Classification of simple Harish-Chandra modules over $\mathbb{Q}$ - Virasoro algebra

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Abstract

We give a complete classification of simple Harish-Chandra modules over $\mathbb{Q}$ - Virasoro algebra.

1 Introduction

A classification of the simple Harish-Chandra modules over complex Virasoro algebra $\mathcal{V}$ was given by O.Mathieu in remarkable paper [7]. According to this classification each simple Harish-Chandra $\mathcal{V}$-module is either highest (lowest) weight module or module from the so-called intermediate series.

There exist a lot of different generalizations of the classical Virasoro algebra and centerless Virasoro algebra (Witt algebra) studied by several authors (see for example [4, 5, 9, 10, 11]). Naturally there appears a question to give a classification of Harish-Chandra modules over all those algebras.

In the subsequent paper we consider the following generalization of the Virasoro algebra which is called $\mathbb{Q}$ - Virasoro algebra. We define $\mathfrak{G}$ to be a Lie algebra with the base $\{e(x)\,|\,x \in \mathbb{Q}\} \cup \{c\}$ and the Lie brackets given by

$$[e(x), e(y)] = (y-x)e(x+y) + \delta_{x,-y}\frac{x^3-x}{12}c; \quad x, y \in \mathbb{Q}$$

$$[c, e(x)] = 0, \quad x \in \mathbb{Q}$$

(here $\delta_{x,-y}$ is a Kronecker symbol). It is a natural generalization of the higher rank Virasoro algebras considered in [10].

An intermediate series for $\mathfrak{G}$ is a family of modules $V(a,b)$, $a, b \in \mathbb{C}$ defined as follows: $V(a,b)$ has the base v_x, $x \in \mathbb{Q}$ and the action of generating elements of $\mathfrak{G}$ is defined by

$$e(x)v(y) = (ax + y + b)v(x+y), \quad x, y \in \mathbb{Q}. \tag{1}$$

For $a \neq 0, 1$ or $b \notin \mathbb{Q}$ the module $V(a,b)$ is simple. In the opposite case it has two simple subquotients and one of them is a trivial $\mathfrak{G}$-module. We will also call the intermediate series the family of all simple subquotients of $V(a,b)$.

The main result of this paper states that:

Theorem 1. *Each simple Harish-Chandra module over $\mathfrak{G}$ is a module of intermediate series.*

We note that this result is slightly different from Mathieu theorem since it would be shown that non-trivial highest or lowest weight module over $\mathfrak{G}$ is not Harish-Chandra module.

Let us briefly describe the structure of the paper. In section 2 we collect all necessary preliminaries. In section 3 we describe the support of simple Harish-Chandra modules over $\mathfrak{G}$. This is an essential part of the proof of main theorem 1 that presented in section 4.

2 Preliminaries

Let $\mathbb{C}$ denotes field of complex numbers, $\mathbb{Q}$ denotes field of rational numbers, $\mathbb{Z}$ denotes ring of integers and $\mathbb{N}$ denotes set of positive integers. We will also denote by $\mathbb{Z}_+$ the set of non-negative integers and by $\mathbb{Q}_+$ the set of non-negative rationals.

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Consider a $\mathbb{Q}$ - Virasoro algebra $\mathfrak{G}$ and keep the notations from section 1. We denote by $\mathfrak{H}$ the Cartan subalgebra of $\mathfrak{G}$ generated by $e(0)$ and c and by $\mathfrak{H}_1$ the subalgebra generated by $e(0)$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ set

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}.$$

A $\mathfrak{G}$-module V will be called a weight module provided

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda.$$

If $V_\lambda \neq 0$ we will call λ the weight of V and V_λ — the corresponding weight space. Since element c is central it acts by maltiplication on scalar on each weight indecomposable module. We will call the corresponding scalar the charge of V. Thus for indecomposable V we will also call the weight of V the corresponding element $\lambda \in \mathfrak{H}_1^*$. From now on weight will mean weight with respect to $e(0)$. For a weight $\mathfrak{G}$-module V we set $\operatorname{Supp} V$ to be the set of all non-zero weights of V. Finally a weight $\mathfrak{G}$-module V will be called Harish-Chandra module provided $\dim V_\lambda < \infty$ for all $\lambda \in \operatorname{Supp} V$. Since $\mathfrak{H}_1$ is one dimensional we identify it with $\mathbb{C}$ in standard way: we define $\lambda \in \mathbb{C}$ to be the eigenvalue of $e(0)$.

