On 3-regular subgraphs of 4-regular graphs with multiple edges *

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Abstract

For any 4-regular graph G (possibly with multiple edges and loops), we [1] proved recently that, if the number N of distinct Euler orientations of G is such that $N \not\equiv\ 1 \pmod 3$, then G has a 3-regular subgraph. Using this approach here we prove a conjecture of Carsten Thomassen [9] that any 4-regular connected graph with multiple edges and loops has a 3-regular subgraph, whenever the number of vertices of the graph is even.

1 Introduction

The Berge-Sauer conjecture (see [4,5]) says that any simple (no multiple edges and loops) 4-regular graph contains a 3-regular subgraph. This conjecture was proved in [7,9]. In [2,3] the Chevalley-Warning theorem was used to extend this result to the graphs with multiple edges, that are 4-regular plus an edge. Our main result of the previous paper [1] Theorem 2.3, presents

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the sufficient condition for a 4-regular graph with multiple edges to have a 3-regular subgraph. It gives the infinite families of the 4-regular graphs for which we can write out exactly the numbers of distinct Euler orientations. The conjecture of Thomassen [9] says that any 4-regular connected graph with multiple edges and loops has a 3-regular subgraph, whenever the number of vertices is even. In the mentioned above paper [1] we reduce this conjecture to the same conjecture for graphs with only multiple edges. Here we give a proof of this conjecture. So, any 4-regular connected graph G with multiple edges and loops has a 3-regular subgraph, if the number of vertices is even. This result was obtained when the author visited the Department of Mathematics of the University of Bielefeld. I would like to thank Professor Rudolf Ahlswede and Professor C. M. Ringel for the support of this research.

2 Preliminary Results

We consider the finite undirected graphs $G = (V, E)$ with multiple edges and without loops. A graph G is k-regular if every vertex of G is incident with exactly k edges.

Definition 1 *Let G be a 4-regular graph with the n vertices: $v_1, v_2, \cdots, v_n$ and $2n$ edges. An Euler orientation of G is a partition $E = \bigcup_{i=1}^{n} A_i$ of the set E of edges such that:* (i) *any two A_i are disjoint,* (ii) *each A_i has exactly two edges both incident with v_i.*

Two partitions A_i and $A'_i,\ i = 1, ..., n$, of E are different if there is i such that $A_i \neq A'_i$. Denote by $N = N(G)$ the number of different Euler orientations of the graph G. It is well known (see, for example [6]) that any connected graph, where every vertex has an even degree, has an Euler orientation. The following theorem [1] connects the number $N(G)$ of Euler orientations of a 4-regular graph G with the property of G to have a 3-regular subgraph.

Theorem 1 *A 4-regular graph G with n vertices has a 3-regular subgraph if N, the number of distinct Euler orientations of G, is such that $N \not\equiv 1$ (mod 3).*

Note that the condition $N \not\equiv 1 \pmod 3$ in Theorem 1 is only sufficient, but not necessary (see [1]).
Now we need two standard transformations for graphs which we considered in [1].

Lemma 1 *Let G_1 and G_2 be any graphs and let N_1 and N_2 be the numbers of distinct Euler orientations of G_1 and G_2 respectively. Choose any two edges e_1 and e_2 of the graphs G_1 and G_2 respectively and define a new graph $G = G_1 \oplus G_2$ (we say, G is a pasting of G_1 and G_2) according to Fig. 1. Then: (a) The number N of distinct Euler orientations of G must be as follows: $N = N_1 N_2 / 2$. (b) If G_1 and G_2 are 4-regular graphs, then at least one of the graphs G_1, G_2 and G has a 3-regular subgraph.*

