STABLE EQUIVALENCE AND REPRESENTATION TYPE

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In this note we discuss various finiteness conditions which are preserved under a stable equivalence $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$ between two artin algebras. More precisely, we prove:

- Λ has strongly unbounded representation type if and only if Γ has strongly unbounded representation type.
- Λ has no continuous pure-injective modules if and only if Γ has no continuous pure-injective modules.
- The Krull-Gabriel dimensions of Λ and Γ coincide.

We shall apply the machinery developed in [12]. There it was shown that Λ is (generically) wild if and only if Γ is (generically) wild.

1. Basic properties of stable equivalences

In this section we collect some basic properties of stable equivalences. Using generic modules, we prove at the end that a stable equivalence preserves strongly unbounded representation type. We begin with some notation and recall briefly the definitions which are needed in this paper.

Let Λ be an artin algebra. Denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and let $\operatorname{mod}\Lambda$ be the full subcategory of finitely presented Λ-modules. A sequence of morphisms $0 \to L \to M \to N \to 0$ in $\operatorname{Mod}\Lambda$ is *pure-exact* if the induced sequence $0 \to \operatorname{Hom}_\Lambda(X,L) \to \operatorname{Hom}_\Lambda(X,M) \to \operatorname{Hom}_\Lambda(X,N) \to 0$ is exact for every X in $\operatorname{mod}\Lambda$. In this case the morphism $L \to M$ is called a *pure monomorphism*, and L is *pure-injective* if every pure monomorphism $L \to M$ splits. For example, every finitely presented Λ-module is pure-injective. We denote by $\operatorname{Pinj}\Lambda$ the full subcategory of pure-injective Λ-modules and the *Ziegler spectrum* $\operatorname{Zsp}\Lambda$ is the set of isomorphism classes of indecomposable pure-injective Λ-modules. A Λ-module M is *endofinite* if the length of M, when regarded in the natural way as an $\operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module, is finite. Note that every endofinite module is pure-injective. An endofinite module is called *generic* if it is indecomposable but not finitely presented [4]. Given a finitely presented Λ-module X we denote by $\operatorname{Add}X$ ($\operatorname{add}X$) the smallest full subcategory of $\operatorname{Mod}\Lambda$ which contains X and is closed under taking arbitrary (finite) coproducts and direct summands. The *stable category* $\underline{\operatorname{Mod}}_X\,\Lambda$ has the same objects as $\operatorname{Mod}\Lambda$ but $\underline{\operatorname{Hom}}_X(M,N)$ is the group of all Λ-morphisms $M \to N$ modulo the subgroup of those which factor through an object in $\operatorname{Add}X$. For example, one obtains the usual stable module category $\underline{\operatorname{Mod}}\,\Lambda$ by taking $X = \Lambda$. Analogously, the categories $\underline{\operatorname{Pinj}}_X\,\Lambda$ and $\underline{\operatorname{mod}}_X\,\Lambda$ are defined.

We shall need the following lemma.

Lemma 1.1. *Every Λ-module M has a decomposition $M = M_X \coprod X_M$ such that X_M is a maximal direct summand of M which belongs to* $\operatorname{Add}X$.

Proof. The union of a chain of pure submodules is again a pure submodule. Also, $\operatorname{Add}X$ is closed under taking direct limits and every module in $\operatorname{Add}X$ is pure-injective since X is endofinite [10, Corollary 9.8]. Thus, by Zorn's lemma there is a maximal pure submodule X_M of M contained in $\operatorname{Add}X$ which is a direct summand. □

The next proposition collects some basic properties of a stable equivalence between two artin algebras.

Proposition 1.2. *An equivalence* $f\colon \underline{\mathrm{mod}}_X \Lambda \to \underline{\mathrm{mod}}_Y \Gamma$ *has the following properties:*

(1) f *extends to an equivalence* $f'\colon \underline{\mathrm{Pinj}}_X \Lambda \to \underline{\mathrm{Pinj}}_Y \Gamma$.

(2) f *induces a bijection* $\operatorname{Zsp}\Lambda \setminus \operatorname{add} X \to \operatorname{Zsp}\Gamma \setminus \operatorname{add} Y$, $M \mapsto M_f$, *such that* $f'M \simeq M_f$ *in* $\underline{\mathrm{Pinj}}_Y \Gamma$.

