Quaternion Generation of the 2-Torsion Part of the Brauer Group of a Local Quintic (Type of Splitting: (1,1,1,1,1))

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1 Introduction

Let X be a smooth projective connected variety over k of characteristic zero, and let $K = k(X)$ be its function field. Then $K|k$ is a function field, that is, K is a separable and finitely generated field extension of k. Let $\mathrm{Br}_{nr}(K|k)$ be the unramified Brauer group of K (with respect to k). It is known (see, e.g., [6]) that there is a natural inclusion of $\mathrm{Br}\ X$ in $\mathrm{Br}(K)$ (here $\mathrm{Br}\ X$ denotes the Brauer group of X [5]) and this inclusion identifies $\mathrm{Br}\ X$ with the unramified Brauer group $\mathrm{Br}_{nr}(K|k)$. (Below we shall write $\mathrm{Br}\ X$ instead of $\mathrm{Br}_{nr}(K|k)$, keeping in mind this identification).

The group $\mathrm{Br}\ X$ is of great importance for many problems (we mention the problems of rationality of varieties, the problem of Brauer-Manin obstructions to various Hasse principles and finally the classical problem of the description of finite-dimensional division algebras over function fields (see [6],[10], [11]).

In this paper we are interested in the problem of the description of $\mathrm{Br}\ X$ for smooth geometrically connected projective curves over local fields, more precisely, for finite extensions k of $\boldsymbol{Q}_p$. The classical class field theory gives us a complete description of the Brauer group $\mathrm{Br}(k)$ of k (and also of finite dimensional central division algebras over k). But if $k(X)$ is a function field of a smooth geometrically connected projective variety over k, the situation is much more complicated. In spite of many results obtained in connection with the developing of multidimensional class field theory ([8],[9]), unfortunately, we do not have a complete description as in the classical situation even in the case of curves X (a "complete" description here means a representation of each element of $\mathrm{Br}(k(X))$ by the corresponding central division algebra over $k(X)$, in other words, it means a description of all central division algebras over $k(X)$). The concept of local-to-global principles is to try to describe global division algebras in terms of local ones. In the case of curves X over k this requires to investigate the groups $\mathrm{Br}(k_v(X_v))$ for all valuations v of k, where k_v is the completion of k at v and $X_v = X \times_k k_v$. By Tsen's theorem, $\mathrm{Br}(k_v(X_v))$ is trivial

in case of a complex v, and its structure is well-known for real v according to results in [1], [2], [3] and [7].

Thus, in this situation, the case of non-Archimedean v is the main unknown case. In the following, by a local field we will understand a finite extension of the field of p-adic numbers $\boldsymbol{Q}_p$. Let now X be a smooth geometrically connected projective curve over a local field k. Then it follows from [4], that there exists an exact sequence

$$0 \longrightarrow \mathrm{Br}\ X \longrightarrow \mathrm{Br}(k(X)) \xrightarrow{\phi} \coprod_{P \in X} \chi(G_P) \xrightarrow{\psi} \chi(G) \longrightarrow 0.$$

Here G denotes the Galois group of a separable closure of k, and G_P denotes the Galois group of a separable closure of the residue class field of $P \in X$, while $\chi(G_P)$ and $\chi(G)$ are the groups of continuous characters of G_P and G respectively (for the definitions of ϕ and ψ see [4]). In view of this sequence the group Im ϕ is known, so the group Br X is of main interest for the description of $\mathrm{Br}(k(X))$. The structure of Br X is well-known in the case of curves of genus zero [4], but even in the case of elliptic curves the complete structure of Br X is unknown. More generally, one can consider a smooth connected projective model X of an affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a polynomial with coefficients in k without multiple roots, and one can ask for the structure of Br X. One of the aims of our few papers is to answer some questions related to this. Since Br X is an Abelian periodic group, the main problem of its description is to describe its m-torsion part ${}_m\mathrm{Br}\ X$ as an abstract Abelian group for any m. Among those groups, ${}_2\mathrm{Br}\ X$ is of particular importance, because its elements are represented by central division algebras with involutions, and this group is, according to a result of Merkurjev, related to the theory of quadratic forms by the isomorphism ${}_2\mathrm{Br}(k(X)) \cong I^2/I^3$, where I is the fundamental ideal of the Witt ring of $k(X)$. Another important problem is to find an explicit representation of all elements of ${}_m\mathrm{Br}\ X$ (or at least for generators) by central division algebras.

Of course, solutions to these two problems strongly depend on the curve X. Both problems have trivial solutions in the case of curves of genus zero. For genus one curves one has more complicated situation. If X is an elliptic curve, both problems were solved completely in [13] for non-dyadic fields k and $m = 2$. There are also a few preliminary results in the case of dyadic fields and in the case of principal homogeneous spaces which will be published elsewhere. For hyperelliptic curves only the case of non-dyadic local curves with good reduction is considered (see [14]).

In this paper we begin a systematic study of the non-dyadic case of a quintic X without any restriction on its reduction. More precisely, our main object is the group ${}_2\mathrm{Br}\ X$ for X being a smooth projective model of the affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a monic polynomial of fifth degree with coefficients in k without multiple roots. The representation of generators of ${}_2\mathrm{Br}\ X$ by quaternion algebras depends on the irreducible factors of $f(x)$ over k. We will consider all the possible decomposition cases of $f(x)$ step by step. In this paper we consider the case $(1, 1, 1, 1, 1)$, i.e., $f(x)$ is a product of linear factors. The main result is a complete list of quaternion algebras representing generators of the group ${}_2\mathrm{Br}\ X$. Furthermore, we give a negative answer (see part 3 of this sequence of papers on the type of splitting (1,2,2), remark 1, p. 2 and remark 2, p. 9), with respect to the following question: Does any central unramified $k(X)$-division algebra of exponent 2 have a maximal subfield which comes from some finite extension of k? Note that, in case where X is an arbitrary elliptic curve or a hyperelliptic curve with

good reduction, the answer is "yes" (see [13], [14]).

2 Preliminary results

Let k be a local field of characteristic 0 with residue field $\bar{k}$. Let $\pi \in k$ denote a prime element, and let O_k, O_k^*, πO_k be its valuation ring, the group of units and the maximal ideal of the valuation ring. For $u \in O_k^*$ let $\bar{u}$ denote its class *modulo* πO_k. We will consider also a hyperelliptic curve C defined over k by the equation $y^2 = f(x)$, where $f(x) \in O_k[x]$ is a monic polynomial without multiple roots and $\deg f = n$. Let us assume also that $f(x) = \prod_{i=1}^{m} f_i(x)$, $f_i(x) \in O_k[x]$ are monic irreducible polynomials. Let α be a fixed unit of O_k which is not a square in k. $\bar{g}(x)$ will denote the reduction of a polynomial $g(x) \in O_k$. For $a, b \in k^*$ we write $a \sim b$ if a and b belong to the same class *modulo* $(k^*)^2$.

In our further considerations we will need the following lemmas.

Lemma 1

$$|\,{}_2\mathrm{Br}\, C| = 32.$$

For the proof see theorem 1 (p. 236) and §4 (p. 237) in [14].

Lemma 2 *Let $g(x) \in k[x]$ be a monic divisor of $f(x)$, $g(x) \notin k$ and let either n be odd or $\deg g(x)$ be even. Then for any $a \in k^*$ the quaternion algebra*

$$\left(\frac{a\ ,\ g(x)}{k(C)}\right)$$

is unramified and not isomorphic to the scalar algebra

$$\left(\frac{\pi\ ,\ \alpha}{k(C)}\right).$$

The proof is similar to the proof of lemma 7 in [14], p. 242.

Lemma 3 *Let $g(x) \in O_k[x]$ be a monic divisor of $f(x)$. If $\bar{g} \notin \bar{k}[x]^2$ and $\overline{f(x)/g(x)} \notin \bar{k}[x]^2$, then the quaternion algebra*

$$\left(\frac{\pi\ ,\ g(x)}{k(C)}\right)$$

is non-trivial.

For the proof see [14], proposition 4, p. 244.

Lemma 4 *Let $g(x) \in O_k[x]$ be a monic divisor of $f(x)$. If the quaternion algebra*

$$\bar{A} = \left(\frac{\bar{\alpha}\ ,\ \bar{g}(x)}{\bar{k}(x, \sqrt{\bar{f}(x)})}\right)$$

is non-trivial, then the algebra

$$A = \left(\frac{\alpha\ ,\ g(x)}{k(C)}\right)$$

is also non-trivial.

Proof. If $A \sim 1$, then $k(x)(\sqrt{f(x)})$ is a splitting field for the algebra

$$A_0 = \left(\frac{\alpha\ ,\ g(x)}{k(x)} \right),$$

therefore this field is isomorphic to a maximal subfield of A_0. Then

$$f = \varepsilon_1^2 \alpha + \varepsilon_2^2 g - \varepsilon_3^2 \alpha g,\ \varepsilon_i \in k(x).$$

If $\varepsilon_i = p_i/q_i,\ p_i \in k[x],\ q_i \in k[x]$ let $s_i = \mathrm{cont}_\pi\, p_i - \mathrm{con_t}\, \pi\, q_i$, where we denote by $\mathrm{cont}_\pi\, p$ the exponent of the highest π power dividing all coefficients of p. p We have

$$f = \pi^{2s_1} \frac{{p_1'}^2}{{q_1'}^2} \alpha + \pi^{2s_2} \frac{{p_2'}^2}{{q_2'}^2} g - \pi^{2s_3} \frac{{p_3'}^2}{{q_3'}^2} \alpha g,$$

where p_i' and q_i' are primitive polynomials. If h is the lowest common multiple of $q_1',\ q_2',\ q_3'$ and primitive, then

$$h^2 f = \pi^{2s_1} \mu_1^2 \alpha + \pi^{2s_2} \mu_2^2 g - \pi^{2s_3} \mu_3^2 \alpha g \quad (*)$$

and all the polynomials μ_i, h are primitive. In particular, $\mu_i, h \in O_k[x]$. By comparing the π-contents of both sides of the previous equality we obtain that not all s_i can be positive. If all of them are equal to 0, then we have

$$\bar{f} = \bar{\varepsilon}_1^2 \bar{\alpha} + \bar{\varepsilon}_2^2 \bar{g} - \bar{\varepsilon}_3^2 \bar{\alpha} \bar{g}$$

and $\bar{A} \sim 1$. If $s_{i_0} = \min_{1 \le i \le 3} \{s_i\} < 0$, then, after comparing the π-contents of both sides of (*) we obtain that there exists $j_0 \neq i_0$ such that $s_{i_0} = s_{j_0}$. If $s_1 = s_2 < s_3$, then after taking the reduction $\bar{\mu}_1^2 \bar{\alpha} + \bar{\mu}_2^2 \bar{g} = 0$. If $s_1 = s_3 < s_2$, then $\bar{\mu}_1^2 \bar{\alpha} - \bar{\mu}_3^2 \bar{\alpha} \bar{g} = 0$. If $s_2 = s_3 < s_1$, then $\bar{\mu}_2^2 \bar{g} - \bar{\mu}_3^2 \bar{\alpha} \bar{g} = 0$. Finally, let $s_1 = s_2 = s_3$, so $\bar{\mu}_1^2 \bar{\alpha} + \bar{\mu}_2^2 \bar{g} - \bar{\mu}_3^2 \bar{\alpha} \bar{g} = 0$. In all these cases we have that there exists $\theta \in \bar{A}$ such that the reduced norm of θ is zero, and this is a contradiction with $\bar{A} \not\sim 1$.

Lemma 5 *Let* V *be a variety defined over a finite field* $\boldsymbol{F}_q$. *Then for any positive integer* N *there exists a positive integer* n_0 *such that for any* $n > n_0$ *the subset* $V(\boldsymbol{F}_{q^n})$ *of all* $\boldsymbol{F}_{q^n}$*-rational points of* V *consists of more than* N *elements.*

The proof is straightforward in view of a theorem of Lang and Weil [15].

Lemma 6 *Let* K *be an algebraically closed field and* $f(x), g(x) \in K[x]$. *Then the system*

$$\begin{cases} y^2 = f(x) \\ z^2 = g(x) \end{cases}$$

defines a variety V *in* $\boldsymbol{P}^3(K)$ *if and only if* $f(x) \notin K[x]^2$, $g(x) \notin K[x]^2$ *and* $f(x)g(x) \notin K[x]^2$.

Proof. It is sufficient to prove that the K-algebra $K[V] = K[x, y, z]/(y^2 - f(x), z^2 - g(x))$ has no zero divisors. Let us consider the 4-dimensional commutative algebra A over the field $K(x)$ with the basis $1, i, j, ij$ and with the multiplication rule $i^2 = f, j^2 = g, ji = ij$.

The associativity of the multiplication of the basis elements is evident. Let us consider the map

$$\phi : K[V] \longrightarrow A, \quad \phi([p(x,y,z)]) = p(x,i,j), \quad p \in K[x,y,z].$$

Since $\phi((y^2-f(x), z^2-g(x))) = 0$, the map ϕ is correctly defined. Furthermore, ϕ is a ring homomorphism. Let us show that ϕ is an embedding. Indeed, let $p \in K[x,y,z]$. We have $p(x,y,z) = p_0(x)+p_1(x)y+p_2(x)z+p_3(x)yz+q(x,y,z)(y^2-f(x))+r(x,y,z)(z^2-g(x))$. Then $\phi([p]) = p_0 + p_1 i + p_2 j + p_3 ij$. If $\phi([p]) = 0$, then $p_0 = p_1 = p_2 = p_3 = 0$ and $p \in (y^2 - f(x), z^2 - g(x))$, so $[p] = 0$. Since $K[V] \in A$ it is enough to prove that A has no zero divisors. Let $I = \{a \in A, ab = 0 \text{ for some } b \in A, b \neq 0\}$ and $a \in I, a \neq 0$. If $a = a_0 + a_1 i + a_2 j + a_3 ij$, then

$$a(\alpha + \beta ij) = (\alpha a_0 + \beta a_3 fg) + (\alpha a_1 + \beta a_2 g)i + (\alpha a_2 + \beta a_1 f)j + (\alpha a_3 + \beta a_0)ij \in I.$$

$$ai = a_1 f + a_0 i + a_3 fj + a_2 ij \in I.$$

$$aj = a_2 g + a_3 gi + a_0 j + a_1 ij \in I.$$

$$aij = a_3 fg + a_2 gi + a_1 fj + a_0 ij \in I.$$

Since $a \neq 0$, then we have that among of a_l there exists a non-zero coefficient. Then, after replacement of a by ai, aj, or aij if necessary, we can assume that $a_0 \neq 0$. If $a_3 \neq 0$ let us replace a by $a' = a(\alpha + \beta ij)$, where $\beta = -\alpha a_3/a_0$. After the replacement we have $a'_3 = 0$. If $a'_0 = 0$, then $a_0^2 = a_3^2 fg$. Hence $fg \in K[x]^2$ and we have a contradiction. Thus $a'_0 \neq 0$ and one can assume from the very beginning that $a = 1 + a_1 i + a_2 j$. If $ab = 0, b \in I$, then, in view of the previous arguments, we can suppose that $b = 1 + x_1 i + x_2 j$. Therefore, $(1+a_1 i+a_2 j)(1+x_1 i+x_2 j) = (1+a_1 x_1 f+a_2 x_2 g)+(a_1+x_1)i+(a_2+x_2)j+(a_2 x_1+a_1 x_2)ij = 0$, so

$$\begin{cases} x_1 = -a_1, \\ x_2 = -a_2, \\ a_1 a_2 = 0, \\ a_1^2 f + a_2^2 g = 1. \end{cases}$$

If $a_1 = 0$, then $g \in K[x]^2$ and if $a_2 = 0$, then $f \in K[x]^2$. The lemma is proved.

We often will use the following well known fact.

Remark 1 *Let F be an arbitrary field, and let $F\langle\star\rangle$ denote the power series extension field of F (in one variable, which we do not specify). Then if A is a central simple F-algebra, the index of A is equal to the index of $A \otimes_F F\langle\star\rangle$.*

3 The reduction theorem and the cases of consideration.

From now on we will assume that $f(x) \in k[x]$ completely splits over k.

Let $k(C)$ be the function field of C. If f is of odd degree, we may assume without loss of generality that $f(x) \in O_k[x]$ and $f(x)$ is monic. We also may assume that $f(x) = x(x - \pi^\varepsilon u)g(x)$, where $0 \leq \varepsilon \leq 1$, $u \in O_k^*$, and $g(x) \in O_k[x]$.

Proof: By a linear transformation we obtain that $f(x)$ is divisible by x, hence

$$f(x) = x(x - \pi^l u) \prod_{i=1}^{n-2} (x - \pi^{k_i} u_i),$$

where $0 \leq l \leq k_1 \leq k_n$ and $u, u_1, \ldots, u_n \in O_k^*$. Let $l = 2m + \varepsilon$ with $\varepsilon \in \{0, 1\}$ and $x = \pi^{2m} x'$. Then

$$y^2 = \pi^{10m} x'(x' - \pi^\varepsilon u) \prod_{i=1}^{n-2} (x' - \pi^{k_i - 2m} u_i),$$

and the transformation $y = y'\pi^{5m}$ will lead us to the equation of the desired type.

Let now f be of degree 5. The aim of this section is to prove that by means of some special transformation of indeterminates we may restrict our consideration to some special forms of $f(x)$. More precisely, we prove the

Theorem 1 *Let $f(x)$ be as above, such that, in addition $\bar{f}(x)$ has multiple roots over $\bar{k}$ (bad reduction). Then $f(x)$ can be transformed by an appropriate transformation $x \mapsto x + m$, where $m \in \pi O_k$ into one of the following forms.*

I. $f(x) = x(x - \pi u)(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3)$,
where $u, u_i \in O_k^*$, $1 \leq k_1 \leq k_2 \leq k_3$. If $k_i = 1$, then $\bar{u}_i \neq \bar{u}$.
If $k_i = k_j$, $i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

I$'_*$. $f(x) = x(x - \pi u)(x - \pi^{k_1} u_1)(x - \pi^{k_1}(u_1 + \pi^s v))(x - \pi^{k_3} u_3)$,
where $1 < k_1 < k_3$, $u, v, u_1, u_3 \in O_k^*$ and $s > 0$.

I$'_{**}$. $f(x) = x(x - \pi u)(x - \pi(u + \pi^s v))(x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3)$,
where $1 \leq k_2 \leq k_3$, $1 < k_3$, $u, v, u_2, u_3 \in O_k^*$.
If $k_2 = 1$, then $\bar{u} \neq \bar{u}_2$, if $k_2 = k_3$, then $\bar{u}_2 \neq \bar{u}_3$, and $s > 0$.

II. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x - \pi^{k_3} u_3)(x + e)$,
where $1 \leq k_1 \leq k_2 \leq k_3$, $u_i, e \in O_k^*$, and if $k_i = k_j$, $i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

II$'$. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_1}(u_1 + \pi^s v))(x - \pi^{k_3} u_3)(x + e)$,
where $1 \leq k_1 < k_3$, $u_i, v \in O_k^*$, $s > 0$.

III. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x + e)(x + e - \pi^l v)$,
where $1 \leq k_1 \leq k_2$, $u_i, e \in O_k^*$, $l > 0$, and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

IV. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x + e_1)(x + e_2)$,
where $1 \leq k_1 \leq k_2$, $u_i, e_i \in O_k^*$, $\bar{e}_1 \neq \bar{e}_2$, and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

V. $f(x) = x(x - \pi^{k_1} v_1)(x + u)(x + u - \pi^{k_2} v_2)(x + e)$,
where $k_1, k_2 \geq 1$, $u, e, v_i \in O_k^*$, $\bar{u} \neq \bar{e}$.

VI. $f(x) = x(x - \pi^\kappa u)(x + e_1)(x + e_2)(x + e_3)$,
where $\kappa \geq 1$, $u, e_i \in O_k^*$, $\bar{e}_i \neq \bar{e}_j$ if $i \neq j$.

The proof of the theorem is based on the following lemmas.

Lemma 7 *Let $\bar{f}(x) = x^5$. Then by an appropriate transformation $x \mapsto x + n, n \in \pi O_k$ the polynomial $f(x)$ can be transformed to one of the forms* $\mathrm{I}, \mathrm{I}'_*, \mathrm{I}'_{**}$.

Proof. Assume that $f(x)$ cannot be transformed to the form I. Then by an appropriate transformation it can be transformed to a polynomial of the form

$$x(x - \pi u)(x - \pi^{k_1}u_1)(x - \pi^{k_2}u_2)(x - \pi^{k_3}u_3) \qquad (**)$$

where $u, u_i \in O_k^*,\ 0 < k_1 \le k_2 \le k_3 > 1$, all elements $\bar{u}_i$ such that $k_i = k_3$ are pairwise unequal and in the case $k_1 = k_2 = 1$ we have $\bar{u}_1 \ne \bar{u}_2$. Indeed, by above considerations

$$f(x) = x(x - \pi v)(x - \pi^{k_1}v_1)(x - \pi^{k_2}v_2)(x - \pi^{k_3}v_3),$$

where $v, v_i \in O_k^*$ and $0 < k_1 \le k_2 \le k_3$. If $k_3 = 1$, then the change of variables $x \mapsto x + \pi v_3$ transforms $f(x)$ to the polynomial

$$x(x - \pi(v - v_3))(x - \pi(v_1 - v_3))(x - \pi(v_2 - v_3))(x - \pi(-v_3)).$$

By assumption this polynomial is not of the form I, so at least one of the elements $\overline{v - v_3}$, $\overline{v_1 - v_3}$, $\overline{v_2 - v_3}$ is zero and this means that for the transformed polynomial after renumeration the exponent k_3 is greater than 1. Thus we may assume from the beginning that $k_3 > 1$.

Let now $k_2 = k_3$. Then the transformation $x \mapsto x + \pi^{k_3}v_3$ leads us to the polynomial

$$x(x - \pi(v - \pi^{k_3-1}v_3))(x - \pi^{k_1}(v_1 - \pi^{k_3-k_1}v_3))(x - \pi^{k_3}(-v_3))(x - \pi^{k_3}(v_2 - v_3)).$$

If $k_1 \ne k_3$, then this polynomial is of the form $(**)$. If $k_1 = k_3$, then we have the polynomial

$$x(x - \pi(v - \pi^{k_3-1}v_3))(x - \pi^{k_3}(v_1 - v_3))(x - \pi^{k_3}(-v_3))(x - \pi^{k_3}(v_2 - v_3)).$$

If $\bar{v}_1 - \bar{v}_3 \ne \bar{v}_2 - \bar{v}_3$, then this polynomial is also of the form $(**)$. If $\bar{v}_1 - \bar{v}_3 = \bar{v}_2 - \bar{v}_3 = 0$ then we have either the above case $k_1 \ne k_2 = k_3$ or the case $k_1 < k_2 < k_3$, and in the last case our polynomial is again of the form $(**)$. Thus $\bar{v}_1 - \bar{v}_3 = \bar{v}_2 - \bar{v}_3 \ne 0$. In this case it is enough to use the transformation $x \mapsto x + \pi^{k_3}(v_1 - v_3)$. Hence in all possible cases $f(x)$ can be transformed to a polynomial of the form $(**)$. Thus we have: in the case $k_1 = k_2 = 1$ we can transform $f(x)$ to

$$x(x - \pi u)(x - \pi(u + \pi^s v))(x - \pi u_2)(x - \pi^{k_3}u_3), \qquad (1)$$

where $s > 0,\ k_3 > 1,\ u, u_i, v \in O_k^*,\ \bar{u} \ne \bar{u}_2$.

In case $k_2 > 1,\ k_1 = 1$ the polynomial $f(x)$ can be transformed to

$$x(x - \pi u)(x - \pi(u + \pi^s v))(x - \pi^{k_2}u_2)(x - \pi^{k_3}u_3), \qquad (2)$$

(if $k_2 = k_3,\ \bar{u}_2 \ne \bar{u}_3,\ s > 0$).

The remaining case is

$$x(x - \pi u)(x - \pi^{k_1}u_1)(x - \pi^{k_1}(u_1 + \pi^s v))(x - \pi^{k_3}u_3), \qquad (3)$$

where $1 < k_1 < k_3,\ s > 0$. Combining together the cases 1 and 2 we will obtain the case of the form I'_{**} and case 3 is the case of the form I'_*.

Lemma 8 *$\bar{f}(x) = x^4(x+\bar{e})$, where $\bar{e} \neq 0$. Then by an appropriate transformation $x \mapsto x+m$, $m \in \pi O_k$ $f(x)$ can be transformed to one of the forms* $\mathrm{II}, \mathrm{II}'$.

Proof. Let

$$f(x) = x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)(x+e),$$

where $u_i, e \in O_k^*$ and $0 < k_1 \leq k_2 \leq k_3$.

Firstly, let $k_3 = 1$ and all $\bar{u}_i$ be pairwise unequal. Then $f(x)$ is of the form II. Otherwise there exist $i \neq j$ such that $u_i \neq u_j$ but $\bar{u}_i = \bar{u}_j$. Then the transformation $x \mapsto x + \pi u_i$ leads us to the case where $k_3 > 1$. Let now $k_3 > 1$. Then there are the following possibilities for k_1, k_2, k_3.

(i) $k_1 = k_2 = 1$,

(ii) $1 = k_1 < k_2$,

(iii) $1 < k_1 = k_2 = k_3$,

(iv) $1 < k_1 < k_2$,

(v) $1 < k_1 = k_2 < k_3$.

Let us consider all the possibilities step by step.

In case i), if $\bar{u}_1 \neq \bar{u}_2$, then $f(x)$ is of the form II. Otherwise, after the transformation $x \mapsto x + \pi u_1$ we will have a polynomial of the form II'.

In case ii), if $k_2 = k_3$ and $\bar{u}_2 = \bar{u}_3$, then the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to a polynomial of the form II. In case ii), if $k_2 \neq k_3$ or $k_2 = k_3$ but $\bar{u}_2 \neq \bar{u}_3$, then $f(x)$ is of the form II.

In case iii), if u_i $(i = 1, 2, 3)$ are pairwise unequal, then $f(x)$ is of the form II. Otherwise, without loss of generality, let us suppose $\bar{u}_1 = \bar{u}_3$. Then the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to the polynomial

$$x(x-\pi^{k_3}(-u_3))(x-\pi^{k_3}(u_1-u_3))(x-\pi^{k_3}(u_2-u_3))(x+e+\pi^{k_3}u_3).$$

This polynomial either is of the form II or can be transformed to one of the cases iv), v).

In case iv), if either $k_2 \neq k_3$ or $k_2 = k_3$ but $\bar{u}_2 \neq \bar{u}_3$ and $f(x)$ is of the form II. Otherwise, the transformation $x \mapsto x + \pi^{k_3}u_3$ leads us to case iv), where $k_2 \neq k_3$.

In case v), it is clear that $f(x)$ is either of the form II or II'.

Lemma 9 *By means of an appropriate transformation $x \mapsto x+m$, $m \in \pi O_k$, the polynomial $f(x)$ can be transformed into a polynomial of the form:*

III. $x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e)(x+e-\pi^l v)$,

where $u_i, e \in O_k^*$, $l > 0$, $1 \leq k_1 \leq k_2$

and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$

(case $\bar{f}(x) = x^3(x+\bar{e})^2$, $\bar{e} \neq 0$).

IV. $x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x+e_1)(x+e_2)$,

where $u_i, e_i \in O_k^*$, $\bar{e}_1 \neq \bar{e}_2$, $1 \leq k_1 \leq k_2$ and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$

(case $\bar{f}(x) = x^3(x+\bar{e}_1)(x+\bar{e}_2)$, $\bar{e}_i \neq 0$, $\bar{e}_1 \neq \bar{e}_2$).

V. $x(x-\pi^{k_1}v_1)(x+u)(x+u-\pi^{k_2}v_2)(x+e)$,

where $u, e, v_i \in O_k^*,\ k_1, k_2 \geq 1,\ \bar{u} \neq \bar{e}$

(case $\bar{f}(x) = x^2(x+\bar{u})^2(x+\bar{e}),\ \bar{e}\bar{u} \neq 0,\ \bar{u} \neq \bar{e}$*).*

VI. $x(x-\pi^{\kappa}u)(x+e_1)(x+e_2)(x+e_3)$,

where $u, e_i \in O_k^*,\ \bar{e}_i \neq \bar{e}_j\ i \neq j$

(case $\bar{f}(x) = x^2(x+\bar{e}_1)(x+\bar{e}_2)(x+\bar{e}_3),\ \bar{e}_1\bar{e}_2\bar{e}_3 \neq 0,\ \bar{e}_i \neq \bar{e}_j$ *if* $i \neq j$*).*

Proof. Let $\bar{f}(x) = x^3(x+\bar{e})^2$. If either $k_1 \neq k_2$ or $k_1 = k_2$ but $\bar{u}_1 \neq \bar{u}_2$, then $f(x)$ is of the form III. Otherwise, the transformation $x \mapsto x + \pi^{k_2}u_2$ leads us to the case $k_1 \neq k_2$, so we have transformed $f(x)$ to the form III.

Let $\bar{f}(x) = x^3(x+\bar{e}_1)(x+\bar{e}_2)$. Then similarly to the previous case either $f(x)$ is of the form III or one may use the transformation $x \mapsto x + \pi^{k_2}u_2$.

The two last splitting cases of $\bar{f}(x)$ are obvious.

Now our theorem follows from lemmas 7–9 because all possible splitting cases of $\bar{f}(x)$ are considered.

It follows from the theorem that in the case of bad reduction of C given by the equation $y^2 = f(x)$ ($\deg f = 5$), to describe the quaternion generation of ${}_2\mathrm{Br}\, C$ we may restrict our attention to $f(x)$ given by the forms I–VI, I$'_*$, I$'_{**}$, II$'$. For the further considerations we will need the following list of subcases.

The cases of consideration.

I. $f(x) = x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,

where $u, u_i \in O_k^*,\ 1 \leq k_1 \leq k_2 \leq k_3$. If $k_i = 1$, then $\bar{u}_i \neq \bar{u}$.

If $k_i = k_j,\ i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

I1. $k_1 \equiv k_2(2)$.

I2. I2$'$. $k_1 \not\equiv k_2(2),\ k_1 \equiv k_3(2)$.

I2$''$. $k_1 \not\equiv k_2(2),\ k_1 \not\equiv k_3(2),\ k_2 \neq k_3,\ -uu_1u_2 \sim 1$.

I3. $k_1 \not\equiv k_2(2),\ k_1 \not\equiv k_3(2)$ and either $k_2 = k_3$ or $-uu_1u_2 \not\sim 1$.

I$'_*$. $f(x) = x(x-\pi u)(x-\pi^{k_1}u_1)(x-\pi^{k_1}(u_1+\pi^s v))(x-\pi^{k_3}u_3)$,

where $u, v, u_i \in O_k^*,\ 1 < k_1 < k_3$ and $s > 0$.

I$'_{**}$. $f(x) = x(x-\pi u)(x-\pi(u+\pi^s v))(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)$,

where $u, v, u_i \in O_k^*,\ 1 \leq k_2 \leq k_3,\ 1 < k_3$ and $s > 0$.

If $k_2 = 1$, then $\bar{u} \neq \bar{u}_2$, and if $k_2 = k_3$, then $\bar{u}_2 \neq \bar{u}_3$.

I$'_{**}$1. $k_2 \equiv 1(2)$.

I$'_{**}$2. $k_2 \equiv 0(2)$ and either $k_3 \equiv 1(2)$ or ($k_3 \equiv 0(2)$ but $k_2 \neq k_3$ and $-u_2 \sim 1$).

I$'_{**}$3. $k_2 \equiv k_3 \equiv 0(2)$ and either $k_2 = k_3$ or $-u_2 \not\sim 1$.

II. $f(x) = x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)(x+e)$,

where $u_i, e \in O_k^*,\ 1 \leq k_1 \leq k_2 \leq k_3$, if $k_i = k_j,\ i \neq j$, then $\bar{u}_i \neq \bar{u}_j$.

II1. $e \sim 1$.

II1, 1. $k_1 \not\equiv k_2(2)$.

II1, 2. Either $k_1 \equiv k_2(2)$, $k_1 \not\equiv k_3(2)$ or $k_1 \equiv k_2 \equiv k_3(2)$ but $k_2 \neq k_3$ and $u_1 u_2 \sim 1$.

II1, 3. $k_1 \equiv k_2 \equiv k_3(2)$ and either $k_2 = k_3$ or $u_1 u_2 \not\sim 1$.

II2. $e \not\sim 1$.

II2, 1. $k_1 \not\equiv k_2(2)$.

II2, 2. $k_1 \equiv k_2(2)$, $k_1 \not\equiv k_3(2)$ or $k_1 \equiv k_2 \equiv k_3(2)$ but $k_2 \neq k_3$ and $u_1 u_2 \not\sim 1$.

II2, 3. $k_1 \equiv k_2 \equiv k_3(2)$ and either $k_2 = k_3$ or $u_1 u_2 \sim 1$.

II′. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_1}(u_1 + \pi^s v))(x - \pi^{k_3} u_3)(x + e)$,

where $u_i, e, v \in O_k^*$, $1 \leq k_1 < k_3$, $s > 0$.

II′1. $e \sim 1$.

II′2, 1. $e \not\sim 1$, $k_1 \not\equiv k_3(2)$.

II′2, 2. $e \not\sim 1$, $k_1 \equiv k_3(2)$.

III. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x + e)(x + e - \pi^l v)$,

where $u_i, e, v \in O_k^*$, $1 \leq k_1 \leq k_2$, $l > 0$ and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

III1. $-e \not\sim 1$, $k_1 \not\equiv 0(2)$.

III2. $-e \sim 1$, $k_1 \not\equiv 0(2)$.

III3. $-e \not\sim 1$ and either $(k_1 \equiv 0(2),\ k_2 \not\equiv 0(2))$ or
$(k_1 \equiv k_2 \equiv 0(2)$ and $k_1 < k_2,\ -u_1 \sim 1)$.

III4. $-e \sim 1$ and either $(k_1 \equiv 0(2),\ k_2 \not\equiv 0(2))$ or
$(k_1 \equiv k_2 \equiv 0(2)$ and $k_1 < k_2,\ -u_1 \sim 1)$.

III5. $-e \not\sim 1$, $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-u_1 \not\sim 1$.

III6. $-e \sim 1$, $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-u_1 \not\sim 1$.

IV. $f(x) = x(x - \pi^{k_1} u_1)(x - \pi^{k_2} u_2)(x + e_1)(x + e_2)$,

where $u_i, e_i \in O_k^*$, $\bar{e}_1 \neq \bar{e}_2$, $1 \leq k_1 \leq k_2$, and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

IV1. $k_1 \equiv k_2 \equiv 0(2)$ and either $k_1 = k_2$ or $-e_1 e_2 u_1 \not\sim 1$.

IV2. Other cases.

V. $f(x) = x(x - \pi^{k_1} u_1)(x + u)(x + u - \pi^{k_2} u_2)(x + e)$,

where $u, e, v_i \in O_k^*$, $\bar{u} \neq \bar{e}$, $k_1, k_2 > 0$.

V1. $e \not\sim 1$, $e - u \not\sim 1$.

V2. $e \not\sim 1$, $e - u \sim 1$.

V3. $e \sim 1$, $e - u \not\sim 1$.

V4. $e \sim 1$, $e - u \sim 1$.

VI. $f(x) = x(x - \pi^\kappa u)(x + e_1)(x + e_2)(x + e_3)$,
where $u, e_i \in O_k^*$, $\bar{e}_i \neq \bar{e}_j$ if $i \neq j$.

VI1. $e_1e_2e_3 \not\sim 1$.

VI2. $e_1e_2e_3 \sim 1$.

4 Presentation by quaternion algebras.

The aim of this section is to present a list of quaternion algebras over $k(C)$ which are representatives of some system of generators of ${}_2\mathrm{Br}\, C$.

Theorem 2 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* I. *Then the class of (π, α) and the classes of the following quaternion algebras generate ${}_2\mathrm{Br}\, C$.*

I1. $\{(\alpha, l_i)\}$, *where l_i is a monic linear divisor of $f(x)$.*

I2. (α, x), $(\alpha, x - \pi u)$, $(\alpha, x - \pi^{k_1}u_1)$, (π, x).

I3. (α, x), $(\alpha, x - \pi u)$, $(\pi, x - \pi^{k_3}u_3)$, (π, x).

Proof. By lemma 1, the order of ${}_2\mathrm{Br}\, C$ is 32, so to prove the theorem, it is enough to prove that, in each case I1, I2, I3, the classes of the corresponding algebras generate the subgroup H of ${}_2\mathrm{Br}\, C$ of order 16 such that $[(\pi, \alpha)] \notin H$.

In order to prove that some algebra is non-trivial it is enough to prove that some of its completions is non-trivial. We have

$$(\alpha, x)_{x-\pi u} \sim (\alpha, \pi u) \not\sim 1,$$

$$(\alpha, x - \pi^{k_i}u_i)_{x-\pi u} \sim (\alpha, \pi u - \pi^{k_i}u_i) \sim (\alpha, \pi) \not\sim 1 \ (i = 1, 2, 3),$$

$$(\alpha, x - \pi u)_x \sim (\alpha, \pi u) \not\sim 1.$$

Let now $\{i, j, t, m, n\} = \{1, 2, 3, 4, 5\}$. Then $(\alpha, l_il_jl_tl_ml_n) \sim 1$, so that $(\alpha, l_il_jl_tl_m) \sim (\alpha, l_n) \not\sim 1$ and $(\alpha, l_il_jl_t) \sim (\alpha, l_ml_n)$. It follows from the above consideration that to prove that the order of the group H is 16 it is enough to prove that all the algebras (α, l_il_j) are non-trivial. If $l_i = x - \pi u$, then

$$(\alpha, (x - \pi u)l_j)_{x-\pi u} \sim (\alpha, l_tl_ml_n)_{x-\pi u} \sim (\alpha, \pi^3) \not\sim 1.$$

Let us consider the remaining cases, where $l_i, l_j \neq x - \pi u$.

$$\begin{aligned}
(\alpha, x(x - \pi^{k_1}u_1))_{x-\pi^{k_1}u_1} &\sim (\alpha, (x - \pi u)(x - \pi^{k_2}u_2)(x - \pi^{k_3}u_3))_{x-\pi^{k_1}u_1} \\
&\sim (\alpha, \pi^{2k_1+1}) \not\sim 1. \\
(\alpha, x(x - \pi^{k_3}u_3))_{x-\pi^{k_3}u_3} &\sim (\alpha, \pi^{k_1+k_2+1}) \\
&\sim (\alpha, \pi) \not\sim 1. \\
(\alpha, (x - \pi^{k_1}u_1)(x - \pi^{k_2}u_2))_{x-\pi^{k_1}u_1} &\sim (\alpha, \pi^{k_1}u_1(\pi^{k_1}u_1 - \pi u)(\pi^{k_1}u_1 - \pi^{k_3}u_3)) \\
&\sim (\alpha, \pi) \not\sim 1. \\
(\alpha, (x - \pi^{k_1}u_1)(x - \pi^{k_3}u_3))_{x-\pi^{k_1}u_1} &\sim (\alpha, \pi^{k_1}u_1(\pi^{k_1}u_1 - \pi u)(\pi^{k_1}u_1 - \pi^{k_2}u_2))
\end{aligned}$$

$$
\begin{aligned}
& \sim \quad (\alpha,\pi) \not\sim 1.\\
(\alpha, x(x-\pi^{k_2}u_2))_{x-\pi^{k_2}u_2} & \sim \quad (\alpha,\pi^{k_1+k_2+1})\\
& \sim \quad (\alpha,\pi) \not\sim 1.\\
(\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} & \sim \quad (\alpha,\pi^{k_2}u_2(\pi^{k_2}u_2-\pi u)(\pi^{k_2}u_2-\pi^{k_1}u_1))\\
& \sim \quad (\alpha,\pi) \not\sim 1.
\end{aligned}
$$

To finish the consideration of case I1 one needs to observe that by lemma 2 all algebras from the list in case I1 are unramified and non-isomorphic to (π,α).

In case I2, let H be the subgroup of ${}_2\mathrm{Br}\,C$ generated by the classes of algebras (α,x), $(\alpha,x-\pi u)$, $(\alpha,x-\pi^{k_1}u_1)$, (π,x). All these algebras are unramified over $k(C)$. Let us prove that H is of order 16. Just as in the previous case one can prove that the algebras $(\alpha,x-\pi u)$, $(\alpha,x(x-\pi^{k_1}u_1))$, (α,x), $(\alpha,x-\pi^{k_1}u_1)$, $(\alpha,x(x-\pi^{k_1}u_1))$, $(\alpha,x(x-\pi u))$, $(\alpha,(x-\pi u)(x-\pi^{k_1}u_1))$ are non-trivial. To prove that $[(\alpha,x)]$, $[(\alpha,x-\pi u)]$, $[(\alpha,x-\pi^{k_1}u_1)]$ generate a group of order 8 one needs only to prove that $(\alpha,x(x-\pi u)(x-\pi^{k_1}u_1)) \sim (\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)) \not\sim 1$.

In case I2$'$ we have

$$
(\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi^{k_3}u_3} \sim (\alpha,\pi^{k_1+k_3+1}) \not\sim 1.
$$

In case I2$''$ we have $k_2+1<k_3$ and

$$
\begin{aligned}
(\alpha,(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi^{k_2+1}} & \quad \sim\\
(\alpha,(\pi^{k_2+1}-\pi^{k_2}u_2)(\pi^{k_2+1}-\pi^{k_3}u_3)) & \quad \sim \quad (\alpha,\pi) \not\sim 1
\end{aligned}
$$

since $f(\pi^{k_2+1}) \sim -\pi^{k_1+k_2+1}uu_1u_2 \sim -uu_1u_2 \sim 1$.

To prove $(\pi,x) \not\sim 1$, let us consider a curve $(-1)^{k_2}\bar\alpha\bar u\bar u_1x^2-\bar u\bar u_1\bar u_2=y^2$ defined over the residue field $\bar k$ of k. This curve has a smooth point $(\tilde a,\tilde b)$ such that $\tilde a,\tilde b\in\bar k(\tilde a)$, where $\bar k(\tilde a)$ is an extension of $\bar k$ of some prime odd degree. Then, for the unramified extension N over k with residue field $\bar k(\tilde a)$, there is a lift $(a,b)\in N^2$ of $(\tilde a,\tilde b)$ to a point of a curve $(-1)^{k_2}\alpha uu_1x^2-uu_1u_2=y^2$. Set $w=(-1)^{k_2}\alpha a^2$ and $\theta=\pi^{k_2}w$. It is easy to see that $f(\theta)\sim(-1)^{k_2}\alpha uu_1a^2-uu_1u_2\sim 1$. Then it follows that

$$
\left(\frac{\pi\ ,\ x}{k(x,\sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ (-1)^{k_2}w}{N\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.
$$

Hence $(\pi,x)\not\sim 1$. Since the algebra

$$
\left(\frac{\pi\ ,\ x}{k(\sqrt{\alpha})(x,\sqrt{f(x)})}\right)
$$

belongs again to case I2 (with constant field $k(\sqrt{\alpha})$, then $k(\sqrt{\alpha})(\sqrt{f(x)})$ does not split the algebra (π,x) but it splits all the algebras (α,x), $(\alpha,x-\pi u)$, $(\alpha,x-\pi^{k_1}u_1)$, so it follows that the elements $[(\alpha,x)]$, $[(\alpha,x-\pi u)]$, $[(\alpha,x-\pi^{k_1}u_1)]$ and $[(\pi,x)]$ generate a group H of order 16. In view of lemma 2 we have $[(\pi,\alpha)]\notin H$. This finishes the consideration of case I2.

In case I3 one can prove similarly as in case I2 that $[(\alpha, x)]$, $[(\alpha, x-\pi u)]$ generate a group of order 4. As to the non-triviality of (π, x) one can prove this just in the same way as in the previous case if $k_2 < k_3$. If $k_2 = k_3$, consider the curve over $\bar{k}$ defined by the equation

$$(-1)^{k_2}\bar{\alpha}\bar{u}\bar{u}_1\left(x^2 - \frac{(-1)^{k_2}\bar{u}_2}{\bar{\alpha}}\right)\left(x^2 - \frac{(-1)^{k_2}\bar{u}_3}{\bar{\alpha}}\right) = y^2.$$

Then there exists an extension $\bar{k}(\tilde{a})|\bar{k}$ of odd prime degree with a smooth point $(\tilde{a}, \tilde{b})$ such that $\tilde{b} \in \bar{k}(\tilde{a})$. Let $(a, b) \in N^2$ be a lifted point on the curve defined by the equation

$$(-1)^{k_2}\alpha u u_1\left(x^2 - \frac{(-1)^{k_2}u_2}{\alpha}\right)\left(x^2 - \frac{(-1)^{k_2}u_3}{\alpha}\right) = y^2,$$

where $N|k$ is the unramified extension with residue field $\bar{k}(\tilde{a})$ and $w = (-1)^{k_2}\alpha a^2$. Set $\theta = \pi^{k_2}w$. Then $f(\theta) \sim w(w-u_2)(w-u_3)uu_1 \sim 1$ and consequently

$$\left(\frac{\pi\ ,\ x}{k(x, \sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.$$

Thus $(\pi, x) \not\sim 1$.

Let N be an extension of k as above for a curve defined by the equation

$$uu_1((-1)^{k_2}\alpha x^2 - u_2) = y^2.$$

Then such a curve has a point (a, b) over N. If $k_2 < k_3$ for $\theta = \pi^{k_2}(-1)^{k_2}\alpha a^2$ we have $f(\theta) \sim 1$ and

$$\left(\frac{\pi\ ,\ x-\pi^{k_3}u_3}{k(x, \sqrt{f(x)})}\right)_{x-\theta} \sim \left(\frac{\pi\ ,\ \theta}{N(\sqrt{f(\theta)})\langle x-\theta\rangle}\right) \sim \left(\frac{\pi\ ,\ \alpha}{N\langle x-\theta\rangle}\right) \not\sim 1.$$

In case $k_2 = k_3$ one needs to consider, instead of the latter curve, a curve defined by the equation

$$y^2 = (-1)^{k_2}\alpha u u_1((-1)^{k_2}\alpha x^2 - u_2)((-1)^{k_2}\alpha x^2 - u_3)$$

and proceed analogously as in the case $k_2 < k_3$.

One can prove in a similar way that $(\pi, x(x-\pi^{k_3}u_3)) \not\sim 1$. If $k_2 < k_3$, consider a curve $y^2 = \alpha(x^2 + u_2/(uu_1))(x^2 + (u_2-u_3)/(uu_1))$, in case $k_2 = k_3$ consider a curve $y^2 = \alpha x^2 - u_3$ and proceed analogously as in the previous cases.

To finish the consideration of case I3 let us prove that groups $\langle[(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$ and $\langle[(\pi, x)],\ [(\pi, x-\pi^{k_3}u_3)]\rangle$ have trivial intersection.

Since the extension of k by $\sqrt{\alpha}$ leaves us either in case I2 or in case I3, the algebra (π, x) is not split by $k(\sqrt{\alpha})(x, \sqrt{f(x)})$, and, consequently, $[(\pi, x)] \notin \langle[(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. The transformation $x \mapsto x + \pi^{k_3}u_3$ leaves us again either in case I2 or in the case I3, so that $[(\pi, x-\pi^{k_3}u_3)] \notin \langle[(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. Let us consider the algebra $(\pi, x(x-\pi^{k_3}u_3))$. If $k_2 = k_3$, then the extension of constants by $\sqrt{\alpha}$ shows us that $[(\pi, x(x-\pi^{k_3}u_3))] \notin \langle[(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. Let now $k_2 \neq k_3$. Comparing completions of $(\pi, x(x-\pi^{k_3}u_3))$ and $(\alpha, x-\pi u)$

$$(\pi, x(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\pi, \pi^{k_2}u_2(\pi^{k_2}u_2 - \pi^{k_3}u_3)) \sim 1,$$

$$(\alpha, x-\pi u)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}u_2 - \pi u) \sim (\alpha, \pi) \not\sim 1,$$

one can see that $(\pi, x(x-\pi^{k_3}u_3))$ and $(\alpha, x-\pi u)$ are non-isomorphic. Comparing completions of $(\pi, x(x-\pi^{k_3}u_3))$ and (α, x) in $x-\pi u$ one can see that these algebras are also non-isomorphic. And finally we have for $i \in \{1,2\},\ k_i \equiv 0(2)$ the following.

$$\begin{aligned}(\alpha, x(x-\pi u))_{x-\pi^{k_i}u_i} &\sim (\alpha, \pi^{k_i+1}) \not\sim 1 \text{ and}\\ (\pi, x(x-\pi^{k_3}u_3))_{x-\pi^{k_i}u_i} &\sim (\pi, (\pi^{k_i}u_i)^2) \sim 1\end{aligned}$$

Hence $(\alpha, x(x-\pi u)) \not\sim (\pi, x(x-\pi^{k_3}u_3))$. This completes the proof.