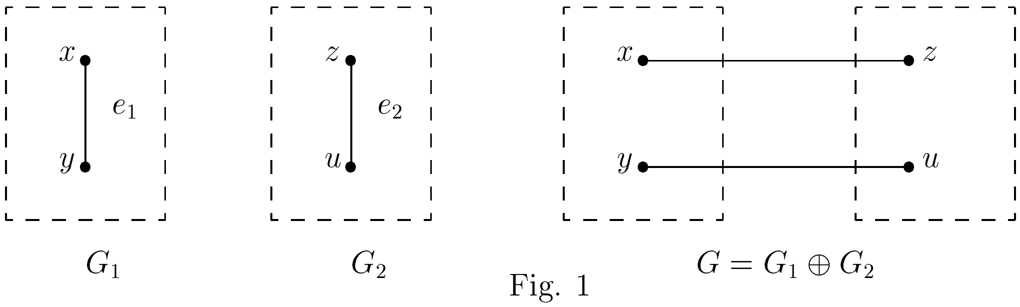


Fig. 1

Assume now that v is an arbitrary point of the 4-regular graph G (see Fig. 2). Let $e_1, e_2, e_3,$ and e_4 be the corresponding edges incident to v.
Now we define a new graph G_v which have instead of the vertex v three vertices u, u' and u'' and four new edges $f_1, f_2, f_3,$ and f_4 (see Fig. 2).

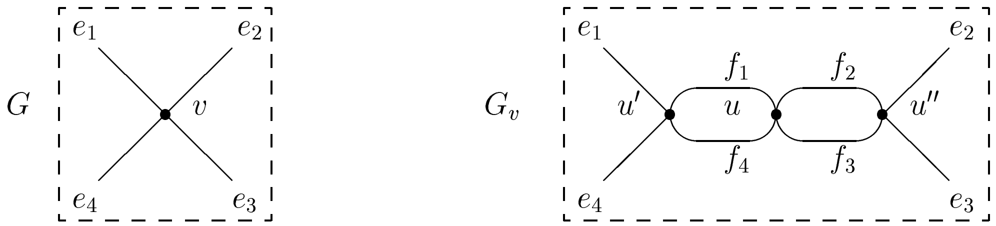


Fig.2

Lemma 2 *Let G be any 4-regular graph with $n \geq 1$ vertices and let the graph G_v be obtained from G according to Fig. 2 for any vertex $v \in V(G)$. Then both graphs G and G_v have or do not have 3-regular subgraphs at the same time.*

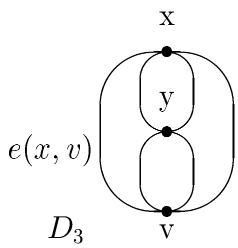
For a graph G and u arbitrary vertices $v_1, \ldots, v_u \in V(G)$ denote by $G_{v_1,\cdots,v_u}$ the graph, obtained from G by the same procedure, as shown on Fig. 2. If G has a loop in vertex v, i.e. if $e_1 = e_2$, then G_v does not have this loop. It gives the following result [1].

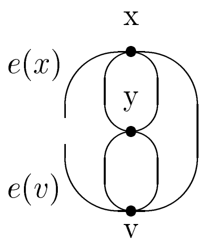
Theorem 2 *Let G be any 4-regular graph G with n vertices and u loops, say $v_1, v_2, \ldots, v_u$, where $n \geq u \geq 1$. Then a 4-regular graph $G_{v_1,v_2,\ldots,v_u}$ without loops has or doesn't have 3-regular subgraphs simultaneously with the original graph G.*

Note [1] that Theorem 2 reduces the conjecture of Thomassen [9] for graphs only with multiple edges. This conjecture asserts that any 4-regular connected graph with an even number of vertices has a 3-regular subgraph. This is clear since, for even n, Theorem 2 gives us another graph with an even number of vertices also.
We need one more concept. Different from the case of an (usual) edge $e = e(x, y)$, when the edge e is incident with two vertices x and y, we shall use the concept of a half-edge $e = e(x)$, when the edge e is incident with only one vertex x. We will say that we cut the edge $e(x, y)$ on two half-edges $e(x)$ and $e(y)$, if we change the edge $e(x, y)$ by two half-edges $e(x)$ and $e(y)$. See Fig. 3 for illustration of this concept.
The following result from [7] is a modification of the result of Alon, Friedland and Kalai [2,3].

Lemma 3 *A 4-regular graph G has a 3-regular subgraph if we cut any one edge of G on two half-edges.*

We illustrate this lemma on Fig.3. Let D_s ($s \geq 1$) denote a 4-regular graph obtained from a cycle with s vertices by replacing every edge by two parallel edges. It is well known [2,3] that D_s has no a 3-regular subgraph if s is an odd number. Consider the graph D_3 (Fig. 3.a). If we cut the edge $e(x,v)$ on two half-edges $e(x)$ and $e(v)$ (Fig. 3.b), then this resulting graph contains a 3-regular subgraph (Fig. 3.c).