(3) $M \in \operatorname{Zsp}\Lambda \setminus \operatorname{add} X$ *is a generic module if and only if* M_f *is generic.*

Proof. (1) The canonical functor $\operatorname{mod}\Lambda \to \underline{\mathrm{mod}}_X \Lambda$ extends uniquely to a functor $p_\Lambda\colon \operatorname{Mod}\Lambda \to \varinjlim \underline{\mathrm{mod}}_X \Lambda$ which commutes with direct limits. Here, $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ denotes the category $\operatorname{Flat}((\underline{\mathrm{mod}}_X \Lambda)^{\mathrm{op}}, \mathrm{Ab})$ of flat functors $(\underline{\mathrm{mod}}_X \Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ which is characterized by the following three properties [12]:

(i) $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ contains, up to equivalence, $\underline{\mathrm{mod}}_X \Lambda$ as a full subcategory.

(ii) $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ is an additive category with direct limits.

(iii) Every object in $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ is a direct limit of objects in $\underline{\mathrm{mod}}_X \Lambda$.

It follows from Proposition 2.2 and Proposition 3.2 in [12] that p_Λ induces an equivalence $\underline{\mathrm{Pinj}}_X \Lambda \to \operatorname{Pinj}(\varinjlim \underline{\mathrm{mod}}_X \Lambda)$ where the notion of purity in $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ is defined as for $\operatorname{Mod}\Lambda$. The assertion in (1) is now a consequence since f extends to an equivalence $\varinjlim \underline{\mathrm{mod}}_X \Lambda \to \varinjlim \underline{\mathrm{mod}}_Y \Gamma$.

(2) Every indecomposable pure-injective module has a local endomorphism ring. Therefore the canonical functor $\operatorname{Mod}\Lambda \to \underline{\mathrm{Mod}}_X \Lambda$ induces an injective map from $\operatorname{Zsp}\Lambda \setminus \operatorname{add} X$ into the set of isomorphism classes of indecomposable objects in $\underline{\mathrm{Pinj}}_X \Lambda$. The map is surjective by the preceding lemma. Using (1) we find for every $M \in \operatorname{Zsp}\Lambda \setminus \operatorname{add} X$ a module M_f such that $f'M$ and M_f are isomorphic in $\underline{\mathrm{Pinj}}_Y \Gamma$. It follows that the assignment $M \mapsto M_f$ gives a bijection between $\operatorname{Zsp}\Lambda \setminus \operatorname{add} X$ and $\operatorname{Zsp}\Gamma \setminus \operatorname{add} Y$.

(3) An object N in $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ is said to be *endofinite* if $\operatorname{Hom}(A, N)$ is of finite length over $\operatorname{End}(N)^{\mathrm{op}}$ for all $A \in \underline{\mathrm{mod}}_X \Lambda$. Given a Λ-module M, it is easily checked that $p_\Lambda M$ is endofinite if M is endofinite. Also the converse is true: if $p_\Lambda M$ is endofinite, then M is endofinite [12, Corollary 3.3]. Moreover, $p_\Lambda M$ belongs to $\underline{\mathrm{mod}}_X \Lambda$ iff M is finitely presented. The assertion now follows from the construction of the bijection $M \mapsto M_f$. □

We now combine the preceding proposition with a result of Crawley-Boevey. Recall that an artin algebra Λ is said to be of *strongly unbounded representation type* if, for infinitely many $n \in \mathbb{N}$, there are infinitely many non-isomorphic indecomposable Λ-modules of length n.

Theorem 1.3. *Let* Λ *and* Γ *be artin* R*-algebras over a commutative local artinian ring* R *such that* $R/\operatorname{rad} R$ *is infinite. Suppose there exist an equivalence between* $\underline{\mathrm{mod}}_X \Lambda$ *and* $\underline{\mathrm{mod}}_Y \Gamma$ *for some modules* $X \in \operatorname{mod}\Lambda$ *and* $Y \in \operatorname{mod}\Gamma$. *Then* Λ *is of strongly unbounded representation type if and only if* Γ *is of strongly unbounded representation type.*

Proof. In [4, Theorem 7.3], Crawley-Boevey proved that an artin R-algebra is of strongly unbounded representation type iff there exists a generic Λ-module. The assertion therefore follows from Proposition 1.2. □

2. Continuous pure-injective modules

It is well-known that an artin algebra Λ is of finite representation type if and only if every pure-injective Λ-module is a coproduct of indecomposable modules [2]. In this section we exhibit the following weaker conditions:

(i) Any pure-injective Λ-module is the pure-injective envelope of a coproduct of indecomposable pure-injective modules.

(ii) Any non-zero pure-injective Λ-module has an indecomposable direct summand.

(iii) There are no continuous pure-injective Λ-modules.

The first two conditions are equivalent and (iii) is a reformulation of (ii) since a non-zero pure-injective module M is said to be *continuous* if M has no indecomposable direct summand. We refer the reader to [13, 9] for a detailed discussion of these conditions. For example, a hereditary artin algebra is of finite or tame representation type iff (i) – (iii) hold [9]. Our main result in this section is the following.