In cases I'_* and I'_{**} we have

Theorem 3 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* I'. *Then the class of (π, α) and the classes of the following quaternion algebras generate ${}_2\mathrm{Br}\, C$.*

I'_*. *$\{(\alpha, l_i)\}$, where l_i is a monic linear divisor of f.*

$\mathrm{I}'_{**}1$. *$\{(\alpha, l_i)\}$, where l_i is a monic linear divisor of f.*

$\mathrm{I}'_{**}2$. *$(\alpha, x),\ (\alpha, x-\pi u),\ (\alpha, x-\pi u'),\ (\pi, x),\ u' = u + \pi^s v$.*

$\mathrm{I}'_{**}3$. *$(\alpha, x),\ (\alpha, x-\pi u),\ (\pi, x-\pi^{k_3}u_3),\ (\pi, x)$.*

Proof. Similarly as in case I one can prove that $(\alpha, l_i) \not\sim 1$ for a monic linear divisor of $f(x)$. In case I'_* if $l_i = x - \pi u$ and $i \neq j$ we have, similarly as in case I, $(\alpha, l_i l_j) \not\sim 1$. Furthermore,

$$(\alpha, x(x-\pi^{k_1}u_1))_x \sim (\alpha, \pi^{k_1+k_3+1}), (\alpha, x(x-\pi^{k_1}u_1))_{x-\pi^{k_3}u_3} \sim (\alpha, \pi^{k_1+k_3}).$$

One of the algebras $(\alpha, \pi^{k_1+k_3})$ and $(\alpha, \pi^{k_1+k_3+1})$ is non-trivial, so that we have $(\alpha, x(x-\pi^{k_1}u_1)) \not\sim 1$. Similarly, one can prove that $(\alpha, x(x-\pi^{k_1}(u_1+\pi^s v))) \not\sim 1$. We have also

$$(\alpha, x(x-\pi^{k_3}u_3))_x \sim (\alpha, \pi^{2k_1+1}) \not\sim 1$$

and

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_1}u'_1))_{x-\pi^{k_1}u_1} \sim (\alpha, x(x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u_1} \not\sim 1.$$

For the algebra $(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))$ we have

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u_1} \sim (\alpha, x(x-\pi^{k_1}u'_1)(x-\pi u))_{x-\pi^{k_1}u_1} \sim (\alpha, \pi^{s+1}),$$

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3))_{x-\pi^{k_1}u'_1} \sim (\alpha, \pi^{2k_1+s}) \sim (\alpha, \pi^s).$$

One of the algebras $(\alpha, \pi^s),\ (\alpha, \pi^{s+1})$ is non-trivial, therefore we find that

$$(\alpha, (x-\pi^{k_1}u_1)(x-\pi^{k_3}u_3)) \not\sim 1.$$

In a similar way one proves that $(\alpha, (x-\pi^{k_1}u'_1)(x-\pi^{k_3}u_3)) \not\sim 1$. Now similarly to case I one can complete the proof.

Let us consider case I'_{**}. We have

$$(\alpha, (x-\pi u)(x-\pi u'))_{x-\pi u} \sim (\alpha, x(x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi u} \sim (\alpha, \pi^3) \not\sim 1.$$

We have also $(\alpha, x(x-\pi u)) \not\sim 1$, since

$$(\alpha, x(x-\pi u))_{x-\pi u} \sim (\alpha, \pi^{s+1}) \text{ and } (\alpha, x(x-\pi u))_{x-\pi u'} \sim (\alpha, \pi^s).$$

Similarly, $(\alpha, x(x-\pi u')) \not\sim 1$.
Considering the algebra $(\alpha, (x-\pi u)(x-\pi^{k_2}u_2))$, we have $(\alpha, (x-\pi u)(x-\pi^{k_2}u_2)) \not\sim 1$, since

$$(\alpha, (x-\pi u)(x-\pi^{k_2}u_2))_{x-\pi u} \sim (\alpha, \pi^{s+1}) \text{ and } (\alpha, (x-\pi u)(x-\pi^{k_2}u_2))_{x-\pi u'} \sim (\alpha, \pi^s).$$

Similarly, $(\alpha, (x-\pi u')(x-\pi^{k_2}u_2)) \not\sim 1$.
Furthermore, $(\alpha, (x-\pi u)(x-\pi^{k_3}u_3)) \not\sim 1$, since

$$(\alpha, (x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi u} \sim (\alpha, \pi^{s+1}) \text{ and } (\alpha, (x-\pi u)(x-\pi^{k_3}u_3))_{x-\pi u'} \sim (\alpha, \pi^s).$$

Similarly $(\alpha, (x-\pi u')(x-\pi^{k_3}u_3)) \not\sim 1$.
We have also $(\alpha, x(x-\pi^{k_2}u_2)) \not\sim 1$, since

$$(\alpha, x(x-\pi^{k_2}u_2))_{x-\pi^{k_2}u_2} \sim (\alpha, (x-\pi u)(x-\pi u')(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}),$$

and k_2 is odd.
Then we have

$$(\alpha, x(x-\pi^{k_3}u_3))_x \sim (\alpha, (x-\pi u)(x-\pi u')(x-\pi^{k_2}u_2))_x \sim (\alpha, \pi^{k_2})$$

and k_2 is odd.
Finally, $(\alpha, (x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3)) \not\sim 1$, since

$$(\alpha, (x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_{x-\pi^{k_2}u_2} \sim (\alpha, x(x-\pi u)(x-\pi u'))_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}) \not\sim 1.$$

It is clear from the above considerations that similarly to case I one can complete the proof in case I'_{**}.

Let us consider the next case. We can conclude that the group $\langle [(\alpha, x)],\ [(\alpha, x-\pi u)],\ [(\alpha, x-\pi u')]\rangle$ is of order 8 in case $\mathrm{I}'_{**}2$ and that $\langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$ is of order 4 in case $\mathrm{I}'_{**}3$. The non-triviality of the algebra (π, x) in both cases can be established by considerations similar as in the proof of the theorem 2. As to the algebra $(\pi, x-\pi^{k_3}u_3)$ in case $\mathrm{I}'_{**}3$ one can use the replacement $x' = x-\pi^{k_3}u_3$ and reduces this case to the case of the algebra (π, x). The non-triviality of the algebra $(\pi, x(x-\pi^{k_3}u_3))$ in case $\mathrm{I}'_{**}3$ can be obtained by using similar arguments as in case I. To complete the proof in case $\mathrm{I}'_{**}2$, let us observe that the extension of constants by $\sqrt{\alpha}$ does not split (π, x), but splits any algebra of the form $(\alpha, g(x))$.

To complete the proof of the theorem in case $\mathrm{I}'_{**}3$ we need to prove that the groups $\langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$ and $\langle [(\pi, x)],\ [(\pi, x-\pi^{k_3}u_3)]\rangle$ have a trivial intersection. But in case $\mathrm{I}'_{**}3$, $k_2 = k_3$, the extension of constants by $\sqrt{\alpha}$ leaves us in case $\mathrm{I}'_{**}3$ and this completes the proof in the case $k_2 = k_3$. If we are in case $\mathrm{I}'_{**}3$, $k_2 \neq k_3$ then the similar extension of constants leads us to case $\mathrm{I}'_{**}2$, so that $[(\pi, x)] \notin \langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. After the replacement $x' = x-\pi^{k_3}u_3$ we are left again in case $\mathrm{I}'_{**}3$, hence $[(\pi, x-\pi^{k_3}u_3)] \notin \langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$. The proof of the fact that $[(\pi, x(x-\pi^{k_3}u_3))] \notin \langle [(\alpha, x)],\ [(\alpha, x-\pi u)]\rangle$ is just the same as the corresponding part of the proof of the theorem 2. This completes the proof of the theorem.

In case II we have

Theorem 4 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* II. *Then the class of (π, α) and the classes of the following algebras generate ${}_2\mathrm{Br}\, C$.*

II1, 1. $(\alpha, x),\ (\alpha, x - \pi^{k_1}u_1),\ (\alpha, x - \pi^{k_2}u_2),\ (\pi, x)$.

II1, 2. $(\alpha, x),\ (\alpha, x - \pi^{k_1}u_1),\ (\pi, x - \pi^{k_1}u_1),\ (\pi, x)$.

II1, 3. $(\alpha, x),\ (\pi, x - \pi^{k_1}u_1),\ (\pi, x - \pi^{k_2}u_2),\ (\pi, x)$.

II2, 1. $(\alpha, x),\ (\alpha, x - \pi^{k_3}u_3),\ (\pi, x + e),\ (\pi, x)$.

II2, 2. $(\alpha, x),\ (\pi, x - \pi^{k_1}u_1),\ (\pi, x + e),\ (\pi, x)$.

II2, 3. $(\pi, x - \pi^{k_1}u_1),\ (\pi, x - \pi^{k_2}u_2),\ (\pi, x - \pi^{k_3}u_3),\ (\pi, x)$.

Proof. Observe first of all that in case II $(\alpha, x + e) \sim 1$ for any $e \in O_k^*$ and $(\pi, x + e) \not\sim 1$ if and only if $e \in O_k^* \setminus (O_k^*)^2$. Indeed, one needs only to prove that for any irreducible polynomial $p(x) \in k[x]$ we have $(\alpha, x + e)_{p(x)} \sim 1$, and for any irreducible $p(x)$ we have $(\pi, x+e)_{p(x)} \sim 1$ if $e \in (O_k^*)^2$ but there exists $p(x)$ such that $(\pi, x+e)_{p(x)} \not\sim 1$ if $e \notin (O_k^*)^2$.

Assume firstly that θ is a root of $p(x)$. Then

$$(\alpha, x + e)_{p(x)} \sim (\alpha, \theta + e) \sim (\alpha, \theta(\theta - \pi^{k_1}u_1)(\theta - \pi^{k_2}u_2)(\theta - \pi^{k_3}u_3))$$

and if $\theta \notin \pi O_k$, then $(\alpha, x + e)_{p(x)} \sim (\alpha, \theta^4) \sim 1$. Let $\theta \in \pi O_k$, then

$$(\alpha, x + e)_{p(x)} \sim (\alpha, \theta + e) \sim (\alpha, e) \sim 1.$$

Assume that $e \in (O_k^*)^2$, then similar arguments show that $(\pi, x + e) \sim 1$. In the case $e \notin (O_k^*)^2$ we have $(\pi, x + e)_x \sim (\pi, e) \not\sim 1$.

Consider case II1. Let $l_0 = x,\ l_j = x - \pi^{k_j}u_j,\ j = 1, 2, 3$. Since $\bar{f}(x)/\bar{l}_j(x),\ \bar{l}_j(x) \notin \bar{k}[x]^2,\ j = 0, 1, 2, 3$, then by lemma 3 $(\pi, l_j) \not\sim 1$. Furthermore,

$$\left(\frac{\bar{\alpha}\ ,\ \bar{l}_i(x)}{\bar{k}(x)(\sqrt{\bar{f}(x)})}\right) = \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x + \bar{e}})}\right).$$

Since in case II1 $e \sim 1$ we have

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x + \bar{e}})}\right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x\rangle}\right) \not\sim 1.$$

Hence

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x)(\sqrt{x + \bar{e}})}\right) \not\sim 1$$

and, in view of lemma 4 $(\alpha, l_i) \not\sim 1,\ (i = 0, 1, 2, 3)$. Thus all the algebras listed in cases II1, 1, II1, 2, II1, 3 are non-trivial. Since $(\alpha, x + e) \sim 1$, then $(\alpha, l_0l_1l_2) \sim (\alpha, l_3) \not\sim 1$, and hence to prove that $\langle[(\alpha, l_0)],\ [(\alpha, l_1)],\ [(\alpha, l_2)]\rangle$ is a group of order 8 it is enough to prove that $(\alpha, l_0l_1) \not\sim 1,\ (\alpha, l_0l_2) \not\sim 1,\ (\alpha, l_1l_2) \not\sim 1$. We have

$$(\alpha, l_0l_1)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}\pi^{k_1}) \sim (\alpha, \pi) \not\sim 1,$$

$$(\alpha, l_0l_2)_{x-\pi^{k_2}u_2} \sim (\alpha, l_1l_3)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_1}\pi^{k_2}) \not\sim 1,\ (\alpha, l_1l_2)_x \sim (\alpha, \pi^{k_1}\pi^{k_2}) \not\sim 1.$$

It is clear that the extension of constants by $\sqrt{\alpha}$ leaves us in case II1, and this implies that $[(\pi,x)] \notin \langle [(\alpha,l_0)],\ [(\alpha,l_1)],\ [(\alpha,l_2)]\rangle$, so that case II1, 1 is done.

In case II1, 2, let us show firstly that $(\alpha, x(x-\pi^{k_1}u_1)) \not\sim 1$ and $(\pi, x(x-\pi^{k_1}u_1)) \not\sim 1$. If $k_2 \not\equiv k_3(2)$ we have

$$(\alpha, x(x-\pi^{k_1}u_1))_x \sim (\alpha, (x-\pi^{k_2}u_2)(x-\pi^{k_3}u_3))_x \sim (\alpha, \pi^{k_2+k_3}) \not\sim 1.$$

If $k_2 \equiv k_3(2)$ we have

$$f(\pi^{k_2+1}) \sim \pi^{k_2+1}(-\pi^{k_1}u_1)(-\pi^{k_2}u_2)\pi^{k_2+1} \sim u_1u_2 \sim 1$$

and this implies

$$(\alpha, x(x-\pi^{k_1}u_1))_{x-\pi^{k_2+1}} \sim (\alpha, \pi^{k_1+k_2+1}) \not\sim 1,$$

so that $(\alpha, x(x-\pi^{k_1}u_1)) \not\sim 1$.

To prove $(\pi, x(x-\pi^{k_1}u_1)) \not\sim 1$, let us find a polynomial $p(x)$ such that $(\pi, x(x-\pi^{k_1}u_1))_{p(X)} \not\sim 1$. Let $p(x)$ be a minimal polynomial of element $\theta = \pi^{k_3}w$, where w is a unit from the algebraic closure of k. Then

$$f(\theta) \sim \pi^{k_3}w(-\pi^{k_1}u_1)(-\pi^{k_2}u_2)\pi^{k_3}(w-u_3) \sim u_1u_2w(w-u_3).$$

Assume now that $k(w)|k$ is an unramified prime odd degree extension (or its degree is 1). Then

$$(\pi, x(x-\pi^{k_1}u_1))_{p(x)} \sim \left(\frac{\pi\ ,\ (-1)^{k_1+k_3+1}u_1w}{k(w)\langle x-\theta\rangle\sqrt{u_1u_2w(w-u_3)}}\right).$$

And, if $u_1u_2w(w-u_3) \sim 1$ and $(-1)^{k_1+k_3+1}u_1w \sim \alpha$, then $(\pi, x(x-\pi^{k_1}u_1))_{p(x)} \not\sim 1$. To find such a w, let us consider a curve over $\bar{k}$ given by the equation $y^2 = \bar{u}_1\bar{u}_2x^2 + (-1)^{k_1+k_3}\bar{u}_2\bar{u}_3\bar{\alpha}$.

There exists a finite odd degree extension of $\bar{k}$, where our curve has a smooth point $(\tilde{a},\tilde{b})$ which can be lifted to a point (a,b) on the curve $y^2 = u_1u_2x^2 + (-1)^{k_1+k_3}u_2u_3\alpha$. And we will be done if put $w = (-1)^{k_1+k_3+1}u_1\alpha a^2$. It follows immediately from the above considerations that $\langle [(\alpha,x)],\ [(\alpha, x-\pi^{k_1}u_1)]\rangle$ and $\langle [(\pi,x)],\ [(\pi, x-\pi^{k_1}u_1)]\rangle$ are groups of order 4. Since the extension of constants by $\sqrt{\alpha}$ leaves us in the case II1, 2 these two groups have the trivial intersection and this completes the consideration of case II1, 2.

In case II1, 3 to prove that $\langle [(\pi,x)],\ [(\pi, x-\pi^{k_1}u_1)]\ [(\pi, x-\pi^{k_2}u_2)]\rangle$ is a group of order 8 in view of $(\pi, x(x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)) \sim (\pi, x-\pi^{k_3}u_3) \not\sim 1$ it is enough to establish that the algebras $(\pi, x(x-\pi^{k_1}u_1))$, $(\pi, x(x-\pi^{k_2}u_2))$, $(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))$ are non-trivial. If $k_2 < k_3$, then the non-triviality of the first algebra can be checked similarly as in the previous case. Let now $k_2 = k_3$ and assume that w is a unit from the algebraic closure of k with the above mentioned properties, $\theta = \pi^{k_3}w$, and let $p(x)$ be the minimal polynomial of θ. Then we have $f(\theta) \sim w(\pi^{k_3-k_1}w - u_1)(w-u_2)(w-u_3)$ and

$$(\pi, x(x-\pi^{k_1}u_1))_{p(x)} \sim \left(\frac{\pi\ ,\ w(w\pi^{k_3-k_1}-u_1)}{k(w)\langle x-\theta\rangle(\sqrt{w(\pi^{k_3-k_1}w-u_1)(w-u_2)(w-u_3)})}\right),$$

and if

$$\begin{cases} w(\pi^{k_3-k_1}w-u_1) \sim \alpha \\ (w-u_2)(w-u_3) \sim \alpha, \end{cases}$$

then $(\pi, x(x-\pi^{k_1}u_1))_{p(x)} \not\sim 1$.

Assume that $k_1 < k_3$, then one can take $w = -\alpha u_1 a^2$, where (a, b) is a point of a curve $y^2 = \alpha(x^2 + u_2/(\alpha u_1))(x^2 + u_3/(\alpha u_1))$.

If $k_1 = k_2 = k_3$, then we have a system

$$\begin{cases} w(w-u_1) \sim \alpha \\ (w-u_2)(w-u_3) \sim \alpha\,. \end{cases}$$

The existence of w in this case can be established by inspection of points on the curve defined by the following system in view of lemmas 5 and 6.

$$\begin{cases} y^2 = \bar{\alpha} x(x-\bar{u}_1) \\ z^2 = \bar{\alpha}(x-\bar{u}_2)(w-\bar{u}_3). \end{cases}$$

If $k_2 \neq k_3$, then the non-triviality of the algebra $(\pi, x(x-\pi^{k_2}u_2))$ can be checked as in the previous cases.

Let $k_2 = k_3$ and $k_1 < k_2$. If, as above, $\theta = w\pi^{k_3}$, then we have

$$f(\theta) \sim \pi^{k_3} w(-\pi^{k_1}u_1)\pi^{k_3}(w-u_2)\pi^{k_3}(w-u_3) \sim -wu_1(w-u_2)(w-u_3)$$

and for the minimal polynomial $p(x)$ of θ over k

$$(\pi, x(x-\pi^{k_2}u_2))_{p(x)} \sim \left(\frac{\pi\ ,\ w(w-u_2)}{k(w)\langle p\rangle(\sqrt{f(\theta)})}\right).$$

The non-triviality of the algebra $(\pi, x(x-\pi^{k_2}u_2))_{p(x)}$ now will follow from the existence of solution of the system

$$\begin{cases} w(w-u_2) \sim \alpha \\ -u_1(w-u_3) \sim \alpha \end{cases}$$

and $k(w)|k$ is of odd prime degree or its degree is 1. The existence of such w as usual can be deduced by inspection of some points of the reduced curve.

Let us show that $(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)) \not\sim 1$. If $k_2 \neq k_3$, then

$$(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))_x \sim (\pi, \pi^{k_1+k_2}u_1u_2) \sim (\pi, u_1u_2) \not\sim 1.$$

Let $k_2 = k_3$ and $k_1 < k_3$ (if $k_1 = k_2$, then one can prove everything as in one of the previous cases since $(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2)) \sim (\pi, x(x-\pi^{k_3}u_3))$).

Similarly as in the above considerations one can find a unit w such that $k(w)|k$ is of odd prime degree and such that

$$\begin{cases} w(w-u_3) \sim \alpha \\ -u_1(w-u_2) \sim \alpha. \end{cases}$$

Then, for the minimal polynomial $p(x)$ of $\theta = \pi^{k_3}w$ we have

$$(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))_{p(x)} \not\sim 1,$$

so that $(\pi, (x-\pi^{k_1}u_1)(x-\pi^{k_2}u_2))$ is also non-trivial.

Thus, $\langle [(\pi, x)],\ [(\pi, x - \pi^{k_1}u_1)],\ [(\pi, x - \pi^{k_2}u_2)]\rangle$ is a group of order 8. Since the extension of constants by $\sqrt{\pi}$ leaves us in case II1, 3, we conclude that $[(\alpha, x)]$ does not belong to this group and the proof of case II1 is finished.

In case II2 the algebras (π, x), $(\pi, x - \pi^{k_i}u_i)$, $(\pi, x(x+e))$ and $(\pi, (x+e)(x - \pi^{k_i}u_i))$ are non-trivial by the lemma 3.

Let us consider case II2, 1. There exists $i \in \{1, 2\}$ such that k_i is odd. Then we have

$$(\alpha, x)_{x-\pi^{k_i}u_i} \not\sim 1,\ \ (\alpha, x - \pi^{k_3}u_3)_{x-\pi^{k_i}u_i} \not\sim 1,$$

$$(\alpha, x(x - \pi^{k_3}u_3))_x \sim (\alpha, (x+e)(x - \pi^{k_1}u_1)(x - \pi^{k_3}u_3))_x \sim (\pi, \pi^{k_1+k_2}) \not\sim 1,$$

so that $\langle [(\alpha, x)],\ [(\alpha, x - \pi^{k_3}u_3)]\rangle$ is of order 4. From the above considerations it follows that $\langle [(\pi, x)],\ [(\pi, x+e)]\rangle$ is also of order 4. After extension of constants to $k(\sqrt{\alpha})$ we are in case II1, 1, so that

$$(\pi, x) \otimes k(\sqrt{\alpha})(x, \sqrt{f(x)}) \not\sim 1 \text{ and}$$

$$(\pi, x(x+e)) \otimes k(\sqrt{\alpha})(x, \sqrt{f(x)}) \sim (\pi, x) \otimes k(\sqrt{\alpha})(x, \sqrt{f(x)}) \not\sim 1,$$

and therefore $[(\pi, x)],\ [(\pi, x(x+e))] \notin \langle [(\alpha, x)],\ [(\alpha, x - \pi^{k_3}u_3)]\rangle$. To complete the inspection of case II2, 1 we need only to prove that $[(\pi, x+e)] \notin \langle [(\alpha, x)],\ [(\alpha, x - \pi^{k_3}u_3)]\rangle$.

Let k_j is even, $j \in \{1, 2\}$. Then

$$(\pi, x+e)_{x-\pi^{k_j}u_j} \sim (\pi, e) \not\sim 1,\ (\alpha, x)_{x-\pi^{k_j}u_j} \sim (\alpha, \pi^{k_j}u_j) \sim 1,$$

$$(\alpha, x - \pi^{k_3}u_3)_{x-\pi^{k_j}u_j} \sim 1,\ (\alpha, x(x - \pi^{k_3}u_3))_{x-\pi^{k_j}u_j} \sim 1,$$

and case II2, 1 is done.

In case II2, 1, to see that the group $\langle [(\pi, x - \pi^{k_1}u_1)],\ [(\pi, x+e)],\ [(\pi, x)]\rangle$ has order 8, it is enough to prove that $(\pi, x(x - \pi^{k_1}u_1)(x+e)) \not\sim 1$ and $(\pi, x(x - \pi^{k_1}u_1)) \not\sim 1$. This follows from the above considerations. But the extension of constants by $\sqrt{\alpha}$ leads us to case II1, 2 and we saw that in this case $(\pi, x+e) \sim 1$, $(\pi, x(x - \pi^{k_1}u_1)) \not\sim 1$ and therefore $(\pi, x(x - \pi^{k_1}u_1)(x+e)) \not\sim 1$ over $k(C)$.

Let us prove now that $(\alpha, x) \not\sim 1$. If $k_2 \not\equiv k_3(2)$ and k_i ($i \in \{2, 3\}$) is odd, then $(\alpha, x)_{x-\pi^{k_i}u_i} \sim (\alpha, \pi^{k_i}) \not\sim 1$, therefore, without loss of generality, we can assume k_1, k_2, k_3 to be even ($k_2 < k_3$). Then

$$f(\pi^{k_2+1}) \sim \pi^{k_2+1}(-\pi^{k_1}u_1)(-\pi^{k_2}u_2)\pi^{k_2+1}e \sim u_1u_2e \sim 1 \text{ and}$$

$$(\alpha, x)_{x-\pi^{k_2+1}} \sim \left(\frac{\alpha\ ,\ \pi^{k_2+1}}{k\langle x - \pi^{k_2+1}\rangle(\sqrt{f(\pi^{k_2+1})})}\right) \sim \left(\frac{\alpha\ ,\ \pi}{k\langle x - \pi^{k_2+1}\rangle}\right) \not\sim 1.$$

To complete the consideration of case II2, 2, we need to prove that $[(\alpha, x)] \notin \langle [(\pi, x - \pi^{k_1}u_1)],\ [(\pi, x+e)],\ [(\pi, x)]\rangle$.

First of all, $(\alpha, x) \not\sim (\pi, x)$, because the extension of constants by $\sqrt{\alpha}$ leads us to case II1, 2, where $(\pi, x) \not\sim 1$. Furthermore, the same extension of constants shows that $(\alpha, x) \not\sim (\pi, x(x+e))$, since in case II1, 2 we have $(\pi, x+e) \sim 1$. By the same way we have

$$(\alpha, x) \not\sim (\pi, x - \pi^{k_1}u_1),\ (\alpha, x) \not\sim (\pi, (x - \pi^{k_1}u_1)(x+e)),$$

$$(\alpha, x) \not\sim (\pi, x(x - \pi^{k_1}u_1)) \text{ and } (\alpha, x) \not\sim (\pi, x(x+e)(x - \pi^{k_1}u_1)).$$

Consider the algebra $(\pi, x+e)$. If, among k_1, k_2, k_3, there exists an even k_i, then $(\alpha, x)_{x-\pi^{k_i}u_i} \sim 1$, but $(\pi, x+e)_{x-\pi^{k_i}u_i} \sim (\pi, e) \not\sim 1$. Let now all k_i be odd. Then, in view of $k_2 + 1 < k_3$, we have $f(\pi^{k_2+1}) \sim u_1u_2e \sim 1$ and $(\alpha, x)_{x-\pi^{k_2+1}} \sim (\alpha, \pi^{k_2+1}) \sim 1$, $(\pi, x+e)_{x-\pi^{k_2+1}} \sim (\pi, e) \not\sim 1$ and again $(\alpha, x) \not\sim (\pi, x+e)$. This completes the consideration of case II2, 2.

Consider case II2, 3. Let $l_0 = x$, $l_j = x - \pi^{k_j}u_j$ $(j = 1, 2, 3)$. Let us observe that all the algebras listed in II2, 3 are non-trivial because of lemma 3, so to prove that the group generated by them is of order 16, it is enough to check that all of them are pairwise non-isomorphic.

First of all it is true for $(\pi, l_i(x+e))$ because of lemma 3. For the algebra (π, l_0l_3), if $k_2 \neq k_3$, we have $(\pi, l_0l_3)_x \sim (\pi, u_1u_2e) \not\sim 1$. If $k_2 = k_3$, $k_1 < k_2$ one can find a unit w such that $k(w)|k$ is of odd prime degree and

$$\begin{cases} w(w-u_3) \sim \alpha \\ -u_1e(w-u_2) \sim \alpha. \end{cases}$$

Then, for the minimal polynomial $p(x)$ of $\pi^{k_3}w$ over k, one has $(\pi, l_0l_3)_{p(x)} \not\sim 1$. In the case $k_1 = k_2 = k_3$ one can use a unit w such that

$$\begin{cases} w(w-u_3) \sim \alpha \\ (w-u_1)(w-u_2) \sim \alpha. \end{cases}$$

Then again $(\pi, l_0l_3)_{p(x)} \not\sim 1$. Similar arguments prove that (π, l_0l_1) and (π, l_0l_2) are non-trivial in the case $k_2 < k_3$. Indeed, in the first case one can use a unit w such that

$$\begin{cases} w \sim -u_1\alpha \\ w - u_3 \sim -u_2 \end{cases}$$

and in the second case one can use a unit w such that

$$\begin{cases} w \sim -u_2\alpha \\ w - u_3 \sim -u_1. \end{cases}$$

Then, in the both cases, for the minimal polynomial $p(x)$ of $\pi^{k_3}w$ over k one will have $(\pi, l_0l_1)_{p(x)} \not\sim 1$ and $(\pi, l_0l_2)_{p(x)} \not\sim 1$. Let now $k_2 = k_3$. Then, in the case of the algebra (π, l_0l_1), we need to use a unit w,

$$\begin{cases} w \sim -\alpha u_1 \\ (w-u_2)(w-u_3) \sim 1 \end{cases}$$

if $k_1 < k_2$ and

$$\begin{cases} w(w-u_1) \sim \alpha \\ (w-u_2)(w-u_3) \sim 1 \end{cases}$$

if $k_1 = k_2 = k_3$. For the algebra (π, l_0l_2) one can work similarly.

Consider the algebras (π, l_1l_3), (π, l_2l_3), (π, l_1l_2). After replacing $x' = x - \pi^{k_3}u_3$, we will again be in case II2, 3 and with new notations these algebras (π, l_1l_3), (π, l_2l_3) will look like the algebras $(\pi, l_0'l_1')$, $(\pi, l_0'l_2')$, where $l_0' = x'$ and $l_i' = l_i + \pi^{k_3}u_3$. But the last algebras are non-trivial in view of the above arguments. In the case $k_2 = k_3$ for the

algebra $(\pi, l_1 l_2)$ one can renumerate l_1, l_2, l_3 such that $l_2' = l_3,\ l_3' = l_2,\ l_1' = l_1$, so that $(\pi, l_1 l_2) = (\pi, l_1' l_3')$ and the last algebra is non-trivial. Finally, let $k_1 < k_3$. Then

$$(\pi, l_1 l_2)_{x-\pi^{k_1}u_1} \sim (\pi, l_0 l_3(x+e))_{x-\pi^{k_1}u_1} \sim (\pi, e) \not\sim 1.$$

Now, the theorem is proved.

If $f(x)$ is of the form II$'$ we have the following statement.

Theorem 5 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* II$'$. *Then the class of (π, α) and the classes of the following algebras generate ${}_2\mathrm{Br}\, C$.*

II$'$1. $(\alpha, x),\ (\alpha, x-\pi^{k_1}u_1),\ (\pi, x-\pi^{k_1}u_1),\ (\pi, x)$.

II$'$2, 1. $(\alpha, x),\ (\pi, x-\pi^{k_1}u_1),\ (\pi, x),\ (\pi, x+e)$.

II$'$2, 2. $(\pi, x-\pi^{k_1}u_1),\ (\pi, x-\pi^{k_1}u_1'),\ (\pi, x+e),\ (\pi, x),\ u_1' = u_1 + \pi^s v$.

Proof. As in the proof of the theorem 4 one can easily prove that $(\alpha, x+e) \not\sim 1$ for any $e \in O_k^*$ and $(\pi, x+e) \not\sim 1$ if and only if $e \in O_k^* \setminus (O_k^*)^2$. Similarly to this proof one can check that the algebras $(\pi, x),\ (\pi, x-\pi^{k_1}u_1),\ (\pi, x-\pi^{k_1}u_1'),\ (\pi, x-\pi^{k_3}u_3)$ are non-trivial.

If $e \sim 1$, then one has as in the proof of the theorem 4, that the algebras $(\alpha, x - \pi^{k_1}u_1),\ (\alpha, x)$ are non-trivial. Hence all the algebras (except (α, x) in case II$'2, 1$) listed in case II$'$ are non-trivial.

Consider case II$'$1. First of all, observe that $(\alpha, l_0 l_1) \not\sim 1$. Indeed, if $k_1 \not\equiv k_3(2)$, then

$$(\alpha, l_0 l_1)_x \sim (\alpha, (x-\pi^{k_1}u_1')l_3(x+e))_x \sim (\alpha, \pi^{k_1+k_3}) \not\sim 1.$$

If $k_1 \equiv k_3(2)$, then since $f(\pi^{k_1+1}) \sim e \sim 1$ we have

$$(\alpha, l_0 l_1)_{x-\pi^{k_1+1}} \sim \left(\frac{\alpha\ ,\ \pi}{k\langle x - \pi^{k_1+1}\rangle}\right) \not\sim 1.$$

As to the algebra $(\pi, l_0 l_1)$, one can find a unit w generating an odd degree extension over k such that

$$\begin{cases} w \sim (-1)^s \alpha u_1 \\ w - v \sim (-1)^s \alpha u_1. \end{cases}$$

Let $\theta = \pi^{k_1}(u_1 + \pi^s v),\ p = Irr_{k(w)|k}(\theta)$. Then

$$\begin{aligned} f(\theta) &\sim \pi^{k_1}u_1(\pi^{k_1}u_1 + \pi^{k_1+s}w - \pi^{k_1}u_1)(\pi^{k_1}u_1 + \pi^{k_1+s}w - \pi^{k_1}u_1 - \pi^{k_1+s}v) \\ &\sim w(w-v) \sim 1 \end{aligned}$$

and

$$(\pi, l_0 l_1)_p \sim (\pi, \pi^s w u_1) \sim (\pi, (-1)^s u_1 w) \sim (\pi, \alpha) \not\sim 1.$$

Thus, the groups $\langle[(\alpha, l_0)],\ [(\alpha, l_1)]\rangle$ and $\langle[(\pi, l_0)],\ [(\pi, l_1)]\rangle$ are of order 4. And since the extension of constants by $\sqrt{\alpha}$ leaves us in case II$'$1, these two groups have trivial intersection, so case II$'$1 is done.

Now we are in case II$'$2. The extension of k by $\sqrt{\alpha}$ leads us to case II$'$1 and splits the algebra $(\pi, x+e)$, hence the group $\langle[(\pi, l_0)],\ [(\pi, l_1)],\ [(\pi, x+e)]\rangle$ is of order 8.

Now if $i \in \{1,3\}$ and k_i is odd, then $(\alpha,x)_{x-\pi^{k_i}u_i} \sim (\alpha,\pi^{k_i}) \not\sim 1$ and therefore $(\alpha,x) \not\sim 1$. To complete the consideration of case II$'2,1$, we only need to prove that $[(\alpha,x)] \notin \langle [(\pi,l_0)],\ [(\pi,l_1)],\ [(\pi,x+e)]\rangle$. The extension of constants by $\sqrt{\alpha}$ shows us that $[(\alpha,x)] \notin \langle [(\pi,l_0)],\ [(\pi,l_1)]\rangle$ and $(\alpha,x) \not\sim (\pi,l_0(x+e))$, $(\alpha,x) \not\sim (\pi,l_1(x+e))$ and $(\alpha,x) \not\sim (\pi,l_0l_1(x+e))$. Furthermore, for even $k_i \in \{k_2,k_3\}$ $(\alpha,x)_{l_i} \sim 1$ and $(\pi,x+e)_{l_i} \not\sim 1$, so $(\alpha,x) \not\sim (\pi,x+e)$. Case II$'2,1$ is done.

Consider case II$'2.2$. In this case it is enough to prove that all listed algebras with the algebra $(\pi, x-\pi^{k_3}u_3)$ are pairwise non-isomorphic. Furthermore, $(\pi,l_i(x+e)) \not\sim 1$ and $(\pi,l_1'(x+e)) \not\sim 1$ as was noted before. We also have

$$(\pi,l_0l_3)_x \sim (\pi,l_1l_1'(x+e))_x \sim (\pi,e) \not\sim 1,$$

so that $(\pi,l_0l_3) \not\sim 1$. One can also prove similarly to the previous case that $(\pi,l_0l_1) \not\sim 1$ and $(\pi,l_0l_1') \not\sim 1$. To prove that $(\pi,l_1l_3) \not\sim 1$ (similarly $(\pi,l_1'l_3) \not\sim 1$) it is enough to observe that, after the replacement $x' = x-\pi^{k_3}u_3$, our algebra looks like $(\pi,\tilde{l_0}\tilde{l_1})$, where $\tilde{l_0} = x'$ and $\tilde{l_1} = l_1 + \pi^{k_3}u_3$. Finally,

$$(\pi,l_1l_1')_{l_1} \sim (\pi,l_0l_3(x+e))_{l_1} \sim (\pi,e) \not\sim 1.$$

Hence the theorem is proved.

If $f(x)$ is of the form III we have

Theorem 6 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* III. *Then the class of (π,α) and the classes of the following algebras generate ${}_2\mathrm{Br}\,C$.*

III1. $(\alpha,x),\ (\alpha,x-\pi^{k_2}u_2),\ (\pi,x+e),\ (\pi,x+e-\pi^lv)$.

III2. $(\alpha,x),\ (\alpha,x-\pi^{k_2}u_2),\ (\pi,x+e),\ (\alpha,x+e)$.

III3. $(\alpha,x),\ (\pi,x),\ (\pi,x+e),\ (\pi,x+e-\pi^lv)$.

III4. $(\alpha,x),\ (\pi,x),\ (\pi,x+e),\ (\alpha,x+e)$.

III5. $(\pi,x),\ (\pi,x-\pi^{k_2}u_2),\ (\pi,x+e),\ (\pi,x+e-\pi^lv)$.

III6. $(\pi,x),\ (\pi,x-\pi^{k_2}u_2),\ (\pi,x+e),\ (\alpha,x+e)$.

Proof. Let $G_1 = \langle [(\pi,x+e)],\ [(\pi,x+e')]\rangle$, where $e' = e-\pi^lv$. Let us prove that, in cases III1, III3, III5, the group G_1 is of order 4. Indeed, in these cases $(\pi,x+e) \not\sim 1$ and $(\pi,x+e') \not\sim 1$ according to lemma 3. Furthermore,

$$(\pi,(x+e)(x+e))_{x+e} \sim (\pi,x(x-\pi^{k_1}u_1)(x-\pi^{k_1}u_1))_{x+e} \sim (\pi,-e) \not\sim 1.$$

Now let $G_2 = \langle [(\pi,x+e)],\ [(\alpha,x+e)]\rangle$, then in the cases III2, III4, III6 G_2 is also of order 4. Indeed, $(\alpha,x+e) \not\sim 1$ by lemma 4 and $(\pi,x+e) \not\sim 1$ by lemma 3. And $(\alpha,x+e) \not\sim (\pi,x+e)$ since $k(\sqrt{\alpha})(x,\sqrt{f(x)})$ does not split $(\pi,x+e)$.

In cases III1, III2 let $G_3 = \langle [(\alpha,x)],\ [(\alpha,x-\pi^{k_2}u_2)]\rangle$. Then G_3 is of order 4. Indeed,

$$(\alpha,x)_{x-\pi^{k_1}u_1} \sim (\alpha,\pi^{k_1}) \not\sim 1,\ (\alpha,x-\pi^{k_2}u_2)_{x-\pi^{k_1}u_1} \sim (\alpha,\pi) \not\sim 1,$$

$$(\alpha,x(x-\pi^{k_2}u_2))_x \sim (\alpha,(x-\pi^{k_1}u_1)(x+e)(x+e'))_x \sim (\alpha,-\pi^{k_1}u_1ee') \not\sim 1.$$

In cases III3, III4 let $G_4 = \langle [(\alpha, x)],\ [(\pi, x)] \rangle$ and let us show that G_4 is of order 4. For the algebra (α, x) we have: if k_2 is odd, then

$$(\alpha, x)_{x-\pi^{k_2}u_2} \sim (\alpha, \pi^{k_2}u_2) \not\sim 1.$$

If k_2 is even, then

$$f(\pi^{k_1+1}) \sim \pi^{k_1+1}(-\pi^{k_1}u_1)\pi^{k_1+1} \sim -u_1 \sim 1$$

and

$$(\alpha, x)_{x-\pi^{k_1+1}} \sim \left(\frac{\alpha\ ,\ \pi}{k\langle x - \pi^{k_1+1} \rangle} \right) \not\sim 1.$$

As for the algebra (π, x) one can find a unit w such that $k(w)|k$ is an unramified odd prime degree extension and

$$\begin{cases} w \sim \alpha \\ w - u_1 \sim 1. \end{cases}$$

In view of $f(\pi^{k_1}w) \sim w - u_1$ we have

$$(\pi, x)_{p(x)} \sim \left(\frac{\pi\ ,\ \pi^{k_1}w}{k(w)\langle p \rangle} \right) \not\sim 1,$$

where $p(x)$ is a minimal polynomial of $\pi^{k_1}w$ over k. Now, since $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ does not split (π, x), we have $(\pi, x) \not\sim (\alpha, x)$.

Finally, let $G_5 = \langle [(\pi, x)],\ [(\pi, x - \pi^{k_2}u_2)] \rangle$ in cases III5 and III6. We prove that G_5 is of order 4. To prove that (π, x) is non-trivial it is enough to repeat the previous arguments with the last w. Similarly, with the same w in the case $k_1 < k_2$, we have for the algebra $(\pi, x - \pi^{k_2}u_2)$

$$(\pi, x - \pi^{k_2}u_2)_{x-\pi^{k_1}w} \sim (\pi, x)_{x-\pi^{k_1}w} \not\sim 1.$$

If $k_1 = k_2$, let w satisfy the following property

$$\begin{cases} w - u_2 \sim \alpha \\ w(w - u_1) \sim \alpha. \end{cases}$$

Then $(\pi, x - \pi^{k_2}u_2)_{x-\pi^{k_1}w} \not\sim 1$. For the algebra $(\pi, x(x - \pi^{k_2}u_2))$ in the case $k_1 \neq k_2$ we have

$$(\pi, x(x - \pi^{k_2}u_2))_x \sim (\pi, (x - \pi^{k_1}u_1)(x + e)(x + e))_x \sim (\pi, -u_1) \not\sim 1.$$

If $k_1 = k_2$, one can find a unit w with the property

$$\begin{cases} w(w - u_2) \sim \alpha \\ w - u_1 \sim \alpha. \end{cases}$$

Then $(\pi, x(x - \pi^{k_2}u_2))_{x-\pi^{k_1}w} \not\sim 1$. Thus G_5 is of order 4.

To complete the proof of the theorem it remains to show that all the following groups are trivial

1. $G_1 \cap G_3$ in case III1,

2. $G_2 \cap G_3$ in case III2,

3. $G_1 \cap G_4$ in case III3,

4. $G_2 \cap G_4$ in case III4,

5. $G_1 \cap G_5$ in case III5,

6. $G_2 \cap G_5$ in case III6.

In case III6 the extension of constants by $\sqrt{\pi}$ leaves us in this case, so that $[(\alpha, x)] \notin G_5$ and $[(\alpha, x) \otimes (\pi, x+e)] \notin G_5$. Let $g(x) \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)$. Then $(\pi, g(x)(x+e)) \not\sim 1$ in view of $\bar{g}(x)(x+\bar{e})$, $\bar{f}(x)/(\bar{g}(x)(x+\bar{e})) \notin \bar{k}[x]^2$ and by lemma 3. This implies that $G_2 \cap G_5$ is trivial.

In case III4 since the extension of constants by $\sqrt{\alpha}$ leaves us in case III4, then to prove that $G_2 \cap G_4$ is trivial it is enough to prove that $(\alpha, x(x+e)) \not\sim 1$ and $(\pi, x(x+e)) \not\sim 1$. But $(\alpha, x(x+e)) \not\sim 1$, since $(\bar{\alpha}, x(x+\bar{e})) \not\sim 1$ and $(\pi, x(x+e)) \not\sim 1$, since $x(x+\bar{e})$, $\bar{f}/(x(x+\bar{e})) \notin \bar{k}[x]^2$.

In case III2, since the extension of constants by $\sqrt{\alpha}$ leaves us in this case, we have $[(\pi, x+e)]$, $[(\pi\alpha, x+e)] \notin G_3$. And it remains to show that $[(\alpha, x+e)] \notin G_3$. But for any $g \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)\}$ $(\alpha, g(x+e)) \not\sim 1$ since $(\bar{\alpha}, \bar{g}(x+\bar{e})) \not\sim 1$.

In case III1, the extension of constants by $\sqrt{\alpha}$ leads us to the case III2, since $[(\pi, x+e)] \notin G_3$. The same is true for $[(\pi, x+e')]$. Now it remains to prove that $[(\pi, (x+e)(x+e'))] \notin G_3$. We have $(\alpha, x)_{x-\pi^{k_1}u_1} \sim (\alpha, x-\pi^{k_2}u_2)_{x-\pi^{k_1}u_1} \not\sim 1$ but $(\pi, (x+e)(x+e'))_{x-\pi^{k_1}u_1} \sim (\pi, ee') \sim 1$. Furthermore,

$$(\alpha, x(x-\pi^{k_2}u_2))_x \sim (\alpha, (x-\pi^{k_1}u_1)(x+e)(x+e'))_x \sim (\alpha, -\pi^{k_1}u_1 ee') \not\sim 1$$

and $(\pi, (x+e)(x+e'))_x \sim (\pi, ee') \sim 1$. Thus $[(\pi, (x+e)(x+e'))] \notin G_3$.

In case III3, the extension of constants by $\sqrt{\alpha}$ leads us to the case III4. So this extension does not split the algebras $(\pi, x+e)$, $(\pi, x+e')$. Let us show that $(\alpha, x) \not\sim (\pi, (x+e)(x+e'))$. In case k_2 is odd we have

$$(\pi, (x+e)(x+e'))_{x-\pi^{k_2}u_2} \sim 1 \text{ and } (\alpha, x)_{x-\pi^{k_2}u_2} \not\sim 1.$$

If k_2 is even, we have

$$(\pi, (x+e)(x+e'))_{x-\pi^{k_1+1}} \sim 1 \text{ and } (\alpha, x)_{x-\pi^{k_1+1}} \not\sim 1.$$

Thus $[(\alpha, x)] \notin G_1$. We have

$$(\pi, x) \not\sim (\pi, x+e) \text{ and } (\pi, x) \not\sim (\pi, x+e'),$$

since $x(x+\bar{e}), \bar{f}(x)/(x(x+\bar{e})) \notin \bar{k}[x]^2$. Let w be a unit such that $(\pi, x)_{x-\pi^{k_1}w} \not\sim 1$, then $(\pi, (x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$. This implies $[(\pi, x)] \notin G_1$. The remaining possibility is $[(\pi\alpha, x)] \in G_1$. Observe firstly that $(\pi\alpha, x)_{x-\pi^{k_1}w} \not\sim 1$, hence $(\pi\alpha, x) \not\sim (\pi, (x+e)(x+e'))$. Furthermore, if $(\pi\alpha, x) \sim (\pi, x+e)$, then $(\alpha, x) \sim (\pi, x(x+e))$. But $k(\sqrt{\alpha})(x, \sqrt{f(x)})$ splits (α, x) and does not split $(\pi, x(x+e))$. Similarly, $(\pi\alpha, x) \not\sim (\pi, x+e')$, so $G_1 \cap G_4$ is trivial.

Now pass to the remaining case III5. If $g \in \{x, x-\pi^{k_2}u_2, x(x-\pi^{k_2}u_2)\}$, then $(\pi, g(x+e) \not\sim 1$ by lemma 3. Thus $[(\pi, x+e)] \notin G_5$. Similarly $[(\pi, x+e')] \notin G_5$. Let $(\pi, g) \sim (\pi, (x+e)(x+e'))$. On the other hand, if w is a unit such that $(\pi, x)_{x-\pi^{k_1}w} \not\sim 1$, then,

since $(\pi,(x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$, we have $g \neq x$. In the same way one can prove that $g \neq x-\pi^{k_2}u_2$. Finally, if $g = x(x-\pi^{k_2}u_2)$, then if $k_1 = k_2$ for a unit w such that $(\pi, x(x-\pi^{k_2}u_2))_{x-\pi^{k_1}w} \not\sim 1$, we have $(\pi,(x+e)(x+e'))_{x-\pi^{k_1}w} \sim 1$, and if $k_1 \neq k_2$, then $(\pi,(x+e)(x+e'))_x \sim 1$ and $(\pi, x(x-\pi^{k_2}u_2))_x \not\sim 1$. Thus $[(\pi,(x+e)(x+e'))] \notin G_4$. The theorem is proved.

If $f(x)$ is of the form IV we have the following statement.

Theorem 7 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* IV. *Then the class of (π,α) and the classes of the following algebras generate ${}_2\mathrm{Br}\,C$.*

IV1. $(\pi,x),\ (\pi,x-\pi^{k_2}u_2),\ (\pi,x+e_1),\ (\pi,x+e_2)$.

IV1. $(\pi,x),\ (\pi,x+e_1),\ (\pi,x+e_2),\ (\alpha,x)$.

Proof. Observe that $G = \langle [(\pi,x)],\ [(\pi,x+e_1)],\ [(\pi,x+e_2)],\ \rangle$ is of order 8. First of all $x, x+\bar{e}_1, x+\bar{e}_2, x(x+\bar{e}_1), x(x+\bar{e}_2), (x+\bar{e}_1)(x+\bar{e}_2), \bar{f}(x)/x, \bar{f}(x)/(x+\bar{e}_1), \bar{f}(x)/(x+\bar{e}_2), \bar{f}(x)/(x(x+\bar{e}_1)), \bar{f}(x)/(x(x+\bar{e}_2)), \bar{f}(x)/((x+\bar{e}_1)(x+\bar{e}_2)) \notin \bar{k}[x]^2$ and by lemma 3. $(\pi,x) \not\sim 1$, $(\pi,x+e_1) \not\sim 1$, $(\pi,x+e_2) \not\sim 1$, $(\pi,x(x+e_1)) \not\sim 1$, $(\pi,x(x+e_2)) \not\sim 1$, $(\pi,(x+e_1)(x+e_2)) \not\sim 1$, so G is of order 8.

If one of the k_i is odd, we have $(\alpha,x)_{x-\pi^{k_i}u_i} \not\sim 1$. If $k_1 \equiv k_2 \equiv 0(2)$, $k_1 < k_2$ and $-u_1e_1e_2 \in (k^*)^2$, then $k_1 < k_1+1 < k_2$ and $(\alpha,x)_{x-\pi^{k_1+1}} \not\sim 1$, since $f(\pi^{k_1+1}) \sim -u_1e_1e_2$. Thus in case IV2 $(\alpha,x) \not\sim 1$. Since the extension of constants by $\sqrt{\alpha}$ leaves us in case IV, we conclude that $[(\alpha,x)] \notin G$.