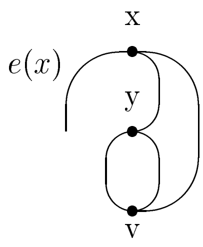


Fig. 3.a Fig. 3.b Fig. 3.c

We need one more result which follows directly from Lemmas 1 and 3.

Lemma 4 *Let G_1 and G_2 be two arbitrary 4-regular graphs. Then their pasting,*
i.e. a graph $G = G_1 \oplus G_2$, (see Fig. 1) always has a 3-regular subgraph.

3 On Validity of Thomassen's Conjecture

The next theorem is the main result of this paper.

Theorem 3 *Let G be an arbitrary 4-regular connected graph with multiple edges*
and with n vertices. If n is even, then G has a 3-regular subgraph.

Proof. Assume in contrary that the statement of the theorem is not right. Among all 4-regular connected graphs with multiple edges and with even numbers of vertices (which have no 3-regular subgraphs) we choose a graph G with the smallest, say n, number of vertices. It is clear that $n \geq 4$ (the only 4-regular connected graph with 2 vertices has 3-regular subgraphs). Assume that G has at least one pair of multiple edges. For simple graphs (i.e. for 4-regular graphs without multiple edges) the result is known [8,10]. Let e_1 and e_2 be one of such pairs of multiple edges which connect the vertices w and z; see Fig. 4.a. Now instead of the graph G we consider a 4-regular graph G_1 obtained from G according to Fig. 4.b. As G_1 has $n-2$ vertices, this graph has a 3-regular subgraph, say H_1. We claim that G has a 3-regular subgraph also. Our proof will consist from the different cases which correspond to the cases of equalities between the vertices v, u, x and y.

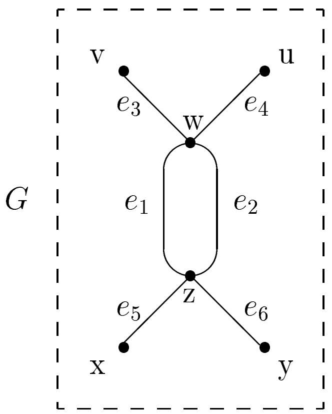


Fig. 4.a

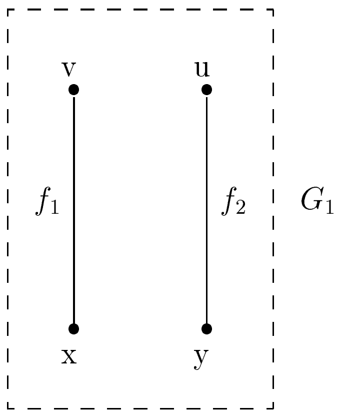


Fig. 4.b

The case $v \neq u \neq x \neq y$. First note that H_1, a 3-regular subgraph of G_1, should contain at least one of the edges f_1 and f_2 (see Fig. 4.b) (if not, then H_1 is a 3-regular subgraph of G). If f_1 or (and) f_2 is an edge of H_1, then H_1 is easily imbedded to the graph H, a 3-regular subgraph of G. Indeed, if H_1 contains f_1, then the graph H with vertices $V(H) = V(H_1) \cup \{w, z\}$ and edges $E(H) = E(H_1) \cup \{e_1, e_2, e_3, e_5\} \setminus \{f_1\}$ is a 3-regular subgraph of G. If the both edges f_1 and f_2 are edges of H_1, then the graph H with the sane set of vertices as in the previous case and with the set of edges $E(H) = E(H_1) \cup \{e_1, e_3, e_4, e_5, e_6\} \setminus \{f_1, f_2\}$ is a 3-regular subgraph of G also. So we have a contradiction that G has the minimal number of vertices.

The case $v = u$ or $x = y$.

Instead of G consider a 4-regular graph G_1 (see Fig. 5) which has $n-2$ vertices. But according to Lemma 2 both graphs G and G_1 have or do not have 3-regular subgraphs at the same time. Therefore G_1 does not have a 3-regular subgraph, i.e. we obtain a contradiction to the choice of G.

