Decay rates of the inverse of nonsymmetric tridiagonal and band matrices

Reinhard Nabben
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld
Germany

Abstract

It is well known that the inverse $C = [c_{i,j}]$ of an irreducible nonsingular symmetric tridiagonal matrix is a Green matrix, i.e. it satisfies $c_{i,j} = u_i v_j$ for $i \leq j$ and two sequences of real numbers $\{u_i\}$ and $\{v_i\}$. A similar result holds for nonsymmetric matrices A. There the inverse is given by four sequences $\{u_i\}, \{v_i\}, \{y_i\}$, and $\{y_i\}$. Here we characterize certain properties of A in terms of the u_i, v_i, x_i, and y_i namely that A is an M–matrix or positive definite. We also establish a relation of zero row sums and zero column sums of A and pairwise constant u_i, v_i, x_i and y_i. Moreover we consider decay rates for the entries of the inverse of tridiagonal and block tridiagonal (banded) matrices. For diagonally dominant matrices we show that the entries of the inverse strictly decay along a row or column. We give a sharp decay result for tridiagonal irreducible M–matrices and tridiagonal positive definite matrices. We also give a decay rate for arbitrary banded M–matrices.

1 Introduction

In many mathematical problems which gives rise to a linear system of equations the system matrix is tridiagonal or block tridiagonal. For example the numerical solution of partial differential equations mostly leads to tridiagonal (one dimensional problems) and block tridiagonal (higher dimensional problems). Therefore these classes of matrices have been extensively studied. A review of this topic is given by Meurant [Meu] for symmetric matrices. One of the most important results is established by Gantmacher and Krein [GK], who proved that the inverse of a symmetric irreducible tridiagonal matrix is a so-called Green matrix which is given by two sequences $\{u_i\}$ and $\{v_i\}$ of real numbers. A similar result is established by Ikebe. He proved that the inverse of a nonsymmetric irreducible tridiagonal matrix has a similar structure as a symmetric one and is given by four sequences $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, and $\{y_i\}$. We call such a matrix a generalized Green matrix.

Considering (block) tridiagonal matrices and their inverses there are different kinds of problems. On one hand one wants to find explicit formulas for the inverses of (block) tridiagonal matrices and on the other hand one wants to characterize matrices whose inverses are (block) tridiagonal. Here we continue considering tridiagonal matrices and their inverses. We consider generalized Green matrices and show that the inverse of a generalized Green matrix is tridiagonal. Thus, with Ikebe's result we obtain a characterization of irreducible nonsingular tridiagonal matrices. Moreover we characterizes tridiagonal M–matrices and tridiagonal symmetric positive definite matrices in terms of the $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, and $\{y_i\}$. We explain the connection of zero row sums and zero columns of a tridiagonal matrix and pairwise constant u_i, v_i, x_i and y_i.

It has been observed that for large classes of banded matrices the entries of the inverse tend to zero as $|i-j|$ becomes larger. The rate of decay is important to construct sparse approximations of the inverse as preconditioners. In [D] and [DMS] it is shown that the entries of the inverse of a symmetric positive matrix are bounded in an exponentially decaying manner along a row or column. For nonsymmetric tridiagonal M–matrices which are strictly diagonally dominant by rows and by columns we establish that the entries of the inverse indeed decay along each row and column away from the diagonal. We also give a bound for this decay. This result generalize a result by Concus, Golub, and Meurant [CGM] for symmetric matrices.

For arbitrary tridiagonal M–matrices or positive definite matrices the entries of the inverse do not decay a long a row in general. However, here we establish a sharp decay rate related to two diagonal entries for the inverses of tridiagonal matrices. This result can be proved easily and continues the number of elegant results on tridiagonal matrices. For symmetric matrices the result is related to Vassilevski [Vas] result for which Cauchy-Bunyakowski-Schwarz constants are used. Therefore we establish a new proof of Vassilevski's result and obtain a new way to compute or estimate Cauchy-Bunyakowski-Schwarz constants for tridiagonal matrices. Moreover we establish a decay rate for the inverses of banded M–matrices.

2 Inverses of tridiagonal matrices

One of the most important results on symmetric tridiagonal matrices is the result by Gantmacher and Krein [GK] which describes the structure of the inverse of these matrices.

Theorem 2.1 *Let $A \in \mathbb{R}^{n,n}$ be symmetric, irreducible and nonsingular. Then A is tridiagonal if and only if $A^{-1}$is a Green matrix.*

A Green matrices $C = [c_{i,j}] \in \mathbb{R}^{n,n}$ is given by two sequences $\{u_i\}_{i=1}^n$, $\{v_i\}_{i=1}^n$ of numbers such that

$$C = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1v_n & u_2v_n & \cdots & u_nv_n \end{bmatrix}.$$

or

$$c_{i,j} = \begin{cases} u_iv_j & i \geq j, \\ u_iv_j & i \leq j. \end{cases}$$

Green matrices can be described more elegantly as the Hadamard product (elementwise product) of a so-called *type D* and a *flipped type D matrix*, (see [Mar] and [MNNST]) i.e.

$$C = \begin{bmatrix} u_1 & u_1 & \cdots & u_1 \\ u_1 & u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_2 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \cdots & v_n \end{bmatrix}. \tag{2.1}$$

For nonsymmetric tridiagonal matrices $A \in \mathbb{R}^{n,n}$ with

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix} \tag{2.2}$$

Ikebe [I] proved that the inverse of A has a similar structure as for the symmetric case.

Theorem 2.2 *Let $A \in \mathbb{R}^{n,n}$ be irreducible and tridiagonal. If A is nonsingular then there exist four sequences $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, $\{y_i\}$ such that $A^{-1} =: C = [c_{i,j}]$ is given by*

$$c_{i,j} = \begin{cases} u_i v_j & i \geq j, \\ x_i y_j & i \leq j. \end{cases}$$

In the following we discuss inverses of nonsymmetric tridiagonal matrices. Therefore we define a generalized Green matrix related to Ikebe's result.