Consider case IV1. If $k_1 \neq k_2$, we have

$$(\pi, x(x-\pi^{k_2}u_2))_x \sim (\pi,(x-\pi^{k_1}u_1)(x+e_1)(x+e_2))_x \sim (\pi,-u_1e_1e_2) \not\sim 1.$$

If $k_1 = k_2$, let w be a unit such that $k(w)|k$ is an unramified prime degree extension and

$$\begin{cases} w(w-u_2) \sim \alpha \\ w-u_1 \sim \alpha e_1e_2. \end{cases}$$

Then $(\pi, x(x-\pi^{k_2}u_2))_{x-\pi^{k_2}w} \not\sim 1$. This implies $(\pi,x) \not\sim (\pi, x-\pi^{k_2}u_2)$. By lemma 3, the algebra $(\pi, x-\pi^{k_2}u_2)$ is also non-isomorphic to $(\pi,x+e_1), (\pi,x+e_2), (\pi,x(x+e_2)), (\pi,x(x+e_2)), (\pi,x(x+e_1)(x+e_2))$. Finally, let $k_1 \neq k_2$. Then

$$(\pi,(x-\pi^{k_2}u_2)(x+e_1)(x+e_2))_{x-\pi^{k_1}w} \sim (\pi, x(x-\pi^{k_1}u_1))_{x-\pi^{k_1}w} \not\sim 1$$

if w is a unit with the above properties and

$$\begin{cases} w \sim \alpha e_1e_2 \\ w-u_1 \sim e_1e_2. \end{cases}$$

If $k_1 = k_2$, we only need to change the latter conditions by

$$\begin{cases} w(w-u_1) \sim \alpha \\ w-u_2 \sim \alpha e_1e_2. \end{cases}$$

In any case, $(\pi, x-\pi^{k_2}u_2) \not\sim (\pi,(x+e_1)(x+e_2))$. This implies $[(\pi,x-\pi^{k_2}u_2)] \notin G$ and completes the proof of the theorem.

If $f(x)$ is of the form V we have

Theorem 8 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* V. *Then the class of (π, α) and the classes of the following algebras generate ${}_2\mathrm{Br}\, C$.*

V1. (π, x), $(\pi, x - \pi^{k_1} v_1)$, $(\pi, x + u)$, $(\pi, x + u - \pi^{k_2} v_2)$.

V2. (π, x), $(\pi, x - \pi^{k_1} v_1)$, $(\pi, x + u)$, $(\alpha, x + u)$.

V3. (π, x), (α, x), $(\pi, x + u)$, $(\pi, x + u - \pi^{k_2} v_2)$.

V2. (π, x), (α, x), $(\pi, x + u)$, $(\alpha, x + u)$.

Proof. Let l_1, l_2, l_3, l_4, l_5 be the different linear monic divisors of $f(x)$. If $l_i \neq x + e$ we have, by lemma 3 and in view of $\bar{l}_i, \bar{f}(x)/\bar{l}_i \notin \bar{k}[x]^2$, that $(\pi, l_i) \not\sim 1$. For the same reason, $(\pi, l_i l_j) \not\sim 1$ if $\bar{l}_i \neq \bar{l}_j$. Let now $e \not\sim 1$, then $(\pi, x(x - \pi^{k_1} v_1))_{x + \alpha v_1 \pi^{k_1+2}} \not\sim 1$, since $f(-\alpha v_1 \pi^{k_1+2}) \sim \alpha e \sim 1$. Thus if $e \not\sim 1$ then $(\pi, x(x - \pi^{k_1} v_1)) \not\sim 1$. The change of variables $x' = x + u$ leads us to the equation

$$y^2 = x'(x' - \pi^{k_2} v_2)(x' - u)(x' - u - \pi^{k_1} v_1)(x' + (e - u)),$$

and we have $(\pi, (x+u)(x+u-\pi^{k_2} v_2)) = (\pi, x'(x' - \pi^{k_2} v_2))$, so, by the previous arguments, $(\pi, (x + u)(x + u - \pi^{k_2} v_2)) \not\sim 1$. Observe also that, if $e \sim 1$, then

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(x, \sqrt{\bar{f}(x)})} \right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x \rangle} \right) \not\sim 1.$$

It follows by lemma 4 that in the case $e \sim 1$ $(\alpha, x) \not\sim 1$. Similarly if $e - u \sim 1$, then $(\alpha, x + u) \not\sim 1$.

Now, to complete the consideration of case VI we need only to prove that $(\pi, l_i l_j l_r) \not\sim 1$, where l_i, l_j, l_r are not equal to each other and to $x + e$ and $(\pi, x(x - \pi^{k_1} v_1)(x + u)(x + u - \pi^{k_2} v_2)) \not\sim 1$ if $e \not\sim 1$. But the last statement is equivalent to $(\pi, x + e) \not\sim 1$ which holds in view of $(\pi, x + e)_x \not\sim 1$. As to the previous one, we have $\bar{l}_i \bar{l}_j \bar{l}_r$, $\bar{f}(x)/(\bar{l}_i \bar{l}_j \bar{l}_r) \notin \bar{k}[x]^2$, so lemma 3 can be applied.

In case V2 it is enough to observe that, using the above arguments, the group $\langle [(\pi, x)],\ [(\pi, x - \pi^{k_1} v_1)],\ [(\pi, x + u)] \rangle$ is of order 8, and the field $k(\sqrt{\pi})(x, \sqrt{f(x)})$ does not split the algebra $(\alpha, x + u)$.

The case V3 can be handled similarly as case V2.

We saw that the group $\langle [(\pi, x)],\ [(\pi, x + u)] \rangle$ is of order 4. To prove that also $\langle [(\alpha, x)],\ [(\alpha, x + u)] \rangle$ is of order 4 we need to prove that $(\alpha, x(x + u)) \not\sim 1$. But this is true in view of lemma 4, since

$$\left(\frac{\bar{\alpha}\ ,\ x(x + \bar{u})}{\bar{k}(x, \sqrt{\bar{f}(x)})} \right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x \rangle} \right) \not\sim 1.$$

Finally, since the extension of constants by $\sqrt{\alpha}$ leaves us in case V4, the groups $\langle [(\pi, x)],\ [(\pi, x + u)] \rangle$ and $\langle [(\alpha, x)],\ [(\alpha, x + u)] \rangle$ have trivial intersection. The theorem is proved.

If $f(x)$ is of the form VI we have

Theorem 9 *Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is of the form* VI. *Then the class of (π, α) and the classes of the following algebras generate ${}_2\mathrm{Br}\, C$.*

VI1. $\{(\pi, l_i)\}_i$, *l_i is a monic linear divisor of f.*

VI2. $\{(\pi, l_i)\}_i$, *l_i is a monic linear divisor of f, $l_i \neq x - \pi^k u$ and (α, x).*

Proof. In case VI1 it is enough to check that for any monic linear divisor of $f(x)$ $(\pi, l_i) \not\sim 1$ and $(\pi, l_i l_j) \not\sim 1$ if $i \neq j$. We have $(\pi, l_i) \not\sim 1$ by lemma 3. For the same reason, $(\pi, l_i l_j) \not\sim 1$ if $l_i l_j \neq x(x - \pi^\kappa u)$. If $l_i l_j = x(x - \pi^\kappa u)$ we have $(\pi, x(x - \pi^\kappa u))_x \sim (\pi, e_1 e_2 e_3) \not\sim 1$.

In case V2, arguments similar as above show that the group H generated by $[(\pi, l_i)]$ is of order 8, where l_i runs through the set of all monic linear divisors not equal to $x - \pi^\kappa u$. To complete the consideration of this case, let us show that $(\alpha, x) \not\sim 1$. If κ is odd we have $(\alpha, x)_{x - \pi^\kappa u} \not\sim 1$, if not, then $(\alpha, x)_{x-\pi} \not\sim 1$, since $f(\pi) \sim \pi(\pi - \pi^\kappa u) e_1 e_2 e_3 \sim e_1 e_2 e_3 \sim 1$. Finally, the extension of constants by $\sqrt{\alpha}$ leaves us in case VI2, so all non-trivial algebras from H are non-trivial after this extension but this is not true for (α, x). The theorem is proved.

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Quaternion Generation of the 2-Torsion Part of the Brauer Group of Local Quintic (Type of Splitting (1,1,1,2))

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1 Introduction

This paper is a continuation of [3], where we considere a quintic given by an affine equation $y^2 = f(x)$, and $f(x)$ has the splitting type $(1,1,1,1,1)$. Here we consider the case where the splitting type of $f(x)$ is $(1,1,1,2)$. We use the same notations as in the previous paper.

2 Preliminary results

Let k be a local field of characteristic 0 with residue field $\bar{k}$. Let $\pi \in k$ denote a prime element, and let O_k, O_k^*, πO_k be its valuation ring, the group of units and the maximal ideal of the valuation ring. For $u \in O_k^*$ let $\bar{u}$ denote its class *modulo* πO_k. We will consider also a hyperelliptic curve C defined over k by the equation $y^2 = f(x)$, where $f(x) \in O_k[x]$ is a monic polynomial without multiple roots and $deg\, f = n$. Without loss of generality one can assume that $f(x) = \prod_{i=1}^{m} f_i(x)$, where $f_i(x) \in O_k[x]$ are monic irreducible polynomials. This paper is devoted to the case $\deg f_1 = \deg f_2 = \deg f_3 = 1, \deg f_4 = 2$ and the reduction is bad. Let α be a fixed unit of O_k which is not a square in k. $\bar{g}(x)$ will denote the reduction of a polynomial $g(x) \in O_k$. For $a, b \in k^*$ we write $a \sim b$ if a and b belong to the same class *modulo* $(k^*)^2$.

In our further considerations we will need lemmas from [3].

The evident list of all cases under consideration according to the reduction type of $f(x)$ is as follows.

I. $f(x) = (x^2 - \pi^\kappa u)(x - e_1)(x - e_2)(x - e_3)$,
$\pi^\kappa u \not\sim 1,\ u, e_i \in O_k^*,\ \kappa > 0,\ \bar{e}_i \neq \bar{e}_j$ if $i \neq j$.

II. $f(x) = (x^2 - \pi^\kappa u)(x - e)(x - e')(x - e_1)$,
$\pi^\kappa u \not\sim 1,\ u, e, e', e_1 \in O_k^*,\ \kappa > 0,\ \bar{e} = \bar{e}' \neq \bar{e}_1$.

III. $f(x) = (x^2 - \pi^\kappa u)(x - e_1)(x - e_2)(x - e_3)$,
$u \in O_k^*,\ \kappa > 0,\ \pi^\kappa u \not\sim 1,\ \bar{e}_1 = \bar{e}_2 = \bar{e}_3 \neq 0$.

IV. $f(x) = (x^2 + ax + b)x(x - \pi^\kappa u)(x + e)$,
$u, e \in O_k^*,\ \kappa > 0,\ x^2 + \bar{a}x + \bar{b}$ is irreducible over $\bar{k}$.

V. $f(x) = (x^2 + ax + b)x(x - \pi^{k_1}u_1)(x - \pi^{k_2}u_2)$,
$u_i \in O_k^*,\ 0 < k_1 \le k_2$, and if $k_1 = k_2$, then $\bar{u}_1 \ne \bar{u}_2$,
$x^2 + \bar{a}x + \bar{b}$ is irreducible over $\bar{k}$.

VI. $f(x) = (x^2 - \pi^{\kappa}u)(x - \pi^m v)(x + e_1)(x + e_2)$,
$\pi^{\kappa}u \not\sim 1,\ \kappa, m > 0,\ u, v, e_i \in O_k^*,\ \bar{e}_1 \ne \bar{e}_2$.

VII. $f(x) = (x^2 - \pi^{\kappa}u)(x - \pi^m v)(x + e)(x + e')$,
$\pi^{\kappa}u \not\sim 1,\ \kappa, m, l > 0,\ u, v, e \in O_k^*,\ \bar{e} = \bar{e}'$.

VIII. $f(x) = (x^2 - \pi^{\kappa}u)(x - \pi^{m_1}v_1)(x - \pi^{m_2}v_2)(x + e)$,
$\pi^{\kappa}u \not\sim 1,\ 0 < m_1 \le m_2,\ \kappa > 0,\ u, v_i, e \in O_k^*$.

IX. $f(x) = (x^2 - \pi^{\kappa}u)(x - \pi^{m_1}v_1)(x - \pi^{m_2}v_2)(x - \pi^{m_3}v_3)$,
$\pi^{\kappa}u \not\sim 1,\ 0 < m_1 \le m_2 \le m_3,\ \kappa > 0,\ u, v_i \in O_k^*$.

Let us first prove some preliminary results.

Lemma 1

$$| \,{}_2\mathrm{Br}\, C \,| = 16.$$

For the proof see theorem 1 (p. 236) and §4 (p. 237) in [2].

Lemma 2 *Let $u \in O_k^*$ be a unit such that $u \not\sim 1$, then*
i) *for any $w \in O_k^*$ we have $w \pm \sqrt{u} \sim 1$ in $k(\sqrt{u})$ iff $w^2 - u \sim 1$ in k,*
ii) *for any local field k there exists $w \in O_k^*$ such that $w^2 - u \not\sim 1$,*
iii) *for any $v \in O_k^*$ there exists $w \in O_k^*$ such that $v^2 - w \in O_k^*$ and $v^2 - w \not\sim 1$.*

Proof. i) follows from the surjectivity on units of the norm homomorphism $N_{k(\sqrt{u})|k}$, and for ii) and iii) see [1], lemma 1.

Lemma 3 *Let $g(x) \in k[x]$ be a monic polynomial, and let either n be odd or $\deg g(x)$ be even. Then, for any $b \in k(C)^*$, the quaternion algebra*

$$\left(\frac{b\ ,\ g(x)}{k(C)}\right)$$

is not isomorphic to the scalar algebra

$$\left(\frac{\pi\ ,\ \alpha}{k(C)}\right).$$

To check this fact it is enough to observe that

$$\left(\frac{b\ ,\ g(x)}{k(C)}\right)_{\infty} \sim 1 \text{ and } \left(\frac{\pi\ ,\ \alpha}{k(C)}\right)_{\infty} \not\sim 1.$$

Lemma 4 *Let K be an algebraically closed field and $f(x), g(x), h(x) \in K[x]$. Then the system*

$$\begin{cases} y^2 = f(x), \\ z^2 = g(x), \\ t^2 = h(x) \end{cases}$$

defines a variety V *in* $\boldsymbol{P}^4(K)$ *if and only if all polynomials* $f(x)$, $g(x)$, $h(x)$, $f(x)g(x)$, $f(x)h(x)$, $g(x)h(x)$ *and* $f(x)g(x)h(x)$ *are not in* $K[x]^2$.

Proof. Consider the homomorphism

$$\phi: \ K[x,y,z,t] \longrightarrow K(x, \sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)}),$$

defined by the rule

$$x \mapsto x, \ y \mapsto \sqrt{f(x)}, \ z \mapsto \sqrt{g(x)}, \ t \mapsto \sqrt{h(x)}.$$

Consider the ideal $I = (y^2 - f(x), z^2 - g(x), t^2 - h(x)) \subset Ker\,\phi$. Let $F = F_0(x) + F_1(x)y + F_2(x)z + F_3(x)t + F_4(x)yz + F_5(x)yt + F_6(x)zt + G(x,y,z,t) \in Ker\,\phi$, where $G(x,y,z,t) \in I$. So $\phi(F) = F_0(x) + F_1(x)\sqrt{f} + F_2(x)\sqrt{g} + F_3(x)\sqrt{h} + F_4(x)\sqrt{fg} + F_5(x)\sqrt{fh} + F_6(x)\sqrt{gh} = 0$. Let us check that the extension

$$K(x)(\sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)})|K(x)$$

is of degree 8. Indeed, $K(x)(\sqrt{f(x)})|K(x)$ is of degree 2. If $\sqrt{g(x)} \in K(x)(\sqrt{f(x)})$, then $g(x) = a(x)^2 + b(x)^2 f(x) + 2a(x)b(x)\sqrt{f(x)}$, where $a(x), b(x) \in K(x)$. So $a(x)b(x) = 0$ and either $g(x) = a(x)^2$ or $g(x) = b(x)^2 f(x)$. And this is impossible since $g(x) \notin K[x]^2$ and $f(x)g(x) \notin K[x]^2$. Hence $K(x)(\sqrt{f(x)}, \sqrt{g(x)})|K(x)$ is of degree 4. Let now $\sqrt{h(x)} \in K(x)(\sqrt{f(x)}, \sqrt{g(x)})$, i.e., $\sqrt{h(x)} = a_0(x) + a_1(x)\sqrt{f(x)} + a_2(x)\sqrt{g(x)} + a_3(x)\sqrt{f(x)g(x)}$. We have $h = (a_0^2 + a_1^2 f + a_2^2 g + a_3^2 fg) + 2(a_0a_1 + a_2a_3 g)\sqrt{f} + 2(a_0a_2 + a_1a_3 f)\sqrt{g} + 2(a_0a_3 + a_1a_2)\sqrt{fg}$. Therefore

$$\begin{cases} a_0a_1 + a_2a_3 g = 0, \\ a_0a_2 + a_1a_3 f = 0, \\ a_0a_3 + a_1a_2 = 0. \end{cases}$$

We have $h(x) \notin K(x)(\sqrt{f(x)g(x)})$, so that one of a_1, a_2, say, a_1 is not equal to 0. Then

$$\begin{cases} a_2 = -a_0a_3/a_1, \\ a_0(a_3^2 g - a_1^2) = 0, \\ a_3(a_1^2 f - a_0^2) = 0, \end{cases}$$

Furthermore, $a_3^2 g - a_1^2 \neq 0$, since $a_1 \neq 0$ and $g(x) \notin K[x]^2$. Then $a_0 = a_2 = a_3 = 0$ and $\sqrt{h(x)} \in K(x)(\sqrt{f(x)})$. This contradiction proves that the extension

$$K(x)(\sqrt{f(x)}, \sqrt{g(x)}, \sqrt{h(x)})|K(x)$$

is of degree 8. The evident basis of this extension is 1, $\sqrt{f(x)}$, $\sqrt{g(x)}$, $\sqrt{h(x)}$, $\sqrt{f(x)g(x)}$, $\sqrt{f(x)h(x)}$, $\sqrt{g(x)h(x)}$, so $F_i(x) = 0$ for any i and $Ker\,\phi = I$. Then I is prime and V is a variety.

Lemma 5 *Let $f(x) = (x^2 - \pi^\kappa u)g(x)$, $g(x) \in O_k[x]$ without multiple roots, $\bar{g}(x) \notin \bar{k}[x]^2$, $u \in O_k^*$, $\kappa > 0$, $\pi^\kappa u \not\sim 1$ and $E = g(0) \in O_k^*$. Then the following algebra is nontrivial, unramified and not isomorphic to the scalar algebra (π, α).*

1. $(-\pi E u, x)$ *if* $\kappa \equiv 1 \pmod 2$.
2. $(\alpha, x - \pi^{\kappa/2} w)$ *if* $\kappa \equiv 0 \pmod 2$ *and* $E \sim 1$, *where* $w \in O_k^*$, $w^2 - u \not\sim 1$.
3. $(\pi, x^2 - \pi^\kappa u)$ *if* $\kappa \equiv 0 \pmod 2$ *and* $E \not\sim 1$.

Proof.

$$(-\pi E u, x)_x \sim \left(\frac{-\pi E u \ , \ x}{k\langle x \rangle (\sqrt{-\pi E u})} \right) \sim 1,$$

so this algebra is unramified. Let K be an unramified odd degree extension of k such that there exists a unit $\tau \in O_K^*$ with the property $\tau \not\sim 1$, $g(\tau) \sim 1$ and $p(x) = \mathrm{Irr}_{K|k}(\tau)$. Such K exists in view of lemma 6 from [3]. Then

$$(-\pi E u, x)_p \sim \left(\frac{-\pi E u \ , \ \tau}{K\langle p \rangle (\sqrt{g(\tau)})} \right) \sim \left(\frac{\pi \ , \ \tau}{K\langle p \rangle} \right) \not\sim 1.$$

The unit w in 2. exists because of lemma 2, ii). We have

$$(\alpha, x - \pi^{\kappa/2} w)_{x - \pi^{\kappa/2} w} \sim \left(\frac{\alpha \ , \ x - \pi^{\kappa/2} w}{k\langle x - \pi^{\kappa/2} w \rangle (\sqrt{(\pi^\kappa w^2 - \pi^\kappa u) E})} \right) \sim$$

$$\sim \left(\frac{\alpha \ , \ x - \pi^{\kappa/2} w}{k\langle x - \pi^{\kappa/2} w \rangle (\sqrt{\alpha})} \right) \sim 1,$$

so the algebra $(\alpha, x - \pi^{\kappa/2} w)$ is unramified. Furthermore,

$$(\alpha, x - \pi^{\kappa/2} w)_{x^3 - \pi} \sim \left(\frac{\alpha \ , \ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3 - \pi \rangle (\sqrt{(\pi^{2/3} - \pi^\kappa u) E})} \right) \sim \left(\frac{\alpha \ , \ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3 - \pi \rangle} \right) \not\sim 1.$$

From [3], Lemma 2 we conclude that the algebra $(\pi, x^2 - \pi^\kappa u)$ is unramified. Moreover,

$$(\pi, x^2 - \pi^\kappa u)_{x - \pi^{\kappa/2} w} \sim \left(\frac{\pi \ , \ \alpha}{k\langle x - \pi^{\kappa/2} w \rangle (\sqrt{\alpha E})} \right) \sim \left(\frac{\pi \ , \ \alpha}{k\langle x - \pi^{\kappa/2} w \rangle} \right) \not\sim 1.$$

To complete the proof, it is enough to check that all algebras are not isomorphic to the scalar algebra, and this is true because of lemma 3 of this paper.

3 Presentation by algebras

Theorem 1 *Let f be as in cases* I *or* II. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras.*

- *Case* I.
  1. $(e_1 e_1 e_3 u \pi, x)$, $(\pi, x - e_1)$, $(\pi, x - e_2)$, *if* $\kappa \equiv 1 \pmod 2$.

2. $(\alpha, x-\pi^{\kappa/2}w),\ (\pi, x-e_1),\ (\pi, x-e_2),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1e_2e_3 \sim 1.$

3. $(\pi, x^2-\pi^{\kappa}u),\ (\pi, x-e_1),\ (\pi, x-e_2),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1e_2e_3 \not\sim 1.$

- *Case* II.

1. $(e_1u\pi, x),\ (\pi, x-e),\ (\pi, x-e'),$ *if* $\kappa \equiv 1 \pmod 2,\ e-e_1 \not\sim 1.$
2. $(e_1u\pi, x),\ (\pi, x-e),\ (\alpha, x-e),$ *if* $\kappa \equiv 1 \pmod 2,\ e-e_1 \sim 1.$
3. $(\alpha, x-\pi^{\kappa/2}w),\ (\pi, x-e),\ (\pi, x-e'),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1 \sim 1,\ e-e_1 \not\sim 1.$
4. $(\alpha, x-\pi^{\kappa/2}w),\ (\pi, x-e),\ (\alpha, x-e),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1 \sim 1,\ e-e_1 \sim 1.$
5. $(\pi, x^2-\pi^{\kappa}u),\ (\pi, x-e),\ (\pi, x-e'),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1 \not\sim 1,\ e-e_1 \not\sim 1.$
6. $(\pi, x^2-\pi^{\kappa}u),\ (\pi, x-e),\ (\alpha, x-e),$ *if* $\kappa \equiv 0 \pmod 2,\ -e_1 \not\sim 1,\ e-e_1 \sim 1,$

where $w \in O_k^*,\ w^2-u \not\sim 1.$

Proof. In case I all algebras $(\pi, x-e_i)$ and $(\pi, (x-e_i)(x-e_j)), i \neq j$ are unramified and not isomorphic to (π, α) in view of lemma 2 from [3]. They are nontrivial by lemma 3 from [3].

In case II let us prove that if $e-e_1 \not\sim 1$, then the algebras $(\pi, x-e),\ (\pi, x-e')$ and $(\pi, (x-e)(x-e'))$ are unramified, nontrivial and not isomorphic to (π, α). If $e-e_1 \sim 1$, then the same is true for the algebras $(\pi, x-e),\ (\alpha, x-e)$ and $(\pi\alpha, x-e)$. Indeed, if $e-e_1 \not\sim 1$, then $(\pi, (x-e)(x-e'))_{x-e} \sim (\pi, (x^2-\pi^{\kappa}u)(x-e_1))_{x-e} \sim (\pi, e-e_1) \not\sim 1$. Let now $e-e_1 \sim 1$, then

$$(\bar{\alpha}, x-\bar{e})_{x-\bar{e}} \sim \left(\frac{\bar{\alpha}\ ,\ x-\bar{e}}{\bar{k}\langle x-\bar{e}\rangle(\sqrt{\bar{e}-\bar{e}_1})}\right) \not\sim 1,$$

and in view of lemma 4 from [3] we find $(\alpha, x-e) \not\sim 1$. Using the preceding argument over $k(\sqrt{\pi})$, we obtain over this field $(\pi\alpha, x-e) \sim (\alpha, x-e) \not\sim 1$, and similarly, using $k(\sqrt{\pi\alpha})$, we find $(\pi, x-e) \not\sim 1$.

Since the polynomial f in cases I and II satisfies the conditions of lemma 3, it is enough to prove that the algebra from this lemma is not isomorphic to the algebras already considered in the current proof.

Consider firstly case I. If $\kappa \equiv 1 \pmod 2$, then one can find w_1, w_2 and w_3 from O_K^* (K as above) such that $w_i \sim 1,\ w_i-e_i \not\sim 1,\ (w_i-e_1)(w_i-e_2)(w_i-e_3) \sim 1$. Indeed, for example, in case $i=1$ it is enough to solve the following system

$$\begin{cases} w_1 = x^2, \\ y^2 = \alpha(x^2-e_1) \\ z^2 = \alpha(x^2-e_2)(x^2-e_3), \end{cases}$$

and lemma 6 from [3] works. Let $p_i = \mathrm{Irr}_{K|k}(w_i)$. Then, for any i, we have $(-\pi Eu, x)_{p_i} \sim 1$, $(\pi, x-e_i)_{p_i} \not\sim 1$ and $(\pi, (x-e_1)(x-e_2))_{p_3} \not\sim 1$.

If $E = -e_1e_2e_3 \sim 1$, then the algebra $(\alpha, x-\pi^{\kappa/2}w)$ splits after extension of constants by $\sqrt{\alpha}$, but this is not true for the algebras $(\pi, x-e_i)$ and $(\pi, (x-e_1)(x-e_2))$.

Let $\kappa \equiv 0 \pmod 2,\ -e_1e_2e_3 \not\sim 1$, then $(\pi, x^2-\pi^{\kappa}u) \not\sim (\pi, x-e_i),\ (\pi, (x-e_1)(x-e_2))$ in view of lemma 3 from [3].

Consider now case II. The extension by $\sqrt{\alpha}$ does not split $(-\pi Eu, x)$, therefore $(-\pi Eu, x) \not\sim (\alpha, x-e)$. Let $w \in O_K^*$ such that $w \sim 1,\ w-e \not\sim 1$ and $(w-e)(w-e')(w-$

$e_1) \sim w - e_1 \sim 1$, $p = \mathrm{Irr}_{K|k}(w)$. Then $(-\pi Eu, x)_p \sim 1$ and $(\pi, x-e)_p \not\sim 1$. Changing the prime element $\nu = \pi\alpha$ one can check that $(-\pi Eu, x) \not\sim (\pi\alpha, x-e)$. In the same way we have $(-\pi Eu, x) \not\sim (\pi, x-e')$. Let us show finally that $(-\pi Eu, x) \not\sim (\pi, (x-e)(x-e'))$. If $q = \mathrm{Irr}_{K|k}(w)$, where $w \not\sim 1$, $\bar{w} \neq \bar{e}$ and $w - e_1 \sim 1$, then $(-\pi Eu, x)_q \not\sim 1$, and $(\pi, (x-e)(x-e'))_q \sim (\pi, (w-e)^2) \sim 1$.

Let now $\kappa \equiv 0 \pmod 2$, and $E \sim -e_1 \sim 1$. We have $(\alpha, x - \pi^{\kappa/2}w) \not\sim (\pi, x-e)$, $(\pi, x-e')$, $(\pi, (x-e)(x-e'))$, since the extension of constants by $\sqrt{\pi}$ does not split the first algebra but splits the others. After this extension $(\pi\alpha, x-e) \sim (\alpha, x-e)$. It only remains to check that $(\alpha, x - \pi^{\kappa/2}w) \not\sim (\alpha, x-e)$ if $e - e_1 \sim 1$. This is true since $(\alpha, x - \pi^{\kappa/2}w)_{x^3-\pi} \not\sim 1$, but $(\alpha, (x-e)(x-e'))_{x^3-\pi} \sim 1$.

Finally, let $\kappa \equiv 0 \pmod 2$, $-e_1 \not\sim 1$. We have $(\pi, (x^2 - \pi^\kappa u)(x-e)), (\pi, (x^2-\pi^\kappa u)(x-e')) \not\sim 1$. Moreover, $(\pi, (x^2 - \pi^\kappa u)(x-e)(x-e'))_{x-e} \sim (\pi, e - e_1) \not\sim 1$ if $e - e_1 \not\sim 1$. The extension of constants by $\sqrt{\pi}$ does not split $(\alpha, x-e)$, so that $(\pi, x^2 - \pi^\kappa u) \not\sim (\alpha, x-e)$. After the same extension $(\pi\alpha, x-e) \sim (\alpha, x-e) \not\sim 1$, so $(\pi\alpha, x-e) \not\sim (\pi, x^2 - \pi^\kappa u)$. The theorem is proved.

Without loss of generality one can assume that in case III $f(x) = (x^2 - \pi^\kappa u)(x - e_1)(x-e_2)(x-e_3)$, where $e_1 = e$, $e_2 = e + \pi^{k_1}u_1$, $e_3 = e + \pi^{k_2}u_2$, $e, u_i \in O_k^*$, $0 < k_1 \leq k_2$ and if $k_1 = k_2$, then $\bar{u}_1 \neq \bar{u}_2$.

Theorem 2 *Let f be as in case* III. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras.*

1. $(eu\pi, x)$, $(\alpha, x-e_1)$, $(\alpha, x-e_2)$,
   if $\kappa \equiv 1 \pmod 2$, $k_1 \equiv 1 \pmod 2$.
2. $(eu\pi, x)$, $(\alpha, x-e_1)$, $(\pi, x-e_1)$,
   if $\kappa \equiv 1 \pmod 2$, $k_1 \equiv 0 \pmod 2$ *and either* $k_2 \equiv 1 \pmod 2$ *or* $k_1 < k_2$, $-u_1 \sim 1$.
3. $(eu\pi, x)$, $(\pi, x-e_1)$, $(\pi, x-e_2)$,
   if $\kappa \equiv 1 \pmod 2$, $k_1 \equiv k_2 \equiv 0 \pmod 2$ *and either* $k_1 = k_2$ *or* $-u_1 \not\sim 1$.
4. $(\alpha, x - \pi^{\kappa/2}w)$, $(\alpha, x-e_1)$, $(\alpha, x-e_2)$,
   if $\kappa \equiv 0 \pmod 2$, $-e \sim 1$, $k_1 \equiv 1 \pmod 2$.
5. $(\alpha, x - \pi^{\kappa/2}w)$, $(\alpha, x-e_1)$, $(\pi, x-e_1)$,
   if $\kappa \equiv 0 \pmod 2$, $-e \sim 1$, $k_1 \equiv 0 \pmod 2$ *and either* $k_2 \equiv 1 \pmod 2$ *or* $k_1 < k_2$, $-u_1 \sim 1$.
6. $(\alpha, x - \pi^{\kappa/2}w)$, $(\pi, x-e_1)$, $(\pi, x-e_2)$,
   if $\kappa \equiv 0 \pmod 2$, $-e \sim 1$, $k_1 \equiv k_2 \equiv 0 \pmod 2$ *and either* $k_1 = k_2$ *or* $-u_1 \not\sim 1$.
7. $(\pi, x^2 - \pi^\kappa u)$, $(\alpha, x-e_1)$, $(\alpha, x-e_2)$,
   if $\kappa \equiv 0 \pmod 2$, $-e \not\sim 1$, $k_1 \equiv 1 \pmod 2$.
8. $(\pi, x^2 - \pi^\kappa u)$, $(\alpha, x-e_1)$, $(\pi, x-e_1)$,
   if $\kappa \equiv 0 \pmod 2$, $-e \not\sim 1$, $k_1 \equiv 0 \pmod 2$ *and either* $k_2 \equiv 1 \pmod 2$ *or* $k_1 < k_2$, $-u_1 \sim 1$.

9. $(\pi, x^2 - \pi^\kappa u)$, $(\pi, x - e_1)$, $(\pi, x - e_2)$,
if $\kappa \equiv 0 \pmod 2$, $-e \not\sim 1$, $k_1 \equiv k_2 \equiv 0 \pmod 2$ *and either* $k_1 = k_2$ *or* $-u_1 \not\sim 1$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$.

Let us prove firstly

Lemma 6 *In case* III, *the following algebras are unramified, nontrivial and not isomorphic to* (π, α).

1. $(\alpha, x - e_1)$, $(\alpha, x - e_2)$, $(\alpha, (x - e_1)(x - e_2))$, *if* $k_1 \equiv 1 \pmod 2$.
2. $(\alpha, x - e_1)$, $(\pi, x - e_1)$, $(\alpha\pi, x - e_1)$, *if* $k_1 \equiv 0 \pmod 2$ *and either* $k_2 \equiv 1 \pmod 2$ *or* $(k_2 \equiv 0 \pmod 2$, $k_1 < k_2$ *and* $-u_1 \sim 1)$.
3. $(\pi, x - e_1)$, $(\pi, x - e_2)$, $(\pi, (x - e_1)(x - e_2))$, *if* $k_1 \equiv k_2 \equiv 0 \pmod 2$ *and either* $k_1 < k_2$ *and* $-u_1 \sim 1$ *or* $k_1 = k_2$.

Proof of the lemma. In 1 we have:
$(\alpha, x - e_1)_{x-e_2} \not\sim 1$, $(\alpha, x - e_2)_{x-e_1} \not\sim 1$, $(\alpha, (x - e_1)(x - e_2))_{x-e_2} \sim (\alpha, x - e_3)_{x-e_2} \not\sim 1$.
2. Let $\theta = e + \pi^{k_1} w$. Then $f(\theta) \sim w - u_1$. If w satisfies the condition

$$\begin{cases} w_1 \not\sim 1, \\ w - u_1 \sim 1, \end{cases}$$

then $(\pi, x - e_1)_p \not\sim 1$ and $(\pi\alpha, x - e_1)_p \not\sim 1$, where $p = \mathrm{Irr}_{K|k}(\theta)$. Furthermore, in case $k_2 \equiv 1 \pmod 2$ we have $(\alpha, x - e_1)_{x-e_3} \not\sim 1$ and $(\alpha, x - e_1)_{x-e-\pi^{k_1+1}} \sim (\alpha, \pi^{k_1+1}) \not\sim 1$ since $f(e + \pi^{k_1+1}) \sim -u_1 \sim 1$, otherwise.

Finally, consider the subcase 3. Let $\theta_i = e + \pi^{k_1} w_i$, $i = 1, 2, 3$. Then $f(\theta_i) \sim w_i(w_i - u_1)(w_i - u_2)$, provided that $k_1 = k_2$. If the elements w_i satisfy the conditions

$$\begin{cases} w_1 \not\sim 1, \\ w_2 - u_1 \not\sim 1, \\ w_3 - u_2 \not\sim 1, \\ (w_1 - u_1)(w_1 - u_2) \not\sim 1, \\ w_2(w_2 - u_2) \not\sim 1, \\ w_3(w_3 - u_3) \not\sim 1, \end{cases}$$

then $f(\theta_i) \sim 1$ and $(\pi, x - e_i)_{p_i} \not\sim 1$, where $p_i = \mathrm{Irr}_{K|k}(\theta_i)$. Moreover, $(\pi, (x - e_i)(x - e_j))_{p_l} \sim (\pi, (x^2 - \pi^\kappa u)(x - e_l))_{p_l} \sim (\pi, x - e_l)_{p_l} \not\sim 1$, where $\{i, j, l\} = \{1, 2, 3\}$.

Let now $k_1 < k_2$ be even and $-u_1 \not\sim 1$. Let θ be as in 2. Then $(\pi, x - e_1)_p \not\sim 1$ and $(\pi, (x - e_1)(x - e_2))_p \not\sim 1$, where p is as above. We have also $(\pi, x - e_2)_{x-e_1} \sim (\pi, -u_1) \not\sim 1$. The lemma is proved.

Now to prove the theorem it is enough to check that the algebra from lemma 3 is not isomorphic to all algebras from lemma 5 in the corresponding subcases.

Let firstly $\kappa \equiv 1 \pmod 2$. Extension of constants by $\sqrt{\alpha}$ leaves us in case 1 of lemma 3, and in cases 1 and 2 of lemma 4, therefore this extension does not split the algebra $(eu\pi, x)$. On the other hand, it splits all algebras from case 1 of lemma 5. And in case 2 of this lemma one only needs to prove $(eu\pi, x) \not\sim (\pi, x - e_1)$ and $(eu\pi, x) \not\sim (\pi\alpha, x - e_1)$. But

if w, θ and p are as in the proof of this case, then, after the extension indicated above, $(eu\pi, x)_p \sim (\pi, e) \sim 1$. On the other hand, $(\pi\alpha, x - e_1)_p \sim (\pi, x - e_1)_p \not\sim 1$.

Let us take a unit $\eta \in O_K^*$ with the usual properties satisfying the condition $\bar{\eta} \neq \bar{e},\ \eta \not\sim 1,\ \eta - e \sim 1$. Then $f(\eta) \sim \eta - e \sim 1$ and $(eu\pi, x)_q \not\sim 1$, where $q = \mathrm{Irr}_{K|k}(\eta)$. We also have $(\pi, x - e_i)_q \sim 1$ and $(\pi, (x - e_1)(x - e_2))_q \sim 1$. Thus, the case $\kappa \equiv 1 \pmod 2$ is done.

If κ is even and $-e \sim 1$, then, for any $A \in \{\pi, \alpha, \pi\alpha\}$, we have $(A, x - e_i)_{x^3 - \pi} \sim (A, -e) \sim 1$, $(A, (x - e_1)(x - e_1))_{x^3 - \pi} \sim 1$, but $(\alpha, x - \pi^{\kappa/2} w)_{x^3 - \pi} \not\sim 1$.

Finally, let κ be even, $-e \not\sim 1$, then $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{\kappa/2} w} \not\sim 1$ and $(\alpha, x - e_i)_{x - \pi^{\kappa/2} w} \sim 1$, $(\alpha, (x - e_1)(x - e_2))_{x - \pi^{\kappa/2} w} \sim 1$. If p_i are the points from the proof of lemma 5 in which the corresponding algebras are nontrivial, then $(\pi, x^2 - \pi^\kappa u)_{p_i} \sim 1$ and we are done.

Let us pass to the next two cases.

Theorem 3 *Let f be as in cases* IV *or* V. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebras (π, α), $(\pi, x^2 + ax + b)$, and of the following algebras.*

- *Case* IV.
  1. (π, x), $(\pi, x - \pi^\kappa u)$, *if* $be \not\sim 1$.
  2. (π, x), (α, x), *if* $be \sim 1$.
- *Case* V.
  1. (α, x), $(\alpha, x - \pi^{k_1} u_1)$, *if* $k_1 \equiv 1 \pmod 2$.
  2. (α, x), (π, x), *if* $k_1 \equiv 0 \pmod 2$ *and either* $k_2 \equiv 1 \pmod 2$ *or* $k_1 < k_2$ *and* $-bu_1 \sim 1$.
  3. (π, x), $(\pi, x - \pi^{k_1} u_1)$, *if* $k_1 \equiv k_2 \equiv 0 \pmod 2$ *and either* $k_1 = k_2$ *or* $-bu_1 \not\sim 1$.

Proof. In case IV $(\pi, x) \not\sim 1$, $(\pi, x - \pi^\kappa u) \not\sim 1$ by lemma 3 from [3]. For the same reason, $(\pi, x) \not\sim 1$ in V.2, V.3 and $(\pi, x - \pi^{k_1} u_1) \not\sim 1$ in V.3. If $be \not\sim 1$ in IV, then, for $w \in O_K^*$, we have $f(\pi^\kappa w) \sim \alpha w(w - e)$ and $(\pi, x(x - \pi^\kappa u))_{\mathrm{Irr}_{K|k}(\pi^\kappa w)} \sim (\pi, w(w - e)) \not\sim 1$, provided that $w(w - e) \not\sim 1$. If $be \sim 1$, then

$$\left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}(\tilde{C})} \right)_x \sim \left(\frac{\bar{\alpha}\ ,\ x}{\bar{k}\langle x \rangle} \right) \not\sim 1,$$

so $(\alpha, x) \not\sim 1$. Replacing $\nu = \pi\alpha$, one can check $(\pi\alpha, x) \not\sim 1$.

If $k_1 \equiv 1 \pmod 2$ in case V, then $(\alpha, x)_{x - \pi^{k_1} u_1} \not\sim 1$ and $(\alpha, x - \pi^{k_1} u_1)_x \not\sim 1$. Moreover, $(\alpha, x(x - \pi^{k_1} u_1))_{x - \pi^{k_1} u_1} \sim (\alpha, (x^2 + ax + b)(x - \pi^{k_2} u_2))_{x - \pi^{k_1} u_1} \sim (\alpha, b(u_1 - \pi^{k_2 - k_1} u_2)\pi^{k_1}) \not\sim 1$.

Now we are in the subcase V.2. If $k_2 \equiv 1 \pmod 2$, then $(\alpha, x)_{x - \pi^{k_2} u_2} \not\sim 1$. In case $k_2 \equiv 0 \pmod 2$ we have $f(\pi^{k_1 + 1}) \sim -bu_1 \sim 1$ and $(\alpha, x)_{x - \pi^{k_1 + 1}} \not\sim 1$. Replacing the prime element one can check that $(\alpha\pi, x) \not\sim 1$.

Consider the last subcase in V. We have $f(\pi^{k_1} w) \sim bw(w - u_1)(w - \pi^{k_2 - k_1} u_2)$ and $(\pi, x(x - \pi^{k_1} u_1))_{\mathrm{Irr}_{K|k}(\pi^{k_1} w)} \sim (\pi, w(w - u_1)) \not\sim 1$, provided that

$$\begin{cases} w(w - u_1) \not\sim 1, \\ b(w - \pi^{k_2 - k_1} u_2) \not\sim 1. \end{cases}$$

It remains to prove that the algebra (π, x^2+ax+b) is nontrivial and not isomorphic to the algebras listed above. In case IV we have $(\pi, x^2+ax+b) \not\sim 1$, $(\pi, x(x^2+ax+b)) \not\sim 1$, $(\pi, (x-\pi^\kappa u)(x^2+ax+b)) \not\sim 1$ by lemma 3 from [3]. In IV.1 $(\pi, x(x-\pi^\kappa u)(x^2+ax+b)) \not\sim 1$ for the same reason. Extension of constants by $\sqrt{\pi}$ leaves us in case IV.2, so that $(\pi, x^2+ax+b) \not\sim (\alpha, x)$. After this extension, we also have $(\pi\alpha, x) \sim (\alpha, x) \not\sim (\pi, x^2+ax+b)$.

Finally, let us compare the algebra (π, x^2+ax+b) with the others in case V. First of all, $(\pi, x^2+ax+b) \not\sim 1$ and $(\pi, (x^2+ax+b)x(x-\pi^{k_1}u_1)) \not\sim 1$ similarly as in case IV. After extension of constants by $\sqrt{\alpha}$ we are not in the same case because the polynomial x^2+ax+b splits after this extension. But the algebra (π, x^2+ax+b) is still nontrivial in view of lemma 3 from [3]. So this algebra is not isomorphic to (α, x), $(\alpha, x-\pi^{k_1}u_1)$ and $(\alpha, x(x-\pi^{k_1}u_1))$. To complete the proof one only needs to check that (π, x^2+ax+b) is not isomorphic to (π, x), $(\pi\alpha, x)$ in V.2 and (π, x), $(\pi, x-\pi^{k_1}u_1))$ in V.3. In V.2 we have $f(\pi^{k_1}w) \sim b(w-u_1)$, where w is as above. Let $p = \mathrm{Irr}_{K|k}(\pi^{k_1}w)$. Then $(\pi, x)_p \sim (\pi\alpha, x)_p \sim (\pi, w)$, $(\pi, x^2+ax+b)_p \sim (\pi, b)$. If we require

$$\begin{cases} b(w-u_1) \sim 1, \\ bw \not\sim 1, \end{cases}$$

then $(\pi, x)_p \sim (\pi\alpha, x)_p \not\sim (\pi, x^2+ax+b)_p$.

Consider the subcase V.3. Let $k_1 = k_2$. To prove that $(\pi, x^2+ax+b) \not\sim (\pi, x), (\pi, x-\pi^{k_1}u_1)$ we need to solve the systems

$$\begin{cases} bw_1 \not\sim 1, \\ (w_1-u_1)(w_1-u_2) \not\sim 1 \end{cases} \quad \text{and} \quad \begin{cases} b(w_2-u_1) \not\sim 1, \\ w_2(w_2-u_2) \not\sim 1. \end{cases}$$

By the same way one can prove $(\pi, x^2+ax+b) \not\sim (\pi, x)$, provided that $k_1 < k_2$. Finally, in this case $(\pi, (x^2+ax+b)(x-\pi^{k_1}u_1))_x \sim (\pi, -bu_1) \not\sim 1$.

Theorem 4 *Let f be as in cases* VI *or* VII. *Then the group ${}_2\mathrm{Br}\,C$ is generated by the classes of the algebra (π, α), and of the following algebras.*

- *Case* VI. $(\pi, x+e_1)$, $(\pi, x+e_2)$.
- *Case* VII. $(\pi, x+e)$ *and* $(\alpha, x+e)$ *if* $-e \sim 1$, $(\pi, x+e')$ *if* $-e \not\sim 1$.

and

1. $(\alpha, x^2-\pi^\kappa u)$ *if* $m \geq \kappa/2$, $\kappa \equiv 1 \pmod 2$.
2. $(\pi, x^2-\pi^\kappa u)$ *if* $m \geq \kappa/2$, $\kappa \equiv 0 \pmod 4$.
3. $(\pi E, x-\pi^{\kappa/2}w)$ *if* $m \geq \kappa/2$, $\kappa \equiv 2 \pmod 4$.
4. $(\pi uve_1e_2, x)$ *if* $m < \kappa/2$, $\kappa \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$.
5. $(\pi E, x-\pi^{\kappa/2}w)$ *if* $m < \kappa/2$, $\kappa \equiv 0 \pmod 2$, $m \equiv 1 \pmod 2$.
6. $(\alpha, x-\pi^m v)$ *if* $m < \kappa/2$, $\kappa \equiv m \equiv 1 \pmod 2$, $uve_1e_2 \sim 1$.
7. (α, x) *if* $m < \kappa/2$, $\kappa \equiv m \equiv 1 \pmod 2$, $uve_1e_2 \not\sim 1$.
8. $(\pi, x^2-\pi^\kappa u)$ *if* $m < \kappa/2$, $\kappa \equiv m \equiv 0 \pmod 2$, $-ve_1e_2 \not\sim 1$.

9. $(\alpha, x-\pi^{\kappa/2}w)$ *if* $m<\kappa/2$, $\kappa\equiv m\equiv 0\,(\mathrm{mod}\,2)$, $-ve_1e_2\sim 1$,

where $w\in O_k^*$, $w^2-u\not\sim 1$, E *has to be chosen such that* $f(\pi^{\kappa/2}w)\sim\pi E$, *and in case* VII *the units* e_1 *and* e_2 *must be omitted in all algebras and conditions listed above.*

Proof. The algebras $(\pi, x+e_1)$, $(\pi, x+e_2)$, $(\pi,(x+e_1)(x+e_2))$ are nontrivial in case VI. The same is true for the algebras $(\pi, x+e)$ and $(\pi, x+e')$ in case VII. In the last case we also have

$$\left(\frac{\bar\alpha\ ,\ x+\bar e}{\bar k(\tilde E)}\right)_{x+\bar e}\sim\left(\frac{\bar\alpha\ ,\ x+\bar e}{\bar k(\sqrt{x})}\right)_{x+\bar e}\sim\left(\frac{\bar\alpha\ ,\ x+\bar e}{\bar k\langle x+\bar e\rangle(\sqrt{-\bar e})}\right)\not\sim 1,$$

provided that $-e\sim 1$. Moreover, then the extension by $\sqrt{\alpha}$ does not split $(\pi, x+e)$, hence $(\pi\alpha, x+e)\not\sim 1$. If $-e\not\sim 1$, then $(\pi,(x+e)(x+e'))_{x+e}\sim(\pi,(x^2-\pi^\kappa u)(x-\pi^m v))_{x+e}\sim(\pi,-e)\not\sim 1$.

Let us now consider the remaining algebra simultaneously in both cases.

1. $(\alpha, x^2-\pi^\kappa u)_{x-\pi^m v}\not\sim 1$. Furthermore, $(\alpha, x^2-\pi^\kappa u)\not\sim(\pi, x+e_i), (\pi,(x+e_1)(x+e_1))$ in the case VI, since the last two algebras do not split after extension by $\sqrt{\alpha}$. For the same reason, $(\alpha, x^2-\pi^\kappa u)\not\sim(\pi, x+e)$, $(\pi, x+e')$, and $(\pi\alpha, x+e)$ in case VII. We also have $(\alpha, x+e)_{x-\pi^m v}\sim 1$, so that $(\alpha, x+e)\not\sim(\alpha, x^2-\pi^\kappa u)$ and $(\pi,(x+e)(x+e'))_{x-\pi^m v}\sim(\pi, ee')\sim 1$, therefore $(\alpha, x^2-\pi^\kappa u)\not\sim(\pi,(x+e)(x+e'))$.

