COMPARISON OF a-FUNCTIONS

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ABSTRACT. We show agreement of Lusztig's a-function with the a-function in [3].

1. INTRODUCTION

In this note, W denotes a finite, simply-laced Weyl group, or the affine Weyl group $E_9 = \hat{E}_8$. Let Γ_g be the set of simple generators, which are parametrized by the nodes of the associated Coxeter graph Γ.

Every $w \in W$ may be written as a product $s_1 s_2 s_3 \cdots s_n$ of simple generators. If n is minimal, we call this product "reduced" and define $l(w) = n$ and $supp(w)$ to be the set of generators which appear in the product.

Let H be the Iwahori-Hecke algebra associated to W. This is an algebra over $\mathbb{Q}[q^{1/2}, q^{-1/2}]$ with generators T_s for each $s \in \Gamma_g$ satisfying the relations $T_s^2 = (q-1)T_s + q$, $T_s T_t = T_t T_s$ if $st = ts$, and $T_s T_t T_s = T_t T_s T_t$ if $sts = tst$, for distinct s, $t \in \Gamma_g$. Also, let $v = q^{1/2}$. This algebra has a basis T_w, $w \in W$, where we have $T_w = T_{s_1} \cdots T_{s_n}$ whenever $s_1 \cdots s_n$ is a reduced expression for w. There is a unique automorphism $^- : H \to H$ which sends v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$.

In [4], Kazhdan and Lusztig construct a canonical basis c_w (denoted C'_w in [4]) of H characterized by the property that $c_w = \bar{c}_w = v^{-l(w)} \sum_{y \leq w} P_{y,w}(q) T_y$, where $\leq$ denotes the Bruhat order, and $P_{y,w}(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w}(q) = 1$.

Define $\bar{H} = H / < c_{sts} >_{s,t \in \Gamma_g, sts = tst, s \neq t}$.

For any $s \in \Gamma_g$, let b_s be the projection of $c_s = v^{-1}(T_s + 1)$ in $\bar{H}$. In [2], it is shown that $\bar{H}$ is an algebra over $\mathbb{Q}[q^{1/2}, q^{-1/2}]$ generated by b_s with the relations: 1. $b_s^2 = (v + v^{-1})b_s$, 2. $b_s b_t = b_t b_s$, if $st = ts$, and 3. $b_s b_t b_s = b_s$ if $sts = tst$, $s \neq t$.

Let W_c consist of those elements of W whose reduced expressions avoid substrings of the form sts, where s and t are non-commuting generators. For $w \in W_c$, one can define $b_w = b_{s_1} \cdots b_{s_n}$ where $w = s_1 \cdots s_n$ is any reduced expression for w. According to [1], the elements b_w, $w \in W_c$, form a basis of $\bar{H}$.

Associated with these two bases are left, right, and two-sided cell structures, and a certain "a-function". The purpose of this note is to compare these structures and, in particular, show that the a-functions agree on W_c.

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2. Preliminaries

2.1. **Cells.** See [4] for a definition of Kazhdan-Lusztig left, right, and two-sided cells. Let $a : W \to \mathbb{Z}$ denote the a-function in [5].

See [3] for a definition of monomial left, right, and two-sided cells. We denote by $\overset{m}{\leq}_L$, $\overset{m}{\leq}_R$, and $\overset{m}{\leq}_{LR}$ the left, right, and two-sided preorders as used in [3]. We denote by $\overset{m}{\sim}_L$, $\overset{m}{\sim}_R$, and $\overset{m}{\sim}_{LR}$ the notions of left, right, and two-sided equivalence as used in [3]. We denote by $L_m(w)$, $R_m(w)$, and $T_m(w)$ the left, right, and two-sided monomial cells which contain w. Let $a_m : W_c \to \mathbb{Z}$ denote the a-function in [3].