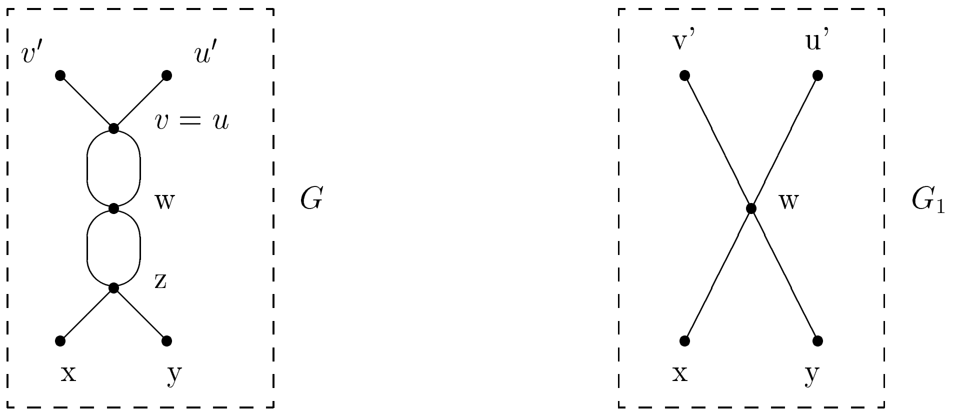


Fig. 5

The case $v = x$ (or $u = y$, or $v = y$, or $u = x$).

This case is illustrated on Fig. 6. The graph G_1 has $n-2$ vertices and should have a 3-regular subgraph H_1. Let us show that in all possible cases this subgraph might be imbedded to the graph H, a 3-regular sugraph of G. First consider the case when H_1 contains the edge f_1. Then the graph H with the set of vertices $V(H) = V(H_1) \cup \{w, z\}$ and the set of edges $E(H) = E(H_1) \cup \{e_1, e_2, e_4, e_5\} \setminus \{f_1\}$ is a 3-regular subgraph of G. Now assume that H_1 contain both of the edges f_1 and f_2.

Then the graph H with the same set of vertices as in the previous case and with the set of edges $E(H) = E(H_1) \cup \{e_1, e_3, e_4, e_5, e_6\} \setminus \{f_1, f_2\}$ is a 3-regular subgraph of G also. It contradicts to the choice of G.

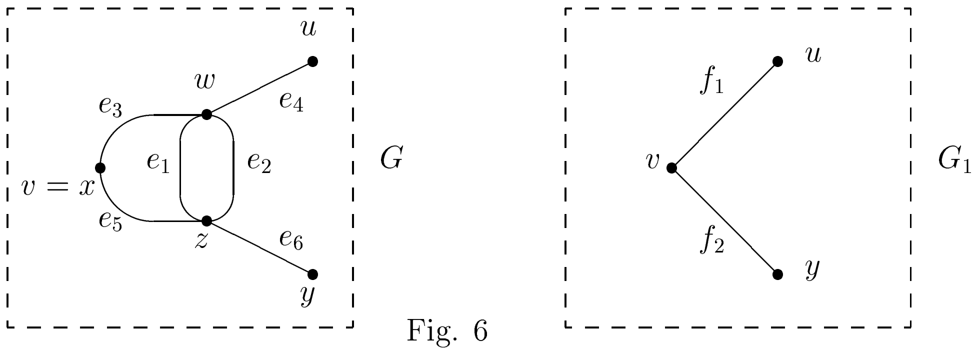


Fig. 6

The case $v = x$ and $u = y$ (or $v = y$ and $x = u$).
Instead of G we consider the graph G_1 (Fig. 7), which has $n-2$ vertices and therefore has a 3-regular subgraph. In all possible cases the graph H_1 might be imbedded to the graph H, a 3-regular subgraph of G. If H_1 contains the edge f_1 then H has the set of vertices $V(H) = V(H_1) \cup \{w, z\}$ and the set of edges $E(H) = E(H_1) \cup \{e_1, e_2, e_3, e_6\} \setminus \{f_1\}$. If H_1 contains both of the edges f_1 and f_2, then H has the same set of vertices as in the previous case and the set of edges $E(H) = E(H_1) \cup \{e_1, e_3, e_4, e_5, e_6\} \setminus \{f_1, f_2\}$. So we again have a contradiction.