2. Let $w\in O_K^*$ such that

$$\begin{cases} w^2-u\not\sim 1,\\ a(w-v\pi^{m-\kappa/2})\not\sim 1,\end{cases}\quad\text{where } a=\begin{cases} e_1e_2 \text{ in VI},\\ 1 \text{ in VII}.\end{cases}$$

Then $f(\pi^{\kappa/2}w)\sim 1$ and $(\pi, x^2-\pi^\kappa u)_p\not\sim 1$, where $p=\mathrm{Irr}_{K|k}(\pi^{\kappa/2}w)$. Let now $w_0\in O_K^*$ such that

$$\begin{cases} w_0^2-u\sim 1,\\ a(w_0-v\pi^{m-\kappa/2})\sim 1,\end{cases}$$

Then $f(\pi^{\kappa/2}w)\sim 1$ and $(\pi, x^2-\pi^\kappa u)_{p_0}\sim 1$, where $p_0=\mathrm{Irr}_{K|k}(\pi^{\kappa/2}w_0)$. In case $e_i\sim 1$ we have $(\pi, x^2-\pi^\kappa u)_p\not\sim(\pi, x+e_i)_p$ and, if $e_i\not\sim 1$, then $(\pi, x^2-\pi^\kappa u)_{p_0}\not\sim(\pi, x+e_i)_{p_0}$. In any case $(\pi, x^2-\pi^\kappa u)\not\sim(\pi, x+e_i)$. By the same way one can check that $(\pi, x^2-\pi^\kappa u)\not\sim(\pi,(x+e_1)(x+e_2))$, $(\pi, x+e)$, $(\pi, x+e')$ and $(\pi,(x+e)(x+e'))$. Finally, $(\pi, x^2-\pi^\kappa u)\not\sim(\pi\alpha, x+e)$, $(\alpha, x+e)$ since two last algebras do not split in $k(\sqrt{\pi})$.

3.

$$(\pi E, x-\pi^{\kappa/2}w)_{x^2-\pi^\kappa u}\sim\left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle D(x^2-\pi^\kappa u)\rangle}\right)\not\sim 1.$$

To finish the consideration of this subcase, it is enough to observe that the completion in $x^2-\pi^\kappa u$ splits any other algebra from the list.

4. $f(0)\sim\pi uve_1e_2$, so the algebra $(\pi uve_1e_2, x)$ is unramified. Let w from O_K^* and $q=\mathrm{Irr}_{K|k}(\pi^m w)$. Then $f(\pi^m w)\sim e_1e_2(w-v)$ in VI or $w-v$ in VII. Furthermore, $(\pi uve_1e_2, x)_q\sim(\pi uve_1e_2, w)$. If we require $w\not\sim 1$ and $e_1e_2(w-v)\sim 1$, then $(\pi uve_1e_2, x)_q\not\sim 1$. Similarly, if $we_i\not\sim 1$ and $e_1e_2(w-v)\sim 1$, then $(\pi uve_1e_2, x)_q\not\sim(\pi, x+e_i)_q$. Similarly, we can prove that $(\pi uve_1e_2, x)$ is not isomorphic to the remaining algebras.

5. This case can be considered similarly to the case 3.

6. $f(0) \sim uve_1e_2$ (or uv in VII) ~ 1. Therefore, $(\alpha, x-\pi^m v)_x \sim (\alpha, \pi^m) \not\sim 1$. The extension by $\sqrt{\alpha}$ does not split the algebras $(\pi, x+e_i)$, $(\pi, x+e)$, $(\pi, x+e')$, $(\pi, (x+e_1)(x+e_2))$, $(\pi\alpha, x+e)$. So it is sufficient to check that $(\alpha, x-\pi^m v) \not\sim (\alpha, x+e), (\pi, (x+e)(x+e'))$. But this is true in view of $(\alpha, x+e)_x \sim 1$ and $(\pi, (x+e)(x+e'))_x \sim 1$.
7. The algebra (α, x) is unramified since $f(0) \sim \alpha$. $(\alpha, x)_{x-\pi^m v} \sim (\alpha, \pi^m v) \not\sim 1$. In case VI the extension by $\sqrt{\alpha}$ does not split the algebras $(\pi, x+e_i)$, $(\pi, (x+e_1)(x+e_2))$ since this extension leaves us in case VI. The same holds for the algebras $(\pi, x+e)$, $(\pi\alpha, x+e)$ and $(\pi, x+e')$. Indeed, after this extension, $(\pi\alpha, x+e) \sim (\pi, x+e) \not\sim 1$ since we are in VII again. And $(\pi, x+e')$ is also nontrivial after the extension because the units e and e' are symmetrical. Finally, $(\alpha, x+e)_{x-\pi^m v} \sim (\alpha, e) \sim 1$ and $(\pi, (x+e)(x+e'))_{x-\pi^m v} \sim 1$.
8. $f(\pi^{\kappa/2}w) \sim -ve_1e_2(w^2-u) \sim 1$, provided that $w^2-u \not\sim 1$. Thus $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \not\sim 1$. The algebras $(\pi, (x^2-\pi^\kappa u)(x+e_i))$, $(\pi, (x^2-\pi^\kappa u)(x+e))$, $(\pi, (x^2-\pi^\kappa u)(x+e'))$, $(\pi, (x^2-\pi^\kappa u)(x+e_1)(x+e_2))$ are nontrivial by lemma 3 from [3]. We only need to prove $(\pi, x^2-\pi^\kappa u) \not\sim (\pi, (x+e)(x+e')), (\alpha, x+e), (\pi\alpha, x+e)$. But the two last algebras do not split after extension by $\sqrt{\pi}$ because this extension leaves us in case VII, $-e \sim 1$. Finally, $(\pi, (x+e)(x+e'))_{x-\pi^{\kappa/2}w} \sim 1$.
9. $f(\pi^{\kappa/2}w) \sim \alpha$, hence the algebra $(\alpha, x-\pi^{\kappa/2}w)$ is unramified. Let $q = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^m \sqrt[3]{\pi})$. Then $f(\pi^m \sqrt[3]{\pi}) \sim -ve_1e_2 \sim 1$. Hence

$$(\alpha, x-\pi^{\kappa/2}w)_q \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

Since the extension by $\sqrt{\pi}$ leaves us in the current subcase, to complete the proof one need only to check that $(\alpha, x-\pi^{\kappa/2}w) \not\sim (\alpha, x+e)$. But this holds since $(\alpha, x+e)_q \sim (\alpha, e) \sim 1$.

Theorem 5 *Let f be as in case* VIII. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α), and of the following algebras ($l_i = x - \pi^{m_i} v_i$).*

- (π, l_1), $(\pi, x+e)$ *if* $e \not\sim 1$.
- (π, l_1), (α, l_1) *if* $e \sim 1$.

and

1. $(\alpha, x^2-\pi^\kappa u)$ *if* $\kappa/2 < m_1$, $e \sim 1$, $\kappa \equiv 1 \pmod 2$.
2. $(\pi, x^2-\pi^\kappa u)$ *if* $\kappa/2 < m_1$, $e \sim 1$, $\kappa \equiv 0 \pmod 2$ *and* $-1 \sim 1$.
3. $(\alpha, x-\pi^{\kappa/2}w)$ *if* $\kappa/2 < m_1$, $e \sim 1$, $\kappa \equiv 0 \pmod 2$ *and* $-1 \not\sim 1$.
4. $(\pi, x^2-\pi^\kappa u)$ *if* $\kappa/2 < m_1$, $e \not\sim 1$, $\kappa \equiv 0 \pmod 2$ *and* $-1 \not\sim 1$.
5. (α, l_1) *if* $\kappa/2 < m_1$, $e \not\sim 1$, $\kappa \equiv 0 \pmod 2$ *and* $-1 \sim 1$.
6. (α, l_1) *if* $\kappa/2 < m_1$, $e \not\sim 1$, $\kappa \equiv 1 \pmod 2$.
7. $(\pi, x^2-\pi^\kappa u)$ *if* $\kappa/2 = m_1$ *and either* $m_1 < m_2$ *or* $\bar{v}_1 \neq \bar{v}_2$.
8. $(\pi, x^2-\pi^\kappa u)$ *if* $\kappa/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e(v_1^2-u) \not\sim 1$.
9. (α, l_1) *if* $\kappa/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e \not\sim 1$, $v_1^2 - u \not\sim 1$.

10. $(\alpha, x-\pi^{\kappa/2}w)$ *if* $\kappa/2 = m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $e \sim 1$, $v_1^2 - u \sim 1$.

11. $(\alpha, x^2-\pi^{\kappa}u)$ *if* $m_1 < \kappa/2 \le m_2$, $\kappa \equiv 1 \pmod 2$.

12. $(\pi, x^2-\pi^{\kappa}u)$ *if* $m_1 < \kappa/2 \le m_2$, $\kappa \equiv 0 \pmod 2$ *and* $m_1 + \kappa/2 \equiv 0 \pmod 2$.

13. $(\pi E, x-\pi^{\kappa/2}w)$ *if* $m_1 < \kappa/2 \le m_2$, $\kappa \equiv 0 \pmod 2$ *and* $m_1 + \kappa/2 \equiv 1 \pmod 2$.

14. $(\pi E, x-\pi^{\kappa/2}w)$ *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 1 \pmod 2$, $\kappa \equiv 0 \pmod 2$.

15. $(\alpha, x^2-\pi^{\kappa}u)$ *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 1 \pmod 2$, $\kappa \equiv 1 \pmod 2$ *and* $-ev_1v_2u \sim 1$.

16. (α, x) *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 1 \pmod 2$, $\kappa \equiv 1 \pmod 2$ *and* $-ev_1v_2u \not\sim 1$.

17. $(\pi F, x)$ *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 0 \pmod 2$, $\kappa \equiv 1 \pmod 2$, *where* $F \in O_k^*$, $\pi F \sim f(0)$.

18. $(\pi, x^2-\pi^{\kappa}u)$ *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 0 \pmod 2$, $\kappa \equiv 0 \pmod 2$, $ev_1v_2 \not\sim 1$.

19. $(\alpha, x-\pi^{\kappa/2}w)$ *if* $m_2 < \kappa/2$, $m_1 + m_2 \equiv 0 \pmod 2$, $\kappa \equiv 0 \pmod 2$, $ev_1v_2 \sim 1$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$ *and* E *is chosen such that* $f(\pi^{\kappa/2}w) \sim \pi E$.

Proof. In view of lemma 3 from [3], the algebras (π, l_1) and $(\pi, l_1(x+e))$ are nontrivial. Furthermore, $(\pi, x+e)_{l_1} \not\sim 1$, provided that $e \not\sim 1$. After extension by $\sqrt{\alpha}$ we have $(\pi\alpha, l_1) \sim (\pi, l_1) \not\sim 1$. Finally, $f(\sqrt[3]{\pi}) \sim e$, so in case $e \sim 1$

$$(\alpha, l_1)_{x^3-\pi} \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle x^3-\pi\rangle}\right) \not\sim 1.$$

Consider now the third generator.
1. The algebra $(\alpha, x^2-\pi^{\kappa}u)$ is nontrivial since $(\alpha, x^2-\pi^{\kappa}u)_{l_1} \not\sim 1$. $(\alpha, x^2-\pi^{\kappa}u)_{x^3-\pi} \sim 1$, on the other hand $(\alpha, l_1)_{x^3-\pi} \not\sim 1$, therefore these algebras are not isomorphic. The algebra $(\alpha, x^2-\pi^{\kappa}u)$ is not isomorphic to (π, l_1), $(\pi\alpha, l_1)$ because the extension by $\sqrt{\alpha}$ does not split any of two last algebras.
2. $(\pi, x^2-\pi^{\kappa}u)_{l_1} \sim (\pi, -u) \not\sim 1$. By lemma 3 from [3], $(\pi, x^2-\pi^{\kappa}u) \not\sim (\pi, l_1)$. Since, after extension by $\sqrt{\pi}$, one has $(\pi\alpha, l_1) \sim (\alpha, l_1) \not\sim 1$, it follows that $(\pi, x^2-\pi^{\kappa}u)$ is not isomorphic to these algebras.
3. Because of $f(\pi^{\kappa/2}w) \sim \alpha$ the algebra $(\alpha, x-\pi^{\kappa/2}w))$ is unramified. $(\alpha, x-\pi^{\kappa/2}w))_{x^3-\pi} \not\sim 1$. Let $q = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^s)$, where $s = \kappa/2 + 1/3$. Then $f(\pi^s) \sim 1$, therefore

$$(\alpha, x-\pi^{\kappa/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{s}}{k(\sqrt[3]{\pi})\langle q\rangle}\right),$$

so $(\alpha, x-\pi^{\kappa/2}w)_q \not\sim (\alpha, l_1)_q$. To complete the consideration of this subcase, it is enough to observe that, after extension by $\sqrt{\pi}$, we have $(\pi, l_1) \sim 1 \not\sim (\alpha, x-\pi^{\kappa/2}w)$ and $(\pi\alpha, l_1) \sim (\alpha, l_1) \not\sim (\alpha, x-\pi^{\kappa/2}w)$.
4. $f(\pi^{\kappa/2}w)) \sim e(w^2-u) \sim 1$ and $(\pi, x^2-\pi^{\kappa}u)_{x-\pi^{\kappa/2}w} \not\sim 1$. By lemma 3 from [3] we obtain $(\pi, x^2-\pi^{\kappa}u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$. Finally, $(\pi, x^2-\pi^{\kappa}u)_{l_1} \sim 1 \not\sim (\pi, x+e)_{l_1}$.
5. Let

$$\varepsilon = \begin{cases} 1, \text{ if } \kappa \equiv 0 \pmod 4, \\ 2, \text{ if } \kappa \equiv 2 \pmod 4, \end{cases}$$

$q = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^{\kappa/2+\varepsilon/3})$. Then $f(\pi^{\kappa/2+\varepsilon/3}) \sim 1$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ \pi^{\kappa/2+\varepsilon/3}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

The algebra (α, l_1) is not isomorphic to the remaining algebras, since the extension by $\sqrt{\pi}$ leaves us in the current subcase, and therefore it does not split this algebra.
6. Since extension by $\sqrt{\alpha}$ leaves us in case VIII, the algebras (π, l_1) and $(\pi, l_1(x+e))$ do not split after this extension, so they are not isomorphic to (α, l_1). Furthermore, if $m_1 < m_2$ or $\bar{v}_1 \neq \bar{v}_2$, then $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{m_1})$ and $(\alpha, l_1)_{l_1} \sim (\alpha, (x^2 - \pi^\kappa u) l_2(x+e))_{l_1} \sim (\alpha, \pi^{m_1+1})$. One of these two algebras is nontrivial. Thus, $(\alpha, l_1) \not\sim 1$. Let now $m_1 = m_2$ and $v_2 = v_1 + \pi^s \tau,\ s > 0, \tau \in O_k^*$. Then $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{m_1+s})$ and $(\alpha, l_1)_{l_1} \sim (\alpha, \pi^{m_1+s+1})$, so that the algebra (α, l_1) is nontrivial again. It remains to check that $(\alpha, l_1) \not\sim (\pi, x+e)$. But $(\pi, x+e)_{l_1} \not\sim 1$ and $(\pi, x+e)_{l_2} \not\sim 1$. On the other hand, among the algebras $(\alpha, l_1)_{l_1}$ and $(\alpha, l_1)_{l_2}$ there is a trivial one.
7. $f(\pi^{\kappa/2} w) \sim (w^2 - u)(w - v_1)(w - v_2 \pi^{m_2 - m_1})$. If we require

$$\begin{cases} w^2 - u \not\sim 1, \\ (w - v_1)(w - v_2 \pi^{m_2 - m_1}) \not\sim 1, \end{cases}$$

$w \in O_K^*,\ \bar{w} \neq \bar{v}_1, \bar{v}_2$, $p = \mathrm{Irr}_{K|k}(\pi^{\kappa/2} w)$, then $f(\pi^{\kappa/2} w) \sim 1$ and $(\pi, x^2 - \pi^\kappa u)_p \not\sim 1$. As we already had seen above, $(\pi, x^2 - \pi^\kappa u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$ by lemma 3 from [3]. If

$$\begin{cases} w_0^2 - u \sim 1, \\ (w_0 - v_1)(w_0 - v_2 \pi^{m_2 - m_1}) \sim 1, \end{cases}$$

$w_0 \in O_K^*,\ \bar{w}_0 \neq \bar{v}_1, \bar{v}_2$, $p_0 = \mathrm{Irr}_{K|k}(\pi^{\kappa/2} w_0)$, then $f(\pi^{\kappa/2} w_0) \sim 1$ and $(\pi, x^2 - \pi^\kappa u)_{p_0} \sim 1$. But $(\pi, x+e)_{p_0} \sim (\pi, e) \not\sim 1$, provided that $e \not\sim 1$. Thus, in case $e \not\sim 1$ we are done. Case $e \sim 1$ can be considered similarly as subcase 2.
8. Let $e \not\sim 1,\ v_1^2 - u \sim 1$. There exists $w \in O_k^*$ such that $w^2 - u \not\sim 1$. So $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{\kappa/2} w} \sim (\pi, w^2 - u) \not\sim 1$ since $f(\pi^{\kappa/2} w) \sim (w^2 - u)e \sim 1$. If $e \sim 1,\ v_1^2 - u \not\sim 1$, then $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{m_1} v_1} \sim (\pi, v_1^2 - u) \not\sim 1$. Let us check that the algebra $(\pi, x^2 - \pi^\kappa u)$ is not isomorphic to the other algebras. If $e \not\sim 1$, then $(\pi, x^2 - \pi^\kappa u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$ as above. We also have $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{m_1} v_1} \sim (\pi, v_1^2 - u) \sim 1$ and $(\pi, x+e)_{x - \pi^{m_1} v_1} \sim (\pi, e) \not\sim 1$. Let now $e \sim 1$. Then one can argue as in subcase 2.
9. Let

$$\varepsilon = \begin{cases} 1, \text{ if } \kappa \equiv 0 \pmod 4, \\ 2, \text{ if } \kappa \equiv 2 \pmod 4, \end{cases}$$

$\theta = \pi^{\kappa/2}(v_1 + (\sqrt[3]{\pi})^\varepsilon)$, $q = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim (v_1^2 - u)e \sim 1$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1+\varepsilon}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

$(\alpha, l_1) \not\sim (\pi, l_1), (\pi, x+e), (\pi, l_1(x+e))$ since extension by $\sqrt{\pi}$ leaves us in the current subcase and therefore this extension does not split the algebra (α, l_1).
10. The algebra $(\alpha, x - \pi^{\kappa/2} w)$ is unramified. $f(\sqrt[3]{\pi}) \sim e \sim 1$ so $(\alpha, x - \pi^{\kappa/2} w)_{x^3 - \pi} \not\sim 1$. Since after the extension by $\sqrt{\pi}$ we are in the current subcase again, it is enough to

prove only that $(\alpha, x-\pi^{\kappa/2}w) \not\sim (\alpha, l_1)$. To check this fact, assume $\theta_0 = \pi^{\kappa/2}(v_1 + \sqrt[3]{\pi})$, $q_0 = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta_0)$. Then $f(\theta_0) \sim 1$ and

$$(\alpha, l_1)_{q_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2+1}}{k(\sqrt[3]{\pi})\langle q_0\rangle}\right) \text{ and } (\alpha, x-\pi^{\kappa/2}w)_{q_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2}}{k(\sqrt[3]{\pi})\langle q_0\rangle}\right).$$

11. $(\alpha, x^2-\pi^\kappa u)_{l_2} \sim (\alpha, \pi^\kappa) \not\sim 1$. If $e \sim 1$, then $\sqrt{\alpha}$ does not split (π, l_1) and $(\pi\alpha, l_1)$ so that these algebras are not isomorphic to $(\alpha, x^2-\pi^\kappa u)$. Moreover, $(\alpha, x^2-\pi^\kappa u)_{x^3-\pi} \not\sim (\alpha, l_1)_{x^3-\pi}$. Let $e \not\sim 1$. Then we have $(\alpha, x^2-\pi^\kappa u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$. Finally, $(\alpha, x^2-\pi^\kappa u)_{l_1} \sim 1$, but $(\pi, x+e)_{l_1} \not\sim 1$.
12. Let $w \in O_K^*$ such that

$$\begin{cases} w^2 - u \not\sim 1, \\ -v_1 e(w - v_2\pi^{m_2-\kappa/2}) \not\sim 1, \end{cases}$$

then $f(\pi^{\kappa/2}w) \sim 1$ and $(\pi, x^2-\pi^\kappa u)_{\mathrm{Irr}_{K|k}}(\pi^{\kappa/2}w) \not\sim 1$. If $e \not\sim 1$, then $(\pi, x^2-\pi^\kappa u) \not\sim (\pi, l_1), (\pi, l_1(x+e))$. We also have $(\pi, x^2-\pi^\kappa u)_{l_1} \sim 1$ and $(\pi, x+e)_{l_1} \not\sim 1$. If $e \sim 1$, then the extension by $\sqrt{\pi}$ leaves us in the subcase 12, so we can argue as in subcase 2.
13. The algebra $(\pi E, x-\pi^{\kappa/2}w)$ is unramified by the choice of E.

$$(\pi E, x-\pi^{\kappa/2}w)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi E\ ,\ \pi^{\kappa/2}(\sqrt{u}-w)}{k(\sqrt{u})\langle x^2-\pi^\kappa u\rangle(\sqrt{\pi(x^2-\pi^\kappa u)})}\right) \sim$$

$$\sim \left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle\sqrt{\pi(x^2-\pi^\kappa u)}\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\pi, l_1)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ -\pi^{m_1}v_1}{k(\sqrt{u})\langle\sqrt{\pi(x^2-\pi^\kappa u)}\rangle}\right) \sim 1.$$

The same is true for the algebras $(\pi, x+e)$, $(\pi, l_1(x+e))$, (α, l_1) and $(\pi\alpha, l_1)$.
14. This case is totally similar to the previous one.
15. $f(0) \sim 1$, so $(\alpha, x^2-\pi^\kappa u)_x \sim (\alpha, \pi^\kappa) \not\sim 1$. Extension by $\sqrt{\alpha}$ does not split any of (π, l_1), $(\pi\alpha, l_1)$, $(\pi, l_1(x+e))$ in the corresponding cases, therefore $(\alpha, x^2-\pi^\kappa u)$ is not isomorphic to any of them. If $e \sim 1$, then $(\alpha, l_1)_{x^3-\pi} \not\sim (\alpha, x^2-\pi^\kappa u)_{x^3-\pi}$. Finally, if $e \not\sim 1$, then $(\pi, x+e)_{l_1} \not\sim (\alpha, x^2-\pi^\kappa u)_{l_1}$.
16. $f(0) \sim -ev_1v_2u \sim \alpha$. Thus the algebra (α, x) is unramified. We also have

$$(\alpha, x)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi^\kappa u}}{k(\sqrt{\pi u})\langle x^2-\pi^\kappa u\rangle(\sqrt{-\alpha(x^2-\pi^\kappa u)})}\right) \sim$$

$$\sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle\sqrt{-\alpha(x^2-\pi^\kappa u)}\rangle}\right) \not\sim 1.$$

As it was already noticed ,the other algebras are trivial after completion in $x^2-\pi^\kappa u$.

17. The algebra $(F\pi, x)$ is unramified by the choice of F. Let $\eta \in O_K^*$, K be as above, $\eta \not\sim 1$, $\eta + e \sim 1$, and $r = \mathrm{Irr}_{K|k}(\eta)$. Then

$$(F\pi, x)_r \sim \left(\frac{F\pi\ ,\ \eta}{K\langle r\rangle} \right) \not\sim 1,$$

since $f(\eta) \sim \eta + e \sim 1$. On the other hand, $(\alpha, l_1)_q \sim (\alpha, \eta) \sim 1$ and $(\pi, x+e)_q \sim (\pi, \eta + e) \sim 1$. Let firstly $m_1 < m_2$ or $\bar{v}_1 \neq \bar{v}_2$. Then, for $w \in O_K^*$, $\bar{w} \neq \bar{v}_1, \bar{v}_2$, we have $f(\pi^{m_2} w) \sim -e(w - v_2)(v_1 - w\pi^{m_2 - m_1})$. Let $p = \mathrm{Irr}_{K|k}(\pi^{m_2} w)$. Then $(\pi, l_1)_p \sim (\pi, (-1)^{m_1+1}(v_1 - w\pi^{m_2-m_1}))$ and $(F\pi, x)_p \sim (\pi, (-F)^{m_2} w)$. We require

$$\begin{cases} -e(w - v_2)(v_1 - w\pi^{m_2-m_1}) \sim 1, \\ -F^{m_2} w(v_1 - w\pi^{m_2-m_1}) \not\sim 1. \end{cases}$$

Then $(\pi, l_1)_p \not\sim (F\pi, x)_p$. Similarly one can check $(\pi\alpha, l_1) \not\sim (F\pi, x)$ and $(\pi, l_1(x+e)) \not\sim (F\pi, x)$. Consider now the case $m_1 = m_2$, $v_2 = v_1 + \pi^r \mu, r > 0, \mu \in O_k^*$. Suppose $w \in O_K^*$, $\bar{w} \neq \bar{\mu}$, $s = \mathrm{Irr}_{K|k}(\pi^{m_2}(v_1 + \pi^r w))$. Then $f(\pi^{m_2}(v_1 + \pi^r w)) \sim ew(w - \mu)$, $(\pi, l_1)_s \sim (\pi, (-1)^{m_1+r} w)$ and $(F\pi, x)_s \sim (\pi, (-F)^{m_1} v_1)$. If we require

$$\begin{cases} ew(w - \mu) \sim 1, \\ (-1)^r F^{m_1} v_1 w \not\sim 1, \end{cases}$$

then $(\pi, l_1)_s \not\sim (F\pi, x)_s$. For the remaining algebras the arguments are similar.
18. If $w \in O_k^*$, $w^2 - u \not\sim 1$, then $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{\kappa/2} w} \not\sim 1$ in view of $f(\pi^{\kappa/2} w) \sim e v_1 v_2 (w^2 - u) \sim 1$. Let η and r be as in the previous case, then $(\pi, l_1)_r \not\sim 1$, $(\pi, l_1(x+e))_r \not\sim 1$ and $(\pi\alpha, l_1)_r \not\sim 1$, but $(\pi, x^2 - \pi^\kappa u)_r \sim 1$. Extension by $\sqrt{\pi}$ leaves us in the current case, so it does not split (α, l_1), provided that $e \sim 1$, but it splits $(\pi, x^2 - \pi^\kappa u)$. Finally, $(\pi, x+e)_{l_1} \not\sim (\pi, x^2 - \pi^\kappa u)_{l_1}$ if $e \not\sim 1$.
19. $f(\pi^{\kappa/2} w) \sim e v_1 v_2 (w^2 - u) \sim \alpha$, hence the algebra $(\alpha, x - \pi^{\kappa/2} w)$ is unramified. If

$$\varepsilon = \begin{cases} 1, \text{ if } m_2 \equiv 0 \,(\mathrm{mod}\, 2), \\ 2, \text{ if } m_2 \equiv 1 \,(\mathrm{mod}\, 2), \end{cases}$$

then $f(\pi^{m_2 + \varepsilon/3}) \sim e v_1 v_2 \sim 1$. Assume $p = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^{m_2+\varepsilon/3})$. Then

$$(\alpha, x - \pi^{\kappa/2} w)_p \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_2+\varepsilon}}{k(\sqrt[3]{\pi})\langle p\rangle} \right) \not\sim 1.$$

Since extension by $\sqrt{\pi}$ leaves us in the case 19, this extension does not split the algebra $(\alpha, x - \pi^{\kappa/2} w)$. Hence $(\alpha, x - \pi^{\kappa/2} w) \not\sim (\pi, l_1)$, $(\pi, x+e)$, $(\pi, l_1(x+e))$. Since after this extension we have $(\pi\alpha, l_1) \sim (\alpha, l_1)$, it suffices to show that $(\alpha, x - \pi^{\kappa/2} w) \not\sim (\alpha, l_1)$, provided that $e \sim 1$. Let $p_0 = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^{m_2 + 1/3})$. Then $f(\pi^{m_2+1/3}) \sim 1$ as above and

$$(\alpha, x - \pi^{\kappa/2} w)_{p_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_2+1}}{k(\sqrt[3]{\pi})\langle p_0\rangle} \right), \qquad (\alpha, l_1)_{p_0} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1}}{k(\sqrt[3]{\pi})\langle p_0\rangle} \right).$$

Because of $m_1 \equiv m_2 \,(\mathrm{mod}\, 2)$ we have $(\alpha, x - \pi^{\kappa/2} w)_{p_0} \not\sim (\alpha, l_1)_{p_0}$ and the theorem is proved.

Without loss of generality one can assume that in case IX either $0 < \kappa < 4$ and $\kappa/2 \leq m_1$ (case IX.1) or $m_1 = 1$, $\kappa > 2$ (case IX.2). Let $l_i = x - \pi^{k_i} v_i$, $i = 1, 2, 3$.

Theorem 6 *Let f be as in case* IX.1 *and as soon as $m_1 = m_2 = m_3$, then $\bar{v}_3 \neq \bar{v}_i$, $i = 1, 2$. Then the group ${}_2\mathrm{Br}\, C$ is generated by the class of the algebra (π, α) and of the following algebras.*

1. $(\alpha, x^2 - \pi^\kappa u)$, (α, l_1), (α, l_2), *if*
   $\kappa \equiv 1 \pmod 2$, $m_1 \equiv 0 \pmod 2$.
2. $(\pi E, p_w)$, (α, l_1), (α, l_2), *if*
   $\kappa \equiv 0 \pmod 2$, $m_1 \equiv 1 \pmod 2$.
3. $(\pi E, p_w)$, (α, l_2), (π, l_2), *if*
   $\kappa \equiv m_1 \equiv 0 \pmod 2$, $m_1 < m_2$, $uv_1 \sim 1$.
4. $(\alpha, x^2 - \pi^\kappa u)$, (α, l_2), (π, l_2), *if*
   $\kappa \equiv m_1 \equiv 1 \pmod 2$, $m_1 < m_2$, $uv_1 \sim 1$.
5. $\kappa \equiv m_1 \equiv 1 \pmod 2$, *and if* $m_1 < m_2$, *then* $uv_1 \not\sim 1$.
   - **a)** $(\alpha, x^2 - \pi^\kappa u)$, (π, l_1), (α, l_1),
     if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi = -u(v_1 - v_3 \pi^{m_3 - m_1}) \sim 1$.
   - **b)** $(\alpha, x^2 - \pi^\kappa u)$, (π, l_1), (π, l_2), *otherwise.*
6. $\kappa \equiv m_1 \equiv 0 \pmod 2$, *and if* $m_1 < m_2$, *then* $uv_1 \not\sim 1$.
   - **a)** $(\pi E, p_w)$, (π, l_1), (α, l_1),
     if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ *and* $\xi \sim 1$.
   - **b)** $(\pi E, p_w)$, (π, l_1), (π, l_2), *otherwise,*

where $w \in O_K^*$ satisfies the following conditions. If $m_1 > 1$, then $w \in O_k^*$, $w^2 - u \not\sim 1$, $E = \alpha w$, and $p_w = x - \pi w$. If $m_1 = 1$, then

- *for $1 < m_2$, then*
$$\begin{cases} w^2 - u \not\sim 1, \\ w(w - v_1) \sim 1. \end{cases}$$
- *for $1 = m_2 < m_3$, then*
$$\begin{cases} (w^2 - u)(w - v_1)w \not\sim 1, \\ (w^2 - u)(w - v_2)w \not\sim 1, \\ (w^2 - u)(w - v_1)(w - v_2) \not\sim 1. \end{cases}$$
- *for $1 = m_2 = m_3$, then*
$$\begin{cases} (w^2 - u)(w - v_1)(w - v_2) \not\sim 1, \\ (w^2 - u)(w - v_1)(w - v_3) \not\sim 1, \\ (w^2 - u)(w - v_2)(w - v_3) \not\sim 1 \end{cases}$$

for a suitable unramified odd degree extension $K|k$, $p_w = \mathrm{Irr}_{K|k}(\pi w)$, $E = N_{K|k}(E_0)$, and $E_0 \in O_K^*$ such that $f(\pi w) \sim \pi E_0$ in K.

It is convenient to divide the proof into several parts. We assume in the following lemmas that all conditions of the above theorem are satisfied.

Lemma 7 *If $\kappa \equiv 1 \pmod 2$, then the algebra $(\alpha, x^2 - \pi^\kappa u)$ is unramified and nontrivial. Otherwise, the same is true for the algebra $(\pi E, p_w)$, where w and p_w are from the above theorem.*

We have $(\alpha, x^2 - \pi^\kappa u)_{l_i} \sim (\alpha, \pi^\kappa) \not\sim 1$, since $\kappa < 2m_i,\ i = 1, 2, 3$. If $\kappa = 2$, $m_1 > 1$, then $f(\pi w) \sim \pi w(w^2 - u) \sim \pi E$ and the algebra $(\pi E, x - \pi w)$ is unramified. $(\pi E, x - \pi w)_{l_i} \sim (\pi\alpha w, -\pi w) \not\sim 1$. Let now $\kappa = 2$, $m_1 = 1$. Then

$$f(\pi w) \sim \pi(w^2 - u)(w - v_1)(w - \pi^{m_2 - 1} v_2)(w - \pi^{m_3 - 1} v_3).$$

Let $\bar{w} \neq \bar{v}_1$ and if $m_i = 1$, then $\bar{w} \neq \bar{v}_i$. Then $f(\pi w) \sim \pi E_0,\ E_0 \in O_K^*$, where

$$E_0 = \begin{cases} (w^2 - u)(w - v_1), & \text{if } 1 < m_2, \\ (w^2 - u)(w - v_1)(w - v_2)w, & \text{if } 1 = m_2 < m_3, \\ (w^2 - u)(w - v_1)(w - v_2)(w - v_3), & \text{if } 1 = m_2 = m_3. \end{cases}$$

We have

$$(\pi E, p_w)_{p_w} \sim \left(\frac{\pi E_0\ ,\ p_w}{K(\sqrt{\pi E_0})\langle p_w \rangle} \right) \sim 1,$$

since $E = N_{K|k}(E_0) \sim E_0$ in K. So the algebra $(\pi E, p_w)$ is unramified. Let us require $(\pi E, p_w)_{l_i} \not\sim 1,\ i = 1, 2, 3$. Then

$$(\pi E, p_w)_{l_i} \sim (\pi, N_{K|k}(E_0(w - v_i))),$$

$$(\pi E, p_w)_{l_2} \sim (\pi, N_{K|k}(E_0(w - v_1 \pi^{m_i - 1}))),\ i = 2, 3,$$

and these algebras are nontrivial iff $E_0(w - v_1) \not\sim 1$ and $E_0(w - v_i) \not\sim 1$, provided that $m_i = 1$ or $E_0 w \not\sim 1$, provided that $m_i > 1$, $i = 2, 3$. It just gives the conditions on w from the theorem. These conditions can always be satisfied by lemma 6 from [3]. Hence the lemma is proved.

Lemma 8 *If $\kappa \not\equiv m_1 \pmod 2$, then the classes of the algebras (π, α), (α, l_1), (α, l_2) together with the class of the algebra from lemma 7 give all generators of ${}_2\mathrm{Br}\, C$.*

Proof. Let firstly $\kappa/2 < m_1$. Then

$$(\alpha, l_1)_{l_1} \sim (\alpha, (x^2 - \pi^\kappa u) l_2 l_3)_{l_1} \sim (\alpha, \pi^{\kappa + m_1} l_2(\pi^{m_1} v_1)) \sim (\alpha, \pi l_2(\pi^{m_1} v_1))$$

and $(\alpha, l_1)_{l_2} \sim (\alpha, l_2(\pi^{m_1} v_1))$. Thus exactly one of the algebras $(\alpha, l_1)_{l_1}$ and $(\alpha, l_1)_{l_2}$ is nontrivial. Since the algebra from lemma 7 is nontrivial in l_1, l_2, it is not isomorphic to (α, l_1). Let now $\kappa = 2$, $m_1 = 1$. Then $(\alpha, l_1)_{l_1} \sim (\alpha, \pi l_2(\pi^{m_1} v_1)) \not\sim (\alpha, l_1)_{l_2}$ again.

Consider the algebras (α, l_2) and $(\alpha, l_1 l_2)$. We have

$$(\alpha, l_2)_{l_1} \sim (\alpha, l_2(\pi^{m_1} v_1)),\ (\alpha, l_2)_{l_2} \sim (\alpha, \pi^{\kappa + m_2} l_2(\pi^{m_1} v_1)),\ (\alpha, l_2)_{l_3} \sim (\alpha, \pi^{m_2}),$$

$$(\alpha, l_1 l_2)_{l_2} \sim (\alpha, (x^2 - \pi^\kappa u) l_3)_{l_2} \sim (\alpha, \pi^{\kappa + m_2}),\ (\alpha, l_1 l_2)_{l_3} \sim (\alpha, \pi^{m_1 + m_2}).$$

Hence $(\alpha, l_1 l_2)_{l_2} \not\sim (\alpha, l_1 l_2)_{l_3}$. This means that $(\alpha, l_1 l_2) \not\sim 1$ and $(\alpha, l_1 l_2)$ is not isomorphic to the algebra from lemma 7. Moreover, if $m_1 \equiv m_2 \pmod 2$, then $(\alpha, l_2)_{l_2} \sim (\alpha, \pi l_2(\pi^{m_1} v_1)) \not\sim (\alpha, l_2)_{l_1}$ and $(\alpha, l_2)_{l_2} \sim (\alpha, l_2(\pi^{m_1} v_1)) \sim (\alpha, \pi^{m_1}) \not\sim (\alpha, l_2)_{l_3}$ otherwise, since in this case $m_1 < m_2$. The lemma is proved.

Lemma 9 *If $\kappa \equiv m_1 \pmod 2$, $m_1 < m_2$, $uv_1 \sim 1$, then the classes of the algebras (π, α), (α, l_2), (π, l_2) together with the class of the algebra from lemma 7 give all generators of ${}_2\mathrm{Br}\, C$.*

Proof. Let $\varepsilon \in \{1, 2\}$, $\tau \in O_k^*$, $\theta = \pi^{m_1}(\sqrt[3]{\pi})^{\varepsilon}\tau$, and $p = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim \pi^{\kappa+m_1}uv_1 \sim 1$. Assume $A \in \{\pi, \alpha, \pi\alpha\}$. Then

$$(A, l_2)_p \sim \left(\frac{A\ ,\ \theta}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \sim \left(\frac{A\ ,\ (\sqrt[3]{\pi})^{m_1+\varepsilon}\tau}{k(\sqrt[3]{\pi})\langle p\rangle}\right),$$

and for any A we are always able to find ε and τ such that $(A, l_2)_p \not\sim 1$. Let us now compare the algebras (A, l_2) with the algebra from lemma 7. Let ε and τ be such that $(A, l_2)_p \sim 1$. In view of $\kappa/2 < m_1 + \varepsilon/3$ and $\kappa/2 < m_2$, the algebra from lemma 7 is nontrivial in p because it is nontrivial in l_2. This completes the proof of the lemma.

Lemma 10 *Let $\kappa \equiv m_1 \pmod 2$ and, if $m_1 < m_2$, then $uv_1 \not\sim 1$. Then the classes of the algebras (π, α), (π, l_1), the class of the algebra from lemma 7 and the class of the following algebra give all generators of ${}_2\mathrm{Br}\, C$.*

- *(π, l_2), if $m_1 < m_2$ or $\bar{v}_1 \neq \bar{v}_2$.*
- *(π, l_2), if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ and $\xi \not\sim 1$.*
- *(α, l_1), if $m_1 = m_2$, $\bar{v}_1 = \bar{v}_2$ and $\xi \sim 1$.*

Proof. In this case $\kappa/2 < m_1$. If $m_1 < m_2$, then $(\pi, l_1)_{l_1} \sim (\pi, -\pi^{\kappa}u)$ and $(\pi, l_1)_{l_2} \sim (\pi, -\pi^{m_1}v_1) \sim (\pi, -\pi^{\kappa}v_1)$. Since $uv_1 \not\sim 1$, among the algebras $(\pi, l_1)_{l_1}$ and $(\pi, l_1)_{l_2}$ there is just one trivial algebra. So $(\pi, l_1) \not\sim 1$ and (π, l_1) is not isomorphic to the algebra from lemma 7. Let now $m_1 = m_2$ and $\mu_i \in O_K^*$. Then $f(\pi^{m_1}\mu_i) \sim -u(\mu_i - v_1)(\mu_i - v_2)(\mu_i - v_3\pi^{m_3-m_1})$. If $p_i = \mathrm{Irr}_{K|k}(\pi^{m_1}\mu_i)$, then $(\pi, l_1)_{p_i} \sim (\pi, (-1)^{m_1}(\mu_i - v_1))$. One can find K, μ_1, and μ_2 such that $f(\pi^{m_1}\mu_1) \sim f(\pi^{m_1}\mu_2) \sim 1$ and $(\pi, l_1)_{p_1} \not\sim 1$, $(\pi, l_1)_{p_2} \sim 1$. Since the algebra from lemma 7 is not trivial in p_2 it follows again that $(\pi, l_1) \not\sim 1$ and (π, l_1) is not isomorphic to this algebra.

Let us describe the last generator. Consider firstly the case $m_1 < m_2$. Then we will have $f(\pi^{m_1}\mu_i) \sim -u(\mu_i - v_1)$. If μ_i and p_i are such that

$$-u(\mu_i - v_1) \sim 1 \text{ for } i = 1, 2, 3, 4,$$

and

$$(-1)^{m_1}\mu_1 \sim 1,\ (-1)^{m_1}\mu_2 \not\sim 1\ \mu_3 \not\sim -u\ , \mu_4 \sim -u,$$

then

$$(\pi, l_2)_{p_1} \sim (\pi, (-1)^{m_1}\mu) \sim 1\ (\pi, l_2)_{p_2} \not\sim 1,$$

and

$$(\pi, l_1l_2)_{p_3} \sim (\pi, \mu_3(\mu_3 - v_1)) \sim (\pi, -u\mu_3) \not\sim 1,\ (\pi, l_1l_2)_{p_4} \sim 1.$$

Thus $(\pi, l_2) \not\sim 1$, $(\pi, l_2) \not\sim (\pi, l_1)$, (π, l_2) is not isomorphic to the algebra from lemma 7 and also (π, l_1l_2) is not.

Let $m_1 = m_2$. Then $(\pi, l_2) \not\sim 1$ and (π, l_2) is not isomorphic to the algebra from lemma 7 in view of the symmetry between the algebras (π, l_1) and (π, l_2). Let $\bar{v}_1 \neq \bar{v}_2$.

Then one can find μ_1, μ_2 such that $f(\pi^{m_1}\mu_1) \sim 1,\ i = 1, 2$ and $(\mu_1 - v_1)(\mu_1 - v_2) \not\sim 1$, $(\mu_2 - v_1)(\mu_2 - v_2) \sim 1$. This means that $(\pi, l_1 l_2)_{p_1} \not\sim 1$ and $(\pi, l_1 l_2)_{p_2} \sim 1$.

Let now $v_2 = v_1 + \pi^s \tau$, $q = \mathrm{Irr}_{K|k}(\pi^{m_1}(v_1 + \pi^s \eta)),\ \eta \in O_K^*$. Then $f(\pi^{m_1}(v_1 + \pi^s \eta)) \sim \xi\eta(\eta - \tau)$. If $\xi \not\sim 1$, then we require $\eta(\eta - \tau) \not\sim 1$. In this case $f(\pi^{m_1}(v_1 + \pi^s \eta)) \sim 1$, $(\pi, l_1 l_2)_q \sim (\pi, \eta(\eta - \tau)) \not\sim 1$. Moreover, $(\pi, l_1 l_2)_{l_3} \sim (\pi, (\pi^{m_3 - m_1} v_3 - v_1)(\pi^{m_3 - m_1} v_3 - v_2)) \sim 1$.

Finally, let $\xi \sim 1$. After replacing the prime element $\nu = \pi\alpha$ one can check that the algebra $(\pi\alpha, l_1)$ is nontrivial and not isomorphic to the algebra from lemma 7. To complete the proof it is sufficient to show that the same is true for the algebra (α, l_1). If $\theta_i = \pi^{m_1}(v_1 + (\sqrt[3]{\pi})^i),\ i = 1, 2$, $t_i = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta_i)$, then $f(\theta_i) \sim \xi \sim 1$, and

$$(\alpha, l_1)_{t_i} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m_1 + i}}{k(\sqrt[3]{\pi})\langle t_i \rangle} \right).$$

So, among the algebras $(\alpha, l_1)_{t_i},\ i = 1, 2$, there is exactly one nontrivial algebra. Since the algebra from lemma 7 is nontrivial in t_1 and t_2, the lemma is proved.

Now the proof of the theorem follows immediately from lemmas 7–10.

Let us consider the remaining subcase of case IX.1. In this subcase one can assume that $f(x) = (x^2 - \pi^\kappa u)(x - \pi^m v)(x - \pi^m v_1)(x - \pi^m v_2)$, $v_i = v + \pi^{s_i}\tau_i,\ \tau_i \in O_k^*,\ i = 1, 2$, $0 < s_1 \le s_2$ and if $s_1 = s_2$, then $\bar{\tau}_1 \ne \bar{\tau}_2$. Let $l = x - \pi^m v$, $l_i = x - \pi^m v_i$, and

$$\xi = \begin{cases} u\tau_1, \text{ if } \kappa/2 < m_1, \\ (u - v^2)\tau_1, \text{ otherwise.} \end{cases}$$

Theorem 7 *Let f be as in case* IX.1, $m_1 = m_2 = m_3$, $\bar{v}_1 = \bar{v}_2 = \bar{v}_3$. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras.*

- $A = (\alpha, x^2 - \pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$.
- $A = (\pi E, x - \pi w)$, *if* $\kappa = 2$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$,

$$E = \begin{cases} \alpha w, \text{ if } m > 1, \\ \alpha(w - v), \text{ if } m = 1, \end{cases}$$

and

1. (α, l), (α, l_1), *if*
   $\kappa + m + s_1 \equiv 1 \pmod 2$.
2. (π, l), (π, l_1), *if*
   $\kappa + m + s_1 \equiv 0 \pmod 2$, $s_1 = s_2$.
3. (π, l), (π, l_2), *if*
   $\kappa + m + s_1 \equiv 0 \pmod 2$, $s_1 < s_2$, $\xi \not\sim 1$.
4. (π, l), (α, l), *if*
   $\kappa + m + s_1 \equiv 0 \pmod 2$, $s_1 < s_2$, $\xi \sim 1$,

Proof. If κ is odd, then $(\alpha, x^2-\pi^\kappa u)_l \not\sim 1$ and $(\alpha, x^2-\pi^\kappa u)_{l_i} \not\sim 1,\ i=1,2$. Let $\kappa=2$. Then $f(\pi w) \sim \pi E$ and the algebra A is unramified. We also have $(\pi E, x-\pi w)_l \sim (\pi, w^2-u) \not\sim 1$. By the same argument, $(\pi E, x-\pi w)_{l_i} \not\sim 1$. So A is nontrivial in l, l_1, l_2 in any case.

Observe that $(\alpha,l)_l \sim (\alpha, \pi^{\kappa+s_1+s_2})$, $(\alpha,l)_{l_1} \sim (\alpha,\pi^{m+s_1})$, $(\alpha,l)_{l_2} \sim (\alpha,\pi^{m+s_2})$, $(\alpha,l_1)_l \sim (\alpha,\pi^{m+s_1})$, $(\alpha,l_1)_{l_1} \sim (\alpha,\pi^\kappa)$, $(\alpha,l_1)_{l_2} \sim (\alpha,\pi^{m+s_1})$. Let $\kappa+m+s_1 \equiv 1 \pmod 2$. Then, among the algebras $(\alpha,l)_l$ and $(\alpha,l)_{l_2}$, there exists exactly one nontrivial algebra, so that $(\alpha,l) \not\sim 1$ and $(\alpha,l) \not\sim A$. The same is true for the algebras $(\alpha,l_1)_{l_1}$ and $(\alpha,l_1)_{l_2}$. Finally, $(\alpha,ll_1)_l \sim (\alpha,\pi^{s_1+s_2+1})$ and $(\alpha,ll_1)_{l_2} \sim (\alpha,\pi^{s_1+s_2})$.

Let now $\kappa+m+s_1 \equiv 0 \pmod 2$, $\theta_i = \pi^m(v+\pi^{s_1}\eta_i),\ \eta_i \in O_K^*$, $p_i = \mathrm{Irr}_{K|k}(\theta_i)$,

$$\delta = \begin{cases} -u, \text{ if } \kappa/2 < m, \\ v^2-u, \text{ otherwise.} \end{cases}$$

Then

$$f(\theta_i) \sim \begin{cases} \delta(\eta_i-\tau_1), \text{ if } s_1 < s_2, \\ \delta\eta_i(\eta_i-\tau_1)(\eta_i-\tau_2) \text{ otherwise.} \end{cases}$$

Let us require $f(\theta_i)\sim 1$, then $(\pi,l)_{p_i} \sim (\pi,(-1)^\kappa\eta_i)$. One can check that $A_{p_i} \not\sim 1$. There exist η_1,η_2 such that $(-1)^\kappa\eta_1 \not\sim 1$, $(-1)^\kappa\eta_2 \sim 1$. Then $(\pi,l)_{p_1} \not\sim 1$ and $(\pi,l)_{p_2} \not\sim A_{p_2}$. Consider the case $s_1=s_2$. We have $(\pi,l_1)_{p_i} \sim (\pi,(-1)^\kappa(\eta_i-\tau_1))$. One can always find an element η_i such that

$$\begin{cases} (-1)^\kappa(\eta_1-\tau_1)\sim 1, \\ \delta(-1)^\kappa\eta_1(\eta_1-\tau_2)\sim 1 \end{cases} \Longrightarrow (\pi,l_1)_{p_1} \not\sim A_{p_1}.$$

$$\begin{cases} (-1)^\kappa(\eta_2-\tau_1)\not\sim 1, \\ \delta(-1)^\kappa\eta_2(\eta_2-\tau_2)\not\sim 1 \end{cases} \Longrightarrow (\pi,l_1)_{p_2} \not\sim 1.$$

$$\begin{cases} \eta_3(\eta_3-\tau_1)\not\sim 1, \\ \delta(\eta_3-\tau_2)\not\sim 1 \end{cases} \Longrightarrow (\pi,l_1)_{p_3} \not\sim (\pi,l_1)_{p_3}.$$

$$\begin{cases} \eta_4(\eta_4-\tau_1)\sim 1, \\ \delta(\eta_4-\tau_2)\sim 1 \end{cases} \Longrightarrow (\pi,l_1)_{p_4} \not\sim A_{p_4}\otimes(\pi,l_1)_{p_4}.$$

Let now $\kappa+m+s_1 \equiv 0 \pmod 2$, $s_1<s_2$, $\xi \not\sim 1$. Consider the algebra (π,l_2). If $v'=v+\pi^{s_2}\tau_2$, then $f(x)=(x^2-\pi^\kappa u)(x-\pi^m v')(x-\pi^m(v'+\pi^{s_1}\tau_1'))(x-\pi^m(v'+\pi^{s_2}\tau_2'))$. So $(\pi,l_2)=(\pi,l') \not\sim 1, A$ since we are in the case $\kappa+m+s_1\equiv 0 \pmod 2$ again. Furthermore, $(\pi,ll_2)_{l_1} \sim (\pi,(\pi^{m+s_1}\tau_1)^2)\sim 1$, therefore $(\pi,ll_2)\not\sim A$. We have also $(\pi,ll_2)_l \sim (\pi,\xi)\not\sim 1$.