Let $x \in W$. Define $\mathcal{L}(x) = \{s \in \Gamma_g \mid l(sx) < l(x)\}$. Define $\mathcal{R}(x) = \mathcal{L}(x^{-1})$. Suppose there exists distinct $s, t \in \Gamma_g$ such that $sts = tst$ and $\{s,t\} \cap \mathcal{L}(x) = \{s\}$. Then exactly one of the elements sx or tx solves the equation $\#\{s,t\} \cap \mathcal{L}(w) = 1$ for w. Let ${}^*x$ denote this element. Similarly, define x^*.

Define an equivalence relation $\overset{*}{\sim}_L$ on W to be the transitive closure of the relation ${}^*x \overset{*}{\sim}_L x$. Similarly define $\overset{*}{\sim}_R$. Define $\overset{*}{\sim}_{LR}$ to be the equivalence relation which is obtained from the transitive closure of both $\overset{*}{\sim}_L$ and $\overset{*}{\sim}_R$. We call the equivalence classes of $\overset{*}{\sim}_L$, $\overset{*}{\sim}_R$, and $\overset{*}{\sim}_{LR}$ the left, right, and two-sided star equivalence classes.

2.2. **The set W_c.** We need the following property and its dual (see [1]).

Property 2.1. Let $w \in W_c$, $t \in \Gamma_g$. Assume $wt \notin W_c$. Then we may write $w = w_1 t s w_2$ reduced, t commutes with all $u \in supp(w_2)$, and $(st)^3 = 1$. Furthermore, if $w = w_1' t s' w_2'$ is another such expression, then $s = s'$.

Let P denote the set of subsets of Γ which consist of non-adjacent nodes, including the empty set, $\emptyset$. For any $S \in P$, let $i(S)$ be the product (in any order) of the simple generators corresponding to the nodes in S (with $i(\emptyset) = 1$).

Theorem 2.2. *Let $w \in W_c$ satisfy $w^2 = 1$. There exists a unique $S \in P$ and a unique $x \in W_c$ such that $w = xi(S)x^{-1}$ reduced. Furthermore, $a_m(w) = \#S$.*

Proof. See [3, propositions 3.1.1 and 5.3.1].

3. Kazhdan-Lusztig cells, monomial cells, and star equivalence.

Theorem 3.1. *Left, right, and two-sided monomial cells are equal, respectively, to left, right, and two-sided star equivalence classes.*

Proof. We require a number of lemmas.

Lemma 3.2. *If either x or x^* is in W_c, then both x and x^* are in W_c. If either x or ${}^*x$ is in W_c, then both x and ${}^*x$ are in W_c.*

Proof. By definition, we can assume, by switching x and x^* if necessary, that there exists $s, t \in \Gamma_g$ such that $s \in \mathcal{R}(x)$, $t \notin \mathcal{R}(x)$, $s \notin \mathcal{R}(x^*)$, and $x^* = xt$.

Note that if $x^* \in W_c$, then $x \in W_c$. Conversely, if $x \in W_c$, then $x^* \in W_c$, for if $x^* \notin W_c$, we may write $x = x'ts$ reduced (since $s \in \mathcal{R}(x)$). We conclude that $s \in \mathcal{R}(xt)$, contrary to hypothesis.

A similar argument applies to x and ${}^*x$.

Lemma 3.2 implies that W_c is a union of left star equivalence classes and that W_c is a union of right star equivalence classes and that W_c is a union of two-sided star equivalence classes.

Lemma 3.3. *We have $x \overset{m}{\sim}_R x^*$ and $x \overset{m}{\sim}_L {}^*x$.*

Proof. By definition, we can assume that there exists s, $t \in \Gamma_g$ such that $s \in \mathcal{R}(x)$, $t \notin \mathcal{R}(x)$, $s \notin \mathcal{R}(x^*)$, and $x^* = xt$.

Now, $b_{x^*} = b_x b_t$ so that $x \overset{m}{\leq}_R x^*$. Since $s \in \mathcal{R}(x)$, we can write $x^* = x'st$ reduced. Therefore, $b_x = b_{x^*} b_s$ which shows $x^* \overset{m}{\leq}_R x$. We conclude that $x \overset{m}{\sim}_R x^*$.

The lemma for left equivalence follows in a similar manner.