Definition 2.3 *A matrix $C = [c_{i,j}] \in \mathbb{R}^{n,n}$ is called a generalized Green matrix if there exist four sequences $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, $\{y_i\}$ such that $C = [c_{i,j}]$ is given by*

$$c_{i,j} = \begin{cases} u_i v_j & i \geq j, \\ x_i y_j & i \leq j. \end{cases}$$

Next we give some properties of generalized Green matrices. For $A \in \mathbb{R}^{n,n}$ let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. Moreover, for a generalized Green matrix define

$$d_{i,j} := x_i y_j - v_i u_j \qquad h_{i,j} := u_i v_j - y_i x_j. \tag{2.3}$$

We then have

Theorem 2.4 *Let $A \in \mathbb{R}^{n,n}$ be a generalized Green matrix. Then*

$$\det A = x_1 y_n \prod_{i=1}^{n-1} d_{i+1,i} = u_1 v_n \prod_{i=1}^{n-1} h_{i+1,i} \tag{2.4}$$

$$\det A_{i,i+1} = \frac{\det A}{d_{i+1,i}} \qquad \det A_{i+1,i} = \frac{\det A}{h_{i+1,i}} \tag{2.5}$$

$$\det A_{i,j} = 0 \quad \textit{for} \quad |i-j| > 1. \tag{2.6}$$

Proof. We first consider $\det A$. The case $n = 2$ is obviously true. For $n > 2$ we do Gaussian elimination from the right, i.e. we multiply A from the right with $L = [l_{s,t}] \in \mathbb{R}^{n,n}$ where $l_{s,s} = 1$ for all s, $l_{2,1} = -x_1/x_2$ and $l_{s,t} = 0$ else. Thus

$$AL = \begin{bmatrix} a & b \\ 0 & A_{1,1} \end{bmatrix}$$

with

$$a = x_1 y_1 - \frac{u_1 v_2 x_1}{x_2} = \frac{x_1}{x_2}(x_2 y_1 - u_1 v_2).$$

Using the inductive hypothesis we obtain the first equality of (2.4). Gaussian elimination from the left gives the second equality.

To obtain (2.5) we multiply $A_{i,i+1}$ from the left with $L = [l_{s,t}] \in \mathrm{I\!R}^{\mathrm{n-1,n-1}}$ where

$$\begin{aligned} l_{s,s} &= 1 \quad \text{for all } s \\ l_{s+1,s} &= -\frac{x_s}{x_{s+1}} \quad \text{for} \quad s = 1, \ldots, i-1, i+2, \ldots, n-1 \\ l_{i+1,i} &= -\frac{x_i}{x_{i+2}} \\ l_{s,t} &= 0 \quad \text{else} \end{aligned}$$

we get

$$\det A_{i,i+1} = \frac{\det A}{d_{i+1,i}}.$$

Similarly we obtain $\det A_{i+1,i}$.

Next we first consider $A_{i,j}$ with $j - i > 1$. After multiplying $A_{i,j}$ from the right with $L = [l_{s,t}] \in \mathrm{I\!R}^{\mathrm{n-1,n-1}}$ where

$$\begin{aligned} l_{s,s} &= 1 \quad \text{for all } s \\ l_{s+1,s} &= -\frac{x_s}{x_{s+1}} \quad \text{for} \quad s = 1, \ldots, i \\ l_{s,t} &= 0 \quad \text{else} \end{aligned}$$

we get

$$A_{i,j} L = \begin{bmatrix} \tilde{A}_1 & X \\ 0 & \tilde{A}_2 \end{bmatrix}$$

where $\tilde{A}_1$ is a diagonal matrix whose last diagonal entry is zero. Thus, $\det A_{i,j} = 0$. Similarly we obtain $\det A_{i,j} = 0$ for $i - j > 1$. $\square$

With Ikebe's theorem (Theorem 2.2) and Theorem 2.4 we immediately obtain the following theorem

Theorem 2.5 *Let $A \in \mathbb{R}^{n,n}$ be irreducible and nonsingular. Then A is tridiagonal if and only if A^{-1} is a generalized Green matrix.*

Proof. The sufficient part of Theorem 2.5 is proved by Ikebe [I]. Now assume that A^{-1} is a generalized Green matrix. $A = [a_{i,j}]$ is given by

$$a_{i,j} = (-1)^{i+j} \frac{\det A_{j,i}}{\det A}.$$

With Theorem 2.4 we get that A is tridiagonal. □

Here we should mention that Barrett gives in [B] another characterization of tridiagonal matrices.

Theorem 2.6 *Let $C = [c_{i,j}] \in \mathbb{R}^{n,n}$ with nonzero entries $c_{2,2}, \ldots, c_{n-1,n-1}$. Then C^{-1} is tridiagonal if and only if C has the triangle property, i.e.*

$$c_{i,j} = \frac{c_{i,k} c_{k,j}}{c_{k,k}} \quad \textit{for all} \quad i < k < j \quad i > k > j$$

Note that none of these theorems includes the other. In Theorem 2.5 there is a restriction on irreducibility while in Theorem 2.6 there is a restriction of the diagonal entries of the inverse. In [A] Asplund gives a characterization in terms of vanishing certain minors of the inverse. However the beauty of Ikebe's result and Theorem 2.5 is the explicit form of the inverse which can be used to establish more properties of the inverse of a tridiagonal matrix.

Next we will characterize nonsingular tridiagonal M–matrices. A matrix $A = [a_{i,j}]$ is called a nonsingular M–matrix if $a_{i,j} \leq 0$ for $i \neq j$ and A^{-1} is a nonnegative matrix. We characterize nonsingular tridiagonal M–matrices in terms of the sequences $\{u_i\}, \{v_i\}, \{x_i\}$, and $\{y_i\}$ which gives A^{-1}. To do so we define for $i = 1, \ldots, n-1$

$$\alpha_i := \frac{u_i v_{i+1}}{u_{i+1} v_i} \quad \beta_i := \frac{x_i y_{i+1}}{x_{i+1} y_i}. \tag{2.7}$$

Theorem 2.7 *Let $A \in \mathbb{R}^{n,n}$ be irreducible and nonsingular. Then the following are equivalent:*

1. *A is a tridiagonal M–matrix,*
2. *A^{-1} is a generalized Green matrix where all u_i, v_i, x_i and y_i have the same sign and*

$$\alpha_i \beta_i < 1 \quad \textit{for all } i$$

Proof. $A = [a_{i,j}]$ is given by

$$a_{i,j} = (-1)^{i+j} \frac{\det A^{-1}_{j,i}}{\det A^{-1}}.$$

Thus Theorem 2.4 yields to

$$a_{i,i+1} = -\frac{1}{u_{i+1}v_i - x_i y_{i+1}} \quad a_{i+1,i} = -\frac{1}{x_{i+1}y_i - u_i v_{i+1}}.$$

If A is an M–matrix we have

$$x_{i+1}y_i - u_i v_{i+1} > 0 \quad \text{and} \quad u_{i+1}v_i - x_i y_{i+1} > 0.$$

Therefore

$$\frac{u_i v_{i+1}}{x_{i+1}y_i} < 1 \quad \text{and} \quad \frac{x_i y_{i+1}}{u_{i+1}v_i} < 1.$$

Hence $\alpha_i \beta_i < 1$.

