{geo}}$, then $\bar{f} = f$, hence, $\iota(\rho) = \iota(\bar{f}) = \iota(f) = \iota(\rho_n)$ for large n.

In case f is homotopic to an embedding, so is $\bar{f}$, hence $\iota(\bar{f}) = \iota(f)$ (see 2.10). Thus $\iota(\rho) = \iota(\bar{f}) = \iota(f) = \iota(\rho_n)$ for large n, and we are done. □

6.2 Remark. If Y is a closed smooth manifold and $2\dim(Y) < \dim(X)$, the Whitney embedding theorem implies that any map of Y into X/Γ_{geo} is homotopic to a smooth embedding. Thus in this case $\iota(\rho_n) = \iota(\rho)$ for large n.

6.3. From now on we assume that $G = \mathrm{Isom}_+(\mathbf{H}^n_{\mathbb{F}})$, the group of all orientation-preserving isometries of $\mathbf{H}^n_{\mathbb{F}}$. Let $FD(\pi, G) \subset \mathrm{Hom}(\pi, G)$ be a subspace of faithful discrete representations. As before Y is a finite connected CW-compex.

A group Γ is said to satisfy the condition

(i) if the space $FD(\Gamma, G)/G$ is compact;

(ii) if Γ is a nontrivial group without nontrivial normal nilpotent subgroups.

For example, Γ satisfies (i)-(ii) if Γ is word-hyperbolic and there is a closed oriented $K(\Gamma, 1)$ manifold of dimension $n \geq 3$. As we explained in 4.12, for finitely generated discrete subgroups of G, (ii) is equivalent to being non-virtually nilpotent.

6.4 Corollary. *Let* $\pi = \pi_1(Y,y)$ *and let* $\rho_n \in Hom(\pi, G)$ *be a sequence of representations such that the groups* $\rho_n(\pi)$ *are discrete and torsion free. Let* ρ_n *converge to* ρ *algebraically. Assume* $\rho(\pi)$ *satisfies (ii). Then the set* $\{\iota(\rho_n)\}$ *is finite.*

Proof. This follows from 6.1 and 4.11. □

6.5. Definition. Let $f : X \to Y$ be a map from a topological space X to a set Y. We say that f is a *locally finite-range map* if every point $x \in X$ has a neighborhood U_x such that $f(U_x)$ is a finite set.

6.6. Corollary. *Assume $\pi = \pi_1(Y, y)$ is a torsion free group that satisfies (ii). Then the map $\iota : FD(\pi, G) \to X_B$ is a locally finite-range map. Moreover, if Y is a closed smooth manifold and $2dim(Y) < dim(X)$, then the map $\iota : FD(\pi, G) \to X_B$ is locally constant.*

Proof. This follows from 6.4 and 6.2. □

6.7. Corollary. *Assume $\pi = \pi_1(Y, y)$ is a torsion free group that satisfies (i)-(ii). Then the image $\iota(FD(\pi, G))$ is a finite subset of X_B.*

Proof. Since the G-action on $\mathbf{H}^n_{\mathbb{F}}$ is orientation-preserving, $\iota(g \circ \rho \circ g^{-1}) = \iota(\rho)$ (see 3.3). Therefore, we get a locally finite-range map

$$\overline{\iota} : FD(\pi, G)/G \to X_B.$$

(Indeed, take $[\rho] \in FD(\pi, G)/G$. Since ι is a locally finite-range map we can find U_ρ with $\iota(U_\rho)$ finite. The ι-image of the open set $GU_\rho = \bigcup_{g \in G} gU_\rho g^{-1}$ is a finite set. By the definition of factor-topology the image of GU_ρ in $FD(\pi, G)/G$ is an open set. This is a neighborhood of $[\rho]$ with a finite image.)

Clearly, the compactness of $FD(\pi, G)/G$ implies that $\overline{\iota}$ (and hence ι) is a finite-range map. □

6.8. Theorem. *Assume $\pi = \pi_1(Y, y)$ is a torsion free group that satisfies (i)-(ii). Let n be a positive integer and let $\alpha \in H_m(Y)$ and $\beta \in H_{n-m}(Y)$. Then there is a positive integer K such that*

(1) for any continuous map $f : Y \to N$ of Y into an oriented $\mathbb{F}$-hyperbolic n-manifold N that induces an isomorphism of fundamental groups, and

(2) for any embedding $f : Y \to N$ of Y into an oriented $\mathbb{F}$-hyperbolic n-manifold N that induces a monomorphism of fundamental groups,

$|\langle f_*\alpha, f_*\beta\rangle| \leq K$.

Proof. We apply the corollary 6.7 to the case $\iota = \varepsilon : FD(\pi, G)/G \to \mathrm{Hom}(H_*(Y), H^*(Y))$ to conclude that $\mathrm{Im}(\varepsilon)$ is finite. Since $|\langle f_*\alpha, f_*\beta\rangle| = |(-1)^{n(n-m)}\langle \varepsilon(\alpha), \beta\rangle|$ we are done. □

6.9. Corrolary. *Let M be a closed oriented manifold of dimension $n \geq 2$ such that $\pi_1(M)$ satisfies (i)-(ii) and has no torsion (e.g. M is an oriented aspherical manifold and $\pi_1(M)$ is word-hyperbolic). Then there is a finite subset $F \subset H^k(M)$ that contains the Euler class of any topological embedding of M into a complete oriented $\mathbb{F}$-hyperbolic $(n+k)$-manifold if the embedding induces a monomorphism of the fundamental groups.*

Thus, F contains the Euler class of any oriented vector bundle $\mathbb{R}^k \to E \to M$ if E admits a complete $\mathbb{F}$-hyperbolic structure. In particular, only finitely many oriented $\mathbb{R}^2$-bundles over M admit a complete $\mathbb{F}$-hyperbolic structure.