Finally, let $\kappa+m+s_1 \equiv 0 \pmod 2$, $s_1<s_2$, $\xi\sim 1$. If $\theta_i = \pi^m(v+\pi^{s_1}(\sqrt[3]{\pi})^i)$, $q_i = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta_i)$, $i=1,2$, then $f(\theta)\sim\xi\sim 1$ and

$$(\alpha,l)_{q_i} \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m+s_1+i}}{k(\sqrt[3]{\pi})\langle q_i\rangle}\right), \quad A_{q_i}\not\sim 1.$$

Thus $(\alpha,l)\not\sim 1, A$. To finish the proof one only needs to check that $(\pi\alpha,l)\not\sim 1, A$. But it is evident after replacing the prime element by $\nu=\alpha\pi$.

Theorem 8 *Let f be as in case* IX.2 *Then the group ${}_2\mathrm{Br}\,C$ is generated by the class of the algebra (π,α) and of the following algebras.*

1. $\kappa/2 \le m_2$. *Two generators are* (α, l_1), (α, l_2). *The third one is*

   a) (π, l_2), *if* $\kappa \equiv 1 \pmod 2$,

   b) $(\pi E, x - \pi^{\kappa/2} w)$ *otherwise.*

2. $m_2 < \kappa/2 \le m_3$. *Two generators are* (α, l_1), (α, l_2). *The third one is*

   a) $(\alpha, x^2 - \pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$,

   b) (π, l_3), *if* $\kappa \equiv 0 \pmod 2$, $m_2 + \kappa/2 \equiv 1 \pmod 2$,

   c) $(\pi E, x - \pi^{\kappa/2} w)$, *if* $\kappa \equiv 0 \pmod 2$, $m_2 + \kappa/2 \equiv 0 \pmod 2$,

3. $m_3 < \kappa/2$ *and, if* $m_2 = m_3 = 1$, *then* $\bar v_3 \ne \bar v_1$, $\bar v_3 \ne \bar v_2$. *Two generators are* (α, l_1), (α, l_2). *The third one is*

   a) $(\pi E, x - \pi^{\kappa/2} w)$, *if* $m_2 + m_3 \equiv \kappa \equiv 0 \pmod 2$,

   b) $(\alpha, x - \pi^{\kappa/2} w)$, *if* $m_2 + m_3 \equiv 1 \pmod 2$, $\kappa \equiv 0 \pmod 2$ *and* $-v_1 v_2 v_3 \sim 1$,

   c) $(\pi, x^2 - \pi^\kappa u)$, *if* $m_2 + m_3 \equiv 1 \pmod 2$, $\kappa \equiv 0 \pmod 2$ *and* $-v_1 v_2 v_3 \not\sim 1$,

   d) $(\alpha, x^2 - \pi^\kappa u)$, *if* $m_2 + m_3 \equiv 0 \pmod 2$, $\kappa \equiv 1 \pmod 2$ *and* $u v_1 v_2 v_3 \sim 1$,

   e) $(f(0), x)$, *otherwise.*

4. $m_2 = m_3 = 1 < \kappa/2$, $v_i = v_1 + \pi^{s_i}\tau_i$, $i = 2, 3$, $s_2 \le s_3$ *and, if* $s_1 = s_2$, *then* $\bar\tau_2 \ne \bar\tau_3$. *Two generators are*

   a) (α, l_1), (α, l_2), *if* $s_2 \equiv 0 \pmod 2$,

   b) (π, l_1), (π, l_2), *if* $s_2 = s_3 \equiv 1 \pmod 2$,

   c) (π, l_1), (α, l_1), *if* $s_2 \equiv 1 \pmod 2$, $s_2 < s_3$ *and* $-\tau_2 \sim 1$,

   d) (π, l_1), (π, l_3), *if* $s_2 \equiv 1 \pmod 2$, $s_2 < s_3$ *and* $-\tau_2 \not\sim 1$.

   The third one is

   α) $(\alpha, x^2 - \pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$, $u v_1 \sim 1$,

   β) (α, x), *if* $\kappa \equiv 1 \pmod 2$, $u v_1 \not\sim 1$,

   γ) $(\pi E, x - \pi^{\kappa/2} w)$, *if* $\kappa \equiv 0 \pmod 2$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$, *and* $E \in O_k^*$ *such that* $f(\pi^{\kappa/2} w) \sim \pi E$.

Proof. 1. We have $(\alpha, l_1)_{l_2} \sim (\alpha, -\pi v_1) \not\sim 1$, $(\alpha, l_2)_{l_1} \sim (\alpha, \pi v_1) \not\sim 1$ and $(\alpha, l_1 l_2)_{l_1} \sim (\alpha, (x^2 - \pi^\kappa u) l_3)_{l_1} \sim (\alpha, (\pi v_1)^2 \pi v_1) \not\sim 1$.

Assume firstly $\kappa \equiv 1 \pmod 2$. Suppose $\varepsilon \in \{1, 2\}$ with $\varepsilon \equiv m_2 \pmod 2$ as well as $\theta = \alpha(\sqrt[3]{\pi})^{3m_2 - \varepsilon}$ and $p = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $3\kappa/2 < v_{k(\sqrt[3]{\pi})}(\theta) < 3m_2$. Let $u v_1 \sim 1$. Then we have $f(\theta) \sim u v_1 \sim 1$. Thus

$$(\pi, l_2)_p \sim \left(\frac{\alpha\ ,\ \theta_1}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \alpha d^2}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1.$$

So $(\pi, l_2) \not\sim 1$. If now $u v_1$ is arbitrary, then, after extension by $\sqrt{\alpha}$, we are still in subcase 1.*a* and $u v_1 \sim 1$, therefore the algebra (π, l_2) is nontrivial again does not split by $\sqrt{\alpha}$. Hence $(\pi, l_2) \not\sim (\alpha, l_1), (\alpha, l_2), (\alpha, l_1 l_2)$.

Let now $\kappa \equiv 0 \pmod 2$. We have

$$(\pi E, x - \pi^{\kappa/2} w)_{x^2 - \pi^\kappa u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})} \right) \not\sim 1.$$

Since $(\alpha, l_i)_{x^2 - \pi^\kappa u} \sim 1$ and $(\alpha, l_1 l_2)_{x^2 - \pi^\kappa u} \sim 1$, we are done.
2. $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_1 l_2)_{l_1} \not\sim 1$. If $m_2 > 1$, then $(\alpha, l_2)_{l_1} \not\sim 1$. If $m_2 = 1$, then $(\alpha, l_2)_{l_3} \not\sim 1$. Let us find the last algebra. Consider the case $\kappa \equiv 1 \pmod 2$. $(\alpha, x^2 - \pi^\kappa u)_{l_3} \not\sim 1$. We have also $(\alpha, x^2 - \pi^\kappa u)_{l_1} \sim 1$, $(\alpha, l_1 l_1)_{l_1} \not\sim 1$. If either $m_2 > 1$ or $\bar v_1 \neq \bar v_2$, then $(\alpha, l_1)_{l_2} \not\sim (\alpha, x^2 - \pi^\kappa u)_{l_2}$ and $(\alpha, l_2)_{l_1} \not\sim (\alpha, x^2 - \pi^\kappa u)_{l_1}$. Let $m_2 = 1$ and $\bar v_1 = \bar v_2$. Suppose $\theta = \pi \sqrt[3]{\pi}$. Then $f(\theta) \sim v_1 v_2 \sim 1$. So

$$(\alpha, l_1)_{x^3 - \pi^4} \sim \left(\frac{\alpha\ ,\ -\pi v_1}{k(\sqrt[3]{\pi})\langle x^3 - \pi^4 \rangle} \right) \not\sim 1, \quad (\alpha, l_2)_{x^3 - \pi^4} \not\sim 1$$

but

$$(\alpha, x^2 - \pi^\kappa u)_{x^3 - \pi^4} \sim (\alpha, \theta^2) \sim 1.$$

Let now $\kappa \equiv 0 \pmod 2$ and $m_2 + \kappa/2 \equiv 0 \pmod 2$. Then

$$(\pi E, x - \pi^{\kappa/2} w)_{x^2 - \pi^\kappa u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2 - \pi^\kappa u \rangle} \right) \not\sim 1.$$

To complete the consideration of subcase 2.c, it is enough to observe that all other algebras split in $x^2 - \pi^\kappa u$.

Finally, if $m_2 + \kappa/2 \equiv 1 \pmod 2$, then $f(\pi^{\kappa/2} w_i) \sim v_1 v_2 (w_i^2 - u)(w - \pi^{m_3 - \kappa/2} v_3)$, where $w_i \in O_K^*$ and $\bar w_i \neq \bar v_3$, provided that $m_3 = \kappa/2$. For $p_i = \mathrm{Irr}_{K|k}(\pi^{\kappa/2} w_i)$ we have $(\pi, l_3)_{p_i} \sim (\pi, (-1)^{\kappa/2}(w - \pi^{m_3 - \kappa/2} v_3))$. Let us require

$$\begin{cases} (-1)^{\kappa/2}(w_1 - \pi^{m_3 - \kappa/2} v_3) \not\sim 1, \\ (-1)^{\kappa/2} v_1 v_2 (w_1^2 - u) \not\sim 1, \end{cases} \quad \begin{cases} (-1)^{\kappa/2}(w_2 - \pi^{m_3 - \kappa/2} v_3) \sim 1, \\ (-1)^{\kappa/2} v_1 v_2 (w_2^2 - u) \sim 1. \end{cases}$$

Then $(\pi, l_3)_{p_1} \not\sim 1$, $(\pi, l_3)_{p_2} \sim 1$. Furthermore, the algebras $(\pi, l_1)_{p_1}$ and $(\pi, l_1)_{p_2}$ are either both trivial or both nontrivial. The same is true for the algebras $(\pi, l_2)_{p_1}$, $(\pi, l_2)_{p_2}$ and for the algebras $(\pi, l_1 l_2)_{p_1}$, $(\pi, l_1 l_2)_{p_2}$. So (π, l_3) is not isomorphic to (π, l_i) and $(\pi, l_1 l_2)$.
3. We have $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_2)_{l_3} \not\sim 1$ and $(\alpha, l_1 l_2)_{l_1} \not\sim 1$. If $m_2 + m_3$ and κ are even, then one can argue as in subcase 2.c.

Let $m_2 + m_3 \equiv 1 \pmod 2$, $\kappa \equiv 0 \pmod 2$ and $-v_1 v_2 v_3 \sim 1$. Then $f(\pi^{\kappa/2} w) \sim -v_1 v_2 v_3 (w^2 - u) \sim \alpha$, so the algebra $(\alpha, x - \pi^{\kappa/2} w)$ is unramified. Suppose Then $f(\theta_i) \sim \theta_i^2 \sim 1$ and

$$(\alpha, x - \pi^{\kappa/2} w)_{p_i} \sim \left(\frac{\alpha\ ,\ \theta_i}{k(\sqrt[3]{\pi})\langle p_i \rangle} \right),$$

therefore there is just one nontrivial algebra among $(\alpha, x - \pi^{\kappa/2} w)_{p_i}$, $i = 1, 2$. Since the algebras $(\alpha, l_1)_{p_1}$, $(\alpha, l_1)_{p_2}$ are either simultaneously trivial or simultaneously nontrivial, we have $(\alpha, x - \pi^{\kappa/2} w) \not\sim (\alpha, l_1)$. Similarly, $(\alpha, x - \pi^{\kappa/2} w) \not\sim (\alpha, l_2), (\alpha, l_1 l_2)$.

Let now $-v_1 v_2 v_3 \not\sim 1$. Then we have $f(\pi^{\kappa/2} w) \sim -v_1 v_2 v_3 (w^2 - u) \sim 1$ and $(\pi, x^2 - \pi^\kappa u)_{x - \pi^{\kappa/2} w} \sim (\pi, w^2 - u) \not\sim 1$. For any i $(\pi, x^2 - \pi^\kappa u)_{l_i} \sim 1$. On the other hand, $(\alpha, l_1)_{l_3} \not\sim 1$, $(\alpha, l_1 l_2)_{l_3} \not\sim 1$ and either $(\alpha, l_2)_{l_1} \not\sim 1$ or $(\alpha, l_2)_{l_3} \not\sim 1$.

The case of even κ is done. Let κ be odd. Then $f(0) \sim uv_1v_2v_3\pi^{m_2+m_3}$. In subcase 3.*d*, we have $f(0) \sim 1$ and $(\alpha, x^2 - \pi^\kappa u)_x \not\sim 1$. Moreover, $(\alpha, x^2 - \pi^\kappa u) \not\sim (\alpha, l_i), (\alpha, l_1l_2)$ as in the previous subcase.

Finally, in 3.*e*, the algebra $(f(0), x)$ is unramified. Assume firstly $m_2+m_3 \equiv 0 \pmod 2$, hence $f(0) \sim \alpha$.

$$(\alpha, x)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle -\alpha(x^2-\pi^\kappa u)\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\alpha, l_1)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ -\pi v_1}{k(\sqrt{\pi u})\langle -\alpha(x^2-\pi^\kappa u)\rangle}\right) \sim 1$$

and

$$(\alpha, l_2)_{x^2-\pi^\kappa u} \sim (\alpha, l_1l_2)_{x^2-\pi^\kappa u} \sim 1.$$

Let now $m_2 + m_3 \equiv 1 \pmod 2$. To finish the consideration of case 3, it is sufficient to prove that the algebra $(\pi u v_1v_2v_3, x)$ does not split after extension by $\sqrt{\alpha}$. This extension leaves us in subcase 3.*e*, so we can assume that $-1 \sim u \sim v_i \sim 1$. If $\theta = \pi^{m_3}\sqrt[3]{\pi}w$ and $p = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$, then

$$(\pi, x)_p \sim \left(\frac{\sqrt[3]{\pi}\ ,\ w}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1,$$

since $f(\theta) \sim -v_1v_2v_3 \sim 1$.

4. Consider the case $s_2 \equiv 0 \pmod 2$. We have $(\alpha, l_1)_{l_2} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$, $(\alpha, l_2)_{l_1} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$, and $(\alpha, l_1l_2)_{l_2} \sim (\alpha, l_3(x^2-\pi^\kappa u))_{l_2} \sim (\alpha, \pi^{s_2+1}) \not\sim 1$.

If $s_2 = s_3 \equiv 1 \pmod 2$, let $\theta_i = \pi(v_1 + \pi^{s_2}\mu_i)$, $\mu_i \in O_K^*$, $p_i = \mathrm{Irr}_{K|k}(\theta_i)$, then $f(\theta_i) \sim \mu_i(\mu_i - \tau_2)(\mu_i - \tau_3)$,

$$(\pi, l_1)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right),\quad (\pi, l_2)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i - \tau_2}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right),$$

$$(\pi, l_1l_2)_{p_i} \sim \left(\frac{\pi\ ,\ \mu_i(\mu_i-\tau_2)}{K(\sqrt{f(\theta_i)})\langle p_i\rangle}\right).$$

If we require

$$\begin{cases} \mu_1 \not\sim 1, \\ (\mu_1-\tau_2)(\mu_1-\tau_3) \not\sim 1, \end{cases} \quad \begin{cases} \mu_2 - \tau_2 \not\sim 1, \\ \mu_2(\mu_2-\tau_3) \not\sim 1, \end{cases} \quad \begin{cases} \mu_3(\mu_3-\tau_2) \not\sim 1, \\ \mu_3 - \tau_3 \not\sim 1, \end{cases}$$

then $(\pi, l_1)_{p_1} \not\sim 1$, $(\pi, l_2)_{p_2} \not\sim 1$, and $(\pi, l_1l_2)_{p_3} \not\sim 1$.

Let s_1 be odd, $s_1 < s_2$ and $\theta = \pi(v_1+\pi^{s_2}\mu)$, $\mu \in O_K^*$, $p = \mathrm{Irr}_{K|k}(\theta)$, then $f(\theta) \sim \mu - \tau_2$, $(\pi, l_1)_p \sim (\pi, \mu)$. If

$$\begin{cases} \mu \not\sim 1, \\ \mu - \tau_2 \not\sim 1, \end{cases}$$

then $(\pi, l_1)_p \not\sim 1$. Suppose $\eta = \pi(v_1 + \pi^{s_2}\sqrt[3]{\pi})$, $q = \mathrm{Irr}_{k(\sqrt[3]{(\pi)})|k}(\eta)$. In this case $f(\eta) \sim -\tau_2$ and

$$(\alpha, l_1)_q \sim \left(\frac{\alpha\ ,\ \pi^{s_2+1}\sqrt[3]{\pi}}{k(\sqrt[3]{\pi})|k}\right) \not\sim 1,$$

provided that $-\tau_2 \sim 1$. After replacing the prime element by $\nu = \pi\alpha$ we have $-\tau_2' = -\tau_2\alpha^{-s_2-1} \sim 1$, so we are in the same subcase, hence $(\pi\alpha, l_1) = (\nu, l_1) \not\sim 1$. Let now $-\tau_2 \not\sim 1$. The replacement $v_1' = v_1 + \pi^{s_3}\tau_3$ leaves us in the current subcase, so $(\pi, l_3) = (\pi, l_1') \not\sim 1$. $(\pi, l_1 l_3)_{l_1} \sim (\pi, l_2)_{l_1} \sim (\pi, -\tau_2) \not\sim 1$.

So two generators are found. To find the third one we can argue just the same way as in subcase 3. To complete the proof it remains only to show that the third generator does not belong to the group generated by the first two. This is true for the algebra from α), because it is trivial everywhere in this proof, where the algebras from the group, generated by algebras a)–d), are nontrivial. Finally, the algebras from β and γ) are nontrivial in $x^2 - \pi^\kappa u$, but the algebras from a)–d) are trivial there. The theorem is proved.

References

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[3] G. Margolin, Ulf Rehmann, V. Yanchevskii: *Quaternion generation of the 2-torsion part of the Brauer group of local quintic (the type of splitting (1,1,1,1,1)).*

Quaternion generation of the 2-torsion part of the Brauer group of local quintic (the type of splitting (1,2,2))

Genady Margolin Ulf Rehmann Vyacheslav Yanchevskii

1 Introduction

This paper is a continuation of [3] and [4] where we consider a quintic given by an affine equation $y^2 = f(x)$, and $f(x)$ has the splitting types $(1,1,1,1,1)$ resp. $(1,1,1,2)$. Here we consider the case where the splitting type of $f(x)$ is $(1,2,2)$. We use the same notations as in the previous papers.

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Let k be a local non-dyadic field of characteristic 0 with residue field $\bar{k}$. Let $\pi \in k$ denote a prime element, and let O_k, O_k^*, πO_k be its valuation ring, the group of units and the maximal ideal of the valuation ring. For $u \in O_k^*$ let $\bar{u}$ denote its class *modulo* πO_k. We will consider also a hyperelliptic curve C defined over k by the equation $y^2 = f(x)$, where $f(x) \in O_k[x]$ is a monic polynomial without multiple roots and $deg\, f = n$. Without loss of generality one can assume that $f(x) = \prod_{i=1}^{m} f_i(x)$, where $f_i(x) \in O_k[x]$ are monic irreducible polynomials. This paper is devoted to the case deg $f_1 =$ deg $f_2 = 2$, deg $f_3 = 1$ and the reduction is bad. Let α be a fixed unit of O_k which is not a square in k. $\bar{g}(x)$ will denote the reduction of a polynomial $g(x) \in O_k$. For $a, b \in k^*$ we write $a \sim b$ if a and b belong to the same class *modulo* $(k^*)^2$.

In our further considerations we will need lemmas from [3] and [4].

The evident list of all cases under consideration according to the reduction type of $f(x)$ is as follows $(f(x) = (x^2 - \pi^\kappa u)((x-a)^2 - \delta)(x-e),\ \kappa \geq 0,\ u \in O_k^*,\ a, \delta, e \in O_k,\ \pi^\kappa u, \delta \not\sim 1)$.

I. $\kappa = 0,\ a \in M_k,\ \delta \in O_k^*,\ \bar{u} = \bar{\delta}$.

II. $\kappa > 0,\ \delta \in O_k^*,\ e \in O_k^*$.

III. $\kappa > 0,\ \delta \in O_k^*,\ e \in M_k$.

IV. $\kappa > 0,\ \delta \in M_k,\ a \in O_k^*,\ e \in O_k^*,\ \bar{a} \neq \bar{e},\ \kappa \leq v(\delta)$.

V. $\kappa > 0,\ \delta \in M_k,\ a \in O_k^*,\ e \in M_k$.

VI. $\kappa > 0,\ \delta \in M_k,\ a \in M_k,\ e \in O_k^*,\ \kappa \leq v(\delta)$.

VII. $\kappa > 0,\ \delta \in M_k,\ a \in M_k,\ e \in M_k,\ \kappa \le v(\delta)$.

Lemma 1

$$|\,{}_2\mathrm{Br}\, C| = 8.$$

For the proof see theorem 1 (p. 236) and §4 (p. 237) in [2].

Theorem 1 *Let f be as in case* I. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebras (π, α), $(\pi, x^2 - u)$ and of following algebra.*

1. *$(\pi, x - e)$, if either $e \in O_k^*,\ e^2 - u \not\sim 1$ or $e \in M_k,\ -1 \sim 1$.*
2. *$(\alpha(x^2 - u), x - w)$, otherwise,*

where $w \in O_k^*,\ w^2 - u \not\sim 1$.

Remark 1 *The algebra $(\alpha(x^2 - u), x - w)$ does not split by any quadratic extension of scalars.*

Proof of the theorem. The algebra $(\pi, x^2 - u)$ is nontrivial by lemma 3 from [3]. We also have

$$(\pi, x - e)_{x^2-u} \sim \left(\frac{\pi\ ,\ \sqrt{u} - e}{k(\sqrt{u})\langle x^2 - u\rangle(\sqrt{D(x^2 - u)})} \right),\ D \in k^*.$$

The last algebra is nontrivial iff $\sqrt{u} - e \not\sim 1$ in $k(\sqrt{u})$. And this is true if either $e \in O_k^*,\ e^2 - u \not\sim 1$ or $e \in M_k,\ -1 \sim 1$. The algebra $(\pi, (x^2 - u)(x - e))$ is nontrivial by lemma 3 from [3] again.

Consider now the algebra $A = (\alpha(x^2 - u), x - w)$. It is unramified because $A_{x-w} \sim (\alpha(w^2 - u), x - w) \sim 1$. Let $\theta = (1 + \sqrt[3]{\pi})\sqrt{u}$ and $p = \mathrm{Irr}_{k(\sqrt[3]{\pi}, \sqrt{u})|k}(\theta)$. Then $f(\theta) \sim \sqrt{u} - e \sim 1$, provided that we are in the subcase 2. So

$$A_p \sim \left(\frac{(1 + \sqrt[3]{\pi})^2 u - u\ ,\ \sqrt{u} - w}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p\rangle} \right) \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt{u} - w}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p\rangle} \right) \not\sim 1.$$

On the other hand,

$$(\pi, x^2 - u)_p \sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi}, \sqrt{u})\langle p\rangle} \right) \sim 1.$$

The theorem is proved.

Theorem 2 *Let f be as in cases* II *or* III. *Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras.*

- *Case* I. *The first algebra is $(\pi, x - e)$ and the second one is*
  1. *$(e\xi u\pi, x)$, if $\kappa \equiv 1 \pmod 2$.*
  2. *$(\pi, x^2 - \pi^\kappa u)$, if $\kappa \equiv 0 \pmod 2$ and $e\xi \not\sim 1$.*
  3. *$(\alpha, x - \pi^{\kappa/2} w)$, if $\kappa \equiv 0 \pmod 2$ and $e\xi \sim 1$.*
- *Case* II *($e = \pi^m v$).*

1. $(\pi, x-\pi^m v)$, $(\pi, x^2-\pi^\kappa u)$, *if* $m \le \kappa/2$, $m, \kappa \equiv 0 \pmod 2$ *and either* $\xi v \not\sim 1$ *or* $m = \kappa/2$.
2. $(\pi, x-\pi^m v)$, $(\alpha, x-\pi^{\kappa/2} w)$, *if* $m \le \kappa/2$, $m, \kappa \equiv 0 \pmod 2$, $\xi v \sim 1$ *and* $m < \kappa/2$.
3. $(\pi, (x-a)^2-\delta)$, $(\alpha\xi\eta\pi, x-\pi^{\kappa/2} w)$, *if* $m \le \kappa/2$, $\kappa \equiv 0 \pmod 2$, $m \equiv 1 \pmod 2$. *Here*
$$\eta = \begin{cases} v, & \text{if } m < \kappa/2, \\ v-w, & \text{if } m = \kappa/2. \end{cases}$$
4. $(\pi, x-\pi^m v)$, $(\alpha, x^2-\pi^\kappa u)$, *if* $m \le \kappa/2$, $m, \kappa \equiv 1 \pmod 2$, $-\xi uv \sim 1$.
5. $(\pi, x-\pi^m v)$, (α, x), *if* $m \le \kappa/2$, $m, \kappa \equiv 1 \pmod 2$, $-\xi uv \not\sim 1$.
6. $(\pi, x-\pi^m v)$, $(-\xi uv\pi, x)$, *if* $m \le \kappa/2$, $\kappa \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$.
7. $(\pi, x-\pi^m v)$, $(\alpha, x^2-\pi^\kappa u)$, *if* $m > \kappa/2$, $\kappa \equiv 1 \pmod 2$.
8. $(\pi, x-\pi^m v)$, $(\pi, x^2-\pi^\kappa u)$, *if* $m > \kappa/2$, $\kappa \equiv 0 \pmod 4$.
9. $(\pi, (x-a)^2-\delta)$, $(-\xi w\alpha\pi, x-\pi^{\kappa/2} w)$, *if* $m > \kappa/2$, $\kappa \equiv 2 \pmod 4$.

where $w \in O_k^*$, $w^2-u \not\sim 1$ *and*
$$\xi = \begin{cases} \delta, & \text{if } a \in M_k, \\ \delta - a^2, & \text{otherwise.} \end{cases}$$

Proof. In II and III we have $(\pi, x-e) \not\sim 1$ by lemma 3 from [3].
In II.1 we have $f(0) \sim -e\xi u\pi$, so the algebra $(-e\xi u\pi, x)$ is unramified. Let $K|k$ be an unramified odd degree extension and let $\tau \in O_K^*$ be such that the following conditions hold:
$$\begin{cases} \tau \not\sim 1, \\ \tau - e \sim 1, \\ (\tau-a)^2 - \delta \sim 1. \end{cases}$$
Such K and τ exist in view of lemmas 5 and 6 from [3]. Let $p = \mathrm{Irr}_{K|k}(\tau)$. Then $f(\tau) \sim 1$. Thus
$$(\pi, x-e)_p \sim \left(\frac{\pi\ ,\ \tau - e}{K\langle p\rangle}\right) \sim 1 \text{ and } (-e\xi u\pi, x)_p \sim \left(\frac{\pi\ ,\ \tau}{K\langle p\rangle}\right) \not\sim 1.$$
Consider the subcase II.2. We have $f(w\pi^{\kappa/2}) \sim e\xi(w^2-u) \sim 1$, so $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \not\sim 1$. Finally, $(\pi, x-e) \not\sim (\pi, x^2-\pi^\kappa u)$ by lemma 3 from [3].
If we are in II.3, then $f(w\pi^{\kappa/2}) \sim \alpha$, so that the algebra $(\alpha, x-w\pi^{\kappa/2})$ is unramified. Let $\varepsilon \in \{1,2\}$, $\varepsilon \equiv \kappa/2 \pmod 2$, $\theta = \pi^{\kappa/2-1}(\sqrt[3]{\pi})^\varepsilon$ and $p = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim e\xi \sim 1$ and
$$(\alpha, x-w\pi^{\kappa/2})_p \sim \left(\frac{\alpha\ ,\ \pi^{\kappa/2-1}(\sqrt[3]{\pi})^\varepsilon}{k(\sqrt[3]{\pi})\langle p\rangle}\right) \not\sim 1.$$
To prove $(\pi, x-e) \not\sim (\alpha, x-w\pi^{\kappa/2})$, it is enough to observe that the extension by $\sqrt{\pi}$ leaves us in the subcase II.3, hence this extension does not split the algebra $(\alpha, x-w\pi^{\kappa/2})$.
In III the algebras $(\pi, x-\pi^m v)$ and $(\pi, (x-a)^2-\delta)$ are nontrivial.
III.1. Since $f(\pi^{\kappa/2}w) \sim \xi v \sim 1$, if $m < \kappa/2$, we have $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \not\sim 1$. Let now $\kappa/2 = m$. Then we require for $\tau \in O_K^*$
$$\begin{cases} \tau^2 - u \sim \alpha, \\ v - \tau \sim \alpha\xi, \end{cases}$$

$K|k$ is unramified of odd degree, then $f(\pi^{\kappa/2}\tau) \sim 1$ and $(\pi, x^2 - \pi^\kappa u)_{\mathrm{Irr}(\pi^{\kappa/2}\tau)} \not\sim 1$. Furthermore, $(\pi, x^2 - \pi^\kappa u) \otimes (\pi, x - \pi^m v) \sim (\pi, (x-a)^2 - \delta) \not\sim 1$.
III.2. $f(\pi^{\kappa/2}w) \sim \alpha$, so that $(\alpha, x - \pi^{\kappa/2}w)$ is unramified. Now we can argue as in II.3.
III.3. In view of $f(\pi^{\kappa/2}w) \sim \xi\eta(w^2 - u)\pi$ the algebra $A = (\xi\eta(w^2-u)\pi, x - \pi^{\kappa/2}w)$ is unramified.

$$A_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2 - \pi^\kappa u\rangle(\sqrt{D(x^2-\pi^\kappa u)})} \right) \not\sim 1,\ (D \in k^*).$$

On the other hand,

$$(\pi, (x-a)^2 - \delta)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ -\xi}{k(\sqrt{u})\langle x^2 - \pi^\kappa u\rangle(\sqrt{D(x^2-\pi^\kappa u)})} \right) \sim 1.$$

III.4. We have $f(0) \sim -\xi uv$. Thus $(\alpha, x^2 - \pi^\kappa u)_x \not\sim 1$. Moreover, although the extension by $\sqrt{\alpha}$ leads us to the case $(1,1,1,2)$, this extension does not split the algebra $(\pi, x - \pi^m v)$. Therefore, $(\alpha, x^2 - \pi^\kappa u) \not\sim (\pi, x - \pi^m v)$.
III.5. $f(0) \sim \alpha$, so the algebra (α, x) is unramified. $(\alpha, x)_{x-\pi^m v} \not\sim 1$. The further arguments are similar to the previous subcase.
III.6. The algebra $A = (-\xi uv\pi, x)$ is unramified. Let $\theta = \pi^m\tau$, $\tau \in O_K^*$, $p = \mathrm{Irr}_{K|k}(\theta)$. Then $f(\theta) \sim \xi(v - \tau)$ and $A_p \sim (\pi, \tau)$. If we require

$$\begin{cases} \tau \not\sim 1, \\ \xi(v-\tau) \sim 1, \end{cases}$$

then $A_p \not\sim 1$. And if we require

$$\begin{cases} \tau \not\sim -\xi, \\ \xi(v-\tau) \sim 1, \end{cases}$$

then $A_p \not\sim (\pi, x - \pi^m v)_p$ since in this case $(\pi, x - \pi^m v)_p \sim (\pi, -\xi)$.
III.7. $(\alpha, x^2 - \pi^\kappa u)_{x-\pi^m v} \not\sim 1$. Moreover, the extension by $\sqrt{\alpha}$ does not split the algebra $(\pi, x - \pi^m v)$.
III.8. We have $f(\pi^{\kappa/2}\tau) \sim -\xi\tau(\tau^2 - u)$. If $\tau^2 - u \not\sim 1$ and $-\xi\tau \not\sim 1$, then $(\pi, x^2 - \pi^\kappa u)_{\mathrm{Irr}(\pi^{\kappa/2}\tau)} \not\sim 1$.
III.9. $f(\pi^{\kappa/2}w) \sim -\xi w\alpha\pi$, so the algebra $A = (-\xi w\alpha\pi, x - \pi^{\kappa/2}w)$ is unramified. Finally,

$$A_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2 - \pi^\kappa u\rangle(D(x^2-\pi^\kappa u))} \right) \not\sim 1,$$

but

$$(\pi, (x-a)^2 - \delta)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ -\xi}{k(\sqrt{u})\langle x^2 - \pi^\kappa u\rangle(D(x^2-\pi^\kappa u))} \right) \sim 1.$$

The theorem is proved.

Theorem 3 *Let f be as in case* IV.
($f(x) = (x-e)(x^2 - \pi^\kappa u)((x-v)^2 - \pi^m\gamma)$, $e, u, v, \gamma \in O_k^*$, $\bar{v} \neq \bar{\gamma}$, $0 < \kappa \leq m$.) Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras. The first algebra is

1. $(\pi, x-e)$, *if* $\kappa \equiv 0 \pmod 2$, $-e \not\sim 1$.

2. $(\alpha, x-\pi^{\kappa/2}w)$, *if* $\kappa \equiv 0 \pmod 2$, $-e \sim 1$.

3. $(\pi ue, x)$, *if* $\kappa \equiv 1 \pmod 2$.

The second one is

a. $(\pi, (x-v)^2 - \pi^m\gamma)$, *if* $m \equiv 0 \pmod 2$, $v-e \not\sim 1$.

b. $(\alpha, x-v-\pi^{m/2}\tau)$, *if* $m \equiv 0 \pmod 2$, $v-e \sim 1$.

c. $(\pi(e-v)\gamma, x-v)$, *if* $m \equiv 1 \pmod 2$,

where $w, \tau \in O_k^*$, $w^2-u \not\sim 1$, $\tau^2-\gamma \not\sim 1$.

Proof. If $\kappa \equiv 0 \pmod 2$, then $f(\pi^{\kappa/2}w) \sim -e\alpha$. In case 1 we have $f(\pi^{\kappa/2}w) \sim 1$ and $(\pi, x-e)_{x-\pi^{\kappa/2}w} \sim (\pi, -e) \not\sim 1$. In addition, let us consider the algebra $(\pi, x^2-\pi^\kappa u)$. It is also nontrivial in this place. Indeed, $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \sim (\pi, w^2-u) \not\sim 1$.

In case 2 $f(\pi^{\kappa/2}w) \sim \alpha$ and the algebra $(\alpha, x-\pi^{\kappa/2}w)$ is unramified. Now one can prove $(\alpha, x-\pi^{\kappa/2}w) \not\sim 1$ as in the proof of theorem 2, subcase II.3.

Let now $\kappa \equiv 1 \pmod 2$, $\mu \in O_K^*$, K be as above and

$$\begin{cases} \mu \sim \alpha ue, \\ u-\mu \sim \alpha u. \end{cases}$$

Assume $\theta = \sqrt{\pi^\kappa \mu} \in K(\sqrt{\pi\mu})$ and $q = \mathrm{Irr}_{K(\sqrt{\pi\mu})|k}(\theta)$. Then $f(\theta) \sim e(u-\mu)\pi \sim e\mu(u-\mu) \sim 1$ and

$$(\pi ue, x)_q \sim \left(\frac{\pi ue\ ,\ \theta}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim \left(\frac{ue\tau\ ,\ \sqrt{\pi\mu}}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \not\sim 1.$$

After replacing $x' = x-v$ we have

$$f(x') = (x'-(e-v))((x')^2-\pi^m\gamma)((x'-(-v))^2-\pi^\kappa u).$$

The above arguments show that the algebras from items a, b, c are unramified and nontrivial. (Because nowhere in this arguments we used that $\kappa/2 < m$.) To finish the proof it is enough to check, that these algebras are not isomorphic to the algebras from items $1, 2, 3$ in the corresponding subcases.
$1.a$. We have $(\pi, x-e) \otimes (\pi, (x-v)^2-\pi^m\gamma) \sim (\pi, x^2-\pi^\kappa u) \not\sim 1$.
$2.b$. If A is the algebra from 2 and A' is the one from b, then $A_p \not\sim 1$ and $A'_p \sim (\alpha, -v) \sim 1$, where p is constructed as in the proof of theorem 2, subcase II.3..
$1.b$, $2.a$. For symmetry reasons one needs to consider only one case, say, $2.a$. In this case, for p as above, $A_p \not\sim 1$, but $(\pi, (x-v)^2-\pi^m\gamma)_p \sim (\pi, v^2) \sim 1$.
$3.a$. $(\pi ue, x)_q \not\sim 1$, but $(\pi, (x-v)^2-\pi^m\gamma)_q \sim 1$.
$3.b$. $A'_q \sim (\alpha, -v) \sim 1$.
$3.c$. Finally, we have

$$(\pi(e-v)\gamma, x-v)_q \sim \left(\frac{\pi(e-v)\gamma\ ,\ -v}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim \left(\frac{\tau(e-v)\gamma\ ,\ -v}{K(\sqrt{\pi\mu})\langle q\rangle}\right) \sim 1,$$

and this completes the proof.

Theorem 4 *Let f be as in case* V. *($f(x) = ((x-v)^2 - \pi^m\gamma)(x-\pi^r\delta)(x^2-\pi^\kappa u)$, $v, \gamma.\delta, u \in O_k^*$, $m, r, \kappa > 0$, $\pi^m\gamma, \pi^\kappa u \not\sim 1$.) Then the group ${}_2\mathrm{Br}\,C$ is generated by the classes of the algebra (π, α) and of the following algebras.*
The first algebra is

i) $(\pi, (x-v)^2 - \pi^m\gamma)$, *if* $m \equiv 0 \pmod 2$, $v \not\sim 1$.

ii) $(\alpha, x - v - \pi^{m/2}w)$, *if* $m \equiv 0 \pmod 2$, $v \sim 1$.

iii) $(-v\gamma\pi, x-v)$, *if* $m \equiv 1 \pmod 2$.

The second one is

a. $(\alpha, x - \pi^r\delta)$, *if* $\kappa/2 < r$, $\kappa \equiv 1 \pmod 2$.

b. $(\pi, x - \pi^r\delta)$, *if* $\kappa/2 < r$, $\kappa \equiv 0 \pmod 4$.

c. $(\tau(\tau^2 - u)\pi, x - \pi^{\kappa/2}\tau)$, *if* $\kappa/2 < r$, $\kappa \equiv 2 \pmod 4$.

d. $(\pi\delta u, x)$, *if* $\kappa/2 > r$, $\kappa \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$.

e. $(\pi, x - \pi^r\delta)$, *if* $\kappa \equiv r \equiv 0 \pmod 2$, *and either* $\kappa/2 = r$ *or* $\kappa/2 > r$, $-\delta \not\sim 1$.

f. $(\alpha, x - \pi^{\kappa/2}\tau)$, *if* $\kappa \equiv r \equiv 0 \pmod 2$, $\kappa/2 > r$ *and* $-\delta \sim 1$.

g. $(\xi\alpha\pi, x - \pi^{\kappa/2}\tau)$, *if* $\kappa \equiv 0 \pmod 2$, $r \equiv 1 \pmod 2$, $\kappa/2 \geq r$, *where*

$$\xi = \begin{cases} -\delta, & \text{if } \kappa/2 > r, \\ \tau - \delta - a^2, & \text{if } \kappa/2 = r. \end{cases}$$

h. $(\alpha, x - \pi^r\delta)$, *if* $\kappa \equiv 1 \pmod 2$, $r \equiv 1 \pmod 2$, $\kappa/2 > r$ *and* $\delta u \sim 1$.

k. (α, x), *if* $\kappa \equiv 1 \pmod 2$, $r \equiv 1 \pmod 2$, $\kappa/2 > r$ *and* $\delta u \not\sim 1$,

where $w, \tau \in O_k^*$, $w^2 - \gamma \not\sim 1$, $\tau^2 - u \not\sim 1$.

Proof.
i. $f(\pi^{m/2}w + v) \sim v(w^2 - u) \sim 1$, so $(\pi, (x-v)^2 - \pi^m\gamma)_{x-v-\pi^{m/2}w} \not\sim 1$.
ii. $f(\pi^{m/2}w + v) \sim v(w^2 - u) \sim \alpha$, so the algebra $(\alpha, x - v - \pi^{m/2}w)$ is unramified. We also have $f(v + \sqrt[3]{\pi}) \sim 1$, therefore

$$(\alpha, x - v - \pi^{m/2}w)_{(x-v)^3-\pi} \sim \left(\frac{\alpha\ ,\ \sqrt[3]{\pi}}{k(\sqrt[3]{\pi})\langle (x-v)^3 - \pi\rangle}\right) \not\sim 1.$$

iii. $f(v) \sim -v\gamma\pi$, $(-v\gamma\pi, x-v)$ is unramified. As usual, let $\chi \in O_K^*$ satisfy the conditions

$$\begin{cases} \chi \sim -\alpha v\gamma, \\ \chi - \gamma \sim -\alpha\gamma. \end{cases}$$

Let $\theta = v + \sqrt{\pi^m\chi}$, $q = \mathrm{Irr}_{K(\sqrt{\pi\chi})|k}(\theta)$. Then $f(\theta) \sim \chi v(\chi - \gamma) \sim 1$, hence

$$(-v\gamma\pi, x-v)_q \sim \left(\frac{-v\gamma\chi\ ,\ \sqrt{\pi\chi}}{K(\sqrt{\pi\chi})\langle q\rangle}\right) \not\sim 1.$$

a. $(\alpha, x-\pi^r\delta)_{x-\pi^r\delta} \sim (\alpha, ((x-v)^2-\pi^m\gamma)(x^2-\pi^\kappa u)_{x-\pi^r\delta}) \sim (\alpha, -\pi^\kappa u) \not\sim 1$.
b. We have $f(\mu\pi^{\kappa/2}) \sim -\mu(\mu^2-u)$ for some $\mu \in O_K^*$. Thus $(\pi, x-\pi^r\delta)_{\mathrm{Irr}(\mu\pi^{\kappa/2})} \sim (\pi, \mu) \not\sim 1$, provided that

$$\begin{cases} \mu \not\sim 1, \\ u-\mu^2 \not\sim 1. \end{cases}$$

c. $f(\pi^{\kappa/2}\tau) \sim \tau\alpha\pi$. Furthermore,

$$(\tau\alpha\pi, x-\pi^{\kappa/2}\tau)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ \tau-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^\kappa u\rangle}\right) \not\sim 1.$$

d. If $\nu \in O_K^*$ satisfies the conditions

$$\begin{cases} \nu \sim \alpha\delta u, \\ \nu - u \sim -\alpha u, \end{cases}$$

and $t = \mathrm{Irr}_{K(\sqrt{\pi\nu})|k}(\sqrt{\pi^\kappa\nu})$. Then $f(\sqrt{\pi^\kappa\nu}) \sim -\delta\nu(\nu-u) \sim 1$, so that

$$(\pi\delta u, x)_t \sim \left(\frac{\nu\delta u\ ,\ \sqrt{\pi\nu}}{K(\sqrt{\pi\nu})\langle t\rangle}\right) \not\sim 1.$$

e. If $\kappa/2 = r$, then $f(\pi^{\kappa/2}\lambda) \sim (\lambda-\delta)(\lambda^2-u)$ for some $\lambda \in O_K^*$. Moreover, if $s = \mathrm{Irr}_{K|k}(\pi^{\kappa/2}\lambda)$, then

$$(\pi, x-\pi^r\delta)_s \sim \left(\frac{\pi\ ,\ \lambda-\delta}{K(\sqrt{(\lambda-\delta)(\lambda^2-u)})\langle s\rangle}\right).$$

If we require

$$\begin{cases} \lambda-\delta \not\sim 1, \\ \lambda^2-u \not\sim 1, \end{cases}$$

then $(\pi, x-\pi^r\delta)_s \not\sim 1$.

Let now $\kappa/2 > r$. We have $f(\pi^{\kappa/2}\tau) \sim -\delta(\tau^2-u) \sim 1$. So, $(\pi, x-\pi^r\delta)_{x-\pi^{\kappa/2}\tau} \sim (\pi, -\delta) \not\sim 1$.
f. $f(\pi^{\kappa/2}\tau) \sim -\delta(\tau^2-u) \sim \alpha$ and the algebra $(\alpha, x-\pi^{\kappa/2}\tau)$ is unramified. To prove $(\alpha, x-\pi^{\kappa/2}\tau) \not\sim 1$, one can follow the proof of theorem 2, subcase II.3.
g. Similar as in subcase *c* we have $(\xi\alpha\pi, x-\pi^{\kappa/2}\tau)_{x^2-\pi^\kappa u} \not\sim 1$.
h. $f(0) \sim \pi^{\kappa+r}\delta u \sim 1$, therefore $(\alpha, x-\pi^r\delta)_x \not\sim 1$.
k. $f(0) \sim \pi^{\kappa+r}\delta u \sim \alpha$, hence the algebra (α, x) is unramified. Finally, $(\alpha, x)_{x-\pi^r\delta} \not\sim 1$.

Now, to complete the proof one only needs to check that the algebra from i)–iii) is not isomorphic to the algebra from *a*–*k*. Note, that in all the subcases *a*–*k* the nontriviality of the corresponding algebra is showed by completion at the place with prime element $l \in k[x]$ such that $l = \mathrm{Irr}(\theta)$ and $\theta \in M_{k_{alg}}$. Thus, in *i*) and *ii*) we have $(\pi, (x-v)^2-\pi^m\gamma)_l \sim (\pi, v^2) \sim 1$ and $(\alpha, x-v-\pi^{m/2}w)_l \sim (\alpha, -v) \sim 1$. The nontriviality of the algebra $(-v\gamma\pi, x)$ from iii) is proved by completion at $q = \mathrm{Irr}_{K(\sqrt{\pi\chi})|k}(\theta)$ and $\theta = v+\sqrt{\pi^m\chi}$. Any algebra from *a*–*k* completed at q can be obtained from some algebra over k by extension of scalars. This is trivial in q, since the extension $k(\sqrt{\pi\chi})$ splits any algebra over k. The theorem is proved.

Consider the next case.

Theorem 5 *Let f be as in case* VI. *($f(x)=(x-e)(x^2-\pi^\kappa u)((x-\pi^m v)^2-\pi^r\delta)$, $u,v,\delta,e\in O_k^*$, $0<m, 0<\kappa\le r$, $\pi^\kappa u,\pi^r\delta\not\sim 1$.) Then the group ${}_2\mathrm{Br}\,C$ is generated by the classes of the algebra (π,α) and of the algebras described below.*

If $\kappa, r\equiv 1\,(\mathrm{mod}\,2)$, we assume $2m\le\kappa$, and if $\kappa=r\equiv 0\,(\mathrm{mod}\,2)$, $2m>\kappa$, we assume $\bar u\ne\bar\delta$. Then the first algebra is

i) $(\pi E, x)$, *if* $\kappa\equiv 1\,(\mathrm{mod}\,2)$.

ii) $(\pi, x^2-\pi^\kappa u)$, *if* $\kappa\equiv 0\,(\mathrm{mod}\,2)$ *and* $-e\not\sim 1$.

iii) $(\alpha, x-\pi^{\kappa/2}w)$, *if* $\kappa\equiv 0\,(\mathrm{mod}\,2)$, *either* $\kappa<r$ *or* $\kappa=r>2m$, *and* $-e\sim 1$.

iv) $(\alpha, \mathrm{Irr}_{K|k}(\pi^{\kappa/2}\beta))$, *if* $\kappa\equiv 0\,(\mathrm{mod}\,2)$, $\kappa=r\le 2m$, $-e\sim 1$, *where* $\beta\in O_K^*$ *and*

$$(\beta^2-u)\times\begin{cases}\beta^2-\delta, & \text{if } \kappa<2m,\\ (\beta-v)^2-\delta, & \text{if } \kappa=2m.\end{cases}$$

The second algebra is

a. $(\pi F, x-\pi^m v)$, *if* $r\equiv 1\,(\mathrm{mod}\,2)$.

b. $(\pi, (x-\pi^m v)^2-\pi^r\delta)$ *if* $r\equiv 0\,(\mathrm{mod}\,2)$ *and*

- $\kappa=r$ *and either* $\kappa\le 2m$ *or* $-e\not\sim 1$,
- $2m<\kappa<r$ *and* $-e\not\sim 1$,
- $2m=\kappa<r$ *and* $-e(v^2-u)\not\sim 1$,
- $\kappa<r, \kappa<2m$, $\kappa\equiv 0\,(\mathrm{mod}\,2)$, *and* $ue\not\sim 1$.

c. $(\alpha, x-\pi^m v-\pi^{r/2}\tau)$ *if* $r\equiv 0\,(\mathrm{mod}\,2)$ *and*

- $\kappa=r>2m$, $-e\sim 1$,
- $2m<\kappa<r$ *and* $-e\sim 1$,
- $2m=\kappa<r$ *and* $-e(v^2-u)\sim 1$,
- $\kappa<r, \kappa<2m$, $\kappa\equiv 0\,(\mathrm{mod}\,2)$, *and* $ue\sim 1$.

d. $(ueN_{K|k}(\gamma^2-\delta)\pi, \mathrm{Irr}_{K|k}(\pi^m v+\pi^{r/2}\gamma))$, *if* $r\equiv 0\,(\mathrm{mod}\,2)$, $\kappa<r, \kappa<2m$ *and* $\kappa\equiv 1\,(\mathrm{mod}\,2)$, *where* $\gamma\in O_K^*$, $K|k$ *is of odd degree and unramified,*

$$\gamma^2-\delta\not\sim\begin{cases}1, & \text{if } 2m<r,\\ v^2-\delta, & \text{if } 2m=r,\\ -\delta, & \text{if } 2m>r.\end{cases}$$

Otherwise we have

1. $(\pi, x^2-\pi^\kappa u)$, $(\alpha(x^2-\pi^\kappa u), \pi^{\kappa/2}(x-\pi^{\kappa/2}w))$, *if* $\kappa=r\equiv 0\,(\mathrm{mod}\,2)$, $2m>\kappa$, $\bar u=\bar\delta$.