If $\alpha_i \beta_i < 1$ we have

$$\frac{u_i v_{i+1}}{x_{i+1}y_i} \frac{x_i y_{i+1}}{u_{i+1}v_i} < 1.$$

Since $u_i v_i = x_i y_i$ for all i, i.e.

$$\frac{x_i}{v_i} = \frac{u_i}{y_i}, \quad \text{and} \quad \frac{y_{i+1}}{u_{i+1}} = \frac{v_{i+1}}{x_{i+1}},$$

we obtain

$$\frac{u_i v_{i+1}}{x_{i+1}y_i} = \frac{x_i y_{i+1}}{u_{i+1}v_i}.$$

Therefore

$$x_{i+1}y_i - u_i v_{i+1} > 0 \quad \text{and} \quad u_{i+1}v_i - x_i y_{i+1} > 0.$$

Since A^{-1} is a nonnegative matrix we then obtain that A is an M–matrix. □

Corollary 2.8 *Let $A \in \mathbb{R}^{n,n}$ be symmetric irreducible and nonsingular. Then the following are equivalent:*

1. *A is a tridiagonal M–matrix,*

2. *A^{-1} is a Green matrix given by the sequences $\{u_i\}$ and $\{v_i\}$ where all u_i, v_i and have the same sign and*

$$\frac{u_1}{v_1} < \ldots < \frac{u_n}{v_n}$$

In [MNNST] another characterization of tridiagonal M–matrices is given. There it is used that for each nonsymmetric irreducible tridiagonal M–matrix A there exists a diagonal matrix D such that DA is symmetric. Here we can avoid this symmetrization.

For symmetric positive definite matrices we immediately obtain the following characterization

Corollary 2.9 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular symmetric tridiagonal matrix. A is positive definite if and only if there exists a diagonal matrix $D = [d_{ij}]$ with $|d_{ii}| = 1$ such that*

$$(DAD)^{-1} = \begin{bmatrix} u_1 & u_1 & \cdots & u_1 \\ u_1 & u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_2 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \cdots & v_n \end{bmatrix}. \tag{2.8}$$

where all u_i and v_i in (2.8) have the same sign and

$$0 < \frac{u_1}{v_1} < \frac{u_2}{v_2} < \ldots, < \frac{u_n}{v_n}.$$

These characterizations of tridiagonal M–matrices and tridiagonal positive definite matrices will be useful in the next section.

3 Decay rates

In this section we consider the decay of the elements of the inverse of tridiagonal and banded matrices. Several papers already established results on this topic. In [D] and [DMS] it is shown that the entries of the inverse of a symmetric positive matrix are bounded in an exponentially decaying manner along a row or column. Here we will give some sharp decay rates and we will show that the entries of a nonsymmetric tridiagonal diagonally dominant matrix indeed decay along a row and column.

A matrix $B = [b_{i,j}]$ is diagonally dominant by columns if

$$b_{i,i} \geq \sum_{j \neq i} |b_{i,j}| \quad \text{for all } i.$$

If

$$b_{i,i} \geq \sum_{j \neq i} |b_{j,i}| \quad \text{for all } i.$$

then B is called diagonally dominant by rows. If the above inequalities are strict then B is called strictly diagonally dominant by columns or by rows respectively.

For A as in (2.2) being a symmetric tridiagonal M–matrix which is diagonally dominant by columns (with $a_1 > b_1$ and $a_n > b_{n-1}$) Concus, Golub and Meurant proved in [CGM] that the sequence $\{u_i\}$ is strictly increasing while $\{v_i\}$ is strictly decreasing. Thus the entries of A^{-1} indeed strictly decay along each row or column. Moreover they proved that

$$(A^{-1})_{ij} \leq (A^{-1})_{ii} \rho^{|i-j|}, \tag{3.1}$$

where $1/\rho = \min_{k \geq 2}((a_k + b_{k-1})/b_k))$. Since A is diagonally dominant one has $\rho < 1$.

We will generalize this result to the nonsymmetric case. In the following we also assume for simplicity that A is a tridiagonal Z–matrix, i.e.

$$A = \begin{bmatrix} a_1 & -b_1 & & & \\ -c_1 & a_2 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & a_{n-1} & -b_{n-1} \\ & & & -c_{n-1} & a_n \end{bmatrix} \qquad b_i, c_i \geq 0. \tag{3.2}$$

The entries of the sequences $\{u_i\}, \{v_i\}, \{x_i\}$ and $\{y_i\}$ which give A^{-1}, can be computed as follows

Lemma 3.1 *Let $A \in \mathbb{R}^{n,n}$ be an irreducible tridiagonal Z–matrix. Then*

$$u_1 = 1 \qquad u_2 = \frac{a_1}{b_1}$$

$$
\begin{aligned}
u_i &= \frac{a_{i-1}u_{i-1} - c_{i-2}u_{i-2}}{b_{i-1}} \quad i = 3, \ldots, n \\
x_1 &= 1 \qquad x_2 = \frac{a_1}{c_1} \\
x_i &= \frac{a_{i-1}x_{i-1} - b_{i-2}x_{i-2}}{c_{i-1}} \quad i = 3, \ldots, n \\
v_n &= \frac{1}{a_n u_n - c_{n-1}u_{n-1}} \\
y_i &= \frac{u_i v_i}{x_i} \quad i = n, \ldots, 1 \\
v_i &= \frac{1 + b_i x_i y_{i+1}}{a_i u_i - c_{i-1}u_{i-1}} \quad i = n-1, \ldots, 1
\end{aligned}
$$