One can see that for any indecomposable weight $\mathfrak{G}$-module V and for any $\lambda \in \operatorname{Supp} V$ holds $\operatorname{Supp} V \subset \lambda + \mathbb{Q}$.

A $\mathfrak{G}$-module V will be called dense provided $\operatorname{Supp} V = \lambda + \mathbb{Q}$ for some $\lambda \in \mathfrak{H}_1^*$. A $\mathfrak{G}$-module V will be called pined provided $\operatorname{Supp} V = (\lambda + \mathbb{Q}) \setminus \lambda$ for some $\lambda \in \mathfrak{H}_1^*$. A $\mathfrak{G}$-module V will be caleed cutted provided $\operatorname{Supp} V \subset \lambda + \mathbb{Q}_+$ or $\operatorname{Supp} V \subset \lambda - \mathbb{Q}_+$ for some $\lambda \in \mathfrak{H}_1^*$.

3 Support of simple Harish-Chandra module

In this section we describe the support of Harish-Chandra modules over $\mathfrak{G}$. All this results are inspired by Futorny theorem on the support of irreducible weight module over complex finite-dimensional Lie algebra ([3]). We begin with the following observation:

Let $\mathcal{V}$ denotes the classical Virasoro algebra with the base $e(x)$, $x \in \mathbb{Z}$; c and Lie brackets

$$[e(x), e(y)] = (y-x)e(x+y) + \delta_{x,-y}\frac{x^3-x}{12}c, \quad x, y \in \mathbb{Z},$$
$$[c, e(x)] = 0, \quad x \in \mathbb{Z}.$$

Lemma 1. *Let $L(\lambda)$, $\lambda \in \mathfrak{H}^*$ be simple non-trivial highest weight module over $\mathcal{V}$ ([12, 13]). Then*

$$\sup_{\mu \in \operatorname{Supp} L(\lambda)} \dim L(\lambda)_\mu = \infty.$$

Proof. Let $P(k)$, $k \in \mathbb{N}$ be a partition function ([1]). Fix $N \in \mathbb{N}$.

Denote by $(\cdot,\cdot)_\lambda$ the Shapovalov form on Verma module $M(\lambda)$ ([13]). Following [2] for $d \in \mathbb{C}$ and $r, s \in \mathbb{N}$, $r \neq s$ set

$$\alpha^{\pm}_{r,s}(d) = -\frac{1}{48}((13-d)(r^2+s^2) \pm \sqrt{d^2-26d+25}(r^2-s^2) - 24rs - 2 + 2d).$$

For $r \neq s$, $q \in \mathbb{C}$ let $\psi_{r,s}(q,d) = (q-\alpha^+_{r,s}(d))(q-\alpha^-_{r,s}(d))$ and $\psi_{r,r}(q,d) = q + \frac{1}{24}(r^2-1)(1-d)$.

For $m \in \mathbb{N}$ set $(\cdot,\cdot)_{\lambda,m}$ be a restriction of $(\cdot,\cdot)_\lambda$ on $M(\lambda)_{\lambda(e(0))-m}$. It is well-known (see for example [8]) that radical of $(\cdot,\cdot)_\lambda$ coincides with maximal submodule of $M(\lambda)$. By [2] for $m \in \mathbb{N}$ holds

$$\det(\cdot,\cdot)_{\lambda,m} = \prod_{i=1}^{m} \prod_{r|i, r^2 \leq i} \left(\psi_{r,i/r}(\lambda(e(0)), \lambda(c))\right)^{P(m-i)}.$$

Denote $\mathcal{V}^m$ the Virasoro subalgebra of $\mathcal{V}$ generated by elements $e(mn)$, $n \in \mathbb{Z}$. If we denote by v the canonical generator of $M(\lambda)$ then $U(\mathcal{V}^m)v \simeq M(\lambda^m)$, $\lambda(c) = \lambda^m(c)$. Since $L(\lambda)$ is non-trivial (i.e. $\lambda \neq 0$) holds $\lambda \neq \lambda^m$. Thus there exists such $m \in \mathbb{N}$ that $\det(\cdot,\cdot)_{\lambda^m,N} \neq 0$ and we conclude that $\dim L(\lambda)_{\lambda(e(0))-Nm} \geq P(N) \geq N$. this completes the proof. $\square$

Let V be non-trivial simple Harish-Chandra module over $\mathfrak{G}$. Fix $\lambda \in \operatorname{Supp} V$. Consider a function $f = f(V,\lambda) : \mathbb{Q} \to \mathbb{Z}_+$ defined by:

$$f(q) = \dim V_{\lambda+q}$$

for $q \in \mathbb{Q}$.