2. *The first algebra:* $(eu\pi, x)$, *the second one: either* $(\alpha, x^2-\pi^\kappa u)$, *provided that* $-ue\delta\sim 1$, *or* $(\alpha, x-\pi^m v)$ *otherwise, if* $\kappa, r\equiv 1\,(\mathrm{mod}\,2)$, $\kappa<2m<r$.

3. *The first algebra: either* $(\alpha, x^2-\pi^\kappa u)$*, provided that* $-ue\delta \sim 1$*, or* (α, x) *otherwise, the second one:* $(\pi ue, x-\pi^s)$*,* $\kappa/2 < s < r/2$*, if* $\kappa, r \equiv 1 \pmod 2$*,* $\kappa < r < 2m$*.*

4. *The first algebra from 3, and* $(x, -\pi u(x^2-\pi^\kappa u))$*, if* $\kappa = r \equiv 1 \pmod 2$*,* $\kappa < 2m$*.*

where $E, F, w, \tau \in O_k^*$ *can be found from the conditions* $f(0) \sim \pi E$*,* $f(\pi^m v) \sim \pi F$*,* $w^2 - u \not\sim 1$*,* $\tau^2 - \delta \not\sim 1$*.*

Remark 2 *The algebra* $(\alpha(x^2-\pi^\kappa u), \pi^{\kappa/2}(x-\pi^{\kappa/2}w))$ *from 1. does not split by any quadratic extension of scalars.*

Proof of the theorem.
i). If $\mu \in O_K^*$, then

$$f(\sqrt{\pi^\kappa \mu}) \sim \begin{cases} -e(\mu-u)(\mu-\delta)\,, & \text{if } \kappa = r < 2m, \\ -e\mu(\mu-u) \quad\;\; , & \text{otherwise.} \end{cases}$$

Let $t = \mathrm{Irr}_{K(\sqrt{\pi\mu})|k}(\sqrt{\pi^\kappa\mu})$. If $E\mu \not\sim 1$ in K and $f(\sqrt{\pi^\kappa\mu}) \sim 1$ in K, then

$$(\pi E, x)_t \sim \left(\frac{E\mu\,,\,\sqrt{\pi\mu}}{K(\sqrt{\pi\mu})\langle t\rangle}\right) \not\sim 1.$$

The conditions on K and μ always can be satisfied, provided that in case $\kappa = r < 2m$ we have $\bar{u} \neq \bar{\delta}$.
ii). We have

$$f(\pi^{\kappa/2}\xi) \sim \begin{cases} -e(\xi^2-u), & \text{if } \kappa < r \text{ or } \kappa = r > 2m, \\ -e(\xi^2-u)(\xi^2-\delta), & \text{if } \kappa = r < 2m, \\ -e(\xi^2-u)((\xi-v)^2-\delta), & \text{if } \kappa = r = 2m. \end{cases}$$

Thus if $\kappa < r$ or $\kappa = r > 2m$, then $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \not\sim 1$. Let now $\kappa = r \leq 2m$. Then $(\pi, x^2-\pi^\kappa u)_{\mathrm{Irr}_{K|k}(\pi^{\kappa/2}\xi)} \not\sim 1$, provided that $\xi^2 - u \not\sim 1$ and $\xi^2 - \delta \sim 1$ if $\kappa < 2m$, $(\xi-v)^2 - \delta \sim 1$ if $\kappa = 2m$.
iii). To prove $(\alpha, x-\pi^{\kappa/2}\tau) \not\sim 1$ one can argue as in the proof of theorem 2, subcase II.3.
iv). From the conditions on β, the algebra is unramified. The nontriviality can be proved as in iii).
a. Similarly as in the subcase i) for $t = \mathrm{Irr}_{K(\sqrt{\pi\mu})|k}(\theta)$ and $\theta = \pi^m v + \sqrt{\pi^r\mu}$, we can find K and μ such that $(\pi F, x-\pi^m v)_t \not\sim 1$. Indeed,

$$f(\theta) \sim \begin{cases} -e\mu(\mu-\delta), & \text{if } 2m < \kappa, \\ ue\mu(\mu-\delta), & \text{if } 2m, r > \kappa, \\ -e\mu(\mu-\delta)(v^2-u), & \text{if } \kappa = 2m < r, \end{cases}$$

and the conditions $f(\theta) \sim 1$ and $F\mu \not\sim 1$ can be satisfied in some unramified extension $K|k$ of odd degree.
b,c. By replacing $x' = x - \pi^m v$, case $\kappa = r$ is reduced to ii) and iii). Furthermore,

$$f(\pi^m v + \pi^{r/2}\tau) \sim -e\alpha \times \begin{cases} 1, & \text{if } 2m < \kappa, \\ v^2-u, & \text{if } 2m = \kappa, \\ -u\pi^\kappa, & \text{if } 2m < \kappa. \end{cases}$$

So, in b the algebra $(\pi,(x-\pi^m v)^2-\pi^r\delta)_{x-\pi^m v+\pi^{r/2}\tau}$ is nontrivial, and the algebra from c is unramified. The nontriviality of the last algebra can be checked similarly as in the proof of theorem 2, subcase II.3. Indeed, this algebra is nontrivial in $\mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$, where $\theta=\pi^m v+\pi^{r/2-1}(\sqrt[3]{\pi})^{\varepsilon}$ with $\varepsilon\in\{1,2\}$, and $\varepsilon\equiv r/2\,(\mathrm{mod}\,2)$.

d. We have $f(\pi^m v+\pi^{r/2}\gamma)\sim ue\pi(\gamma^2-\delta)$. Since $K|k$ is of odd degree, $\gamma^2-\delta\sim N_{K|k}(\gamma^2-\delta)$ in K, the algebra $A=(ueN_{K|k}(\gamma^2-\delta)\pi,\mathrm{Irr}_{K|k}(\pi^m v+\pi^{r/2}\gamma))$ is unramified. Let $\mu\in O^*_{K'}$, $K'|k$ be of odd degree and unramified such that

$$\begin{cases}\mu\not\sim ue(\gamma^2-\delta),\\ \mu-u\not\sim -u(\gamma^2-\delta).\end{cases}$$

Let $t=\mathrm{Irr}_{K'(\sqrt{\pi\mu})|k}(\sqrt{\pi^\kappa\mu})$. Then $f(\sqrt{\pi^\kappa\mu})\sim -e\mu(\mu-u)\sim 1$ and

$$A_t\otimes K\sim\left(\frac{ue(\gamma^2-\delta)\mu\ ,\ \prod_{\sigma\in GK|k}(\sqrt{\pi^\kappa\mu}-\pi^m v-\pi^{r/2}\gamma)}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right)\sim\left(\frac{\alpha\ ,\ \sqrt{\pi\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right)\not\sim 1.$$

Let us check now that the algebra A from $i)-iv)$ is not isomorphic to the algebra B from $a-d$.

Let firstly κ and r be odd. Then $2m<\kappa$, $A=(\pi E,x)$, and $A_t\sim 1$. On the other hand,

$$B_t\sim(\pi F,x-\pi^m v)_t\sim\left(\frac{\pi F\ ,\ -\pi^m v}{K(\sqrt{\pi\mu})\langle t\rangle}\right)\sim 1.$$

Let now κ be odd and r be even. We have

$$B_t\sim(\pi,(x-\pi^m v)^2-\pi^r\delta)_t\sim\begin{cases}\left(\frac{\pi\ ,\ \pi^{2m}v^2}{K(\sqrt{\pi\mu})\langle t\rangle}\right),\ \text{if } 2m<\kappa,\\ \left(\frac{\pi\ ,\ \pi^\kappa\mu}{K(\sqrt{\pi\mu})\langle t\rangle}\right),\ \text{otherwise}\end{cases}\sim 1.$$

In c we have $B=(\alpha,x-\pi^m v-\pi^{r/2}\delta)$ and $2m<\kappa$. So

$$B_t\sim\left(\frac{\alpha\ ,\ -\pi^m v}{K(\sqrt{\pi\mu})\langle t\rangle}\right)\sim 1.$$

Finally, if case d takes place, then $2m>\kappa$ and

$$B_t\otimes K'\sim\left(\frac{ue(\gamma^2-\delta)\pi\ ,\ \sqrt{\pi^\kappa\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right)\sim\left(\frac{\alpha ue\mu\ ,\ \sqrt{\pi\mu}}{KK'(\sqrt{\pi\mu})\langle t\rangle}\right)\sim 1,$$

since, from the proof of i), we have $\alpha ue\mu\sim 1$.

Conversely, let κ be even and r be odd. From the proof of a we have $B_t\not\sim 1$, $t=\mathrm{Irr}_{K\sqrt{\pi\mu}|k}(\theta)$, $\theta=\pi^m v+\sqrt{\pi^r\mu}$. On the other hand, if $A=(\pi,x^2-\pi^\kappa u)$, then

$$A_t\sim\left(\frac{\pi\ ,\ \pi^{2m}v^2-\pi^\kappa u}{K(\sqrt{\pi\mu})\langle t\rangle}\right)\sim 1$$

in view of $\min\{2m,r\}<\min\{r,m+r/2\}$. For $A=(\alpha,x-\pi^{\kappa/2}w)$

$$A_t\sim\left(\frac{\alpha\ ,\ \pi^m v-\pi^{\kappa/2}w}{K(\sqrt{\pi\mu})\langle t\rangle}\right)\sim 1,$$

provided that $\bar{w} \neq \bar{v}$, if $\kappa = 2m$,

Finally, let κ and r be even. Extension by $\sqrt{\pi}$ leaves us in case VI. Moreover, if the algebra $(\alpha, x - \pi^{\kappa/2}w)$ is nontrivial (subcase iii)), then it does not split by this extension, since we are still in iii). The same is true for the algebras from iv) and c. So it is enough to prove that in ii), b, we have $(\pi, x - e) \not\sim 1$, and in iii), c we have $(\alpha, x - \pi^{\kappa/2}w) \not\sim (\alpha, x - \pi^m v - \pi^{r/2}\tau)$. Case iv), c is impossible.

In ii), b we have $(\pi, x-e)_{\mathrm{Irr}(\pi^{\kappa/2}\xi)} \sim (\pi, -e) \not\sim 1$. Consider now iii), c. Then $-e \sim 1$. If $\kappa = r$, then $\kappa > 2m$. Since θ from the proof of iii) satisfies the conditions $\theta = \pi^{\kappa/2-1}(\sqrt[3]{\pi})^{\varepsilon}$, $\varepsilon \in \{1, 2\}$, $\varepsilon \equiv m + \kappa/2 \pmod 2$, we have ($q = \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta)$)

$$(\alpha, x - \pi^{\kappa/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m-1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{m}}{k(\sqrt[3]{\pi})\langle q\rangle}\right),$$

so $A_q \not\sim B_q$.

Let now $\kappa < r$. Case $2m < \kappa$ is similar to the previous case. In case $2m = \kappa$, consider $\theta = \pi^m v + \pi^{r/2-1}(\sqrt[3]{\pi})^{\varepsilon}$, $\varepsilon \in \{1, 2\}$, $\varepsilon \equiv \kappa/2 + r/2 \pmod 2$. Then

$$(\alpha, x - \pi^{\kappa/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2-1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right).$$

Finally, in case $2m > \kappa$, let $\theta = \pi^{\kappa/2}(\sqrt[3]{\pi})$. We have

$$(\alpha, x - \pi^{\kappa/2}w)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2}}{k(\sqrt[3]{\pi})\langle q\rangle}\right) \text{ and } (\alpha, x - \pi^m v - \pi^{r/2}\tau)_q \sim \left(\frac{\alpha\ ,\ (\sqrt[3]{\pi})^{\kappa/2+1}}{k(\sqrt[3]{\pi})\langle q\rangle}\right).$$

Consider now cases 1–4 step by step.

1. There exists $a \in O_k^*$ such that $a^2 - 4u \not\sim 1$. If $p = \mathrm{Irr}_{k(\sqrt{u})|k}(\pi^{\kappa/2}(a + \sqrt{u}))$, then

$$(\pi, x^2 - \pi^{\kappa}u)_p \sim \left(\frac{\pi\ ,\ a + 2\sqrt{u}}{k(\sqrt{u})\langle p\rangle}\right),$$

since $f(\pi^{\kappa/2}(a + \sqrt{u})) \sim -e(a^2 + 2a\sqrt{u})^2 \sim 1$ in $k(\sqrt{u})$. So, $(\pi, x^2 - \pi^{\kappa}u)_p \not\sim 1$.

Further, $(\alpha(x^2 - \pi^{\kappa}u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w))_{x-\pi^{\kappa/2}w} \sim (\alpha(w^2 - u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w)) \sim 1$, so that this algebra is unramified. Let us prove the nontriviality. Suppose $\theta = (1 + \sqrt[3]{\pi})\pi^{\kappa/2}\sqrt{u} \in k(\sqrt{u}, \sqrt[3]{\pi})$ and $q = \mathrm{Irr}(\theta)$. Then $\theta^2 - \pi^{\kappa}u \sim (\theta - \pi^m v)^2 - \pi^{\kappa}\delta \sim \sqrt[3]{\pi}$ in $k(\sqrt{u})$. Therefore $f(\theta) \sim 1$ and

$$(\alpha(x^2 - \pi^{\kappa}u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w))_q \sim \left(\frac{\alpha\sqrt[3]{\pi}\ ,\ \pi^{\kappa/2}((1 + \sqrt[3]{\pi})\pi^{\kappa/2}\sqrt{u} - \pi^{\kappa/2}w)}{k(\sqrt{u}, \sqrt[3]{\pi})\langle q\rangle}\right) \sim$$

$$\sim \left(\frac{\sqrt[3]{\pi}\ ,\ \sqrt{u} - w}{k(\sqrt{u}, \sqrt[3]{\pi})\langle q\rangle}\right) \not\sim 1.$$

Thus $\sqrt{\alpha}$ does not split the algebra. To check that $\sqrt{\pi}$ (and $\sqrt{\pi\alpha}$) does not split it either, one only needs to observe that these extensions leave us in subcase VI.1.

2. The first algebra is unramified and nontrivial as in subcase i. We have $f(\pi^m v) \sim -ue\delta$. If $-ue\delta \sim 1$, then $(\alpha, x^2 - \pi^{\kappa}u)_{x-\pi^m v} \not\sim 1$. Otherwise, the algebra $(\alpha, x - \pi^m v)$ is unramified. Moreover, $(\alpha, x)_{x^2-\pi^{\kappa}u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. Finally, the extension by $\sqrt{\alpha}$ leaves us in the subcase VI.2, therefore it does not split the first algebra.

3. If $f(0) \sim -ue\delta \sim 1$, then $(\alpha, x^2-\pi^\kappa u)_x \not\sim 1$. If $-ue\delta \not\sim 1$, then (α, x) is unramified and $(\alpha, x)_{x^2-\pi^\kappa u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. Let $\theta = (\beta - \sqrt{\alpha})\sqrt[3]{\pi}$, where $\beta^2 - \alpha \not\sim 1$, $q = \mathrm{Irr}_{k(\sqrt{\alpha}, \sqrt[3]{\pi})|k}(\theta)$. Then $f(\theta) \sim 1$ and

$$(\pi u e, x - \pi^s)_q \sim \left(\frac{\sqrt[3]{\pi}\,,\ (\beta - \sqrt{\alpha})\sqrt[3]{\pi}}{k(\sqrt{\alpha}, \sqrt[3]{\pi})\langle q \rangle} \right) \not\sim 1.$$

4. $(x, -\pi u(x^2 - \pi^\kappa u))_x \sim (x, \pi^{\kappa+1}u^2) \sim 1$, so the second algebra B is unramified. To prove that B is nontrivial, let us do it for the algebra $B' = B \otimes k(C)(\sqrt{\alpha})$. If $\theta = \pi^{\kappa/2}\mu \in K(\sqrt{\pi\mu})$, $\mu \in O_K^*$, and K is of odd degree and unramified over $k(\sqrt{\alpha})$, then $f(\theta) \sim (\mu - u)(\mu - \delta)$ and

$$B'_{\mathrm{Irr}(\theta)} \sim \left(\frac{\sqrt{\pi\mu}\,,\ \mu - u}{K(\sqrt{\pi\mu})(\sqrt{f(\theta)})\langle \mathrm{Irr}(\theta) \rangle} \right).$$

Anyway, we can find such K and μ that $\mu - u \sim \mu - \delta \not\sim 1$, $\bar{u} \neq \bar{\mu}$, $\bar{\delta} \neq \bar{\mu}$. Then $B'_{\mathrm{Irr}(\theta)} \not\sim 1$. Case VI is done.

It is convenient to divide the last case into several parts.

Lemma 2 *Let case* VII *take place, i.e.*

$$f(x) = (x - \pi^s e)(x^2 - \pi^\kappa u)((x - \pi^m v)^2 - \pi^r \delta),$$

where $e, u, v, \delta \in O_k^*$, $s, \kappa, m, r > 0$, $\pi^\kappa u, \pi^r \delta \not\sim 1, \kappa < r$. *Then the following subcases do not intersect and cover case* VII.

VII_1. $s = 1$, $\kappa > 1$ *and if* $m = 1$, *then* $\bar{v} \neq \bar{e}$.

VII'_1. $s = 1$, $\kappa > 1$, $m = 1$, *and* $\bar{v} = \bar{e}$.

VII_2. $s \geq 2$, $\kappa \geq 4$, $m = 1$.

VII_3. $1 \leq \kappa \leq 3$ *and* $\kappa < 2s$.

Proof. If $s > 1$, $m > 1$, and $\kappa > 3$, then one can replace $x = x'\pi^2$, $y = y'\pi^5$.

Theorem 6 *Let* f *be as in case* VII_1. *Then the group* ${}_2\mathrm{Br}\, C$ *is generated by the classes of the algebra* (π, α) *and of the algebras described below.*

If $\kappa \equiv 1 \pmod 2$, *then assume* $2m < \kappa$, *and if* $\kappa = r \equiv 0 \pmod 2$, $2m > \kappa$, *assume* $\bar{u} \neq \bar{\delta}$. *Then the first algebra is*

i) $(-\pi\alpha N_{K|k}(e - \pi^{\kappa/2-1}\tau), p)$, *if* $\kappa \equiv 0 \pmod 2$,

where $\tau \in O_K^*$ *is such that* $K|k$ *is of odd degree and unramified,*
$p = \mathrm{Irr}_{K|k}(\pi^{\kappa/2}\tau)$ *and* τ *satisfies the conditions:* $(\tau^2 - u) \not\sim 1$, $\phi(\tau) \sim 1$,

$$\phi(x) = \begin{cases} 1, & \text{if } \kappa < r \text{ or } 2m < r, \\ x^2 - \delta, & \text{if } \kappa = r \text{ and } 2m > r, \\ (x - v)^2 - \delta, & \text{if } \kappa = r = 2m. \end{cases}$$

ii) $(\alpha, x^2 - \pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$ *and* $ue\psi \sim 1$, *where*

$$\psi = \begin{cases} 1, & \text{if } 2m < r, \\ v^2 - \delta, & \text{if } 2m = r, \\ -\delta, & \text{if } 2m > r,\ r \text{ is even.} \end{cases}$$

iii) (α, x), *if* $\kappa \equiv 1 \pmod 2$ *and* $ue\psi \not\sim 1$.

The second one is

a. $(-\pi(\mu^2 - \delta)\chi(\mu) N_{L|k}(e - v\pi^{m-1} - \mu\pi^{r/2-1}), q)$, *if* $r \equiv 0 \pmod 2$,

where $\mu \in O_L^*$ *is such that* $L|k$ *is of odd degree and unramified,* $q = \mathrm{Irr}_{L|k}(\pi^m v + \pi^{r/2}\mu)$ *and* μ *satisfies the condition:* $(\mu^2 - \delta) \not\sim 1$ *if* κ *is even, and* $(\mu^2 - \delta)\chi(\mu) \not\sim 1$, *otherwise,*

$$\chi(x) = \begin{cases} 1, & \text{if } 2m < \kappa, \\ v^2 - u, & \text{if } 2m = \kappa < r, \\ -u, & \text{if } 2m > \kappa, \kappa < r, \\ x^2 - u, & \text{if } 2m > \kappa = r, \\ (x+v)^2 - u, & \text{if } \kappa = r = 2m. \end{cases}$$

b. $(\alpha, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $r \equiv 1 \pmod 2$ *and* $\delta(e - \pi^{m-1}v)\chi(\mu) \sim 1$.

c. $(\alpha, x - \pi^m v)$, *if* $r \equiv 1 \pmod 2$ *and* $\delta(e - \pi^{m-1}v)\chi(\mu) \not\sim 1$.

Otherwise we have

1. *The first algebra is from ii) or iii), and the second one is* $(-\pi ue\delta, x - \pi^m v)$, *if* $\kappa, r \equiv 1 \pmod 2$, $\kappa < 2m < r$.

2. *The first algebra is* $(-\pi ue\delta, x)$, *the second one is either* $(\alpha, x^2 - \pi^\kappa u)$, *provided that* $ue \sim 1$ *or* $(\alpha, x - \pi^{(\kappa+1)/2})$, *provided that* $ue \not\sim 1$, *if* $\kappa, r \equiv 1 \pmod 2$, $\kappa < r < 2m$.

3. *If* $\kappa \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$, *and* $2m > \kappa$, *then the first algebra is from ii) or iii), and the second one is*

- $(\pi, (x - \pi^m v)^2 - \pi^r \delta)$ *in the following cases:*
  1. $2m < r$ *and* $ue \not\sim 1$.
  2. $2m = r$, $ue \not\sim 1$, *and* $v^2 - \delta \sim 1$.
  3. $2m > r$, $ue \not\sim 1$, *and* $-1 \not\sim 1$.
- (π, t) *in cases*
  1. $2m = r$, $ue \sim 1$, *and* $v^2 - \delta \not\sim 1$.
  2. $2m > r$, $ue \sim 1$, *and* $-1 \sim 1$,

  where $t = \mathrm{Irr}_{K(\sqrt{\pi w})|k}(\sqrt{\pi^\kappa w})$, K *is as above,* $w \in O_K^* \setminus (O_K^*)^2$, *and* $-e(w - u) \not\sim 1$ *in* K.
- $(\alpha, \mathrm{Irr}_{K|k}(\pi^m v + \pi^{r/2}\lambda))$ *in cases*
  1. $2m < r$, $ue \sim 1$. $v^2 - \delta \not\sim 1$.
  2. $2m > r$, $ue \not\sim -1$,

where $\lambda \in O_K^*$, *and* $\lambda^2 - \delta \not\sim 1$ *in case* $2m < r$, $\lambda^2 - \delta \sim -1$ *in case* $2m > r$. *Note that if* $2m < r$, *then one can suppose* $K = \kappa$, *and if* $2m > r$, *then this is true as soon as* $\bar{k} \neq \boldsymbol{F}_3$.

– $(\alpha, \mathrm{Irr}_K(\pi^m \eta))$ *if* $2m = r$ *and* $ue \sim v^2 - \delta$, *where* $\eta \in O_K^*$, $ue((\eta - v)^2 - \delta) \not\sim 1$.

4. $(-\pi(e - \pi^{\kappa-2}w), x - \pi^{\kappa/2}w)$, $(\alpha(x^2 - \pi^\kappa u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w))$, $w^2 - u \not\sim 1$, *if* $\kappa = r \equiv 0 \pmod 2$, $2m > \kappa$, $\bar{u} = \bar{\delta}$ *and if* $\kappa = 2$, *then* $e^2 - u \sim 1$.

5. $(\pi, x - \pi e)$, $(\pi, x^2 - \pi^\kappa u)$, *if* $\kappa = r = 2$, $m > 1$, $\bar{u} = \bar{\delta}$ *and* $e^2 - u \not\sim 1$.

6. *If* $\kappa = r \equiv 1 \pmod 2$, $2m > \kappa$, *then the first algebra is* $(x, -\pi u(x^2 - \pi^\kappa u))$ *and the second one is*

– $(-\pi ue\delta, x)$, *if* $\bar{u} \neq \bar{\delta}$.

– $(\alpha, x^2 - \pi^\kappa u)$, *if* $\bar{u} = \bar{\delta}$ *and either* $-ue \sim 1$ *or* $2m = \kappa + 1$.

– $(\alpha, \mathrm{Irr}_{k(\sqrt{\pi u})|k}(\theta))$, $\theta = \sqrt{\pi^\kappa u}(1 + \sqrt{\pi u})$, *if* $\bar{u} = \bar{\delta}$, $-ue \not\sim 1$, *and* $2m > \kappa + 1$.

Remark 3 *The algebra* $(\alpha(x^2 - \pi^\kappa u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w))$ *from* 4 *does not split by any quadratic extension of scalars.*

Proof of the theorem.
i). In view of $f(\pi^{\kappa/2}\tau) \sim -\pi\alpha(e - \pi^{\kappa/2}\tau)$ in K the algebra is unramified. We also have $(-\pi\alpha N_{K|k}(e - \pi^{\kappa/2-1}\tau), p)_{x-\pi e} \sim (-\pi\alpha N(e - \pi^{\kappa/2-1}\tau), \prod_\sigma(\pi e - \pi^{\kappa/2}\tau^\sigma))(-\pi\alpha N(e - \pi^{\kappa/2-1}\tau), \pi N(\pi e - \pi^{\kappa/2}\tau)) \not\sim 1$.
ii), iii). $f(0) \sim ue\psi$, so in iii) the algebra (α, x) is unramified. Moreover, in ii) we have $(\alpha, x^2 - \pi^\kappa u)_x \not\sim 1$ and in iii) $(\alpha, x)_{x-\pi e} \not\sim 1$.
a. $f(\pi^m v + \pi^{r/2}\mu) \sim -\pi(e - \pi^{m-1}v - \pi^{r/2-1}\mu)(\mu^2 - \delta)\chi(\mu)$, so the algebra is unramified. If κ is even, then

$$\begin{aligned} & (-\pi\alpha\chi(\mu)N_{L|k}(e - v\pi^{m-1} - \pi^{r/2-1}\mu), q)_{x^2-\pi^\kappa u} \\ \sim & \left(\frac{\pi\ ,\ \prod_\lambda(\pi^{\kappa/2}\sqrt{u} - \pi^m v - \pi^{r/2}\mu^\lambda)}{k(\sqrt{\alpha})\langle x^2 - \pi^\kappa u\rangle(\sqrt{C(x^2 - \pi^\kappa u)})}\right). \end{aligned}$$

The latter algebra is nontrivial if and only if $\chi(\mu)$ is not a square in L. We also have

$$\begin{aligned} & (-\pi(\mu^2 - \delta)\chi(\mu)N_{L|k}(e - v\pi^{m-1} - \pi^{r/2-1}\mu), q)_{x-\pi e} \\ \sim & (-\pi(\mu^2 - \delta)\chi(\mu)N(e - v\pi^{m-1} - \pi^{r/2-1}\mu), \prod_\lambda(\pi e - \pi^m v - \pi^{r/2}\mu^\lambda)) \\ \sim & (\pi, (\mu^2 - \delta)\chi(\mu)), \end{aligned}$$

and this algebra is nontrivial iff $(\mu^2 - \delta)\chi(\mu)$ is not a square in L. Anyway, the algebra from a is nontrivial.
b, *c*. We have $f(\pi^m v) \sim \delta(e - \pi^{\kappa-2}v)\chi(\mu)$ and $(\alpha, x - \pi^m v)_{x-\pi e} \not\sim 1$.

Let us check now that the algebra from i), ii), iii) is not isomorphic to the algebra from a, b, c in the corresponding subcases.

1). Let firstly $\kappa, r \equiv 1 \,(\mathrm{mod}\, 2)$. We have

$$(\alpha, x^2 - \pi^\kappa u)_{x-\pi^m v} \sim 1 \not\sim (\alpha, (x-\pi^m v)^2 - \pi^r \delta)_{x-\pi^m v}.$$

Furthermore,

$$(\alpha, x)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle x^2 - \pi^\kappa u\rangle (\sqrt{D(x^2-\pi^\kappa u)})} \right) \not\sim 1,$$

but

$$(\alpha, x-\pi^m v)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ -\pi^m u}{k(\sqrt{\pi u})\langle x^2 - \pi^\kappa u\rangle (\sqrt{D(x^2-\pi^\kappa u)})} \right) \sim 1.$$

Similarly, one can check that $(\alpha, (x-\pi^m v)^2 - \pi^r \delta)_{x^2-\pi^\kappa u} \sim 1$. Finally, we obtain

$$(\alpha, x-\pi^m v)_{(x-\pi^m v)^2-\pi^r \delta} \sim (\alpha, \sqrt{\pi\delta}) \not\sim 1,$$

and $(\alpha, x^2 - \pi^\kappa u)_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
2). $\kappa \equiv 0 \,(\mathrm{mod}\, 2)$, $r \equiv 1 \,(\mathrm{mod}\, 2)$. Then the algebra from i) is nontrivial in $x - \pi e$, but $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)_{x-\pi e} \sim 1$. Moreover, $(-\pi\alpha N_{K|k}(e - \pi^{\kappa/2-1}\tau), p)_{(x-\pi^m v)^2-\pi^r\delta} \sim$

$$\sim \left(\frac{-\delta\alpha N(e-\pi^{\kappa/2-1}\tau)\ ,\ \prod_\sigma (\pi^m v + \sqrt{\pi^r\delta} - \pi^{\kappa/2}\tau^\sigma)}{k(\sqrt{\pi\delta})\langle (x-\pi^m v)^2 - \pi^r\delta\rangle (\sqrt{C((x-\pi^m v)^2-\pi^r\delta)})} \right).$$

One can choose τ such that if $2m = \kappa$, then $\bar\tau \neq \bar v$. For this τ, the latter algebra is trivial in view of $v_K(\pi^m v - \pi^{\kappa/2}\tau^\sigma) < r/2$. On the other hand, $(\alpha, x-\pi^m v)_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$.
3). $\kappa \equiv 1 \,(\mathrm{mod}\, 2)$, $r \equiv 0 \,(\mathrm{mod}\, 2)$. The algebra from a is nontrivial at $x - \pi e$, but $(\alpha, x^2 - \pi^\kappa u)$ is trivial at this place. We also have $(\alpha, x)_{x^2-\pi^\kappa u} \not\sim 1$ and $(-\pi\alpha N_{L|k}(e - v\pi^{m-1} - \pi^{r/2-1}\mu), q)_{x^2-\pi^\kappa u} \sim$

$$\sim \left(\frac{-\alpha u(e-v\pi^{m-1})\ ,\ \prod_\lambda (\sqrt{\pi^\kappa u} - \pi^m v - \pi^{r/2}\mu^\lambda)}{k(\sqrt{\pi u})\langle x^2-\pi^\kappa u\rangle (\sqrt{D(x^2-\pi^\kappa u)})} \right) \sim$$

$$\sim \left(\frac{-\alpha u(e-v\pi^{m-1})\ ,\ -\pi^m v}{k(\sqrt{\pi u})\langle x^2-\pi^\kappa u\rangle (\sqrt{D(x^2-\pi^\kappa u)})} \right) \sim 1.$$

4). $\kappa, r \equiv 0 \,(\mathrm{mod}\, 2)$. We have

$$(-\pi\alpha N_{K|k}(e - \pi^{\kappa/2-1}), p)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ \prod_\sigma (\tau^\sigma - \sqrt{u})}{k(\sqrt{\alpha})\langle x^2-\pi^\kappa u\rangle (\sqrt{C(x^2-\pi^\kappa u)})} \right).$$

This algebra is nontrivial since $\tau^2 - u \not\sim 1$ in K. So the algebra from i) is nontrivial either at $x - \pi e$ or at $x^2 - \pi^\kappa u$. But the algebra from a is nontrivial at one of these places, and trivial at the other. Therefore, these algebras are not isomorphic.

Let us consider the remaining cases.
1. Let w be a unit from a sufficiently large unramified extension K of odd degree over k, such that

$$\begin{cases} w \not\sim -ue\delta, \\ w - u \sim -e, \end{cases}$$

and $t = \mathrm{Irr}_{K(\sqrt{\pi w})|k}(\sqrt{\pi^\kappa w})$. Then $f(\sqrt{\pi^\kappa w}) \sim -e(w-u) \sim 1$ and

$$(-\pi ue\delta, x - \pi^m v)_t \sim \left(\frac{-wue\delta\ ,\ \sqrt{\pi w}}{K(\sqrt{\pi w})\langle t\rangle} \right) \not\sim 1.$$

To finish the proof of this subcase it is enough to check that extension of scalars by $\sqrt{\alpha}$ does not split the algebra $(-\pi ue\delta, x-\pi^m v)$. Indeed, this extension leaves us in the subcase VII.7.

2. All algebras are unramified. Since $f(\pi^{(\kappa+1)/2}) \sim ue$ we obtain $(x^2 - \pi^\kappa u)_{x-\pi^{(\kappa+1)/2}} \not\sim 1$. Furthermore,

$$(\alpha, x - \pi^{(\kappa+1)/2})_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi u}}{k(\sqrt{\pi u})\langle x^2 - \pi^\kappa u\rangle(\sqrt{D(x^2-\pi^\kappa u)})} \right) \not\sim 1.$$

The nontriviality of the algebra $(-\pi ue\delta, x)$ can be checked similarly as in the previous subcase. Finally, the extension by $\sqrt{\alpha}$ leaves us in the same subcase, so it does not split the latter algebra.

3. The nontriviality of the algebras from ii) and iii) was already shown. Let firstly $2m < r$. Then $f(\pi^m v + \pi^{r/2}\lambda) \sim ue(\lambda^2 - \delta)$, so if $ue \sim 1$, $\lambda^2 - \delta \not\sim 1$, then $(\alpha, \mathrm{Irr}(\pi^m v + \pi^{r/2}\delta))$ is unramified. In this case the first algebra is $(\alpha, x^- \pi^\kappa u)$. This is trivial at $x - \pi e$, but $(\alpha, \mathrm{Irr}(\pi^m v + \pi^{r/2}\delta))_{x-\pi e} \not\sim 1$.

If $ue \not\sim 1$, then $f(\pi^m v + \pi^{r/2}\lambda) \sim 1$, provided that $\lambda^2 - \delta \not\sim 1$. Therefore, $(\pi, (x - \pi^v)^2 - \pi^r\delta)_{\mathrm{Irr}(\pi^m v + \pi^{r/2}\lambda)} \not\sim 1$. The first algebra is (α, x), $(\alpha, x)_{x-\pi e} \not\sim 1$. On the other hand, $(\pi, (x-\pi^v)^2 - \pi^r\delta)_{x-\pi e} \sim 1$.

Let now $2m = r$. We have $f(\pi^m \eta) \sim ue((\eta - v)^2 - \delta)$, so the algebra $(\alpha, \mathrm{Irr}(\pi^m\eta))$ is unramified and nontrivial at $x - \pi e$. Since $ue(v^2 - \delta) \sim 1$, the first algebra is trivial at $x - \pi e$.

If $ue \not\sim 1$, $v^2 - \delta \sim 1$, then $(\pi, (x - \pi^m v)^2 - \pi^r\delta)_{\mathrm{Irr}(\pi^m\eta)} \not\sim 1$. In this case the first algebra is nontrivial at $x - \pi e$, and this is not true for the second one.

Suppose $ue \sim 1$ and $v^2 - \delta \not\sim 1$. We have $f(\sqrt{\pi^\kappa w}) \sim -e(w-u) \sim \alpha$ in $K(\sqrt{\pi w})$. So

$$(\pi, t)_t \sim \left(\frac{w\ ,\ t}{K(\sqrt{\pi w}, \sqrt{\alpha})} \right) \langle t\rangle \sim 1,$$

and the algebra (π, t) is unramified and not isomorphic to (α, x) at $x - \pi e$. It is easy to check that $f(\pi^{m-1/3}) \sim ue \sim 1$. Hence $(\pi, t)_{\mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\pi^{m-1/3})} \sim (\sqrt[3]{\pi}, w) \not\sim 1$.

Finally, let $2m > r$. One can consider this case similarly as the previous ones.

4. We have $f(\pi^{\kappa/2} w) \sim -\pi(e - \pi^{\kappa-2} w)$ and $\alpha((\pi^{\kappa/2}w)^2 - \pi^\kappa u) \sim 1$, so both these algebras are unramified. $(-\pi(e - \pi^{\kappa-2}), x - \pi^{\kappa/2}w)_{x^2-\pi^\kappa u} \sim (\pi, w - \sqrt{u}) \not\sim 1$. Let us check that the second algebra is nontrivial, not isomorphic to the first one, and does not split by any quadratic extension of the scalars. To prove this it is sufficient to show that

$$B = (\alpha(x^2 - \pi^\kappa u), \pi^{\kappa/2}(x - \pi^{\kappa/2}w))_\otimes k(\sqrt{\pi}, \sqrt{\alpha})(C) \not\sim 1.$$

Assume $\nu = \sqrt{\pi}$, $\theta = \nu^\kappa\sqrt{u}(1+\nu)$. Then $f(\theta) \sim 1$ in view of $\delta = u + \nu^{2l}\mu, l > 0$ and if $\kappa = 2$, then $e - \sqrt{u} \not\sim 1$. Thus

$$B_{x-\theta} \sim \left(\frac{\nu\ ,\ w - \sqrt{u}}{k(\sqrt{\pi}, \sqrt{\alpha})\langle x - \theta\rangle} \right) \not\sim 1.$$

5. In this case, $(\pi, x-\pi e)_{x^2-\pi^2 u} \sim (\pi, e-\sqrt{u}) \not\sim 1$ and $(\pi, x^2-\pi^2 u)_{x-\pi e} \sim (\pi, e^2-u) \not\sim 1$. Moreover, $((\pi, x-\pi e)_{\otimes}(\pi, x^2-\pi^2 u))_{x-\pi e} \sim (\pi, (x-\pi^m v)^2 - \pi^r \delta)_{x-\pi e} \sim (\pi, e^2-\delta) \not\sim 1$.
6. We have $(x, -\pi u(x^2-\pi^\kappa u))_x \sim (x, \pi^{\kappa+1} u^2) \sim 1$, therefore the algebra $(x, -\pi u(x^2 - \pi^\kappa u))$ is unramified. Let K, w, and t are as in subcase 1, but

$$\begin{cases} uew(w-\delta) \not\sim 1, \\ -u(w-u) \not\sim 1. \end{cases}$$

Then $f(\sqrt{\pi^\kappa w}) \sim -ew(w-u)(w-\delta) \sim 1$ and

$$(x, -\pi u(x^2-\pi^\kappa u))_t \sim \left(\frac{\sqrt{\pi w}\,,\ -u(w-u)}{K(\sqrt{\pi w}\langle t\rangle)} \right) \not\sim 1.$$

Consider the second algebra. For $\bar{u} \neq \bar{\delta}$ one always can find w and K such that

$$\begin{cases} -ue\delta w \not\sim 1, \\ -ew(w-u)(w-\delta) \sim 1. \end{cases}$$

Then $f(\sqrt{\pi^\kappa w}) \sim 1$ and $(-\pi ue\delta, x)_t \sim (-ue\delta w, \sqrt{\pi w}) \not\sim 1$. So the second algebra is nontrivial. We also can find such K, w, and t that

$$\begin{cases} w \sim -ue\delta, \\ w-u \not\sim -u, \\ w-\delta \not\sim -\delta. \end{cases}$$

For such an element w the first algebra is nontrivial at t and the second one is trivial at t.

Let now $\bar{u} = \bar{\delta}$ and $\theta_w = \sqrt{\pi^\kappa u}(1+\sqrt{\pi u}w)$. Then $f(\theta_w) \sim -ew(uw - \pi^{m-(\kappa+1)/2})$ (if $2m = \kappa+1$ let $\bar{u}\bar{w} \neq \bar{v}$). In case $2m > \kappa+1$ we have $f(\theta_w) \sim -ue \sim 1$ for any w, and otherwise we can find w such that $f(\theta_w) \sim 1$. Thus, for $q = \mathrm{Irr}_{K(\sqrt{\pi w})|k}(\theta_w)$

$$(\alpha, x^2-\pi^\kappa u)_q \sim \left(\frac{\alpha\ ,\ 2w\sqrt{\pi u}}{K(\sqrt{\pi w})\langle q\rangle} \right) \not\sim 1.$$

Let $\bar{u} = \bar{\delta}$, $2m > \kappa+1$, and $-ue \not\sim 1$. Then, for $\theta = \theta_w, K = k, w = 1$ we find $f(\theta) \sim -ue \sim \alpha$, and the algebra (α, q) is unramified.

$$(\alpha, q)_{x^2-\pi^\kappa u} \sim \left(\frac{\alpha\ ,\ (\sqrt{\pi^\kappa u} - \sqrt{\pi^\kappa u}(1+\sqrt{\pi u}))(\sqrt{\pi^\kappa u} + \sqrt{\pi^\kappa u}(1-\sqrt{\pi u}))}{k(\sqrt{\pi u})\langle \sqrt{D(x^2-\pi^\kappa u)}\rangle} \right) \not\sim 1.$$

Both algebras, $(\alpha, x^2-\pi^\kappa u)$ and (α, q), are not isomorphic to the first algebra from the list. Indeed, extension by $\sqrt{\alpha}$ leaves us in the current case and subcase. So it does not split the first algebra.

Hence the theorem is proved.

Theorem 7 *Let f be as in case* VII'_1, *i.e.*

$$f(x) = (x - \pi v(1+\pi^l \tau))(x^2 - \pi^\kappa u)((x-\pi v)^2 - \pi^r \delta), \quad 1 < \kappa \leq r,\ \pi^\kappa u, \pi^r \delta \not\sim 1.$$

Then the group ${}_2\mathrm{Br}\, C$ is generated by the classes of the algebra (π, α) and of the following algebras.

The first algebra A is

i) $(\pi E(w), x - \pi^{\kappa/2} w)$, *if* $\kappa \equiv 0 \pmod 2$,

ii) $(\alpha, x^2 - \pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$ *and* $uv \sim 1$,

iii) (α, x), *if* $\kappa \equiv 1 \pmod 2$ *and* $uv \not\sim 1$,

where $w \in O_k^*$, $w^2 - u \not\sim 1$, *and*

$$E(w) = \alpha \left\{ \begin{array}{l} -v, \ \textit{if } \kappa > 2, \\ w - v, \ \textit{if } \kappa = 2. \end{array} \right\} \left\{ \begin{array}{l} 1, \ \textit{if } r > 2, \\ (w-v)^2 - \delta, \ \textit{if } r = 2. \end{array} \right\}.$$

The second algebra B *is*

a. $(\alpha, x - \pi v)$,

if $r \equiv 1 \pmod 2$, $l \equiv 0 \pmod 2$, $\tau v \delta \xi \not\sim 1$, *where*

$$\xi = \left\{ \begin{array}{l} 1, \ \textit{if } \kappa > 2, \\ v^2 - u \ \textit{if } \kappa = 2. \end{array} \right.$$

b. $(\alpha, (x - \pi v)^2 - \pi^r \delta)$,

if $r \equiv 1 \pmod 2$, $l \equiv 0 \pmod 2$, $\tau v \delta \xi \sim 1$.

c. $(\pi v \delta \tau \xi, x - \pi v)$,

if $r, l \equiv 1 \pmod 2$, $2l + 2 < r$.

d. $(\pi \alpha N_{K|k}(\eta(\gamma)\zeta(\gamma)), p)$,

if either $r, l \equiv 0 \pmod 2$, $2l + 2 \leq r$, *or* $r \equiv 2 \pmod 4$, $2l + 2 > r$, *where*

$$\eta(x) = \left\{ \begin{array}{l} 1, \ \textit{if } \kappa > 2, \\ v^2 - u \ \textit{if } \kappa = 2 < r, \\ (x+v)^2 - \delta, \ \textit{if } \kappa = r = 2, \end{array} \right. \qquad \zeta(x) = \left\{ \begin{array}{l} -v\tau \ \textit{if } 2l + 2 < r, \\ x - v\tau, \ \textit{if } 2l + 2 = r, \\ x, \ \textit{if } 2l + 2 > r, \end{array} \right.$$

$\gamma \in O_K^*$, $K|k$ *is of odd degree and unramified,* $p = \mathrm{Irr}_{K|k}(\pi v + \pi^{r/2}\gamma)$, $\gamma^2 - \delta \not\sim 1$, *and if* $r = 2$, *then* $(\gamma + v)^2 - u \sim 1$.

e. $(\pi, (x - \pi v)^2 - \pi^r \delta)$,

if $r \equiv 0 \pmod 2$, $l \equiv 1 \pmod 2$, *and either* $2l + 2 = r$ *or* $2l + 2 < r$ *and* $-\tau v \xi \not\sim 1$.

f. $(\alpha, x - \pi v - \pi^{r/2}\mu)$,

if $r \equiv 0 \pmod 2$, $l \equiv 1 \pmod 2$, $2l + 2 < r$, *and* $-\tau v \xi \sim 1$, *where* $\mu \in O_k^*$ *and* $\mu^2 - \delta \not\sim 1$.

g. $(\pi, (x - \pi v)^2 - \pi^r \delta)$,

if $r \equiv 0 \pmod 4$, $2l + 2 > r$.

h. $(\alpha, (x - \pi v)^2 - \pi^r \delta)$,

if $r, l \equiv 1 \pmod 2$, $2l + 2 > r$.

Note that in d) if $r > 2$, *then one can assume* $K = k$, $p = x - \pi v - \pi^{r/2}\gamma$.

Proof.
i) We have $f(\pi^{\kappa/2}w) \sim \pi E(w)$, so A is unramified.

$$A_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ w-\sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^\kappa u\rangle(\sqrt{D(x^2-\pi^\kappa u)})} \right) \not\sim 1.$$

ii), iii) Then $f(0) \sim uv$, so $(\alpha, x^2-\pi^\kappa u)_x \not\sim 1$ in case $uv \sim 1$, and (α, x) is unramified otherwise. Finally, $(\alpha, x)_{x^2-\pi^\kappa u} \not\sim 1$.