Proof. Multiplying the i-th row of A with the $(i+1)$-th column of A^{-1} gives the u_i. Multiplying the i-th row of A^{-1} with the $(i-1)$-th column of A gives the x_i. Multiplying the i-th row of A with the i-th column of A^{-1} gives the v_i. Moreover, we use that $u_i v_i = x_i y_i$. □

For the next theorem we define

$$
\frac{1}{\rho_1} := \min_{i \geq 2} \frac{a_i - c_{i-1}}{b_i} \qquad \frac{1}{\rho_2} := \min_{i \geq 2} \frac{a_i - b_{i-1}}{c_i} \tag{3.3}
$$

Theorem 3.2 *Let $A \in \mathbb{R}^{n,n}$ be an irreducible tridiagonal M–matrix.*
If A is diagonally dominant by rows and if $a_1 > b_1$ and $a_n > c_{n-1}$ then $\{u_i\}$ is strictly increasing while $\{y_i\}$ is strictly decreasing. Moreover,

$$
\begin{aligned}
(A^{-1})_{i,j} &\leq \rho_1^{(j-i)}(A^{-1})_{j,j} \quad \text{for } i < j \\
(A^{-1})_{i,j} &\leq \rho_1^{(i-j)}(A^{-1})_{i,i} \quad \text{for } i > j
\end{aligned}
$$

If A is diagonally dominant by columns and if $a_1 > c_1$ and $a_n > b_{n-1}$ then $\{x_i\}$ is strictly increasing while $\{v_i\}$ is strictly decreasing. Moreover,

$$
\begin{aligned}
(A^{-1})_{i,j} &\leq \rho_2^{(j-i)}(A^{-1})_{i,i} \quad \text{for } i < j \\
(A^{-1})_{i,j} &\leq \rho_2^{(i-j)}(A^{-1})_{j,j} \quad \text{for } i > j
\end{aligned}
$$

Proof. First assume that A is diagonally dominant by rows and $a_1 > b_1$, $a_n > b_{n-1}$. It is clear that $u_2 > u_1$. By the induction hypothesis and Lemma 3.1 we get

$$
u_i = \frac{a_{i-1}u_{i-1} - c_{i-2}u_{i-2}}{b_{i-1}} > \frac{a_{i-1} - c_{i-2}}{b_{i-1}} u_{i-1} > u_{i-1}.
$$

To obtain the disered result for the v_i we have to establish a recursive formula. Multiplying the i-th row of A^{-1} with the i-th column of A gives

$$y_i \quad = \quad \frac{1 - c_i u_i v_{i+1}}{a_i x_i - b_{i-1} x_{i-1}}$$

Multiplying the $(i+1)$-th row of A^{-1} with the i-th column of A gives

$$-b_{i-1} x_{i-1} + a_i x_i - c_i x_{i+1} = 0.$$

Multiplying the $(i+1)$-th row of A with the $(i+1)$-th column of A^{-1} gives

$$-c_i u_i v_{i+1} + a_{i+1} x_{i+1} y_{i+1} - b_{i+1} x_{i+1} y_{i+2} = 1.$$

Hence

$$y_i = -\frac{a_{i+1} x_{i+1} y_{i+1} + b_{i+1} x_{i+1} y_{i+2}}{c_i x_{i+1}} = -\frac{a_{i+1}}{c_i} y_{i+1} - \frac{b_{i+1}}{c_i} y_{i+2},$$

and

$$y_{n-1} = -\frac{a_n}{c_{n-1}} y_n$$

We then have $y_{n-1} > y_n$ and by induction $y_i > y_{i+1}$. The decay rates follow immediately from the recursive formulas for the u_i and y_i and the special structure of A^{-1}.

Now, assume that A is diagonally dominant by columns $a_1 > c_1$ and $a_n > b_{n-1}$. Obviously, $x_2 > x_1$. By the induction hypothesis and Lemma 3.1 we get

$$x_i = \frac{a_{i-1} x_{i-1} - b_{i-2} x_{i-2}}{c_{i-1}} > \frac{a_{i-1} - b_{i-2}}{c_{i-1}} x_{i-1} > x_{i-1}.$$

Again, for the v_i we have to establish a recursive formula. Multiplying the i-th row of A with the $(i+1)$-th column of A^{-1} gives

$$a_i u_i + c_{i-1} u_{i-1} + b_i u_{i+1} = 0.$$

Multiplying the $(i+1)$-th row of A^{-1} with the $(i+1)$-th column of A gives

$$b_i x_i y_{i+1} + a_{i+1} u_{i+1} v_{i+1} + c_{i+1} u_{i+1} v_{i+2} = 1.$$

Together with Lemma 3.1 we get

$$v_i = \frac{a_{i+1} u_{i+1} v_{i+1} + c_{i+1} u_{i+1} v_{i+2}}{-b_i u_{i+1}} = -\frac{a_{i+1}}{b_i} v_{i+1} - \frac{c_{i+1}}{b_i} v_{i+2},$$

and

$$v_{n-1} = -\frac{a_n}{b_{n-1}} v_n$$

We then have $v_{n-1} > v_n$ and by induction $v_i > v_{i+1}$. Again, the decay results follow immediately. □

For matrices which are not M–matrices but satisfy the other assumptions of Theorem 3.2 we obtain that the sequences of the absolute values strictly increase or decrease respectively.

Theorem 3.2 generalizes the result for symmetric tridiagonal matrices by Concus, Golub, and Meurant [CGM]. In the following we specify the above result. We will explain the connection of zero row and column sums of A and some pairwise constant u_i, v_i, x_i and y_i. Let for $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $i = 1, \ldots, n$

$$r_i = \sum_{j=1}^{n} a_{ij} \qquad c_i = \sum_{j=1}^{n} a_{j,i}.$$

Then we have

Theorem 3.3 *Let A be an irreducible nonsingular tridiagonal matrix. Then there exist indices $s, t \in \{0, 1, \ldots, n+1\}$ such that*

$$\begin{aligned} r_i &= 0 \quad \textit{for} \ \ i = 1, \ldots, s \\ r_i &= 0 \quad \textit{for} \ \ i = n-t, \ldots, n \end{aligned}$$

if and only if

$$\begin{aligned} u_i &= u_{i+1} \quad \textit{for} \ \ i = 1, \ldots, s \\ y_{i-1} &= y_i \quad \textit{for} \ i = n-t, \ldots, n. \end{aligned}$$