Lemma 3.4. *If $x \overset{m}{\sim}_R y$, then $x \overset{*}{\sim}_R y$. If $x \overset{m}{\sim}_L y$, then $x \overset{*}{\sim}_L y$. If $x \overset{m}{\sim}_{LR} y$, then $x \overset{*}{\sim}_{LR} y$.*

Proof. Suppose $x \overset{m}{\sim}_R y$.

By [3, theorem 4.4.4], we have x, $y \in R_m(d)$ for some involution $d \in W_c$. By theorem 2.2, we can write $d = zi(S)z^{-1}$ reduced, where $S \in P$ and $a_m(d) = \#S$. By [3, lemma 4.4.1], we see that $zi(S) \in R_m(d)$. Therefore, to show that $x \overset{*}{\sim}_R y$, it suffices to show that $x \overset{*}{\sim}_R zi(S)$.

Note that no element of $\mathcal{R}(zi(S)) = S$ is right cancellable (see [3, definition 4.2.4]), for otherwise $zi(S)z^{-1} \notin W_c$. Therefore, by [3, lemma 4.2.5], we can write $x = zi(S)x'$ reduced.

We now prove that $x \overset{*}{\sim}_R zi(S)$ by induction on the length of x', with the case where $l(x') = 0$ being clear.

By [3, lemma 4.4.1], there exists $s \in \mathcal{R}(x)$ which is right cancellable, say by $t \in \Gamma_g$. This means we can write $x = x''ts$ reduced, where $sts = tst$. Since $x \in W_c$, we cannot have $t \in \mathcal{R}(x)$. We conclude that $x^* = xs$. In other words, $x \overset{*}{\sim}_R xs$. The first part of lemma 3.4 follows by induction, since $l(xs) < l(x)$.

The second part of the lemma follows by a similar argument, and the last part follows from the first two parts.

Theorem 3.1 follows from lemma 3.4 and lemma 3.3.

Corollary 3.5. *The intersection of a Kazhdan-Lusztig cell with W_c is a union of monomial cells of the same type (left, right, or two-sided).*

Proof. The corollary follows from theorem 3.1 and results in [4, §4] which imply that Kazhdan-Lusztig cells are unions of star equivalence classes of the same kind.

4. The functions a and a_m.

Theorem 4.1. *For $w \in W_c$, we have $a_m(w) = a(w)$.*

Proof. We first recall some facts about monomial cells.

Let $S, T \in P$. We say that S and T are neighbors if and only if $1+\#S\cap T = \#S = \#T$, and the two nodes in $(S \cup T) \setminus (S \cap T)$ are joined in Γ. Define an equivalence relation $\sim$ on P by taking the transitive closure of the relation $S \sim T$ if S and T are neighbors.

Proposition 4.2. *The map $i(S) \mapsto T_m(i(S))$ induces a bijection between $P/\sim$ and the set of two-sided monomial cells.*

Proof. See [3, theorem 4.5.1].

Let $w = s_1 \cdots s_n$ be a product of distinct generators $s_k \in \Gamma_g$. Note that $c_{s_1} \cdots c_{s_n}$ is stable under $\bar{}$. By the characterization of the Kazhdan-Lusztig canonical basis given in the introduction, we can conclude that $c_w = c_{s_1} \cdots c_{s_n}$. In particular, $c_{i(S)} = b_{i(S)}$ for all $S \in P$. By [6, proposition 1.2] and [5, proposition 2.4], we have $a(i(S)) = \#S$.

Now let $w \in W_c$. By proposition 4.2, we know that $w \overset{m}{\sim}_{LR} i(S)$ for some $S \in P$. By 3.1, we know that $w \overset{*}{\sim}_{LR} i(S)$. We conclude that $a(w) = a(i(S))$. Theorem 4.1 follows.

We note the following analog of [5, corollary 1.9b] whose proof we omit.

Proposition 4.3. *If $x \overset{m}{\leq}_L y$ and $a(x) = a(y)$, then $x \overset{m}{\sim}_L y$. If $x \overset{m}{\leq}_R y$ and $a(x) = a(y)$, then $x \overset{m}{\sim}_R y$.*

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