It easy to check that $f(\pi v) \sim \pi^{r+l+1}\tau v\delta\xi$ and for even r and arbitrary $\mu \in O_K^*$ $f(\pi^v + \pi^{r/2}\mu) \sim (\mu^2-\delta)\eta(\mu)\zeta(\mu)\pi^{\min\{l+1,r/2\}}$.
a. We have $B_{(x-\pi v)^2-\pi^r\delta} \not\sim 1$ and

$$(\alpha, x)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\alpha\ ,\ \pi v}{k(\sqrt{\pi\delta})\langle\ldots\rangle} \right) \sim 1,$$

$$(\alpha, x^2-\pi^\kappa u)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\alpha\ ,\ \xi}{k(\sqrt{\pi\delta})\langle\ldots\rangle} \right) \sim 1,$$

$$(\pi E(w), x-\pi^{\kappa/2}w)_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\delta E(w)\ ,\ \pi v-\pi^{\kappa/2}w}{k(\sqrt{\pi\delta})\langle\ldots\rangle} \right) \sim 1,$$

so $A \not\sim B$.
b. $f(\pi v) \sim 1$, therefore $B_{x-\pi v} \not\sim 1$. On the other hand, $(\alpha, x^2-\pi^\kappa u)_{x-\pi v} \sim 1$. For the remaining A, we have $A_{x^2-\pi^\kappa u} \not\sim 1$, but $B_{x^2-\pi^\kappa u} \sim (\alpha, (\pi\mu)^2) \sim 1$.
c. Let $\theta = \pi v + \sqrt{\pi^r\lambda}$. Then $f(\theta) \sim -v\tau\xi\lambda(\lambda-\delta)$ in $k(\sqrt{\pi\lambda})$ and $B_{\mathrm{Irr}(\theta)} \sim (v\delta\tau\xi\lambda, \sqrt{\pi\lambda})$. The condition $-v\tau\xi\lambda(\lambda-\delta) \sim 1$ and $v\delta\tau\xi\lambda \not\sim 1$ can be satisfied in a sufficiently large unramified extension of odd degree of k. Thus $B \not\sim 1$. We also have $(\alpha, x)_{\mathrm{Irr}(\theta)} \sim (\alpha, \pi v) \sim 1$, $(\alpha, x^2-\pi^\kappa u)_{\mathrm{Irr}(\theta)} \sim (\alpha, (\pi v)^2) \sim 1$, and $(\pi E(w))_{\mathrm{Irr}(\theta)} \sim (\lambda E(w), \pi v - \pi^{\kappa/2}w) \sim 1$. So $A \not\sim B$.
d. B is unramified since $f(\pi v + \pi^{r/2}\gamma) \sim \alpha\eta(\gamma)\zeta(\gamma)\pi$.

$$B_{(x-\pi v)^2-\pi^r\delta} \sim \left(\frac{\pi\ ,\ \prod_\sigma(\gamma^\sigma-\sqrt{\delta})}{k(\sqrt{\delta})\langle\ldots\rangle} \right) \not\sim 1.$$

The algebras (α, x), $(\alpha, x^2-\pi^\kappa u)$ split by $\sqrt{\delta}$, so they are not isomorphic to B. Finally, $(\pi E(w), x-\pi^{\kappa/2}w)_{(x-\pi v)^2-\pi^r\delta} \sim (\pi, \pi v - \pi^{\kappa/2}w + \pi^{r/2}\sqrt{\delta})$. If $r > 2$, then the latter algebra is trivial, and $A \not\sim B$ again. Let $\kappa = r = 2$. Then $A_{x^2-\pi^2u} \not\sim 1$ and $B_{x^2-\pi^2u} \sim (\pi, \prod_\sigma(\gamma^\sigma + v - \sqrt{u})) \sim 1$, since $(\gamma+v)^2 - u \sim 1$ in K.
e. In e) and f) we have

$$f(\pi v + \pi^{r/2}\mu) \sim (\mu^2-\delta)\eta\mu\zeta\mu \sim (\mu^2-\delta)\xi \left\{ \begin{array}{l} -v\tau,\ \text{if } 2l+2 < r, \\ \mu - v\tau,\ \text{if } 2l+2 = r. \end{array} \right\}.$$

If either $2l+2 = r$ or $-\tau v\xi \not\sim 1$, then there exists $\mu \in O_K^*$ such that $\mu^2-\delta \not\sim 1$ and $f(\pi v+\pi^{r/2}\mu) \sim 1$. For such μ we have $B_{\mathrm{Irr}(\pi v+\pi^{r/2}\mu)} \sim (\pi, \mu^2-\delta) \not\sim 1$. Since $r \geq 2l+2 \geq 4$, then $B_x \sim 1$ and $B_{x^2-\pi^\kappa u} \sim 1$. So $B \not\sim A$.
f. In this case $f(\pi v + \pi^{r/2}\mu) \sim \alpha$, so B is unramified. Let $\theta = \pi v(1 + pi^{l+1/3}) \in k(\Pi)$, $Pi = \sqrt[3]{\pi}$. Then $f(\theta) \sim -\tau v\xi(\pi^{2l+8/3}v^2 - \pi^r\delta) \sim 1$.

$$B_{\mathrm{Irr}(\theta)} \sim \left(\frac{\alpha\ ,\ \pi^{l+4/3}v - \pi^{r/2}\mu}{k(\Pi)\langle\mathrm{Irr}(\theta)\rangle} \right) \sim \left(\frac{\alpha\ ,\ \Pi^{3l+4}}{k(\Pi)\langle\mathrm{Irr}(\theta)\rangle} \right) \not\sim 1.$$

Moreover, $A \not\sim B$. Indeed, if κ is even, then $B_{x^2-\pi^\kappa u} \sim 1$ since B splits by $\sqrt{u}$, and if κ is odd, then $(\alpha, x^2-\pi^\kappa u)_{\mathrm{Irr}(\theta)} \sim 1$. Finally, for odd κ, we have $(\alpha, x)_{x^2-\pi^\kappa u} \not\sim 1$ and $B_{x^2-\pi^\kappa u} \sim (\alpha, -\pi v) \sim 1$.
g. Let $\mu \in O_K^*$ such that $\mu^2 - \delta \not\sim 1$, $\mu\eta(\mu) \not\sim 1$. Then $f(\pi v + \pi^{r/2}\mu) \sim 1$ and $B_{\mathrm{Irr}(\pi v+\pi^{r/2}\mu)} \not\sim 1$. But $B_x \sim 1$, so $B \not\sim (\alpha, x^2-\pi^\kappa u)$. The remaining algebra A is not trivial in $x^2-\pi^\kappa u$, but $B_{x^2-\pi^\kappa u} \sim (\pi, (\sqrt{\pi^\kappa u} - \pi v)^2 - \pi^r\delta) \sim 1$ since $r > 2$.
h. $B_{x-\pi v(1+\pi^l\tau)} \sim (\alpha, \pi^{2l+2}v^2\tau^2) - \pi^r\delta \not\sim 1$. We also have $B_x \sim (\alpha, \pi^2 v^2) \sim 1$. If κ is odd, then $B_{x^2-\pi^\kappa u} \sim (\alpha, (\sqrt{\pi^\kappa u} - \pi v)^2 - \pi^r\delta) \sim 1$ since $r > 2$. Finally, if κ is even, then $B_{x^2-\pi^\kappa u} \sim 1$ since B splits by $\sqrt{u}$.

Consider the next case.

Theorem 8 *Let f be as in case* VII_2, *i.e.*

$$f(x) = (x-\pi^s e)(x^2-\pi^\kappa u)((x-\pi v)^2 - \pi^r\delta), \quad s \geq 2,\ 4 \leq \kappa \leq r,\ \ \pi^\kappa u, \pi^r\delta \not\sim 1.$$

Then the group ${}_2\mathrm{Br}\, C$ *is generated by the classes of the algebra* (π, α) *and of the following algebras.*

If $2s < \kappa$, *then the first algebra* A *is*

1. (α, x), *if* $\kappa, s \equiv 1 \pmod 2$ *and* $ue \not\sim 1$.

2. $(\alpha, x^2-\pi^\kappa u)$, *if* $\kappa, s \equiv 1 \pmod 2$ *and* $ue \sim 1$.

3. $(-\pi\alpha e, x - \pi^{\kappa/2}w)$, *if* $\kappa \equiv 0 \pmod 2$, $s \equiv 1 \pmod 2$, *where* $w \in O_k^*$, $w^2 - u \not\sim 1$.

4. $(\pi, x^2-\pi^\kappa u)$, *if* $\kappa, s \equiv 0 \pmod 2$ *and* $-e \not\sim 1$.

5. $(\alpha, x - \pi^{\kappa/2}w)$, *if* $\kappa, s \equiv 0 \pmod 2$ *and* $-e \sim 1$, $w^2 - u \not\sim 1$.

6. $(ue\pi, x)$, *if* $\kappa \equiv 1 \pmod 2$, $s \equiv 0 \pmod 2$.

If $2s \geq \kappa$, *then the first algebra* A *is*

7. $(\alpha, x^2-\pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$.

8. $(\alpha\xi(w)\pi, x - \pi^{\kappa/2}w)$, *if* $\kappa \equiv 2 \pmod 4$, $w^2 - u \not\sim 1$,

$$\xi(x) = \begin{cases} w, & \text{if } 2s > \kappa, \\ w - e, & \text{if } 2s = \kappa, \bar{w} \neq \bar{e}. \end{cases}$$

9. $(\pi, x^2-\pi^\kappa u)$, *if* $\kappa \equiv 0 \pmod 4$.

And the second algebra B *is*

a. $(\alpha, x - \pi v)$, *if* $r \equiv 1 \pmod 2$, $-v\delta \not\sim 1$.

b. $(\alpha, (x-\pi v)^2 - \pi^r\delta)$, *if* $r \equiv 1 \pmod 2$, $-v\delta \sim 1$.

c. $(\pi\alpha v, x - \pi v - \pi^{r/2}\mu)$, *if* $r \equiv 0 \pmod 2$, *where* $\mu \in O_k^*$, $\mu^2 - \delta \not\sim 1$.

Proof. Let $2s < \kappa$. We have $f(0) \sim ue\pi^{\kappa+s}$, $f(\pi v) \sim -v\delta\pi^{r+1}$. If κ is even and $w^2-u \not\sim 1$, then $f(\pi^{\kappa/2}w) \sim -e\alpha\pi^s$. If r is even and $\mu^2-\delta \not\sim 1$, then $f(\pi v+\pi^{r/2}\mu) \sim v\alpha\pi$.
1. $(\alpha, x)_{x-\pi^s e} \not\sim 1$.
2. $(\alpha, x^2-\pi^\kappa u)_x \not\sim 1$.
3. $(-\pi\alpha e, x-\pi^{\kappa/2}w)_{x-\pi^s e} \not\sim 1$.
4. $(\pi, x^2-\pi^\kappa u)_{x-\pi^{\kappa/2}w} \not\sim 1$.
5. $f(\pi^{s+1}) \sim -\pi^s e \sim 1$, so $(\alpha, x-\pi^{\kappa/2}w)_{x-\pi^{s+1}} \sim (\alpha, \pi^{s+1}) \not\sim 1$.
6. $f(\sqrt{\pi^\kappa\gamma}) \sim -e\gamma(\gamma-u)$ in $k(\sqrt{\pi\gamma})$. So $(ue\pi, x)_{\mathrm{Irr}(\sqrt{\pi^\kappa\gamma})} \sim (ue\gamma, \sqrt{\pi\gamma}) \not\sim 1$, provided that $\gamma \in O_K^*$ satisfies the condition $\gamma \not\sim ue$, $\gamma - u \not\sim -u$.

Let now $\kappa \le 2s$. If κ is even and $w^2-u \not\sim 1$, then $f(\pi^{\kappa/2}w) \sim \pi^{\kappa/2}\alpha\xi(w)$.
7. $(\alpha, x^2-\pi^\kappa u)_{x-\pi^s e} \not\sim 1$.
8. $(\alpha\xi(w)\pi, x-\pi^{\kappa/2}w)_{x^2-\pi^\kappa u} \not\sim 1$.
9. Let $\gamma \in O_K^*$, $\gamma^2-u \not\sim 1$, and $\xi(\gamma) \not\sim 1$. Then $f(\pi^{\kappa/2}\gamma) \sim 1$ and $(\pi, x^2-\pi^\kappa u)_{\mathrm{Irr}(\pi^{\kappa/2}\gamma)} \not\sim 1$.
a. $(\alpha, x-\pi v)_{(x-\pi v)^2-\pi^r\delta} \not\sim 1$. On the other hand, $A_{(x-\pi v)^2-\pi^r\delta} \sim 1$.
b. $(\alpha, (x-\pi v)^2-\pi^\delta)_{x-\pi v} \not\sim 1$. $A \not\sim B$ since B is trivial at all the places from 1–9, at which A is nontrivial.
c. $(\pi\alpha v, x-\pi v-\pi^{r/2}\mu)_{(x-\pi v)^2-\pi^r\delta} \sim (\pi, \mu-\sqrt{\delta}) \not\sim 1$. On the other hand, $A_{(x-\pi v)^2-\pi^r\delta} \sim 1$.

The theorem is proved.

Let us pass to the last case.

Theorem 9 *Let f be as in case* VII_3*, i.e.,*

$$f(x) = (x^2-\pi^\kappa u)(x-\pi^s e)((x-\pi^m v)^2-\pi^r\delta),$$

where $\kappa \in \{1,2,3\}, \kappa < 2s, \kappa \le r,\ \pi^\kappa u, \pi^r\delta \not\sim 1$.

Then the group ${}_2\mathrm{Br}C$ is generated by the classes of the algebra (π,α) and of the following algebras.

The first algebra A is

i) $(\alpha, x^2-\pi^\kappa u)$, *if* $\kappa \equiv 1 \pmod 2$.

ii) $(\pi E(w), x-\pi w)$, *if* $\kappa = 2$, *where* $w^2-u \not\sim 1$,

$$E(x) = \begin{cases} \alpha x \text{ if } r > 2, \\ \alpha x(x^2-\delta) \text{ if } r = 2, m > 1, \bar{u} \ne \bar{\delta} \\ x, \text{ if } r = 2, m > 1, \bar{u} = \bar{\delta} \\ \alpha x((x-v)^2-\delta), \text{ if } r = 2, m = 1. \end{cases}$$

1. *If $r < 2s$, $r \le 2m$, then the second algebra B is*

- **a.** $(\alpha, (x-\pi^m v)^2-\pi^r\delta)$, *if* $r \equiv 1 \pmod 2$.
- **b.** $(\pi, (x-\pi^m v)^2-\pi^r\delta)$, *if* $r \equiv 0 \pmod 2$, $\kappa + r/2 \equiv 0 \pmod 2$.
- **c.** $(\pi N_{K|k}(S(\mu)), \mathrm{Irr}_{K|k}(\pi^m v + \pi^{r/2}\mu))$, *if* $r \equiv 0 \pmod 2$, $\kappa + r/2 \equiv 1 \pmod 2$, *and if* $\kappa = r = 2, m > 1$, *then* $\bar{u} \ne d\bar{e}lta$. *Here* $\mu \in O_K^*$, $K|k$ *is of odd degree and unramified,* $\mu^2-\delta \not\sim 1$. *If* $\kappa = r = 2$, *then the additional condition on* μ *is*

$$\begin{cases} \mu^2-u \sim 1, \text{ if } m > 1, \\ (\mu+v)^2-u \sim 1, \text{ if } m = 1. \end{cases}$$

Note that if $\kappa = r = 2$ does not hold, then one can assume $K = k$.

$$S(x) = \alpha \left\{ \begin{array}{l} x, \ \textit{if } 2m > r, \\ x + v, \ \textit{if } 2m = r, \end{array} \right\} \left\{ \begin{array}{l} -u, \ \textit{if } \kappa < r, \\ x^2 - u, \ \textit{if } \kappa = r < 2m, \\ (x+v)^2 - u, \ \textit{if } \kappa = r = 2m, \end{array} \right\}$$

2. *If $\kappa \leq 2m < r$, $m < s$, then the second algebra B is*

a. $(\alpha v \xi \pi, x - \pi^m v - \pi^{r/2}\mu)$, *if* $m + \kappa \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$, *where* $\mu^2 - \delta \not\sim 1$ *and*

$$\xi = \left\{ \begin{array}{l} -u, \ \textit{if } 2m > \kappa, \\ v^2 - u, \ \textit{if } 2m = \kappa. \end{array} \right.$$

b. $(-\delta v \xi \pi, x - \pi^m v)$, *if* $m + \kappa \equiv 0 \pmod 2$, $r \equiv 1 \pmod 2$.

c. $(\alpha, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $m + \kappa, r \equiv 1 \pmod 2$ *and* $-\delta v \xi \sim 1$.

d. $(\alpha, x - \pi^m v)$, *if* $m + \kappa, r \equiv 1 \pmod 2$ *and* $-\delta v \xi \not\sim 1$.

e. $(\pi, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $m + \kappa, r \equiv 0 \pmod 2$, $v\xi = -uv \not\sim 1$.

f. $(\alpha, x - \pi^m v - \pi^{r/2}\mu)$, *if* $m + \kappa, r \equiv 0 \pmod 2$, $v\xi = -uv \sim 1$, $\mu^2 - \delta \not\sim 1$.

3. *Let $2s \leq r$, $s \leq m$, and if $s = m$, then $\bar{v} \neq \bar{e}$. The second algebra B is*

a. $(u\delta\psi\pi, x - \pi^m v)$, *if* $\kappa + s \equiv 0 \pmod 2$, $r \equiv 1 \pmod 2$.

b. $(\alpha, x - \pi^m v)$, *if* $\kappa + s, r \equiv 1 \pmod 2$, $\psi u \delta \not\sim 1$.

c. $(\alpha, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $\kappa + s, r \equiv 1 \pmod 2$, $\psi u \delta \sim 1$.

d. $(-\pi u \alpha \psi(\mu), x - \pi^m v - \pi^{r/2}\mu)$, *if* $\kappa + s \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$, *where* $\mu^2 - \delta \not\sim 1$ *and*

$$\psi(x) = \left\{ \begin{array}{l} -e, \ \textit{if } s < m, \ 2s < r, \\ v - e, \ \textit{if } s = m, \ 2s < r, \\ x - e, \ \textit{if } s < m, \ 2s = r, \\ x + v - e, \ \textit{if } s = m, \ 2s = r, \end{array} \right.$$

if $2s < r$, then $\psi(x) = \psi$ does not depend on x.

e. $(\pi, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $\kappa + s, r \equiv 0 \pmod 2$, $2s < r$, $-u\psi \not\sim 1$.

f. $(\alpha, x - \pi^m v - \pi^{r/2}\mu)$, *if* $\kappa + s, r \equiv 0 \pmod 2$, $2s < r$, $-u\psi \sim 1$, $\mu^2 - \delta \not\sim 1$.

g. $(\pi, x - \pi^s e)$, *if* $\kappa + s, r \equiv 0 \pmod 2$, $2s = r$.

4. *If $2m < \kappa$ (i.e. $m = 1$, $\kappa = 3$, $s \geq 2$), then the second algebra B is*

a. $(\pi\alpha v, x - \pi v - \pi^{r/2}\mu)$, $r \equiv 0 \pmod 2$, $\mu^2 - \delta \not\sim 1$.

b. $(\alpha, (x - \pi v)^2 - \pi^r \delta)$, $r \equiv 1 \pmod 2$, $-v\delta \sim 1$.

c. $(\alpha, x - \pi v)$, $r \equiv 1 \pmod 2$, $-v\delta \not\sim 1$.

5. *If $\kappa < 2m = 2s < r$, $e = v(1 + \pi^l \tau)$, $l > 0$, then the second algebra B is*

a. $(\alpha, x - \pi^m v)$, *if* $\kappa + m + l, r \equiv 1 \pmod 2$, $-uv\tau\delta \not\sim 1$.

b. $(\alpha, (x - \pi^m v)^2 - \pi^r \delta)$, *if* $\kappa + m + l, r \equiv 1 \pmod 2$, $-uv\tau\delta \sim 1$.

c. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l \le r/2$, $\kappa+m+l, r \equiv 0 \pmod 2$, *and either* $m+l < r/2$, $uv\tau \not\sim 1$, *or* $m+l = r/2$.

d. $(\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l < r/2$, $\kappa+m+l, r \equiv 0 \pmod 2$, *and* $uv\tau \sim 1$. $\mu^2 - \delta \not\sim 1$.

e. $(-\pi uv\tau\delta, x-\pi^m v)$, *if* $m+l < r/2$, $\kappa+m+l \equiv 0 \pmod 2$, $r \equiv 1 \pmod 2$.

f. $(\pi T(\mu), x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l \le r/2$, $\kappa+m+l \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$, *where* $\mu^2-\delta \not\sim 1$, *and*

$$T(x) = \begin{cases} \alpha v\tau u, & \text{if } m+l < r/2, \\ -\alpha u(x - v\tau), & \text{if } m+l = r/2. \end{cases}$$

g. $(-\pi u\mu\alpha, x-\pi^m v - \pi^{r/2}\mu)$, *if* $m+l > r/2$, $\kappa + r/2 \equiv 1 \pmod 2$, $r \equiv 0 \pmod 2$.

h. $(\pi, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l > r/2$, $\kappa + r/2, r \equiv 0 \pmod 2$.

j. $(\alpha, (x-\pi^m v)^2 - \pi^r\delta)$, *if* $m+l > r/2$, $r \equiv 1 \pmod 2$, $\kappa + l + m \equiv 0 \pmod 2$.

6. *If* $\kappa = r = 2$, $m, s \ge 2$, $\bar u = \bar\delta$, *then the second algebra* B *is*

a. $(\pi, x^2 - \pi^\kappa u)$, *if* $-1 \sim 1$.

b. $(\alpha(x^2-\pi^\kappa u), x - \pi w)$, *if* $-1 \not\sim 1$, *where* $w^2 - u \not\sim 1$.

Proof.
i),ii). $(\alpha, x^2-\pi^\kappa u)_{x-\pi^s e} \not\sim 1$, provided that κ is odd. Otherwise we have $f(\pi w) \sim \pi E(w)$, so A is unramified. Moreover,

$$(\pi E(w), x-\pi w)_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ w - \sqrt{u}}{k(\sqrt{u})\langle x^2-\pi^\kappa u\rangle (D(x^2-\pi^\kappa u))} \right) \not\sim 1.$$

1.a. $B_{x-\pi^s e} \sim (\alpha, \pi^r) \not\sim 1$. If κ is odd, then $(A\otimes B)_{x^2-\pi^\kappa u} \sim (x-\pi^s e)_{x^2-\pi^\kappa u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$, so $A \not\sim B$. Let κ be even. Then $B_{x^2-\pi^\kappa u} \sim 1$, and $A \not\sim B$ again.
1.b. If r is even, then $f(\pi^m v + \pi^{r/2}\mu) \sim \pi^{\kappa + r/2} S(\mu)$. In case 1.b) one can find K, μ such that $S(\mu) \sim 1$ and $\mu^2 - \delta \not\sim 1$. Therefore $B_{\mathrm{Irr}(\pi^m v + \pi^{r/2}\mu)} \sim (\pi, \mu^2-\delta) \not\sim 1$. For odd κ we have $A_{x^2-\pi^\kappa u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$, but $B_{x^2-\pi^\kappa u} \sim 1$. Let κ be even. Then $r/2$ is even, so $r > \kappa$ and $B_{x^2-\pi^\kappa u} \sim 1$. Anyway, $A \not\sim B$.
1.c. We have $B_{(x-\pi^m v)^2 - \pi^r\delta} \not\sim 1$. On the other hand, for odd κ $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$. Let κ be even. If $r > 2$, then $A_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, -\pi w) \sim 1$. Let $\kappa = r = 2$.

$$B_{x^2-\pi^\kappa u} \sim \left(\frac{\pi\ ,\ \prod_\sigma ((\mu^\sigma + \pi^{m-1} v) - \sqrt{u})}{k(\sqrt{u})\langle \ldots \rangle} \right) \sim 1.$$

Thus, $A \not\sim B$.

It is easy to check that, in 2), $f(\pi^m v) \sim -\pi m + \kappa + r\delta\xi v$ and for even r we have $f(\pi^m v + \pi^{r/2}\mu) \sim (\mu^2 - \delta)\xi v \pi^{m+\kappa}$. Moreover, $E(w) \sim \alpha w$, since $\kappa < r$. So, in 2), $A_{x-\pi^s e} \not\sim 1$ and $A_{x^2-\pi^\kappa u}$ in case i) as well as in case ii).
2.a. Let firstly m be even and κ be odd. Then $B_{x^2-\pi^\kappa u} \sim (\sqrt{\pi u}, -\alpha v)$ and $B_{x-\pi^s e} \sim (\pi, -v)$. There is exactly one nontrivial algebra among the two last algebras. So $B \not\sim 1$ and $A \not\sim B$.

Let now m be odd and κ be even. Then $B_{x-\pi^s e} \sim (\alpha v\xi\pi, -\pi v) \sim (\pi, \alpha\xi)$. Furthermore, the algebra $B_{x^2-\pi^\kappa u}$ is nontrivial iff algebra (π, ξ) is nontrivial over k. Thus, $B \not\sim 1$, $A \not\sim B$ again.

2.b. Assume $\theta = \pi^m v + \sqrt{\pi^r \lambda}$, $\lambda \in O_K^*$, $p = \mathrm{Irr}_{K(\sqrt{\pi\lambda})|k}(\theta)$. We have $f(\theta) \sim v\xi\lambda(\lambda - \delta)$ in $K(\sqrt{\pi\lambda})$.

$$B_p \sim \left(\frac{\sqrt{\pi\lambda}\,,\ -\delta v\xi\lambda}{K(\sqrt{\pi\lambda})\langle p\rangle} \right).$$

One always can find K, λ such that $B_p \not\sim 1$. If κ is odd, then $A_p \sim (\alpha, \pi) \sim 1$ over $K(\sqrt{\pi\lambda})$ and $A \not\sim B$. Let κ be even. $A_p \sim (\alpha\lambda w, \pi^m v + \sqrt{\pi^r\lambda} - \pi w) \sim 1$.

2.c. $B_{x-\pi^m v} \not\sim 1$. In addition, $B_{x-\pi^s e} \sim 1$, so $A \not\sim B$.

2.d. If κ is odd and m is even, then $B_{x^2-\pi^\kappa u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$ and $B_{x-\pi^s e} \sim (\alpha, \pi^m) \sim 1$. Otherwise, $B_{x-\pi^s e} \not\sim 1$ and $B_{x^2-\pi^\kappa u} \sim 1$.

2.e. Let $\mu^2 - \delta \not\sim 1$, then $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $B_{\mathrm{Irr}(\pi^m v + \pi^{r/2}\mu)} \not\sim 1$. $B_{x-\pi^s e} \not\sim 1$.

2.f. Assume $\Pi = \sqrt[3]{\pi}$, $\theta = \pi^m v + \pi^m \Pi$, and $p = \mathrm{Irr}_{k(\Pi)|k}(\theta)$. We have $f(\theta) \sim 1$, $B_p \sim (\alpha, \Pi\pi^m)$, $B_{x-\pi^s e} \sim (\alpha, \pi^m)$. There is exactly one nontrivial algebra among the two last algebras. On the other hand, $A_{x-\pi^s e} \not\sim 1$ and $A_p \not\sim 1$. Indeed, if κ is odd, then $A_p \sim (\alpha, -\pi^\kappa u)$ and $A_p \sim (\alpha w\pi, -\pi w)$. So $b \not\sim 1$ and $B \not\sim A$.

In 3) $f(\pi^m v) \sim \pi^{\kappa+r+s} u\delta\psi$ and $f(\pi^m v + \pi^{r/2}\mu) \sim -\pi^{\kappa+s} u\psi(\mu)(\mu^2 - \delta)$, provided that r is even. If κ is odd, then $A_{x^2-\pi^\kappa u} \sim (\alpha, \sqrt{\pi u}) \not\sim 1$. If κ is even, then $A_{x-\pi^s e} \sim (\pi\alpha w, -\pi w) \not\sim 1$. Anyway, A is nontrivial at two these places. On the other hand, $A_{(x-\pi^m v)^2 - \pi^r\delta} \sim 1$.

3.a. This case can be considered by analogy with 2.b).

3.b. We have $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.

3.c. $B \not\sim 1$ since $B_{x-\pi^m v} \not\sim 1$, and $B \not\sim A$ in view of $B_{x-\pi^s e} \sim 1$.

3.d. $B_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, \mu - \sqrt{\delta}) \not\sim 1$, provided that $\mu^2 - \delta \not\sim 1$. So $B \not\sim 1, A$.

3.e. For $\mu^2 - \delta \not\sim 1$ $f(\pi^m v + \pi^{r/2}\mu) \sim 1$ and $B_{x-\pi^m v - \pi^{r/2}\mu} \not\sim 1$. $B_{x-\pi^s e} \sim 1$.

3.f. Let $\theta_i = \pi^m v + \pi^s \Pi^i \in k(\Pi)$, $\Pi = \sqrt[3]{\pi}$, $p_i = \mathrm{Irr}_{k(\Pi)|k}(\theta_i)$, $i = 1, 2$. Then $f(\theta_i) \sim -\pi^\kappa u\pi^s\psi(\pi^{2s}\Pi^{2i} - \pi^r\delta) \sim -u\psi \sim 1$ and $B_{p_i} \sim (\alpha, \pi^s\Pi^i)$ over $k(\Pi)$. So there is exactly one nontrivial algebra among B_{p_i}, $i = 1, 2$. On the other hand, $A_{p_i} \not\sim 1$. Therefore, $B \not\sim 1, A$.

3.g. Let $\eta_i = \pi^s\gamma_i$, $\gamma_i \in O_K^*$, $\bar{\gamma}_i \neq \bar{e}$, $q_i = \mathrm{Irr}_{K|k}(\eta_i)$, $i = 1, 2$. Then

$$f(\eta_i) \sim -u(\gamma_i - e) \left\{ \begin{array}{l} \gamma_i^2 - \delta, \text{ if } m > s, \\ (\gamma_i - v)^2 - \delta, \text{ if } m = s. \end{array} \right\}$$

We require $\gamma_1 - e \not\sim \gamma_2 - e$ and $f(\eta_i) \sim 1$, $i = 1, 2$. Then $B_{q_i} \sim (\pi, (-1)^s(\gamma_i - e))$ and $A_{q_i} \sim 1$, $i = 1, 2$. Since there is exactly one nontrivial algebra among B_{q_i}, $i = 1, 2$, $B \not\sim 1, A$.

In 4) we have $f(\pi v) \sim -v\delta\pi^{r+1}$ and $f(\pi v + \pi^{r/2}\mu) \sim v(\mu^2 - \delta)\pi$, provided that r is even. In this case $A = (\alpha, x^2 - \pi^3 u)$ and $A_{x-\pi^s e} \not\sim 1$, $A_{x^2-\pi^3 u} \not\sim 1$.

4.a. $B_{x-\pi^s e} \sim (\alpha v\pi, -\pi v) \not\sim 1$, $B_{x^2-\pi^\kappa u} \sim 1$, so that $B \not\sim 1, A$.

4.b. $B \not\sim 1$ in view of $B_{x-\pi v} \not\sim 1$, and $B \not\sim A$ since $B_{x-\pi^s e} \sim 1$.

4.c. Finally, in this case $B_{x-\pi^s e} \not\sim 1$ and $B_{x^2-\pi^3 u} \sim 1$.

In 5) $f(\pi^m v) \sim -\pi^{\kappa+m+l+r} uv\tau\delta$ and if r is even, then

$$f(\pi^m v + \pi^{r/2}\mu) \sim (\mu^2 - \delta) \left\{ \begin{array}{l} \pi^{\kappa+l+m} uv\tau, \text{ if } m + l < r/2, \\ -\pi^{\kappa+l+m} u(\mu - v\tau), \text{ if } m + l = r/2, \\ -\pi^{\kappa+r/2} u\mu, \text{ if } m + l > r/2. \end{array} \right\}$$

We also have $A_{x-\pi^s e} \not\sim 1$ and $A_{x^2-\pi^\kappa u} \not\sim 1$.
5.a. $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
5.b. $B_{x-\pi^m v} \not\sim 1$, $B_{x^2-\pi^\kappa u} \sim 1$.
5.c. If $m+l < r/2$, then $f(\pi^m v + \pi^{r/2}\mu) \sim 1$, provided that $\mu^2 - \delta \not\sim 1$. Otherwise, $f(\pi^m v+\pi^{r/2}\mu) \sim (\mu^2-\delta)(v\tau-\mu)$, and we always can find K, μ such that $f(\pi^m v+\pi^{r/2}\mu) \sim 1$ and $\mu^2 - \delta \not\sim 1$ again. Anyway, $B_{\mathrm{Irr}(\pi^m v+\pi^{r/2}\mu)} \not\sim 1$. $B \not\sim A$ in view of $B_{x^2-\pi^\kappa u} \sim 1$.
5.d. Let $\theta_i = \pi^m v + \pi^(m+l)\Pi^i$, $p_i = \mathrm{Irr}_{k(\Pi)|k}(\theta_i)$, $i = 1, 2$. Then $f(\theta_i) \sim uv\tau \sim 1$ and $B_{p_i} \sim (\alpha, \Pi^{m+l+i})$. So there is exactly one nontrivial algebra among B_{p_i}, $i = 1, 2$. On the other hand, $A_{p_i} \not\sim 1$. Therefore, $B \not\sim 1, A$.
5.e. Let $\theta = \pi^m v + \sqrt{\pi^r\lambda} \in K(\sqrt{\pi\lambda})$ and $p = \mathrm{Irr}(\theta)$. Then $f(\theta) \sim uv\tau\lambda(\lambda-\delta)$ and

$$B_p \sim \left(\frac{\sqrt{\pi\lambda}\,,\ -\lambda uv\tau\delta}{K(\sqrt{\pi\lambda})\langle p\rangle}\right).$$

One always can chooseK and λ such that that $B_p \not\sim 1$. Furthermore, $B \not\sim A$ since $A_p \sim 1$.
5.f. $f(\pi^m v + \pi^{r/2}\mu) \sim \pi T(\mu)$, so B is unramified. $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$, but
$B_{(x-\pi^m v)^2-\pi^r\delta} \sim (\pi, \mu - \sqrt{\delta}) \not\sim 1$.
5.g. $B_{(x-\pi^m v)^2-\pi^r\delta} \not\sim 1$ and $A_{(x-\pi^m v)^2-\pi^r\delta} \sim 1$.
5.h. In view of $f(\pi^m v+\pi^{r/2}\mu) \sim -u\mu(\mu^2-\delta)$ we can find μ such that $f(\pi^m v+\pi^{r/2}\mu) \sim 1$ and $(\mu^2 - \delta) \not\sim 1$. Then $B_{\mathrm{Irr}(\pi^m v+\pi^{r/2}\mu)} \not\sim 1$. Moreover, $B_{x^2-\pi^\kappa u} \sim 1$.
5.j. $B_{x-\pi^s e} \sim (\alpha, \pi^r) \not\sim 1$. Finally, $B \not\sim A$, since $B_{x^2-\pi^\kappa u}$.
6. $A = (\pi w, x - \pi w)$, $w^2 - u \not\sim 1$. So, $A_{x^2-\pi^\kappa u} \not\sim 1$ and $A_{x-\pi^s e} \sim (\pi w, -\pi w) \sim 1$. Let $-1 \sim 1$. Then $B_{x-\pi^s e} \sim (\pi, -u) \not\sim 1$. Thus, $B \not\sim 1, A$. Let now $-1 \not\sim 1$. In this case $B_{x-\pi w} \sim (\alpha(w^2-u), x-\pi w) \sim 1$, so B is unramified. Finally, $B_{x-\pi^s e} \sim (-u\alpha, -\pi w) \not\sim 1$.
The theorem is proved.

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Quaternion generation of the 2-torsion part of the Brauer group of local quintic (the type of splitting (1,4))

Genady Margolin Ulf Rehmann Vyacheslav Yanchevskii

1 Introduction

This paper is a continuation of [3] and [4] and [5], where we consider a quintic given by an affine equation $y^2 = f(x)$, and $f(x)$ has the splitting types $(1,1,1,1,1)$ resp. $(1,1,1,2)$ resp. $(1,2,2)$. Here we consider the case where the splitting type of $f(x)$ is $(1,4)$. We use the same notations as in the previous papers.

2

Let k be a local non-dyadic field of characteristic 0 with residue field $\bar{k}$. Let $\pi \in k$ denote a prime element, and let O_k, O_k^*, πO_k be its valuation ring, the group of units and the maximal ideal of the valuation ring. For $u \in O_k^*$ let $\bar{u}$ denote its class *modulo* πO_k. We will consider also a hyperelliptic curve C defined over k by the equation $y^2 = f(x)$, where $f(x) \in O_k[x]$ is a monic polynomial without multiple roots and $\deg f = n$. Without loss of generality one can assume that $f(x) = \prod_{i=1}^{m} f_i(x)$, where $f_i(x) \in O_k[x]$ are monic irreducible polynomials. This paper is devoted to the case $\deg f_1 = 4$, $\deg f_2 = 1$ and the reduction is bad. Let α be a fixed unit of k which is not a square in k. $\bar{g}(x)$ will denote the reduction of a polynomial $g(x) \in O_k$. For $a, b \in k^*$ we write $a \sim b$ if a and b belong to the same class *modulo* $(k^*)^2$.

In our further considerations we will need lemmas from [3], [4] and [5].

Lemma 1 $|\,{}_2\mathrm{Br}\, C| = 4$.

So it is enough to find one unramified and nontrivial algebra which is not isomorphic to the scalar algebra (π, α). The nonisomorphy of this algebra and (π, α) will usually not be checked since it is an immediate consequence of lemma 3 from [4].

Lemma 2 *By an appropriate replacement of the variable x the polynomial f can be reduced to the form $f(x) = g(x)(x - \pi^l e)$, where $l \geq 0$, $e \in O_k^*$ and g is irreducible, and either*

I. $g(x) = (x^2 - u)^2 + \pi^m v x + \pi^n \delta$, *or*

II. $g(x) = x^4 + \pi^k u x^2 + \pi^m v x + \pi^r \delta$,

where $\kappa, m, n > 0$, $u, v, \delta \in O_k^*$, and, in case I, *$u \notin (O_k^*)^2$.*

Proof. Since $g(x)$ is irreducible, by Hensel's lemma we have only the following two possibilities for $\bar{g}(x)$.
I. $\bar{g}(x) = (x^2 + px + q)^2$, $p, q \in \bar{k}$, $p^2 - 4q \notin \bar{k}^2$,
II. $\bar{g}(x) = (x - \gamma)^4$, $\gamma \in \bar{k}$.

In the second case one can assume that $\gamma = 0$, i.e., $g(x) = x^4 + ax^3 + bx^2 + cx + d$, $a, b, c, d \in M_k$. Moreover, by replacing $x \mapsto x - a/4$ we can remove the monomial ax^3.

In the first case the same replacement gives $(x^2 + px + q)^2 = x^4 + \bar{b}x^2 + \bar{c}x + \bar{d}$, so $p = 0$, $\bar{b} = 2q$, $\bar{d} = q^2$. Suppose $b = -2u$, $\bar{u} = -q$, then $d = u^2 + \lambda$, $\lambda \in M_k$, and $g(x) = (x^2 - u)^2 + cx + \lambda$. hence the lemma is proved.

Theorem 1 *Let the curve C correspond to case* I *from lemma 2. Then the following algebra A is nontrivial, unramified, and not isomorphic to the scalar algebra (π, α). (We define $\Delta = g(\sqrt{u}) = \pi^m v\sqrt{u} + \pi^n \delta \in k(\sqrt{u})$).*

1. $v(\Delta) \equiv 1\,(2)$, $\pi\Delta\xi(\sqrt{u}) \sim 1$ *in* $k(\sqrt{u})$, *where*

$$\xi(x) = \begin{cases} x, & \text{if } l > 0, \\ x - e, & \text{if } l = 0. \end{cases}$$

$A = (\pi, x^2 - u)$.

2. $v(\Delta) \equiv 1\,(2)$, $\pi\Delta\xi(\sqrt{u}) \not\sim 1$.
$A = (\alpha(x^2 - u), \pi(x - \beta))$, $\beta \in O_k^*$, $\beta^2 - u \not\sim 1$.

3. $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{u}) \not\sim 1$ *in* $k(\sqrt{u})$.
$A = (\pi, x - \pi^l e)$.

4. $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{u}) \sim 1$.
$A = (\alpha p(x), x - \beta)$, *where* $p = \mathrm{Irr}_{k(\sqrt{u}|k)}(\theta)$, $\theta = \sqrt{u}(1 + \pi^s w)$, $w \in O^*_{k(\sqrt{u})}$ *such that* $\mu + 4u^2 w^2 \in O^*_{k(\sqrt{u})} \setminus (O^*_{k(\sqrt{u})})^2$, $\Delta = \pi^{2s}\mu$.

Proof.
1. We have $f(\sqrt{u}) \sim \Delta\xi(\sqrt{u})$, so A is unramified. Let $K|k$ be unramified extension of odd degree and $\eta \in O_K^*$ such that

$$\begin{cases} \xi(\eta) \sim 1, \\ \eta^2 - u \not\sim 1. \end{cases}$$

Then $f(\eta) \sim \xi(\eta) \sim 1$ and

$$A_{\mathrm{Irr}_{K|k}(\eta)} \sim \left(\frac{\pi\ ,\ \eta^2 - u}{K\langle\star\rangle}\right) \not\sim 1.$$

2. We have $A_{x-\beta} \sim (\alpha^2, \pi(x - \beta)) \sim 1$ and

$$A_{x^2-u} \sim \left(\frac{x^2 - u\ ,\ \pi(\beta - \sqrt{u})}{k(\sqrt{u})(\sqrt{\Delta\xi(\sqrt{u})})\langle x^2 - u\rangle}\right) \sim 1,$$

since $\Delta\xi(\sqrt{u}) \sim \pi(\beta - \sqrt{u})$ in $k(\sqrt{u})$.

Thus, A is unramified. If $\eta \in O_K^*$ satisfies the conditions

$$\begin{cases} \xi(\eta) \sim 1, \\ \eta^2 - u \sim 1, \end{cases}$$

then $f(\eta) \sim \xi(\eta) \sim 1$ and

$$A_{\mathrm{Irr}_{K|k}(\eta)} \sim \left(\frac{\alpha(\eta^2 - u)\ ,\ \pi(\eta - \beta)}{K\langle\star\rangle} \right) \sim (\alpha, \pi) \not\sim 1.$$

3. Let μ, s, θ, p be from item 4 of this theorem. Then $g(\theta) \sim \mu + 4u^2w^2 \not\sim 1$, and $f(\theta) \sim g(\theta)\xi(\sqrt{u}) \sim 1$. So

$$A_p \sim \left(\frac{\pi\ ,\ \theta - \pi^l e}{k(\sqrt{u})\langle p\rangle} \right) \sim \left(\frac{\pi\ ,\ \xi(\sqrt{u})}{k(\sqrt{u})\langle p\rangle} \right) \not\sim 1.$$

4. In view of $\alpha p(\beta) \sim \alpha(\beta - \sqrt{u} - \pi^s\sqrt{u}w)(\beta + \sqrt{u} + \pi^s\sqrt{u}w^\sigma) \sim \alpha(\beta^2 - u) \sim 1$ we have $A_{x-\beta} \sim 1$. Furthermore, $f(\theta) \sim \mu + 4u^2w^2 \not\sim 1$, therefore $f(\theta) \sim \beta - \sqrt{u}$ in $k(\sqrt{u})$ and

$$A_p \sim \left(\frac{p\ ,\ \beta - \sqrt{u}}{k(\sqrt{u})(\sqrt{f(\theta)})\langle p\rangle} \right) \sim 1.$$

Thus A is unramified. We prove its nontriviality. Assume that $\zeta = \sqrt{u}(1+\Pi) \in k(\sqrt{u}, \Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\zeta) \sim (u^2(2+\Pi)^2\Pi^2 + \pi^{2s}\mu)\xi(\sqrt{u}) \sim 1$ and

$$A_{\mathrm{Irr}_{k(\sqrt{u},\Pi)}}(\zeta) \sim \left(\frac{\alpha p(\zeta)\ ,\ \zeta - \beta}{k(\sqrt{u}, \Pi)\langle\star\rangle} \right) \sim \left(\frac{p(\zeta)\ ,\ \beta - \sqrt{u}}{k(\sqrt{u}, \Pi)\langle\star\rangle} \right) \not\sim 1,$$

since $p(\zeta) \sim (\zeta - \theta)(\zeta - \theta^\sigma) \sim 2\sqrt{u}(\Pi\sqrt{u} - \pi^s w\sqrt{u}) \sim \Pi$.

The theorem is proved.

Let us pass to case II. It is convenient to introduce the following notations.

$$d = \min\left\{ \frac{\kappa}{2}, \frac{m}{3}, \frac{n}{4} \right\} \in \frac{1}{4}\boldsymbol{Z} \bigcup \frac{1}{3}\boldsymbol{Z},$$

$$h(t) = t^4 + \pi^{\kappa-2d}ut^2 + \pi^{m-3d}vt + \pi^{n-4d}\delta \in k(\pi^d)[t].$$

Then, for a suitable unramified $K|k$ of odd degree and $w \in O_K^*$, we have

$$g(\pi^d w) \sim \pi^{4d}h(w) \sim h(w) \in O^*_{K(\pi^d)}.$$

In addition, $\bar{h}(t) \in \bar{k}[t]$ is a monic polynomial which is of degree 4 and does not coincide with t^4.

Theorem 2 *Assume we are in case* II, $d \le l$, *and* $\bar{h}(t) \notin \bar{k}[t]^2$. *Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar algebra.*

1. $d \in 2\boldsymbol{Z}$, $A = (\pi, x - \pi^l e)$.

2. $d \in 1+2\boldsymbol{Z}$, $A=(\pi, \mathrm{Irr}_{K|k}(\pi^d w))$, $w \in O_K^*$ *such that*

$$\begin{cases} \xi(w) \not\sim 1, \\ h(w) \not\sim 1, \end{cases} \quad \textit{where } \xi(x) = \begin{cases} x, & \textit{if } d < l \\ x-e, & \textit{if } d = l. \end{cases}$$

3. $d \in 2/3+\boldsymbol{Z}$, $A=(\pi, x-\pi^l e)$.

4. $d \in 1/3+\boldsymbol{Z}$, $A=(\pi, \mathrm{Irr}_{K(\sqrt[3]{\pi})|k}(\pi^d w))$, K *and* w *as in 2.*

5. $d \in 1/4+\boldsymbol{Z} \cup 3/4+\boldsymbol{Z}$, $A=(\alpha, x-\pi^l e)$.

6. $d \in 1/2+\boldsymbol{Z}$, $A=(-\pi \mathrm{Irr}_{K(\sqrt{\pi})|k}(\pi^d w), \alpha x)$, $w \in O_K^*$, $h(w) \not\sim 1$.

Proof. Since $\bar{h}(t) \notin \bar{k}[t]^2$, the equation $y^2 = a\bar{h}(t)$, $a \in \bar{k}^*$ gives a variety in $\boldsymbol{A}^2(\bar{k}_s)$.

Therefore, by the Lang-Weil theorem, for any $N \in \boldsymbol{N}$ there exists an extension of sufficiently large degree $\tilde{K}|\bar{k}$ such that $|V(\tilde{K})| > N$. So there exist $\tilde{w}_i \in \tilde{K}$, $i=1,2$ such that $\bar{h}(\tilde{w}_1) \sim 1$, $\bar{h}(\tilde{w}_2) \not\sim 1$ in $\tilde{K}$. After lifting $\tilde{K}$ to K and $\tilde{w}_i$ to $w_i \in O_K^*$ ($\bar{\tilde{K}} = \tilde{K}$, $\bar{w}_i = \tilde{w}_i$) we have $h(w_1) \sim 1$, $h(w_2) \not\sim 1$. If w is required to have some additional property (for example, $\xi(w) \not\sim 1$), then one can consider a variety V in $\boldsymbol{A}^3(\bar{k}_s)$ given by

$$\begin{cases} y^2 = a\bar{h}(t), \\ z^2 = \bar{\alpha}\bar{\xi}(t) \text{ (lemma 6 from [3])}. \end{cases}$$

1. Let K, w be as in item 2 of this theorem. Then $f(\pi^d w) \sim h(w)\xi(w) \sim 1$, so

$$A_{Irr(\pi^d w)} \sim \left(\frac{\pi\ ,\ \xi(w)}{K\langle Irr(\pi^d w)\rangle} \right) \not\sim 1.$$

2. $f(\pi^d w) \sim h(w)\xi(w)\pi \sim \pi$, so A is unramified. We also have

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma \in G(K|k)} (\pi^l e - \pi^d w^\sigma)) \sim (\pi, N_{K|k}(\xi(w))) \not\sim 1.$$

3. $f(\pi^d w) \sim wh(w)$ in $K(\sqrt[3]{\pi})$. Let $w, h(w) \not\sim 1$, then

$$A_{Irr(\pi^d w)} \sim \left(\frac{\sqrt[3]{\pi}\ ,\ w}{K(\sqrt[3]{\pi})\langle\star\rangle} \right) \not\sim 1.$$

4. $f(\pi^d w) \sim \pi w h(w) \sim \pi$ and A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma \in G(K|k)} (x^3 - \pi^{3d}(w^\sigma)^3))_{x-\pi^l e} \sim (\pi, -\pi N_{K|k}(w)) \not\sim 1.$$

5. In this case $n \equiv 1\,(2)$ and $d = n/4 < \min\{\kappa/2, m/3, l\}$.
Let $d < d' < \min\{\kappa/2, m/3, l\}$, $d' \in \boldsymbol{Q}$ such that $d' = p/q$, $p, q \in \boldsymbol{N}$, $(p,q) = 1$, $p, q \equiv 1\,(2)$. Under these assumptions $f(\pi^{d'}\delta) \sim \pi^{(p+nq)/q} \sim 1$ in $k(\pi^{1/q})$. Therefore

$$A_{Irr(\pi^{d'}\delta)} \sim (\alpha, \pi^{d'}\delta) \sim \left(\frac{\alpha\ ,\ \pi^{1/q}}{k(\pi^{1/q})\langle\star\rangle} \right) \not\sim 1.$$

6. We have

$$A_x \sim (-\pi \prod_\sigma -\pi^{2d}(w^\sigma)^2, \alpha x) \sim 1.$$

Also we have $f(\pi^d w) \sim \pi^d w h(w) \sim \alpha w \pi^d$ in $k(\sqrt{\pi})$, hence

$$A_{\mathrm{Irr}(\pi^d w)} \sim \left(\frac{-\pi \mathrm{Irr}(\pi^d w)\,,\ \alpha w \pi^d}{K(\sqrt{\pi})(\sqrt{f(\pi^d w)})\langle\star\rangle} \right) \sim 1.$$

Thus, A is unramified. A is nontrivial in view of $f(\pi^{2/7}) \sim \pi^{2/7} - \pi^l e \sim 1$ in $k(\pi^{1/7})$, so

$$A_{x^7-\pi^2} \sim \left(\frac{-\pi \prod_\sigma (\pi^{4/7} - \pi^{2d}(w^\sigma)^2)\,,\ \alpha \pi^{2/7}}{k(\pi^{1/7})\langle x^7 - \pi^2 \rangle} \right) \sim \left(\frac{-\pi^{1/7}\,,\ \alpha}{k(\pi^{1/7})\langle x^7 - \pi^2 \rangle} \right) \not\sim 1.$$

The theorem is proved.

Theorem 3 *Assume we are in case* II *and* $d > l$, $\bar{h}(t) \notin \bar{k}[t]^2$. *Then the following algebra* A *is unramified, nontrivial, and not isomorphic to the scalar algebra.*

1. $l \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$, $-e \sim 1$.
$A = (\alpha, \mathrm{Irr}_{K|k}(\pi^d w))$, $h(w) \not\sim 1$.

2. $l \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$, $-e \not\sim 1$.
$A = (\alpha, x^q - \pi^p)$, *where* $l < p/q < d$, $p, q \in \boldsymbol{N}$, $(p, q) = 1$, $p, q \equiv 1\,(2)$.

3. $l \equiv 0\,(2)$, $n \equiv 1\,(2)$. *(This is always true if* $d \in 1/4 + \boldsymbol{Z} \cup 3/4 + \boldsymbol{Z}$*).*
$A = (-\pi\delta e, x)$.

4. $l \equiv 0\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$, $-\delta e \not\sim 1$.
$A = (\alpha, x)$.

5. $l \equiv 0\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$, $-\delta e \sim 1$.
$A = (\alpha, \mathrm{Irr}_{K(\sqrt{\pi})|k}(\pi^d w))$, $h(w) \not\sim 1$.

6. $l \equiv 1\,(2)$, $n \equiv 1\,(2)$, $-\delta e \sim 1$.
$A = (\alpha, x - \pi^l e)$.

7. $l \equiv 1\,(2)$, $n \equiv 1\,(2)$, $-\delta e \not\sim 1$.
$A = (\alpha, x)$.

8. $l \equiv 1\,(2)$, $n \equiv 0\,(2)$, $d \in (1/3)\boldsymbol{Z}$.
$A = (-\alpha e \pi, \mathrm{Irr}_{K(\pi^d)|k}(\pi^d w))$, $h(w) \not\sim 1$.

9. $l \equiv 1\,(2)$, $n \equiv 0\,(2)$, $d \in 1/2 + \boldsymbol{Z}$.
$A = (-\pi\delta e, x)$.