Moreover, there exist indices $\tilde{s}, \tilde{t} \in \{0, 1, \ldots, n+1\}$ such that

$$\begin{aligned} c_i &= 0 \quad \textit{for} \ \ i = 1, \ldots, \tilde{s} \\ c_i &= 0 \quad \textit{for} \ \ i = n-\tilde{t}, \ldots, n \end{aligned}$$

if and only if

$$\begin{aligned} x_i &= x_{i+1} \quad \textit{for} \ \ i = 1, \ldots, \tilde{s} \\ v_{i-1} &= v_i \quad \textit{for} \ i = n-\tilde{t}, \ldots, n. \end{aligned}$$

Proof. The proof follows immediately from similar formulas for the u_i, v_i, x_i and y_i as derived in Lemma 3.1 and in the proof of Theorem 3.2. $\square$

Note that in Theorem 3.3 we do not assume A to be diagonally dominant or positive definite or to be a Z–matrix. For symmetric matrices Theorem 3.3 says that the first and last pairwise constant u_i and v_i gives zero row sums for the first and last rows in A and vice versa. Moreover, if $r_i = 0$ for $i = 1, \ldots, s$ and $r_i = 0$ for $i = n-t, \ldots, n$ then the inverse of A can be partitioned as

$$A^{-1} = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{3.4}$$

where $C_{11} \in \mathrm{IR}^{\mathrm{s,s}}$ and $C_{33} \in \mathrm{IR}^{\mathrm{t+1,t+1}}$. Here C_{11} is a flipped type D matrix while C_{33} is a type D matrix. C_{31} and C_{13} are flat matrices, i.e. all entries of these blocks are equal. Moreover the rows of C_{12} and the columns of C_{23} are flat. This structure is illustrated in the next example.

Example 3.4

$$A = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$10 * A^{-1} = \begin{bmatrix} -11 & -16 & 10 & 12 & 4 & 4 \\ -16 & -16 & 10 & 12 & 4 & 4 \\ 10 & 10 & 0 & 0 & 0 & 0 \\ 12 & 12 & 0 & 6 & 2 & 2 \\ 4 & 4 & 0 & 2 & 4 & 4 \\ 4 & 4 & 0 & 2 & 4 & 14 \end{bmatrix}$$

Combining Theorem 3.3 with Theorem 3.2 we get

Corollary 3.5 *Let A be an irreducible tridiagonal M–matrix. If $r_i = 0$ for $i = 1, \ldots, s$ and $r_i = 0$ for $i = n-t, \ldots, n$ for $s, t \in \{0, 1, \ldots, n+1\}$. Then the sequences*

$\{u_i\}$ and $\{y_i\}$ satisfy

$$\begin{aligned} u_i &= u_{i+1} \text{ for } i = 1, \ldots, s \\ u_i &< u_{i+1} \text{ else} \\ y_{i-1} &= y_i \text{ for } i = n-t, \ldots, n \\ y_{i-1} &> y_i \text{ else.} \end{aligned}$$

If $c_i = 0$ for $i = 1, \ldots, \tilde{s}$ and $c_i = 0$ for $i = n - \tilde{t}, \ldots, n$ for $\tilde{s}, \tilde{t} \in \{0, 1, \ldots, n+1\}$. Then the sequences $\{x_i\}$ and $\{v_i\}$ satisfy

$$\begin{aligned} x_i &= x_{i+1} \text{ for } i = 1, \ldots, \tilde{s} \\ x_i &< x_{i+1} \text{ else} \\ v_{i-1} &= v_i \text{ for } i = n-\tilde{t}, \ldots, n \\ v_{i-1} &> v_i \text{ else.} \end{aligned}$$

For $M-$matrices or symmetric positive definite tridiagonal or banded matrices, which are not diagonally dominant, the elements of A^{-1} do not decrease in general along a row or column. This can be seen in Example 3.6.

Example 3.6

$$A = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 5 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \qquad 10A^{-1} = \begin{bmatrix} 2.75 & 3 & 4 & 1 \\ 3 & 6 & 8 & 2 \\ 4 & 8 & 24 & 6 \\ 1 & 2 & 6 & 4 \end{bmatrix}$$

However we will establish a decay result for tridiagonal $M-$matrices and symmetric positive definite matrices in the following. These results are based on the characterizations of these matrices given in Theorem 2.7 and Corollary 2.9.

Theorem 3.7 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular irreducible tridiagonal matrix. Let $A^{-1} = C = [c_{ij}]$. Then for $k > 0$*

$$c_{i,i+k} = \left(\prod_{j=1}^{k} \alpha_{i+j}^{\frac{1}{2}} \right) (c_{i,i} c_{i+k,i+k})^{\frac{1}{2}}, \tag{3.5}$$

$$c_{i+k,i} = \left(\prod_{j=1}^{k} \beta_{i+j}^{\frac{1}{2}} \right) (c_{i,i} c_{i+k,i+k})^{\frac{1}{2}}, \tag{3.6}$$

where the α_i and β_i are as in (2.7).

Proof. Obviously, $c_{ii}, c_{i+k,i+k}$ and $c_{i,i+k}$ are given by

$$c_{ii} = u_i v_i, \quad c_{i+k,i+k} = u_{i+k} v_{i+k} \quad c_{i,i+k} = u_i v_{i+k}$$

Moreover

$$\frac{u_i}{v_i} = \left(\prod_{j=1}^{k} \alpha_{i+j} \right) \frac{u_{i+k}}{v_{i+k}}.$$

Hence

$$\begin{aligned} c_{i,i+k} &= u_i v_{i+k} \\ &= (u_i v_i u_{i+k} v_{i+k})^{\frac{1}{2}} \left(\frac{u_i v_{i+k}}{v_i u_{i+k}} \right)^{\frac{1}{2}} \\ &= \left(\prod_{j=1}^{k} \alpha_{i+j}^{\frac{1}{2}} \right) (c_{i,i} c_{i+k,i+k})^{\frac{1}{2}}. \end{aligned}$$

Similarly, we obtain (3.6). □

In general one can not say anything about the α_i and β_i in (3.5) and (3.6). But for M–matrices and for symmetric positive definite matrices we have the following decay results

Corollary 3.8 *Let $A \in \mathbb{R}^{n,n}$ be an irreducible tridiagonal M–matrix. Then for $A^{-1} = C = [c_{ij}]$ we have*

$$c_{i,i+k} c_{i+k,i} = \left(\prod_{j=1}^{k} (\alpha_{i+j} \beta_{i+j})^{\frac{1}{2}} \right) (c_{i,i} c_{i+k,i+k}), \tag{3.7}$$

where $\alpha_i \beta_i < 1$ for all i.