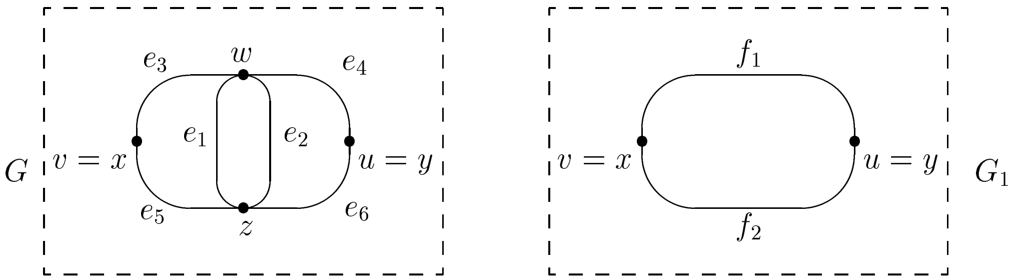


Fig. 7

The case $v = u$ and $x = y$. According to Lemma 2 the graphs G and G_1 (Fig. 8) have or do not have 3-regular subgraphs simultaneously. As G_1 has $n-2$ vertices, it gives a contradiction.

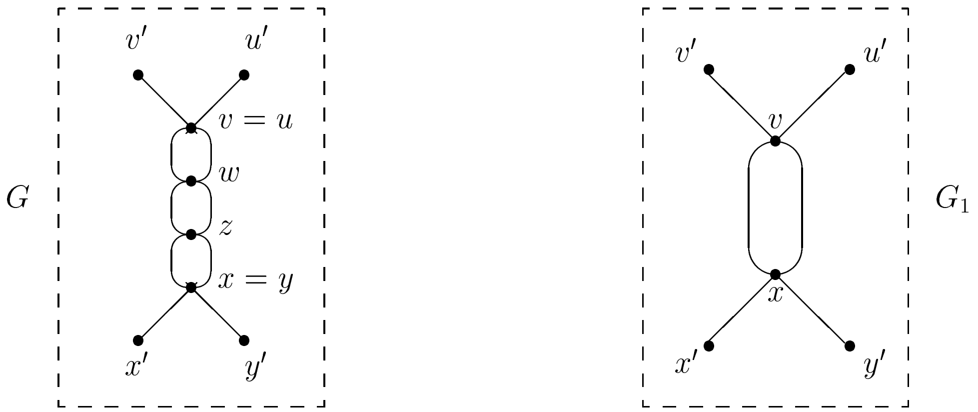


Fig. 8

The case $v = u = x$.
The graph G (Fig. 9) might be presented as pasting (see Fig. 1) of two graphs G_1 and D_3:
$G = G_1 \oplus D_3$ (see Fig. 9). According to Lemma 4 the graph G should have a 3-regular subgraph. We again obtain a contradiction.

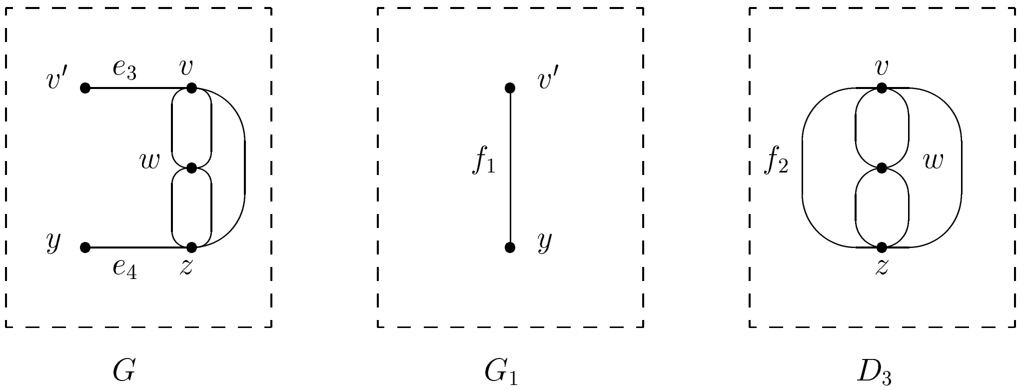


Fig. 9

The case $v = u = x = y$. In this case G is the graph D_3 (see Fig. 9), which has an odd number of vertices. This finishes the proof of Theorem 3.

4 References

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