Proof. 1. $f(\pi^d w) \sim h(w)(-\pi^l e) \sim \alpha$, so A is unramified. $f(\pi^{p/q}) \sim -\pi^l e \sim 1$. Then

$$A_{x^p-\pi^q} \sim \left(\frac{\alpha\ ,\ \prod_\sigma(\pi^{p/q}-\pi^d w^\sigma)}{k(\pi^{1/q})\langle\star\rangle}\right) \sim \left(\frac{\alpha\ ,\ \pi^{1/q}}{k(\pi^{1/q})\langle\star\rangle}\right) \not\sim 1.$$

2. $f(\pi^{p/q}) \sim -e$, so A is unramified. Moreover, $f(\pi^d w) \sim -eh(w) \sim 1$, so that

$$A_{\mathrm{Irr}(\pi^d w)} \sim \left(\frac{\alpha\ ,\ -\pi^p}{k(\pi^d)\langle\star\rangle}\right) \not\sim 1.$$

3. $f(0) \sim -\pi\delta e$, i.e., A is unramified. Let $\tau \in O_K^*$, $\tau \not\sim 1$, $\tau - e \sim 1$. Then $f(\pi^l\tau) \sim \tau - e \sim 1$ and $A_{\mathrm{Irr}(\pi^l\tau)} \sim (-\pi\delta e, \pi^l\tau) \sim (\pi, \tau) \not\sim 1$.

4. $f(0) \sim -\delta e$, so that A is unramified. If w satisfies $h(w) \not\sim \delta$, then $f(\pi^d w) \sim -eh(w) \sim 1$. We have

$$A_{\mathrm{Irr}_{K(\sqrt{\pi})|k}}(\pi^d w) \sim \left(\frac{\alpha\ ,\ \pi^d w}{K(\sqrt{\pi})\langle\star\rangle}\right) \sim (\alpha, \sqrt{\pi}) \not\sim 1.$$

5. In this case $f(\pi^d w) \sim \alpha$ and $f(0) \sim 1$. Then A is unramified and $A_x \sim (\alpha, -\pi^{2d} w) \not\sim 1$.

6. $f(0) \sim 1$, $A_x \sim (\alpha, -\pi^l) \not\sim 1$

7. $A_{x-\pi^l e} \sim (\alpha, \pi^l) \not\sim 1$.

8. $f(\pi^d w) \sim -\alpha e\pi$, i.e., A is unramified. $A_{x-\pi^l e} \sim (-\alpha e\pi, \pi e) \not\sim 1$.

9. Let $\theta = \pi^{d-1/2}\sqrt{\pi w} \in K(\sqrt{\pi w})$. Since $d \in 1/2 + \boldsymbol{Z}$, then $d < m/3$, $h(\sqrt{w}) \sim w^2 + \pi^{\kappa-2d}uw + \pi^{n-4d}\delta = h^0(w)$, and $f(\theta) \sim -h^0(w)we$. Note that, in view of $\bar{h}(t) = \bar{h}^0(t^2)$, we have $\bar{h}^0(t) \notin \bar{k}[t]^2$. So the condition

$$\begin{cases} w \not\sim -\delta e, \\ h^0(w) \not\sim \delta \end{cases}$$

can be satisfied in a suitable K. Under these conditions we have $f(\theta) \sim 1$ and

$$A_{\mathrm{Irr}_{K(\sqrt{\pi w})|k}} \sim \left(\frac{-\pi\delta e\ ,\ w^{d-1/2}\sqrt{\pi w}}{K(\sqrt{\pi w})\langle\star\rangle}\right) \sim \left(\frac{\sqrt{\pi w}\ ,\ -w\delta e}{K(\sqrt{\pi w})\langle\star\rangle}\right) \not\sim 1.$$

Thus, in case $\bar{h}(t) \notin \bar{k}[t]^2$ the proof is completed. Let us make the form of the polynomial $g(t)$ more precise, provided that $\bar{h}(t) \in \bar{k}[t]^2$. If $x^4 + \overline{\pi^{\kappa-2d}u}x^2 + \overline{\pi^{m-3d}v}x + \overline{\pi^{n-4d}\delta} = (x^2 + \alpha x + \beta)^2$, then $\alpha = 0$, $\overline{\pi^{m-3d}v} = 0$, $\overline{\pi^{\kappa-2d}u} = 2\beta$, and $\overline{\pi^{n-4d}\delta} = \beta^2$, $\beta \in \bar{k}^*$. Then $n = 2\kappa$, $m > 3\kappa/2$, $d = \kappa/2 \in (1/2)\boldsymbol{Z}$, $\delta = \gamma^2 + \pi^s\tau$, $u = -2\gamma$, $\tau, \gamma \in O_k^*$, $s > 0$. So $g(x)$ has the following form.

$$g(x) = (x^2 - \pi^k\gamma)^2 + \pi^m vx + \pi^{2\kappa+s}\tau.$$

Let $\Delta = g(\sqrt{\pi^k\gamma}) = \pi^{2\kappa+s}\tau + \pi^m v\sqrt{\pi^k\gamma}\ k(\sqrt{\pi^k\gamma})$. Note that one can always suppose $v(\Delta) = \min\{2\kappa+s, m+\kappa/2\}$. Indeed, let $v(\Delta) > \min\{2\kappa+s, m+\kappa/2\}$. This means that $\kappa \equiv 0\,(2)$, $\gamma = \beta^2 \in (O_k^*)^2$, $2\kappa + s = m + \kappa/2$ and $\bar{\tau} + \bar{v}\bar{\beta} = 0$. Then $\bar{\tau} - \bar{v}\bar{\beta} \neq 0$, and the

replacement β by $-\beta$ gives us the required Δ. Moreover, if $\kappa \equiv 0\,(2)$, then the equality $v(\Delta) = 2m-\kappa$ (or $v(\Delta)/2-\kappa = m-3\kappa/2$) is impossible. Indeed, then $2m-\kappa \le m+\kappa/2$, $m \le 3\kappa/2$.

From now on we will consider only such $g(x)$ and Δ.

Theorem 4 *Let $d < l$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar algebra.*

1. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim \alpha$.
$A = (\pi, x - \pi^l e)$.

2. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim \pi\rho$, $\rho \in O_k^*$.
$A = (\alpha, x - \pi^l e)$.

3. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta \sim \pi^d\beta \sim 1$.
Let $\theta = \pi^d\beta(1+\pi^r w)$, $r = \min\{v(\Delta)/2 - \kappa, m - 3\kappa/2\}$, $w \in O_K^*$ *such that* $\phi(w) \in O_K^* \setminus (O_K^*)^2$, *where*

$$\phi(x) = \begin{cases} 4\gamma^2x^2 + \mu, & \text{if } v(\Delta)/2 - \kappa < m - 3\kappa/2 \\ 4\gamma^2x^2 + v\beta x, & \text{if } v(\Delta)/2 - \kappa > m - 3\kappa/2, \end{cases}$$

$\Delta = \pi^{v(\Delta)}\mu$, *and* $p = \mathrm{Irr}_{K|k}(\theta)$.
Then $A = (\alpha, p)$.

4. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 1\,(2)$.
$A = (\alpha, x - \pi^d\beta)$.

5. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 0\,(2)$, $\beta \sim 1$.
$A = (\alpha, x - \pi^d\beta)$.

6. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \alpha$, $d \equiv 0\,(2)$, $\beta \not\sim 1$.
$A = (\alpha, \mathrm{Irr}_{k(\sqrt[3]{\pi})|k}(\theta'))$, *where* $\theta' = \pi^d\beta(1 + \sqrt[3]{\pi})$.

7. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \pi\rho$, $v(\Delta) \equiv 1\,(2)$.
$A = (\pi\beta\mu, x - \pi^d\beta)$, $\Delta = \pi^{v(\Delta)}\mu$.

8. $\kappa \equiv 0\,(2)$, $\gamma = \beta^2$, $\Delta\pi^d\beta \sim \pi\rho$, $v(\Delta) \equiv 0\,(2)$.
$A = (\pi\beta\alpha, p)$, p *is from item 3.*

9. $\kappa \equiv 1\,(2)$, $m + \kappa/2 < 2\kappa + s$, $\gamma^m v \sim 1$.
$A = (\alpha, x - \pi^l e)$.

10. $\kappa \equiv 1\,(2)$, $m + \kappa/2 < 2\kappa + s$, $\gamma^m v \not\sim 1$.
$A = (\alpha, x^2 - \pi^k\gamma)$.

11. $\kappa \equiv 1\,(2)$, $m + \kappa/2 > 2\kappa + s$.

Let $\theta = \sqrt{\pi^k\gamma}(1 + (\sqrt{\pi\gamma})^s w) \in K(\sqrt{\pi\gamma})$, $w \in O_K^*$ *such that* $\gamma^s(4\gamma^{s+2}w^2 + \tau) \in O_K^* \setminus (O_K^*)^2$, $q = \mathrm{Irr}_{K(\sqrt{\pi\gamma})|k}(\theta)$.

Then $A = (\alpha x, q(0)q(x))$.

12. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $d \equiv 1\,(2)$, $-1 \sim 1$.

Let, similarly as in item 3., $\theta = \pi^d\sqrt{\gamma}(1 + \pi^r w) \in K(\sqrt{\gamma})$, $w \in O^*_{K(\sqrt{\gamma})}$ *such that* $\phi(w) \not\sim 1$ *in* $K(\sqrt{\gamma})$, $p = \mathrm{Irr}_{K(\sqrt{\gamma})|k}(\theta)$.

Then $A = (\pi, p(x))$.

13. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $d \equiv 1\,(2)$, $-1 \not\sim 1$.

Let p *be as in item 12.,* $\mu \in O_k^*$, $\mu^2 - \gamma \not\sim 1$.

Then $A = (\alpha p(x), x - \pi^d\mu)$.

14. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $d \equiv 0\,(2)$, $-1 \not\sim 1$.

$A = (\alpha p(x), x - \pi^d\mu)$, *where* p *and* μ *are as in 13.*

15. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $d \equiv 0\,(2)$, $-1 \sim 1$.

$A = (\pi, x - \pi^l e)$.

16. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $d \equiv 0\,(2)$, $\mu \sim \sqrt{\gamma}$ *in* $k(\sqrt{\gamma})$, $\Delta = \pi^{v(\Delta)}\mu$.

$A = (\pi, x^2 - \pi^k\gamma)$.

17. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $d \equiv 0\,(2)$, $\mu \not\sim \sqrt{\gamma}$.

$A = (\alpha(x^2 - \pi^k\gamma), \pi(x - \pi^d\eta))$, $\eta \in O_k^*$, $\eta^2 - \gamma \not\sim 1$.

18. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $d \equiv 1\,(2)$, $-1 \sim 1$.

$A = (\pi, p(x))$, p *is as in item 12.*

19. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $d \equiv 1\,(2)$, $-1 \not\sim 1$.

$A = (\alpha p(x), x - \pi^d\eta)$, p *is as in item 12.,* $\eta \in O_k^*$, $\eta^2 - \gamma \not\sim 1$.

Proof.

1. $f(\pi^d\beta) \sim \Delta\pi^d\beta \sim 1$, so $A_{x-\pi^d\beta} \sim (\pi, \pi^d\beta) \not\sim 1$.

2. $A_{x-\pi^d\beta} \sim (\alpha, \pi^d\beta) \not\sim 1$.

3. $f(\theta) \sim g(\theta)\theta \sim g(\theta) \sim 4\pi^{2\kappa+2r}\gamma^2w^2(1 + \pi^r w/2)^2 + \pi^{v(\Delta)}\mu + \pi^{m+d+r}v\beta w \sim 4\gamma^2w^2 + \pi^{2[(v(\Delta)/2-\kappa)-r]}\mu + \pi^{(m-3\kappa/2)-r}v\beta w \sim \phi(w) \sim \alpha$. Thus A is unramified. Let $\theta' = \pi^d\beta(1 + \Pi) \in k(\Pi)$, $\Pi = \sqrt[3]{\pi}$, $p' = \mathrm{Irr}(\theta')$. Then $f(\theta') \sim g(\theta') \sim 4\gamma^2 + \pi^{2[(v(\Delta)/2-\kappa)-1/3]}\mu + \pi^{(m-3\kappa/2)-1/3} \sim 1$, and

$$A_{p'} \sim \left(\frac{\alpha\ ,\ \prod_{\sigma\in G(K|k)}(\pi^d\beta(1+\Pi) - \pi^d\beta(1+\pi^r w^\sigma))}{k(\Pi)\langle p'\rangle}\right) \sim (\alpha, \Pi) \not\sim 1.$$

4. $f(\pi^d\beta) \sim \alpha$, so A is unramified. $A_{x-\pi^l e} \sim (\alpha, \pi^d\beta) \not\sim 1$.

5. Let θ', p' be from item 3 of this proof. Then $f(\theta') \sim 1$ and $A_{p'} \not\sim 1$.

6. $f(\theta') \sim g(\theta')(\pi^d\beta - \pi^l e) \sim \pi^d\beta \sim \alpha$, A is unramified. Let θ, p be as in 3. Then $f(\theta) \sim \alpha\phi(w) \sim 1$ and $A_p \not\sim 1$.

7. $f(\pi^d\beta) \sim \pi\mu\beta$, A is unramified. If $\eta \in O_K^*$ satisfies conditions $\eta \sim 1$, $\bar{\eta} \neq \bar{\beta}$, $\eta - \beta \not\sim 1$, then $f(\pi^d\eta) \sim \pi^d\eta \sim 1$ and $A_{\mathrm{Irr}(\pi^d\eta)} \sim (\pi\beta\mu, \eta - \beta) \not\sim 1$.

8. $f(\theta) \sim g(\theta)\pi^d\beta \sim \pi\beta\phi(w)$, so A is unramified.

$$A_{x-\pi^l e} \sim (\pi\alpha\beta, \prod_{\sigma\in G}(\pi^l e - \pi^d\beta(1 + \pi^r w^\sigma))) \sim (\pi\alpha\beta, -\pi\beta) \not\sim 1.$$

9. $f(\sqrt{\pi^k\gamma}) \sim \Delta\sqrt{\pi^k\gamma} \sim \gamma^m v \sim 1$ in $k(\sqrt{\pi\gamma})$. Therefore

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{\alpha\ ,\ \sqrt{\pi\gamma}}{k(\sqrt{\pi\gamma})\langle x^2 - \pi^k\gamma\rangle}\right) \not\sim 1.$$

10. $f(\sqrt{\pi^k\gamma}) \sim \alpha$, A is unramified. $A_{x-\pi^l e} \sim (\alpha, -\pi^k\gamma) \not\sim 1$.

11. We have $g(\theta) \sim \gamma^s(4\gamma^{s+2}w^2 + \tau) \sim \alpha$, then $f(\theta) \sim \alpha\theta$.
A is unramified. Indeed, $A_x \sim (\alpha x, q(0)^2) \sim 1$ and

$$A_q \sim \left(\frac{\alpha\theta\ ,\ q(0)q}{K(\sqrt{\pi\gamma})(\sqrt{\alpha\theta})\langle q\rangle}\right) \sim 1.$$

$A \not\sim (\pi, \alpha)$ in view of $(\pi, \alpha)_{x-\pi^l e} \not\sim 1$, but $A_{x-\pi^l e} \sim (\alpha\pi^l e, q(0)^2) \sim 1$. Finally,

$$A_\infty \sim \left(\frac{\alpha\ ,\ q(0)}{k\langle\sqrt{1/x}\rangle}\right) \not\sim 1,$$

since $v_k(q(0)) \equiv 1\,(2)$.

12. $f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi$, A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma\in G(K|k)} \pi^d\sqrt{\gamma}(-\pi^d\sqrt{\gamma})) \sim (\pi, -\gamma) \not\sim 1.$$

13. $p(\pi^d\mu) \sim \prod_\sigma(\mu - \sqrt{\gamma})(\mu + \sqrt{\gamma}) \sim \mu^2 - \gamma \sim \alpha$, so that $A_{x-\pi^d\mu} \sim (\alpha^2, x - \pi^d\mu) \sim 1$.
$f(\theta) \sim \pi g(\theta)\sqrt{\gamma} \sim \pi(\mu - \sqrt{\gamma})$ in $k(\sqrt{\gamma})$. Then

$$A_p \sim \left(\frac{\alpha p\ ,\ \pi(\mu - \sqrt{\gamma})}{K(\sqrt{\gamma})(\sqrt{f(\theta)})\langle p\rangle}\right) \sim 1.$$

Thus, A is unramified.

$$A_{x-\pi^l e} \sim (\alpha\prod_\sigma \pi^d\sqrt{\gamma}(-\pi^d\sqrt{\gamma}), -\pi^d w) \sim (\pi, -\alpha\gamma) \not\sim 1.$$

14. $f(\theta) \sim g(\theta) \sim \mu - \sqrt{\gamma}$, hence A is unramified. Let $\theta' = \pi^d\sqrt{\gamma}(1+\Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\theta') = g(\theta') \sim 1$ and

$$A_{\mathrm{Irr}_{k(\Pi,\sqrt{\gamma})}(\theta')} \sim \left(\frac{\prod_{\sigma\in G(K(\sqrt{\gamma})|k)}((\sqrt{\gamma}-(\sqrt{\gamma})^\sigma)+\Pi\gamma-(w\sqrt{\gamma})^\sigma\pi^r)\ ,\ \mu-\sqrt{\gamma}}{k(\Pi,\sqrt{\gamma})\langle\star\rangle}\right) \sim$$

$$\sim \left(\frac{\Pi\ ,\ \mu-\sqrt{\gamma}}{k(\Pi,\sqrt{\gamma})\langle\star\rangle}\right) \not\sim 1.$$

15. $f(\theta) \sim g(\theta)\sqrt{\gamma} \sim 1$, then

$$A_p \sim \left(\frac{\pi\ ,\ \sqrt{\gamma}}{K(\sqrt{\gamma})\langle p\rangle}\right) \not\sim 1.$$

16. $f(\pi^d\sqrt{\gamma}) \sim \pi\mu\sqrt{\gamma} \sim \pi$, A is unramified. Let $\zeta \in O_K^*$, $\zeta^2-\gamma \not\sim 1$, $\zeta \sim 1$. Then $f(\pi^d\zeta) \sim g(\pi^d\zeta) \sim 1$ and

$$A_{\mathrm{Irr}(\pi^d\zeta)} \sim \left(\frac{\pi\ ,\ \zeta^2-\gamma}{K\langle\star\rangle}\right) \not\sim 1.$$

17. $A_{x-\pi^d\eta} \sim (\alpha^2, \pi(x-\pi^d\eta)) \sim 1$.

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{x^2-\pi^k\gamma\ ,\ \pi(\eta-\sqrt{\gamma})}{k(\sqrt{\gamma})(\sqrt{f(\pi^d\sqrt{\gamma})})\langle x^2-\pi^k\gamma\rangle}\right) \sim 1,$$

because $f(\pi^d\sqrt{\gamma}) \sim \pi\mu\sqrt{\gamma} \sim \pi(\eta-\sqrt{\gamma})$, i.e., A is unramified.

$$A_\infty \sim \left(\frac{\alpha\ ,\ \pi}{k\langle\sqrt{1/x}\rangle}\right) \not\sim 1.$$

It remains to prove that $A \not\sim (\pi, \alpha)$. Choose ζ as in 16. Then $A_{\mathrm{Irr}(\pi^d\zeta)} \sim (\alpha(\zeta^2-\gamma), \ldots) \sim 1$. On the other hand, $(\pi,\alpha)_{\mathrm{Irr}(\pi^d\zeta)} \not\sim 1$.

18. $f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi$, A is unramified.

$$A_{x-\pi^l e} \sim (\pi, \prod_{\sigma\in G(K(\sqrt{\gamma})|k)} -\pi^d(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma)) \sim (\pi,-\gamma) \not\sim 1.$$

19. We have

$$A_{x-\pi^d\eta} \sim (x-\pi^d\eta, \alpha\prod_\sigma(\eta-(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma)) \sim (x-\pi^d\eta, \alpha(\eta^2-\gamma)) \sim 1.$$

where σ runs over $G(K(\sqrt{\gamma})|k)$.

$f(\theta) \sim \pi\sqrt{\gamma}g(\theta) \sim \pi(\eta-\sqrt{\gamma})$ in $k(\sqrt{\gamma})$. So

$$A_p \sim \left(\frac{p\ ,\ \pi^d(\sqrt{\gamma}-\eta)}{k(\sqrt{\gamma})(\sqrt{\pi(\eta-\sqrt{\gamma})})\langle p\rangle}\right) \sim 1,$$

i.e., A is unramified. To prove the theorem it is enough to check that A is nontrivial.

$$A_{x-\pi^l e} \sim (\alpha\prod_\sigma -\pi^d(\sqrt{\gamma})^\sigma(1+\pi^r w^\sigma), -\pi^d\eta) \sim (-\alpha\gamma, -\pi\eta) \not\sim 1.$$

Theorem 5 *Let $d \geq l$ and if $d = l$, $\pi^k\gamma = (\pi^d\beta)^2 \in k^2$, then $\bar{b} \neq \bar{e}$. Assume*

$$\xi(x) = \begin{cases} -e, & \text{if } d > l, \\ x - e, & \text{if } d = l. \end{cases}$$

Then the following algebra is unramified, nontrivial, and not isomorphic to the scalar algebra.

1. $\pi^k\gamma = (\pi^d\beta)^2 \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $\xi(\beta) \not\sim 1$.
$A = (\pi, x - \pi^l e)$.

2. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $\xi(\beta) \sim 1$.
$A = (\alpha, p(x))$, *where p is defined in item 3 of the previous theorem.*

3. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 1\,(2)$.
$A = (\pi\alpha\xi(\beta), p(x))$.

4. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 1\,(2)$, $\mu \sim \xi(\beta)$, *where* $\Delta = \pi^{v(\Delta)}\mu$.
Let $\Pi = \sqrt[5]{\pi}$, $t \in \{1, 2\}$, $t \equiv d\,(2)$, $\theta' = \pi^d\beta(1 - \Pi^t\alpha\beta) \in k(\Pi)$, $p' = \mathrm{Irr}(\theta')$.
Then $A = (\pi\xi(\beta), p'(x))$.

5. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 1\,(2)$, $\mu \not\sim \xi(\beta)$.
$A = (\alpha, x - \pi^d\beta)$.

6. $\pi^k\gamma \sim 1$, $v(\Delta) \equiv 1\,(2)$, $l \equiv 0\,(2)$.
$A = (\pi\mu\xi(\beta), x - \pi^d\beta)$.

7. $\kappa \equiv 1\,(2)$, $v(\Delta) \equiv 0\,(2)$ *in* $k(\sqrt{\pi\gamma})$, $-e\gamma^l \sim 1$.
Let $\theta = \sqrt{\pi^k\gamma}(1 + (\sqrt{\pi\gamma})^r w) \in K(\sqrt{\pi\gamma})$, $w \in O_K^*$ *such that* $g(\theta) \sim \alpha$ *in* $K(\sqrt{\pi\gamma})$, $p = \mathrm{Irr}(\theta)$.
Then $A = (\alpha, p(x))$.

8. $\kappa \equiv 1\,(2)$, $v(\Delta) \equiv 0\,(2)$ *in* $k(\sqrt{\pi\gamma})$, $-e\gamma^l \not\sim 1$.
Let $\theta' = \sqrt{\pi^k\gamma}(1 + \sqrt[6]{\pi\gamma}) \in k(\sqrt[6]{\pi\gamma})$, $q = \mathrm{Irr}(\theta')$.
Then $A = (\alpha, q(x))$.

9. $\kappa \equiv 1\,(2)$, $v(\Delta) \equiv 1\,(2)$.
$A = (\rho\pi^{(\kappa-1)/2}x, -\pi\gamma(x^2 - \pi^k\gamma))$, $\rho = -ev^{l+m+(\kappa-1)/2}$.

10. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 1\,(2)$, $l = d$, $\xi(\sqrt{\gamma}) \not\sim 1$ *in* $k(\sqrt{\gamma})$.
Let $\theta = \pi^d\sqrt{\gamma}(1 + \pi^r w)$, $w \in O^*_{K(\sqrt{\gamma})}$ *such that* $g(\theta) \in O^*_{K(\sqrt{\gamma})} \setminus (O^*_{K(\sqrt{\gamma})})^2$, $p = \mathrm{Irr}(\theta)$.
Then $A = (\pi, p(x)$.

11. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $l \equiv 0\,(2)$, $l = d$, $\xi(\sqrt{\gamma}) \not\sim 1$ *in* $k(\sqrt{\gamma})$.
$A = (\pi, x - \pi^l e)$.

12. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \sim 1$ *in* $k(\sqrt{\gamma})$ *(the last condition always holds if* $l < d$*).*

$A = (\alpha p, \pi^{l+d}(x - \pi^d\beta))$, *where* $\beta \in O_k^*$, $\beta^2 - \gamma \not\sim 1$, p *is from item 10.*

13. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \not\sim \xi(\sqrt{\gamma})$ *in* $k(\sqrt{\gamma})$.

$A = (\alpha(x^2 - \pi^k\gamma), \pi^{l+d+1}(x - \pi^d\beta))$, *where* $\beta \in O_k^*$, $\beta^2 - \gamma \not\sim 1$.

14. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma})$ *in* $k(\sqrt{\gamma})$, $l \equiv 0(2)$.

$A = (\pi, x^2 - \pi^k\gamma)$.

15. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma}) \sim 1$, $l \equiv 1(2)$.

Let $\Pi = \sqrt[3]{\pi}$, $\theta' = \pi^d\sqrt{\gamma}(1 + \Pi\eta)$, $\eta \in O^*_{k(\sqrt{\gamma})} \setminus (O^*_{k(\sqrt{\gamma})})^2$, $q = \mathrm{Irr}_{k(\Pi,\sqrt{\gamma})|k}(\theta')$.

Then $A = (\pi, q(x))$.

16. $\kappa \equiv 0\,(2)$, $\gamma \not\sim 1$, $v(\Delta) \equiv 1\,(2)$, $\mu \sim \xi(\sqrt{\gamma}) \not\sim 1$, $l \equiv 1(2)$.

$A = (\pi, x - \pi^l e)$.

Proof.

1. If θ, p are the same as in item 3 of the previous theorem, then $g(\theta) \sim \alpha$ and $f(\theta) \sim \alpha\pi^l\xi(\beta) \sim 1$, so $A_p \sim (\pi, \xi(\beta)) \not\sim 1$.

2. $f(\theta) \sim \alpha$, A is unramified. Let $\theta' = \pi^d\beta(1 + \Pi^t)$, $\Pi = \sqrt[5]{\pi}$, $t \in \{1, 2\}$, $t \not\equiv d\,(2)$, $p' = \mathrm{Irr}_{k(\Pi)|k}(\theta')$. Then $f(\theta') \sim \xi(\beta) \sim 1$ and

$$A_{p'} \sim \left(\frac{\alpha\ ,\ \prod_\sigma[\pi^d\beta(1 + \Pi^t) - \pi^d\beta(1 + \pi^r w^\sigma)]}{k(\Pi)\langle p'\rangle}\right) \sim \left(\frac{\alpha\ ,\ \Pi^{d+t}}{k(\Pi)\langle p'\rangle}\right) \not\sim 1.$$

3. A is unramified because of $f(\theta) \sim \pi\alpha\xi(\beta)$. We also have

$$A_{x-\pi^l e} \sim (\pi\alpha\xi(\beta), \prod_\sigma(\pi^l e - \pi^d\beta(1 + \pi^r w^\sigma))) \sim (\pi\alpha\xi(\beta), -\pi\xi(\beta)) \not\sim 1.$$

4. $f(\pi^d\beta) \sim \Delta\pi^l\xi(\beta) \sim 1$, $f(\theta') \sim \pi\xi(\beta)$, so that A is unramified. If $\chi^5 = 1$, χ is a primitive root, then

$$A_{x-\pi^d\beta} \sim (\pi\xi(\beta), \prod_{i=0}^{5}(\pi^d\beta - \pi^d\beta(1 - \Pi^t\chi^{it}\alpha\beta))) \sim (\Pi\xi(\beta), \Pi^{d+t}\alpha)) \not\sim 1.$$

5. $f(\pi^d\beta) \sim \alpha$, A is unramified. $A_{x-\pi^l e} \sim (\alpha, \pi^l\xi(\beta)) \not\sim 1$.

6. $f(\pi^d\beta) \sim \pi\mu\xi(\beta)$. Find $\eta \in O_K^*$ satisfying the following conditions

$$\begin{cases} \bar{\eta} \neq \bar{\beta}, \\ \eta - e \sim 1, \\ \eta \not\sim 1, \text{ if } l < d, \\ \eta - \beta \not\sim 1, \text{ if } l = d. \end{cases}$$

Then $f(\eta\pi^l) \sim (\eta - e)g(\eta\pi^l) \sim 1$ and $A_{\mathrm{Irr}(\eta\pi^l)} \sim (\pi\mu\xi(\beta), \eta - \pi^{d-l}\beta) \not\sim 1$.

7,8. In these cases we have $l < d$, $d \in 1/2 + \boldsymbol{Z}$.

$f(\theta) \sim g(\theta)(-\pi^l e) \sim -e\gamma^l\alpha$ in $K(\sqrt{\pi\gamma})$,

$f(\theta') \sim g(\theta')(-\pi^l e) \sim -e\gamma^l$ in $K(\sqrt{\pi\gamma})$.

If $-e\gamma^l \sim 1$, then $A = (\alpha, p)$ is nontrivial and $p(\theta') \sim \prod_{\sigma\in G(K|k)}(\sqrt{\pi^k\gamma}(1 + \sqrt[6]{\pi\gamma}) - \sqrt{\pi^k\gamma}(1 + (\sqrt{\pi\gamma})^r w^\sigma))(\sqrt{\pi^k\gamma}(1 + \sqrt[6]{\pi\gamma}) + \sqrt{\pi^k\gamma}(1 + (-1)^r\sqrt{(\pi\gamma)^r w^\sigma})) \sim 2\sqrt[6]{\pi\gamma}$, i.e.,

$$A_q \sim \left(\frac{\alpha\ ,\ \sqrt[6]{\pi\gamma}}{k(\sqrt[6]{\pi\gamma})\langle q\rangle}\right) \not\sim 1.$$

Let now $-e\gamma^l \not\sim 1$. Then $A = (\alpha, q)$ is unramified and $q(\theta) \sim -2\sqrt{\pi\gamma}$. So $A_p \not\sim 1$.

9. $A_x \sim (\rho\pi^{(\kappa-1)/2}x, (-\pi\gamma)^2) \sim 1$.

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{-(x^2 - \pi^k\gamma)\ ,\ \rho\sqrt{\pi\gamma}}{k(\sqrt{\pi\gamma})(\sqrt{f(\sqrt{\pi^k\gamma})})\langle x^2 - \pi^k\gamma\rangle}\right).$$

Since $v(\Delta) \equiv 1\,(2)$ in $k(\sqrt{\pi\gamma})$, then

$$2\kappa + s < m + \kappa/2 \text{ and } \Delta \sim \pi^m v\sqrt{\pi^k\gamma} \sim \gamma^{m+(\kappa-1)/2}v\sqrt{\pi\gamma}$$

in $k(\sqrt{\pi\gamma})$.

So $f(\sqrt{\pi^k\gamma}) \sim \rho\sqrt{\pi\gamma}$ in $k(\sqrt{\pi\gamma})$ and $A_{x^2-\pi^k\gamma} \sim 1$. Thus A is unramified. Furthermore,

$$A_\infty \sim \left(\frac{\pi\ ,\ \rho\gamma^{(\kappa-1)/2}}{k\langle\sqrt{1/x}\rangle}\right).$$

If $l \equiv 0\,(2)$, then choose $\eta \in O_K^*$ satisfying the conditions

$$\begin{cases} \eta - e \sim 1, \\ \eta \not\sim \rho\gamma^{(\kappa-1)/2}. \end{cases}$$

Then $f(\pi^l\eta) \sim (\rho\pi^{(\kappa-1)/2}\eta, -\pi\gamma(\pi^{2l}\eta^2 - \pi^k\gamma)) \sim (\pi, \rho\eta\gamma^{(\kappa-1)/2})$, i.e., among the algebras A_∞, $A_{\mathrm{Irr}(\pi^l\eta)}$ there is exactly one nontrivial. Therefore $A \not\sim 1, (\pi, \alpha)$.

Let now $l \equiv 1\,(2)$. One can find $\eta \in O_K^*$ such that

$$\begin{cases} \gamma - \eta \sim 1, \\ \eta \sim -e. \end{cases}$$

Then we have $f(\sqrt{\pi^k\eta}) \sim -\pi^l e \sim -\eta e \sim 1$ in $K(\sqrt{\pi\eta})$ and

$$A_{\mathrm{Irr}(\sqrt{\pi^k\eta})} \sim \left(\frac{\rho\pi^{(\kappa-1)/2}\sqrt{\pi^k\eta}\ ,\ -\pi\gamma\pi^k(\eta - \gamma)}{K(\sqrt{\pi\eta})\langle\star\rangle}\right) \sim \left(\frac{\sqrt{\pi\gamma}\ ,\ \gamma}{K(\sqrt{\pi\eta})\langle\star\rangle}\right) \not\sim 1.$$

On the other hand,

$$(\pi\alpha)_{\mathrm{Irr}(\sqrt{\pi^k\eta})} \sim \left(\frac{\pi\ ,\ \alpha}{K(\sqrt{\pi\eta})\langle\star\rangle}\right) \sim 1.$$

10,11. Since $f(\theta) \sim \pi^l \xi(\sqrt{\gamma})\delta$, $\delta \in O^*_{K(\sqrt{\gamma})} \setminus (O^*_{K(\sqrt{\gamma})})^2$, then, in 10, A is unramified and, in 11,

$$A_p \sim \left(\frac{\pi\ ,\ \pi^l \xi(\sqrt{\gamma})}{K(\sqrt{\gamma})\langle p \rangle} \right) \not\sim 1.$$

Moreover, in the first case, $A_{x-\pi^l e} \sim (\pi, e^2 - \gamma) \not\sim 1$.

12. $p(\pi^d \beta) \sim \beta^2 - \gamma \sim \alpha$, therefore $A_{x-\pi^d\beta} \sim 1$. $f(\theta) \sim g(\theta)\xi(\sqrt{\gamma})\pi^l \sim \delta\pi^l$, $\delta \not\sim 1$ in $k(\sqrt{\gamma})$. Then

$$A_p \sim \left(\frac{p\ ,\ \pi^{l+d}(\pi^d\sqrt{\gamma} - \pi^d\beta)}{K(\sqrt{\gamma})(\sqrt{\pi^l\delta})\langle p \rangle} \right) \sim \left(\frac{p\ ,\ \pi^l(\beta - \sqrt{\gamma})}{K(\sqrt{\gamma})(\sqrt{\pi^l\delta})\langle p \rangle} \right) \sim 1.$$

So, A is unramified. We also have

$$A_\infty \sim \left(\frac{\alpha\ ,\ \pi^{l+d}}{k\langle\sqrt{1/x}\rangle} \right), \quad A_{x-\pi^l e} \sim (\alpha, \pi^d),$$

the latter equality holds because of $p(\pi^l e) \sim 1$. If $l \equiv 1\,(2)$, then exactly one among these algebras is nontrivial, so $A \not\sim 1, (\pi, \alpha)$.

Let $l \equiv 0\,(2)$. Suppose $\theta' = \pi^d\sqrt{\gamma}(1 + \Pi)$, $\Pi = \sqrt[3]{\pi}$. Then $f(\theta') \sim \xi(\sqrt{\gamma}) \sim 1$ and

$$p(\theta') \sim \prod_\sigma (\pi^d\sqrt{\gamma}(1 + \Pi) - (\sqrt{\gamma})^\sigma \pi^d(1 + \pi^r w^\sigma)) \sim \Pi.$$

We have

$$A_{\mathrm{Irr}_{k(\Pi,\sqrt{\gamma})|k}(\theta')} \sim \left(\frac{\Pi\ ,\ \beta - \sqrt{\gamma}}{k(\Pi, \sqrt{\gamma})\langle\star\rangle} \right) \not\sim 1.$$

Note that (π, α) is trivial at this place.

13. $A_{x-\pi^d\beta} \sim (\alpha^2, \ldots) \sim 1$. $f(\pi^d\sqrt{\gamma}) \sim \pi^{l+1}\mu\xi(\sqrt{\gamma})$, then

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{x^2 - \pi^k\gamma\ ,\ \pi^{l+d+1}(\pi^d\sqrt{\gamma} - \pi^d\beta)}{k(\sqrt{\gamma})(\sqrt{f(\pi^d\sqrt{\gamma})})\langle x^2 - \pi^k\gamma\rangle} \right) \sim 1,$$

i.e., A is unramified.

$$A_\infty \sim (\alpha, \pi^{l+d+1}), \quad A_{x-\pi^l e} \sim \left(\alpha \left\{ \begin{array}{ll} 1, & l < d \\ e^2 - \gamma, & l = d \end{array} \right\}, \pi^{d+1} \right).$$

If $l \equiv 1\,(2)$, then we have just one nontrivial algebra, therefore $A \not\sim 1, (\pi, \alpha)$. The same is true in case $l \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \not\sim 1$, since then $l = d$ and $e^2 - \gamma \not\sim 1$. Let $l \equiv 0\,(2)$, $\xi(\sqrt{\gamma}) \sim 1$. For θ' from the previous item we have

$$A_{\mathrm{Irr}_{k(\Pi,\sqrt{\gamma})|k}(\theta')} \sim \left(\frac{(\theta')^2 - \pi^k\gamma\ ,\ \pi^{d+1}(\theta' - \pi^d\beta)}{k(\Pi, \sqrt{\gamma})\langle\star\rangle} \right) \sim \left(\frac{2\Pi\ ,\ \pi^{d+1}(\Pi(\beta - \sqrt{\gamma})}{k(\Pi, \sqrt{\gamma})\langle\star\rangle} \right) \not\sim 1.$$

14,15,16. $f(\pi^d\sqrt{\gamma}) \sim \pi^{l+1}\mu\xi(\sqrt{\gamma}) \sim \pi^{l+1}$. If $\mu \sim \xi(\sqrt{\gamma})$, $l \equiv 0\,(2)$ (case 14), then $(\pi, x^2 - \pi^k u)$ is unramified. If $\mu \sim \xi(\sqrt{\gamma})$, $l \equiv 1\,(2)$, then $f(\sqrt{\pi^k\gamma}) \sim 1$ and

$$(\pi, x - \pi^l e)_{x^2-\pi^k\gamma} \sim \left(\frac{\pi\ ,\ \pi^l\xi(\sqrt{\gamma})}{k(\sqrt{\gamma})\langle x^2 - \pi^k\gamma\rangle}\right) \sim (\pi, \xi(\sqrt{\gamma})) \not\sim 1,$$

provided that $\xi(\sqrt{\gamma}) \not\sim 1$ (case 16). In case 15 we have

$$q(\pi^d\sqrt{\gamma}) \sim \prod_{i=0}^{2}(\pi^d\sqrt{\gamma} - \pi^d\sqrt{\gamma}(1+\chi^i\Pi\eta))(\pi^d\sqrt{\gamma} + \pi^d\sqrt{\gamma}(1+\chi^i\Pi\eta^\sigma)) \sim \prod_{i=0}^{2}\chi^i\Pi\eta \sim \pi\eta,$$

where $\chi^3 = 1$ and χ is a primitive root. So

$$A_{x^2-\pi^k\gamma} \sim \left(\frac{\pi\ ,\ \pi\eta}{k(\sqrt{\gamma})\langle x^2 - \pi^k\gamma\rangle}\right) \not\sim 1.$$

To complete the proof one only needs to check that $A \not\sim 1$ in case 14 and that A is nontrivial in 15. In these cases $f(\theta') \sim \pi^l\xi(\sqrt{\gamma})$. Then $f(\theta') \sim \pi$ in 15, and A is unramified. If case 14 takes place and $l < d$, then $f(\theta') \sim -e \sim 1$ in $k(\sqrt{\gamma})$. Therefore

$$A_q \sim \left(\frac{\pi\ ,\ \pi^k\gamma(1+\Pi\eta)^2 - \pi^k\gamma}{k(\Pi,\gamma)\langle q\rangle}\right) \sim \left(\frac{\Pi\ ,\ \eta}{k(\Pi,\gamma)\langle q\rangle}\right) \not\sim 1.$$

Finally, let $l = d$. Assume $\zeta \in O_K^*$ such that $\bar{\zeta} \neq \bar{e}$, $\zeta - e \sim 1$, and $\zeta^2 - \gamma \not\sim 1$. Then $f(\pi^d\zeta) \sim \zeta - e \sim 1$ and $A_{\mathrm{Irr}_{K|k}(\pi^d\zeta)} \sim (\pi, \zeta^2 - \gamma) \not\sim 1$.

Theorem 6 *Let $d = l$, $\pi^k\gamma = (\pi^d\beta)^2 \in k^2$, and $e = \beta + \pi^n\delta$, $n > 0$, $\delta \in O_k^*$.*

Let firstly $v(\Delta) \equiv 0\,(2)$ and $\theta = \pi^d\beta(1+\pi^r w)$, $w \in O_K^*$ such that $\phi(w) \in O_K^* \setminus (O_K^*)^2$, $p = \mathrm{Irr}_{K|k}(\theta)$, $r = \min\{v(\Delta)/2 - \kappa, m - 3\kappa/2\}$ (see theorem 4, item 3). Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar algebra.

1. $r \leq n$, $r + d \equiv 1\,(2)$. $A = (\alpha N_{K|k}(\chi(w))\pi, p(x))$, *where*

$$\chi(x) = \begin{cases} \beta x, & \text{if } r < n, \\ \beta x - \delta, & \text{if } r = n. \end{cases}$$

2. $r \leq n$, $r + d \equiv 0\,(2)$. $A = (\pi, x - \pi^d e)$.

3. $r > n$, $n + d \equiv 1\,(2)$. $A = (-\pi\delta\alpha, p(x))$.

4. $r > n$, $n + d \equiv 0\,(2)$, $-\delta \not\sim 1$. $A = (\pi, x - \pi^d e)$.

5. $r > n$, $n + d \equiv 0\,(2)$, $-\delta \sim 1$. $A = (\alpha, p(x))$.

Let now $v(\Delta) \equiv 1\,(2)$, $\Delta = \pi^{v(\Delta)}\mu$. Then the following algebra A is unramified, nontrivial, and not isomorphic to the scalar algebra.

6. $n + d \equiv 1\,(2)$, $-\mu\delta \not\sim 1$. $A = (\alpha, x - \pi^d\beta)$.

7. $n + d \equiv 1\,(2)$, $-\mu\delta \sim 1$. $A = (\alpha, x - \pi^d e)$.

8. $n+d \equiv 0\,(2)$, $n \le r$. $A = (-\mu\delta\pi, x - \pi^d\beta)$.

9. $n+d \equiv 0\,(2)$, $n > r$. $A = (-\pi\mu\alpha, x - \pi^d\beta(1+\pi^n w))$, *where* $w \in O_k^*$ *such that* $\delta - \beta w \not\sim 1$. *(If there is no such* w *in* k, *then* $A = (-\pi\mu\alpha, \mathrm{Irr}_{K|k}(\pi^d\beta(1+\pi^n w)))$, $w \in O_K^*$*)*

Proof. Let $v(\Delta) \equiv 0\,(2)$. Then

$$f(\theta) \sim \alpha(\pi^d\beta(1+\pi^r w) - \pi^d(\beta+\pi^n\delta)) \sim \begin{cases} \pi^{d+r}\alpha\chi(w), & \text{if } r \le n, \\ -\pi^{d+n}\alpha\delta, & \text{if } r > n, \end{cases}$$

$\chi(w) \sim N_{K|k}(\chi(w))$ in K.

So, in cases 1,3,5, the algebra A is unramified.

In 4 $f(\theta) \sim -\alpha\delta \sim 1$, then

$$A_p \sim \left(\frac{\pi\ ,\ \pi^d\beta(1+\pi^r w) - \pi^d(\beta+\pi^n\delta)}{K\langle p\rangle}\right) \sim (\pi, -\pi^{n+d}\delta) \not\sim 1.$$

In 2 $f(\theta) \sim \alpha\chi(w)$. The system

$$\begin{cases} \phi(w) \not\sim 1, \\ \chi(w) \not\sim 1 \end{cases}$$

is always solvable in an extension of sufficiently large degree $K|k$. For such w and p we have

$$A_p \sim (\pi, \pi^d\beta(1+\pi^r w) - \pi^d(\beta+\pi^n\delta)) \sim \left(\frac{\pi\ ,\ \chi(w)}{K\langle p\rangle}\right) \not\sim 1.$$

In 1,3

$$p(\pi^d e) \sim \pi^d \prod_{\sigma\in G(K|k)} (\pi^n\delta - \pi^r\beta w^\sigma) \sim \begin{cases} -\pi^{d+r}N_{K|k}(\chi(w)), r \le n, \\ \pi^{d+n}\delta, r = n. \end{cases}$$

Then in

1. $A_{x-\pi^d e} \sim (\alpha N_{K|k}(\chi(w))\pi, -\pi N_{K|k}(\chi(w))) \not\sim 1$.

2. $A_{x-\pi^d e} \sim (-\pi\delta\alpha, \pi\delta) \not\sim 1$. Thus, for the case $v(\Delta) \equiv 0\,(2)$ it is enough to check that in 5 $A \not\sim 1$. To do this, assume $\theta' = \pi^d\beta(1+\Pi^t) \in k(\Pi)$, $\Pi = \sqrt[5]{\pi}$, $t \in \{1,2\}$, $t \not\equiv d\,(2)$, $p' = \mathrm{Irr}_{k(\Pi)|k}(\theta')$. Then $f(\theta') \sim g(\theta') \sim 1$ and

$$p(\theta') \sim \prod_{\sigma\in G(K|k)} (\pi^d\beta(1+\Pi^t) - \pi^d\beta(1+\pi^r w^\sigma)) \sim \pi^d\beta\Pi^t.$$

So,

$$A_{p'} \sim \left(\frac{\alpha\ ,\ \Pi^{d+t}}{k(\Pi)\langle p'\rangle}\right) \not\sim 1.$$

Consider now case $v(\Delta) \equiv 1\,(2)$. We have $f(\pi^d\beta) \sim -\pi^{v(\Delta)}\mu\pi^{d+n}\beta\delta \sim -\mu\delta\pi^{d+n+1}$, therefore, in 6, A is unramified and, in 7, $f(\pi^d\beta) \sim 1$, $A_{x-\pi^d\beta} \sim (\alpha, \pi^{d+n}) \not\sim 1$. Furthermore, in 6, A is nontrivial. Indeed, then $A_{x-\pi^d e} \sim (\alpha, \pi^{d+n}) \not\sim 1$.

In 8, $f(\pi^d\beta) \sim -\mu\delta\pi^{d+n+1}$ and A is unramified. Let $r = \min\{v(\Delta)/2 - \kappa, m - 3\kappa/2\} \in (1/2)\boldsymbol{Z}$, $\theta = \pi^d\beta(1+(-\alpha\mu\delta\pi)^r w) \in K((-\alpha\mu\delta\pi)^r)$, $w \in O^*_{K((-\alpha\mu\delta\pi)^r)}$, $p = \mathrm{Irr}_{K((-\alpha\mu\delta\pi)^r)|k}(\theta)$. Then

$$g(\theta) \sim 4\gamma^2(-\alpha\mu\delta)^{2r}w^2 + \pi^{v(\Delta)-2\kappa-2r}\mu + \pi^{m-3\kappa/2-2r}v\beta w(-\alpha\mu\delta\pi)^r,$$

$$f(\theta) \sim g(\theta)(-\delta + \beta w(-\alpha\mu\delta\pi)^{r-n}) \sim -g(\theta)\delta \left\{ \begin{array}{l} 1, r > n, \\ 1 + \mu\beta w, r = n. \end{array} \right\}$$

Moreover,

$$A_p \sim \left(\frac{-\mu\delta\pi \ , \ \pi^d\beta(-\alpha\mu\delta\pi)^r}{K((-\alpha\mu\delta\pi)^r)(\sqrt{f(\theta)})\langle p\rangle} \right).$$

If $r \in \boldsymbol{Z}$, then

$$A_p \sim \left(\frac{\pi \ , \ (-\alpha)^r(\mu\delta)^d\beta w}{K(\sqrt{f(\theta)})\langle p\rangle} \right).$$

One always can find K, w such that $f(\theta) \sim 1$ and $w \not\sim (-\alpha)^r(\mu\delta)^d\beta$. Then we have $A_p \not\sim 1$.

Consider $r \in 1/2 + \boldsymbol{Z}$. In this case

$$A_p \sim \left(\frac{\alpha \ , \ \sqrt{-\alpha\mu\delta\pi}}{K(\sqrt{-\alpha\mu\delta\pi})(\sqrt{f(\theta)})\langle p\rangle} \right).$$

In addition, $f(\theta) \sim -\delta(4\gamma^2(-\alpha\mu\delta)^{2r}w^2 + \mu)$. So, $A_p \not\sim 1$, provided that $f(\theta) \sim 1$.

9. Since $v(\Delta) = \min\{2\kappa + s, m + \kappa/2\}$, then $v(\Delta) < m + n + \kappa/2$. Suppose $\theta_0 = \pi^d\beta(1 + \pi^n w)$. Then we have $g(\theta_0) \sim \gamma^2(2\pi^n w + \pi^{2n}w^2)^2 + \pi^{v(\Delta)-2\kappa}\mu + \pi^{m-3\kappa/2+n}v\beta w$. In view of $v(\Delta) - 2\kappa < m - 3\kappa/2 + n$ and $v(\Delta) - 2\kappa < 2n$ (the latter inequality holds because of $v(\Delta)/2 - \kappa < r < n$) we have $g(\theta_0) \sim \pi\mu$. Therefore $f(\theta_0) \sim \pi\mu(\beta w - \delta)$ and A is unramified. Finally, $A_{x-\pi^d e} \sim (-\pi\mu\alpha, \delta - \beta w) \not\sim 1$.

The theorem is proved.

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