Proof. The proof follows immediately from Theorem 2.7 and Theorem 3.7. □

Corollary 3.9 *Let $A \in \mathbb{R}^{n,n}$ be an irreducible tridiagonal matrix which is positive definite. Then for $A^{-1} = C = [c_{ij}]$ we have*

$$c_{i,i+k} = \left(\prod_{j=1}^{k} \alpha_{i+j}^{\frac{1}{2}} \right) (c_{i,i} c_{i+k,i+k})^{\frac{1}{2}}, \tag{3.8}$$

where $\alpha_i < 1$ for all i.

Proof. With Theorem 2.9 we have for a symmetric positive definite tridiagonal matrix A that there exists a diagonal matrix D such that $DA^{-1}D$ is given by (2.8) with

$$0 < \frac{u_1}{v_1} < \frac{u_2}{v_2} < \ldots, < \frac{u_n}{v_n}.$$

Thus we have $\alpha_i < 1$ for all i. Since (3.5) of Theorem 3.7 is independent of multiplying A from the left and the right with the same diagonal matrix, we obtain (3.8). □

Note that Corollary 3.9 also include the symmetric M–matrices. Since for M–matrices $\alpha_i\beta_i < 1$ for all i and $\alpha_i < 1$ for symmetric positive definite matrices, (3.7) and (3.8) give a sharp decay result for the entries of the inverse of tridiagonal matrices. Moreover, this results can be proved easily and elegant. The decay rates are given in terms of A^{-1}. However, the next lemma and the next theorem will show the relation of the α_i and β_i to some values determined directly from A.

For $k = 1, \ldots, n-1$ partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.9}$$

where $A_{11} \in \mathrm{I\!R}^{\mathrm{k,k}}$ and $A_{22} \in \mathrm{I\!R}^{\mathrm{n-k,n-k}}$, and splitt A into $A = D_k - N_k$, where D_k is the block diagonal of A. If D_k is nonsingular we define

$$\rho_k = \rho(D_k^{-1}N_k). \tag{3.10}$$

We then have

Lemma 3.10 *Let $A \in \mathrm{I\!R}^{\mathrm{n,n}}$ be tridiagonal and nonsingular. For $k = 1, \ldots, n-1$ let $A = D_k - N_k$ be as in (3.9). Moreover for $A^{-1} = C = [c_{ij}]$ let*

$$\tilde{D}_k = \begin{bmatrix} c_{k,k} & 0 \\ 0 & c_{k+1,k+1} \end{bmatrix} \quad \tilde{N}_k = -\begin{bmatrix} 0 & c_{k,k+1} \\ c_{k+1,k} & 0 \end{bmatrix} \tag{3.11}$$

If D_k and $\tilde{D}_k$ are nonsingular, then with $J = [j_{st}] := D_k^{-1}N_k$ and

$$\hat{J}_k = \begin{bmatrix} 0 & j_{k,k+1} \\ j_{k+1,k} & 0 \end{bmatrix}$$

we have

$$\hat{J}_k = -(\tilde{D}_k^{-1}\tilde{N}_k)^T \tag{3.12}$$

and

$$\sigma(D_k^{-1}N_k) = \sigma(\hat{J}_k) \cup \{0\}. \tag{3.13}$$

Here $\sigma(T)$ denotes the spectrum of the matrix T.

Thus Lemma 3.10 says the following: If we partition A as a 2×2 block matrix then the overlapping 2×2 matrix of the block Jacobian matrix is equal to the negative transposed Jacobian matrix of the related overlapping 2×2 submatrix of A^{-1}.

Proof. We immediately get for $\hat{J}_k$:

$$\hat{J}_k = \begin{bmatrix} 0 & -(A_{11}^{-1})_{kk}a_{k,k+1} \\ -(A_{22}^{-1})_{11}a_{k+1,k} & 0 \end{bmatrix}.$$

Now consider A partioned as in (3.9). A^{-1} is given by

$$A^{-1} = \begin{bmatrix} (A/A_{22})^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix},$$

here (A/A_{11}) denotes the Schur complement of A, i.e. $(A/A_{11}) = A_{22} - A_{12}A_{11}^{-1}A_{21}$. We have

$$-A_{21}A_{11}^{-1} = -a_{k+1,k}[(A_{11})_k^{-T}, 0, \ldots, 0]^T,$$

where $(A_{11})_k^{-1}$, is the last row of A_{11}^{-1}. Hence

$$c_{k+1,k} \;=\; \left[-(A/A_{11})^{-1}A_{21}A_{11}^{-1}\right]_{1,k} = -(A/A_{11})_{1,1}^{-1} * (A_{11}^{-1})_{k,k} * a_{k,k+1}$$

Thus

$$-\frac{c_{k+1,k}}{c_{k+1,k+1}} = (A_{11}^{-1})_{k,k}a_{k,k+1} = -j_{k,k+1}.$$

Similarly we show that

$$-\frac{c_{k,k+1}}{c_{k,k}} = (A_{22}^{-1})_{1,1}a_{k+1,k} = -j_{k+1,k}.$$

Thus

$$\hat{J}_k = -(\tilde{D}^{-1}\tilde{N}_k)^T$$

Equation (3.13) follows from (3.12) and the special structure of J. □

We then obtain

Theorem 3.11 *Let $A \in \mathbb{R}^{n,n}$ be an irreducible tridiagonal M–matrix and let $A^{-1} = C = [c_{ij}]$. For $s = 1, \ldots, n-1$ let ρ_s be as in* (3.10). *Then*

$$c_{i,i+k}c_{i+k,i} = \left(\prod_{l=0}^{k-1} \rho_{i+l}^2\right)(c_{i,i}c_{i+k,i+k}).$$