Lemma 2. *For any $a, b, c \in \mathbb{Q}$, $a < b < c$, $\lambda + b \neq 0$ holds*

$$f(b) \leq \max(f(a), f(c)).$$

Proof. Consider a subalgebra $\mathfrak{G}_1 \subset \mathfrak{G}$ which satisfies following conditions:

1. $\mathfrak{G}_1 \simeq \mathcal{V}$;
2. $\mathfrak{G}_1 = \langle e(mx) \mid m \in \mathbb{Z} \rangle$ for some $x \in \mathbb{Q}$.
3. $a, b \in c + \mathbb{Z}x$.

Then $\langle V_y \mid y \in \mathbb{Z}x \rangle$ is Harish-Chandra module over $\mathfrak{G}_1$. By Mathieu theorem ([7]) any non-trivial simple Harish-Chandra module over $\mathfrak{G}_1$ is either highest or lowest weight module or module from intermediate series. For any simple Harish-Chandra module W over $\mathfrak{G}_1$ the same properti for the function $f(W, \lambda)$, $\lambda \in \operatorname{Supp} W$ is obvious. Since the dimension of the weight space is additive in extension and $\lambda + b \neq 0$ the statement follows. $\square$

Proposition 1. *Let V be a non-trivial cutted module over $\mathfrak{G}$. Then V is not Harish-Chandra module.*

Proof. Suppose that $\operatorname{Supp} V \subset \mu - \mathbb{Q}_+$ for some $\mu \in \mathfrak{H}_1^*$. It follows from the Mathieu theorem that V is a direct sum of highest weight modules with respect to any subalgebra $\mathfrak{G}_1 \subset \mathfrak{G}$, $\mathfrak{G}_1 \simeq \mathcal{V}$.

It follows immediately from lemma 2 that there exists $\lambda \in \operatorname{Supp} V$, $k \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\dim V_{\lambda + x} = k$$

for any $x \in \mathbb{Q}$ with $|x| < \varepsilon$. Choose the minimal such k.

By lemma 1 there exists $y \in \mathbb{Q}$ small enough such that some simple highest weight module over the Virasoro algebra which contains $e(y)$ and with highest weight in $\lambda - \varepsilon + \mathbb{Q}_+$ has a weight spase for some weight in $\lambda - \varepsilon + \mathbb{Q}_+$ of dimension greater then k.

Thus we obtain a contradiction which proves our proposition. $\square$

Theorem 2. *Let V be simple Harish-Chandra $\mathfrak{G}$-module. Then V is either dense or pined.*

Proof. Let $\operatorname{Supp} V \subset \lambda + \mathbb{Q}$ for some $\lambda \in \mathfrak{H}_1^*$. Suppose that $|(\lambda + \mathbb{Q}) \setminus \operatorname{Supp} V| > 1$. Let $x, y \in \mathbb{Q}$, $x < y$ such that $\lambda + x, \lambda + y \notin \operatorname{Supp} V$. Thus it follows from lemma 2 that $\lambda + z \notin \operatorname{Supp} V$ for any $z \in [x, y] \cap \mathbb{Q}$. Finally we easely obtain that V is cutted module. But by proposition 1 this is impossible. $\square$

4 Proof of the main theorem

Denote

$$U_0 = \{u \in U(\mathfrak{G}) \mid [e(0), u] = 0\}.$$

We recall that for any simple Harish-Chandra module V and $\lambda \in \operatorname{Supp} V$ the U_0-module V_λ is simple and conversly each simple finite-dimensional U_0-module can be continued to the unique simple $\mathfrak{G}$-module.