Theorem 2.1. *Let Λ and Γ be artin algebras. Suppose there exist an equivalence between $\underline{\mathrm{mod}}_X\,\Lambda$ and $\underline{\mathrm{mod}}_Y\,\Gamma$ for some modules $X \in \mathrm{mod}\,\Lambda$ and $Y \in \mathrm{mod}\,\Gamma$. Then Λ has no continuous pure-injective modules if and only if Γ has no continuous pure-injective modules.*

Proof. Denote by $f\colon\ \underline{\mathrm{mod}}_X\,\Lambda \to \underline{\mathrm{mod}}_Y\,\Gamma$ an equivalence which extends to an equivalence $f'\colon\ \underline{\mathrm{Pinj}}_X\,\Lambda \to \underline{\mathrm{Pinj}}_Y\,\Gamma$ by Proposition 1.2. Suppose there are no continuous pure-injective Λ-modules and let $N \neq 0$ be a pure-injective Γ-module. We claim that N has an indecomposable direct summand. This is clear if $N \in \mathrm{Add}\,Y$. Otherwise there exists a pure-injective Λ-module $M \neq 0$ having no non-zero direct summand in $\mathrm{Add}\,X$ such that $f'M \simeq N$. By assumption, there exists a decomposition $M = M_1 \coprod M_2$ with M_1 indecomposable. Let $N_1 = (M_1)_f$. Using the equivalence f', we find maps $\varphi\colon N_1 \to N$ and $\psi\colon N \to N_1$ in $\mathrm{Mod}\,\Gamma$ such that $\mathrm{id}_{N_1} - \psi\circ\varphi$ factors through $\mathrm{Add}\,Y$. Thus $\mathrm{id}_{N_1} - \psi\circ\varphi \in \mathrm{rad}\,\mathrm{End}_\Gamma(N_1)$ and therefore φ is a split monomorphism since $\mathrm{End}_\Gamma(N_1)$ is local. It follows that N has an indecomposable direct summand. □

3. Krull-Gabriel dimension

In this section we study the Krull-Gabriel dimension of an artin algebra Λ which Geigle introduced in his thesis [7]. In order to define this concept we recall a few definitions from [6].

Let $\mathcal{C}$ be an abelian category. For example, let $\mathcal{C} = \mathrm{fp}(\mathrm{mod}\,\Lambda, \mathrm{Ab})$ be the category of finitely presented functors $\mathrm{mod}\,\Lambda \to \mathrm{Ab}$ into the category Ab of abelian groups. Recall that a funcor $F\colon \mathcal{A} \to \mathrm{Ab}$ is *finitely presented* if there exist a presentation

$$\mathrm{Hom}(Y, -) \longrightarrow \mathrm{Hom}(Y, -) \longrightarrow F \longrightarrow 0.$$

A full subcategory $\mathcal{S}$ of $\mathcal{C}$ is a *Serre subcategory* provided that for any exact sequence $0 \to X' \to X \to X'' \to 0$ the object X belongs to $\mathcal{S}$ iff X' and X'' belong to $\mathcal{S}$. The corresponding *quotient category* $\mathcal{C}/\mathcal{S}$ is again abelian; it is constructed as follows. The objects of $\mathcal{C}/\mathcal{S}$ are those of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}/\mathcal{S}}(X, Y) = \varinjlim \mathrm{Hom}_{\mathcal{C}}(X', Y/Y')$ with $X' \subseteq X,\ Y' \subseteq Y$ and $X/X', Y' \in \mathcal{S}$. The *Krull-Gabriel dimension* $\mathrm{KGdim}\,\mathcal{C}$ of $\mathcal{C}$ is the least ordinal κ (or ∞ if such an ordinal does not exist) such that there is a filtration

$$0 = \mathcal{C}_{-1} \subseteq \mathcal{C}_0 \subseteq \ldots \subseteq \mathcal{C}_\kappa = \mathcal{C}$$

of Serre subcategories $\mathcal{C}_\alpha \subseteq \mathcal{C}$, indexed by the ordinals $\alpha \leq \kappa$ and satisfying

(i) $\mathcal{C}_a/\mathcal{C}_\beta$ is an abelian length category, if $\alpha = \beta + 1$;

(ii) $\mathcal{C}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{C}_\gamma$, if α is a limit ordinal.

In particular, $\mathrm{KGdim}\,\Lambda = \mathrm{KGdim}\,\mathrm{fp}(\mathrm{mod}\,\Lambda, \mathrm{Ab})$ is the Krull-Gabriel dimension of Λ. Note that $\mathrm{KGdim}\,\Lambda$ equals the m-dimension of the lattice of finitely generated subfunctors of the forgetful functor $\mathrm{Hom}_\Lambda(\Lambda, -)$ in the sense of [13]. We list some of the known properties of this numerical invariant.