Proof. Let $f : M \to N$ be an embedding that induces a monomorphism of the fundamental groups. Consider a covering $p : \tilde{N} \to N$ with $p_*\pi_1(\tilde{N}) = f_*\pi_1(M)$ and lift f to an embedding $\tilde{f} : M \to \tilde{N}$. According to 2.11 we get $e(f) = e(\tilde{f}) = \varepsilon(f)[M]$ and the result follows from the corrolary 6.7. □

6.10. Remark. Observe that the theorem 0.1 (and, hence 0.2 and 0.4) stated for complete hyperbolic manifolds is (trivially) true for any hyperbolic manifold with injective developing map and torsion free fundamental group.

Indeed, recall that a developing map of a hyperbolic manifold N is a local isometry from the universal cover of N onto an open subset of a rank one symmetric space X that is equivariant with respect to the holonomy representation $\pi_1(N) \to \Gamma$ where Γ is the holonomy group. If a developing map is *injective*, it is necessarily a diffeomorphism onto an open subset of X. Then by equivariance the holonomy representation of N is an isomorphism and the holonomy group Γ acts freely and properly discontinuously on the image of the developing map. Hence Γ is a discrete group, and N is

isometric to an open subset of the complete hyperbolic orbifold X/Γ. Note that X/Γ is a hyperbolic manifold iff $\Gamma \cong \pi_1(N)$ is torsion free.

We conclude that any hyperbolic manifold N with injective developing map and torsion free fundamental group is isometric to an open subset of a complete hyperbolic manifold such that the isometry induces an isomorphism of fundamental groups. Given two cycles in N, we can compute their intersection number in the ambient complete hyperbolic manifold and the result follows from 0.1.

§7. Epiloge

We now highlight some applications. For simplicity, we do not state the results in the full generality. Proofs and complete statements can be found in [Bel3].

Given a finite CW-complex K, a *thickening* of K is a compact manifold L with boundary, such that L is homotopy equivalent to K, $\dim(L) \geq \dim(K) + 3$, and the inclusion $\partial L \to L$ induces an isomorphism of fundamental groups. A manifold homeomorphic to the interior of a thickening of K is called an *open thickening* of K.

7.1 Theorem. *Let K be a finite CW-complex. Then the set of diffeomorphism classes of (possibly incomplete) hyperbolic n-manifolds that are open thickenings of K is finite, provided $n > \max\{2dim(K), 4\}$.*

In fact, 7.1 is true for any (X, G)-structure as in §1 (i.e. we assume X is contractible and G is a Lie group acting effectively and analitically on X). The reason is that, for $n > \max\{2\dim(K), 4\}$, open thickenings are classified up to diffeomorphism by the isomorphism classes of the pullbacks to K of their tangent bundles. Similar result is true for thickenings provided $n > \max\{2\dim(K), 5\}$.

Here is an important example. Let K be a finite $K(\Gamma, 1)$ CW-complex. and Γ be a convex-cocompact group of isometries of a negatively curved symmetric space X of real dimension n. Assume the set of discontinuity $\Omega(\Gamma) \subset \partial_\infty X$ is simply-connected and $n \geq \dim(K) + 3$. Then $(X \cup \Omega(\Gamma))/\Gamma$ is a thickenings of K.

7.2. Corollary. *Let K be a finite $K(\Gamma, 1)$ CW-complex and X be a rank one symmetric space of real dimension $n > \max\{2dim(K), 5\}$. For $i \in \{1, 2\}$, let ρ_i be a faithful representation of Γ into the group of isometries of X such that $\rho_i(\Gamma)$ is a convex-cocompact group with simply-connected set of discontinuity.*

Then ρ_1 and ρ_2 are topologically congugate on $X \cup \partial_\infty X$ whenever ρ_1 and ρ_2 lie in the same connected component of the representation variety $Hom(\Gamma, Isom(X))$.

Thickenings of closed manifolds are easy to understand: they are the so-called block bundles. Roughly speaking, k-block bundles over a manifold M are regular neighborhoods of PL-embeddings of M into manifolds of dimension $\dim(M) + k$ (like vector bundles are regular neighborhoods of smooth embeddings). Any thickening of a closed manifold M contains a PL-embedded copy of M; the thickening is homeomorphic to the regular neighborhood of the embedding.

Block bundles, like vector bundles, are classified up to a finite number of possibilities by characteristic classes: Pontrjagin classes, Euler class, and one other class (I call it Hirsch-Haefliger class) that lives in $H^{2k-2}(K, \mathbb{Q})$. The Hirsch-Haefliger class can be nonzero only if k is odd. As of this writing, we cannot control the Hirsch-Haefliger class, so we force it to be zero assuming $H^{2k-2}(K, \mathbb{Q})=0$. Presumably, this is unnecessary; we hope to return to this matter in the future.

7.2. Theorem. *Let M be a closed oriented m-manifold with $\pi_1(M)$ torsion free. Let $k \geq 3$ be a positive integer such that $m + k \geq 6$ and either*

- *k is even and $\pi_1(M)$ is a non-virtually-nilpotent group that does not split over a virtually nilpotent group (e.g. M is an oriented aspherical manifold of dimension $m \geq 3$ with $\pi_1(M)$ word-hyperbolic), or*
- *k is odd and the $(2k-2)$th Betti number of M is zero (e.g. $2k - 2 > m$).*

Then the set of diffeomorphism classes of thickenings of M whose interiors are oriented hyperbolic $(m + k)$-manifolds with injective developing map is finite.

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