Proof. For all s equation (3.11) gives

$$\tilde{D}_s = \begin{bmatrix} u_s v_s & 0 \\ 0 & u_{s+1}v_{s+1} \end{bmatrix} = \begin{bmatrix} x_s y_s & 0 \\ 0 & x_{s+1}y_{s+1} \end{bmatrix} \quad \tilde{N}_k = -\begin{bmatrix} 0 & u_s v_{s+1} \\ x_s y_{s+1} & 0 \end{bmatrix},$$

thus

$$(\rho(\tilde{D}_s^{-1}\tilde{N}_k))^4 \quad = \quad \frac{u_s v_{s+1} x_s y_{s+1}}{u_s v_s u_{s+1} v_{s+1}} \frac{x_s y_{s+1} u_s v_{s+1}}{x_s y_s x_{s+1} y_{s+1}} = \alpha_s \beta_s.$$

Hence with Lemma 3.10 we obtain

$$(\alpha_s\beta_s)^{\frac{1}{2}} = \rho_s^2.$$

□

Moreover for symmetric positive definite matrices we obtain

Corollary 3.12 *Let $A \in \mathbb{R}^{n,n}$ be tridiagonal symmetric positive definite and let $A^{-1} = C = [c_{ij}]$. Then*

$$c_{i,i+k} = \left(\prod_{l=0}^{k-1} \rho_{i+l}\right)(c_{i,i}c_{i+k,i+k})^{\frac{1}{2}}.$$

A similar result is proved by Vassilevski [Vas], (see also [A] p.368). Vassilevski proved for tridiagonal symmetric positive definite matrices A and their inverses $C = [c_{ij}]$ that

$$c_{i,i+k} = \left(\prod_{l=0}^{k-1} \gamma_{i+l}\right)(c_{i,i}c_{i+k,i+k})^{\frac{1}{2}} \tag{3.14}$$

where

$$\gamma_s = \sup_{v \in \mathbb{R}^{s,s}, w \in \mathbb{R}^{n-s,n-s}} \frac{v^T A_{12} w}{(v^T A_{11} v w^T A_{22} w)^{\frac{1}{2}}}. \tag{3.15}$$

and A is partitioned as in (3.9) for all s. The constants γ_s are known as the Cauchy-Bunyakowski-Schwarz constants. Since A is positive definite, $\gamma_s < 1$. Originally in [Vas] the result (3.14) is obtained from a more general result for symmetric positive definite block tridiagonal matrices. There norms of the blocks are compared and the equality in (3.14) becomes an inequality ($\leq$).

The proof given in [Vas] is very long and technical, at least for the point case. But it is easy to prove that the ρ_s in Corollary 3.12 and the γ_s in (3.15) are the same. Thus, our approach for the nonsymmetric case gives even for the symmetric matrices a much simpler proof of Vassilevski result.

Moreover Theorem 3.12 and Lemma 3.10 give another way to compute or estimate the Cauchy-Bunyakowski-Schwarz constants.

Using comparison theorems for regular splittings we obtain from Theorem 3.12 and Theorem 3.11

Corollary 3.13 *Let $A \in \mathbb{R}^{n,n}$ be tridiagonal and irreducible. Let $A^{-1} = C = [c_{ij}]$. Let $A = D - N$, where $D = diag(A)$ and let $\rho = \rho(D^{-1}N)$. If A is an M–matrix then*

$$c_{i,i+k}c_{i+k,i} \leq \rho^{2k} c_{i,i}c_{i+k,i+k}.$$

If A is symmetric positive definite then

$$c_{i,i+k} \leq \rho^k (c_{i,i}c_{i+k,i+k})^{\frac{1}{2}}. \tag{3.16}$$

Proof. The splittings $A = D_k - N_k$ of (3.9) are regular splittings, i.e. D_k^{-1} and N_k are nonnegative matrices. The same holds for the splitting $A = D - N$. Moreover, we have

$$N_k \leq N \quad \text{for all } k,$$

With Varga's comparison theorem for regular splittings ([Var] p. 90) we obtain

$$\gamma_k = \rho(D_k^{-1}N_k) \leq \rho.$$

Since (3.16) is independent of scaling we obtain the result easily for symmetric positive definite matrices.

□

Note that in both cases $\rho < 1$. Thus the spectral radius of the Jacobi iteration matrix is an upper bound for the decay of the entries of the inverse of a tridiagonal M–matrix or symmetric positive definite matrix.

In the following we will see that this is also true for banded M–matrices. A matrix $A = [a_{i,j}]$ is called a $2p+1$ banded matrix, if $a_{i,j} = 0$ for $|i-j| > p$.

For M–matrices A the eigenvalue λ of smallest absolute value is real and the corresponding eigenvector u is positive. Thus we can find a nonsingular diagonal matrix $\Delta = diag(\delta_i)$ with

$$\Delta e = u \quad \text{such that} \quad \Delta^{-1} A \Delta e = \lambda e, \tag{3.17}$$

where e is the vector of all ones. We then have

Theorem 3.14 *Let A be a $2p+1$ banded M–matrix. Let $A^{-1} = C = [c_{s,t}]$. Then for any s, t with $s \in \{(i-1)p+2, \ldots, ip+1\}$ and $t \in \{(j-1)p+2, \ldots, jp+1\}$ ($i = 1$ if $s = 1$, $j = 1$ if $t = 1$) with $i \neq j$*

$$\delta_s^{-1} c_{s,t} \delta_t \leq \rho^{|i-j|} c_{t,t}$$
$$\delta_s^{-1} c_{s,t} \delta_t \leq \rho^{|i-j|} c_{s,s}$$

here $\rho = \rho(D^{-1}N)$ with $D = diag(A)$ and $N = D - A$.