Example 3.1. (1) $\operatorname{KGdim}\Lambda = 0$ iff Λ is of finite representation type [1].

(2) $\operatorname{KGdim}\Lambda = 1$ is impossible for any artin algebra [11, 8].

(3) $\operatorname{KGdim}\Lambda = 2$ holds for any tame hereditary artin algebra [7].

(4) $\operatorname{KGdim}\Lambda < \infty$ implies that there are no continuous pure-injective Λ-modules [9].

(5) If Λ is a finite dimensional algebra over a field, then several examples suggest that $\operatorname{KGdim}\Lambda < \infty$ is closely related to Λ being of tame domestic representation type.

We continue with two lemmas which are essential for the main result of this section.

Lemma 3.2. *Let $X \in \operatorname{mod}\Lambda$. The canonical functors $i\colon \operatorname{add} X \to \operatorname{mod}\Lambda$ and $p\colon \operatorname{mod}\Lambda \to \underline{\operatorname{mod}}_X \Lambda$ have the following properties:*

(1) *The assignment $F \mapsto F \circ p$ induces an equivalence between $\operatorname{fp}(\underline{\operatorname{mod}}_X \Lambda, \operatorname{Ab})$ and the Serre subcategory $\mathcal{S}$ of all functors in $\operatorname{fp}(\operatorname{mod}\Lambda, \operatorname{Ab})$ vanishing on $\operatorname{add} X$.*

(2) *The assignment $F \mapsto F \circ i$ induces an equivalence between $\operatorname{fp}(\operatorname{mod}\Lambda, \operatorname{Ab})/\mathcal{S}$ and $\operatorname{fp}(\operatorname{add} X, \operatorname{Ab})$.*

Proof. The assertion follows from [10, Lemma 5.2], using the fact that $\operatorname{add} X$ is a covariantly finite subcategory of $\operatorname{mod}\Lambda$. □

Lemma 3.3. $\operatorname{KGdim}\operatorname{fp}(\underline{\operatorname{mod}}_X \Lambda, \operatorname{Ab}) = \operatorname{KGdim}\operatorname{fp}(\operatorname{mod}\Lambda, \operatorname{Ab})$ *if* $\operatorname{add} X \neq \operatorname{mod}\Lambda$.

Proof. We apply the preceding lemma and identify $\operatorname{fp}(\underline{\operatorname{mod}}_X \Lambda, \operatorname{Ab})$ with the Serre subcategory $\mathcal{S}$ of all functors in $\mathcal{C} = \operatorname{fp}(\operatorname{mod}\Lambda, \operatorname{Ab})$ vanishing on $\operatorname{add} X$. The functor

$$F = \operatorname{Hom}_\Lambda(X, -)/\operatorname{rad}\operatorname{Hom}_\Lambda(X, -)$$

is finitely presented and of finite length [3]. Moreover, the Serre subcategory of $\mathcal{C}$ generated by $\mathcal{S}$ and F coincides with $\mathcal{C}$ since $\mathcal{C}/\mathcal{S}$ is generated by F. If $(\mathcal{C}_a)_\alpha$ is a filtration of $\mathcal{C}$ satisfying (i) – (ii), then $(\mathcal{C}_\alpha \cap \mathcal{S})_\alpha$ is a filtration of $\mathcal{S}$ satisfying (i) – (ii). Conversely, let $(\mathcal{S}_\alpha)_\alpha$ be a filtration of $\mathcal{S}$ satisfying (i) – (ii). Denote for each α by $\mathcal{C}_\alpha$ the Serre subcategory of $\mathcal{C}$ which is generated by $\mathcal{S}_\alpha$ and F. It is easily checked that the filtration $(\mathcal{C}_a)_\alpha$ satisfies (i) – (ii). Therefore $\operatorname{KGdim}\mathcal{S} = \operatorname{KGdim}\mathcal{C}$. □

We are now in a position to prove our promised result about the Krull-Gabriel dimension.

Theorem 3.4. *Let Λ and Γ be artin algebras. Suppose there exist an equivalence between $\underline{\operatorname{mod}}_X \Lambda$ and $\underline{\operatorname{mod}}_Y \Gamma$ for some modules $X \in \operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}\Gamma$. Then the Krull-Gabriel dimensions of Λ and Γ coincide.*

Proof. We have $\operatorname{add} X = \operatorname{mod}\Lambda$ iff $\operatorname{add} Y = \operatorname{mod}\Gamma$, and in this case

$$\operatorname{KGdim}\Lambda = 0 = \operatorname{KGdim}\Gamma$$

since both algebras are of finite representation type. Otherwise the assertion follows from Lemma 3.3. □

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