For a subalgebra $\mathfrak{G}_1 \subset \mathfrak{G}$ we will also denote

$$U(\mathfrak{G}_1)_0 = \{u \in U(\mathfrak{G}_1) \mid [e(0), u] = 0\}.$$

Lemma 3. *Let N be simple finite-dimensional U_0-module. Then there exists $\mathfrak{G}_1 \subset \mathfrak{G}$, $\mathfrak{G}_1 \simeq \mathcal{V}$ such that N is simple $U(\mathfrak{G}_1)$-module.*

Proof. Since

$$U_0 = \bigcup_{\mathfrak{G}_1 \subset \mathfrak{G},\, \mathfrak{G}_1 \simeq \mathcal{V}} U(\mathfrak{G}_1)_0$$

the statement follows by standard arguments. □

Lemma 4. *Let V be simple Harish-Chandra module over $\mathfrak{G}$ and $\lambda, \mu \in \operatorname{Supp} V$; $\lambda, \mu \neq 0$. Then*

$$\dim V_\lambda = \dim V_\mu.$$

Proof. Suppose that $\dim V_\lambda > \dim V_\mu$ for some $\lambda, \mu \neq 0$. Let $\mathfrak{G}_1$ be subalgebra of $\mathfrak{G}$ generated by elements $e(k(\lambda - \mu))$, $k \in \mathbb{Z}$. Consider a Harish-Chandra $\mathfrak{G}_1$-module $N = U(\mathfrak{G}_1)V_\lambda$.

Since $\dim V_\lambda > \dim V_\mu$ we apply the Mathieu classification theorem and obtain that there should be a subfactor of N which is highest or lowest weight module. Thus by lemma 1 we conclude that

$$\sup_{\nu \in \mathfrak{H}_1^*} \dim V_\nu = \infty.$$

Let $k \in \mathbb{N}$, $k > \dim V_\lambda$ be the dimension of some V_ν. We can choose k in such a way that there exists $\chi \in \operatorname{Supp} V$ and $\varepsilon > 0$ such that $\dim V_{\chi + x} = k$ for all $|x| < k$. Thus we obtain the same contradiction as in the proof of proposition 1. □

Corollary 1. *Let V be simple Harish-Chandra $\mathfrak{G}$-module. Then*

$$\sup_{\lambda \in \operatorname{Supp} V} \dim V_\lambda < \infty$$

Lemma 5. *Let V be a non-trivial simple Harish-Chandra $\mathfrak{G}$-module, $\lambda \in \operatorname{Supp} V$. Then* $\dim V_\lambda = 1$.

Proof. Suppose that $\dim V_\lambda \neq 1$ for some $\lambda \in \mathfrak{H}_1^*$. By lemma 3 there exists a subalgebra $\mathfrak{G}_1 \subset \mathfrak{G}$, $\mathfrak{G}_1 \simeq \mathcal{V}$ such that V_λ is simple $U(\mathfrak{G}_1)_0$-module.

Using Mathieu classification theorem we immediately obtain that $U(\mathfrak{G}_1)V_\lambda$ should have a subfactor which is non-trivial highest or lowest weight module. By lemma 1 this contradicts with corollary 1. □

Proof of theorem 1. Let V be a non-trivial simple Harish-Chandra $\mathfrak{G}$-module and $\lambda \in \operatorname{Supp} V$. By theorem 2 we have that V is either dense or pined and by lemma 5 we have $\dim V_\lambda = 1$.

Denote by $\mathfrak{G}_n$ a minimal subalgebra of $\mathfrak{G}$ such that

1. $\mathfrak{G}_n \simeq \mathcal{V}$;
2. $\mathfrak{G}_n$ contains all elements $e(k^{-1})$, $k = 1, \ldots, n$.

Clearly

$$\mathfrak{G}_i \subset \mathfrak{G}_{i+1}, i > 0 \quad \text{and} \quad \mathfrak{G} = \bigcup_{i>0} \mathfrak{G}_i.$$

By Mathieu theorem for $i > 0$ the module $U(\mathfrak{G}_i)V_\lambda$ is a module from intermediate series. Fix a basis element $v_{\lambda-\mu}$, $\mu \in \operatorname{Supp} U(\mathfrak{G}_1)V_\lambda$ such that the action of generators of $\mathfrak{G}_1$ is defined by the formula (1). We can continue this base to the base of $U(\mathfrak{G}_2)V_\lambda$ then to the base of $U(\mathfrak{G}_3)V_\lambda$ and so on. Finally we obtain the base $v(x)$, $x \in \mathbb{Q}$ or $x \in \mathbb{Q} \setminus \{z\}$ for some fixed $z \in \mathbb{Q}$ of a $\mathfrak{G}$-module V with the action of $e(y)$, $y \in \mathbb{Q}$ defined by formula (1). Thus V is a module of intermediate series. $\square$

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