Proof. Since $\Delta^{-1} A \Delta e = \lambda e$ we have

$$D^{-1} N e = \rho(D^{-1} N) e. \tag{3.18}$$

We first assume that $s > t$. To find the t-th column of C we have to solve $Ax = e_t$, where e_t is the zero vector except the t-th entry which is 1. To do so we consider with $J := D^{-1} N$

$$x^{(k)} = J x^{(k-1)} + D^{-1} e_j, \qquad x^{(0)} = 0. \tag{3.19}$$

If we define $\varepsilon^{(k)} = x - x^{(k)}$ we get

$$\varepsilon^{(k)} = J \varepsilon^{(k-1)} = J^k \varepsilon^{(0)} = J^k x.$$

Thus

$$||\varepsilon^{(k)}||_\infty \leq ||J^k||_\infty ||x||_\infty \leq ||J||_\infty^k ||x||_\infty = \rho(J)^k c_{tt}.$$

The last equality follows from (3.18) and $||x||_\infty = c_{tt}$ (see [FP]). With (3.19) we obtain for $p \neq 1$

$$x_l^{(i-j)} = 0 \quad \text{for } l \geq (i-1)p + 2$$

and for $p = 1$

$$x_l^{(i-j)} = 0 \quad \text{for } l \geq i$$

Now let

$$\varepsilon^{(k)} = \begin{bmatrix} \tilde{\varepsilon}^{(k)} \\ \hat{\varepsilon}^{(k)} \end{bmatrix},$$

where $\tilde{\varepsilon}^{(k)} \in \mathbb{R}^{(i-1)p+2}$. Similar we partition $x^{(k)}$ and x. Then

$$||\varepsilon^{(i-j)}||_\infty \geq ||\hat{\varepsilon}^{(i-j)}||_\infty = ||\hat{x} - \hat{x}^{(i-j)}||_\infty \geq ||\hat{x}||_\infty - ||\hat{x}^{(i-j)}||_\infty = ||\hat{x}||_\infty.$$

Thus

$$\delta_s^{-1} c_{s,t} \delta_t \leq ||\hat{x}||_\infty \leq \rho(J)^{(i-j)} c_{t,t}$$

Similarly we obtain by solving $x^T A = e_s$

$$\delta_s^{-1} c_{s,t} \delta_t \leq \rho(J)^{(i-j)} c_{s,s}$$

Similarly we prove the case $s < t$. $\square$

Theorem 3.14 can be easily extended to H–matrices A, i.e. matrices for which the comparison matrix $\mathcal{M}(A)$ with $(\mathcal{M}(A))_{ii} = |a_{ii}|, (\mathcal{M}(A))_{i,j} = -|a_{i,j}|$ for $i \neq j$, is an M–matrix. The decay rate is then the spectral radius of the jacobi matrix of $\mathcal{M}(A)$. Moreover Theorem 3.14 can be also formulated for sparse matrices, not only banded matrices, using the notation used by Meurant in [Meu].

We immediately get from Theorem 3.14

Corollary 3.15 *Let A be a $2p+1$ banded M–matrix. Let $A^{-1} = C = [c_{s,t}]$. Then for any s, t with $s \in \{(i-1)p+2, \ldots, ip+1\}$ and $t \in \{(j-1)p+2, \ldots, jp+1\}$ ($i = 1$ if $s = 1$, $j = 1$ if $t = 1$) with $i \neq j$*

$$\begin{aligned} c_{s,t} c_{t,s} &\leq \rho^{2|i-j|}(c_{s,s} c_{t,t}) \\ c_{s,t} c_{t,s} &\leq \rho^{2|i-j|} c_{s,s}^2 \\ c_{s,t} c_{t,s} &\leq \rho^{2|i-j|} c_{t,t}^2 \end{aligned}$$

Corollary 3.16 *Let A be a $2p+1$ banded symmetric M–matrix. Let $A^{-1} = C = [c_{s,t}]$. Then for any s, t with $s \in \{(i-1)p+2, \ldots, ip+1\}$ and $t \in \{(j-1)p+2, \ldots, jp+1\}$ ($i = 1$ if $s = 1$, $j = 1$ if $t = 1$) with $i \neq j$*

$$\begin{aligned} c_{s,t} &\leq \rho^{|i-j|}(c_{s,s}c_{t,t})^{\frac{1}{2}} \\ c_{s,t} &\leq \rho^{|i-j|}c_{s,s} \\ c_{s,t} &\leq \rho^{|i-j|}c_{t,t} \end{aligned}$$

The advantage of the Corollaries 3.15 and 3.16 compared with a Theorem by Meurants ([Meu] Theorem 4.13) is that we just need ρ and diagonal entries of A^{-1} to estimate the decay.

Acknowledgement The author thanks Ludwig Elsner and Shmuel Friedland for helpful comments. An earlier version of this paper in which symmetric matrices are considered, appeared as Prepint 96 – 025 of the Sonderforschungsbereich 343, Universität Bielefeld.

References

[A] O. Axelsson, Iterative Solution methods, Cambridge University Press 1994.

[B] W. W. Barrett *A Theorem on Inverses of Tridiagonal Matrices* Lin. Alg. Appl. 27:211-217(1979).

[CGM] C. Concus, G.H. Golub, and G. Meurant, *Block preconditioning for the conjugate gradient method*, SIAM J. Sci. Stat. Comp. 6, 220–152(1985).

[D] S. Demko, *Inverses of band matrices and local convergence of spline projections*, SIAM J. Numer. Anal. 14:616–619(1977).

[DMS] S. Demko, W.F. Moss, P.W. Smith, *Decay rates for inverses of band matrices*, Math. Comp. 43:491–499(1984)

[FP] M. Fiedler and V. Pták, *Diagonally dominant matrices*, Czechoslovak Math. J. 47:420–433(1967).

[GK] F.R. Gantmacher, M.G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie Verlag Berlin, 1960, in russian 1950.

[I] Y. Ikebe *On Inverses of Hessenberg Matrices* Lin. Alg. Appl. 24:93-97(1979).

[Mar] T.L. Markham, *Nonnegative matrices whose inverses are M-matrices*, Proc. Amer. Math. Soc. 36:326–330(1972).

[Meu] G. Meurant, *A Review on the inverse of symmetric tridiagonal and block tridiagonal matrices*, SIAM J. Matrix Anal. Appl. 13, 707–728 (1992).

[MNNST] J.J. McDonald, R. Nabben, M. Neumann, H. Schneider, M. Tsatsomeros, *Inverse tridiagonal Z–matrices,* submitted.

[Var] R.S. Varga, *Matrix Iterative Analysis*, Prentice Hall, 1962.

[Vas] P.S. Vassilevski, *On Some Ways of Approximating Inverses of Banded Matrices in Connection with Deriving Preconditioners Based on Incomplete Block Factorizations*, Computing 43:277–296(1990).