Higher dimensional isoperimetric inequalities and uncombability of the Hilbert modular group

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Abstract

We prove the uncombability of the Hilbert modular group Γ *, corresponding to a totally real number field* K *of degree* ≥ 2.

Introduction

It has been an open problem which S-arithmetic groups and lattices in Lie groups are (bi)combable or (bi)automatic (for precise definitions see [ECHLPT]). Note the following diagram of inclusions between the classes $(\mathcal{B})\mathcal{C}, (\mathcal{B})\mathcal{A}$ of (bi)combable and (bi)automatic groups:

$$\begin{array}{ccc} \mathcal{C} & \supset & \mathcal{BC} \\ \cup & & \cup \\ \mathcal{A} & \supset & \mathcal{BA} \end{array}$$

It is not known whether or not any of the inclusions are equalities, "... although it is widely believed that there are combable groups which are not automatic"[Fa].

Thurston-Epstein have shown [ECHLPT] that $\mathrm{SL}_n(\mathbb{Z})$ is not combable for $n > 2$. (A purely algebraic proof of the fact that $\mathrm{SL}_n(\mathbb{Z})$ is not bicombable for $n > 2$ is contained in [BV]). Using a different method, Gersten-Short have shown [GS], [Sh] that $\mathrm{SL}_2(\mathcal{O})$ is not bicombable, where $\mathcal{O}$ is the ring of integers in the real quadratic number field $\mathbb{Q}(\sqrt{d})$. In [Pi] it is proven, that these groups are not combable.

Every lattice in $SO(n,1)$ is biautomatic. This follows from Epstein's result that geometrically finite hyperbolic groups are biautomatic [ECHLPT] and from the fact that lattices in $SO(n,1)$ are geometrically finite [Bo].

In [Fa] the complete solution of the bicombability problem is given in case of irreducible lattices in semisimple linear Lie groups. Namely, an irreducible lattice in a semisimple linear Lie group $G \neq SO(n,1)$ is bicombable if and only if it is cocompact.

The aim of this note is to prove the following

Theorem. *Let $\mathcal{O}$ be the ring of integers of the totally real field K of degree n over $\mathbb{Q}$. Then, for $n \geq 2$ the group $\mathrm{SL}_2(\mathcal{O})$ is not combable in the sense of* [ECHLPT]. *In particular it is not automatic .*

We combine approaches of [ECHLPT] and [Pi], where, resectively, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$ and $\mathrm{SL}_2(\mathcal{O})$, $\mathcal{O}$ real and quadratic, had been considered.

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Proof

1 Higher isoperimetric inequalities for combable groups

The following theorem provides the main tool for proving uncombability (for the definitions see below).

Theorem (see th.10.3.5 in [ECHLPT]). *Let Γ be a combable group of isometries acting properly and cocompactly on a contractible riemannian manifold M. Then there is a constant k with the following property. Let b be a Lipschitz $(n-1)$-cycle in $M, n \geq 1$. Then there exists a Lipschitz n-chain c, such that $\partial c = b$ and*

$$\mathrm{mass}_n\ c \leq k \cdot \mathrm{diam}\ b \cdot \mathrm{mass}_{n-1}\ b.$$

Corollary. *Let Γ be a group of isometries acting properly and cocompactly on a contractible riemannian manifold M. Suppose that for a fixed natural n and for each natural m we can find in M a Lipschitz $(n-1)$-cycle b_m (depending on m), whose mass and diameter are bounded above by a polynomial in m. Suppose that for any choice of Lipschitz n-chains c_m in M with $\partial c_m = b_m$ the mass of c_m grows exponentially with m. Then Γ is not combable.*

2 Epstein-Thurston's construction of cycles in a product of hyperbolic halfplanes

2.1 H^n

Let $H = \{z \in \mathbb{C} : z = x + yi, y > 0\}$ be the upper half plane with coordinates x, y and with the metric $\frac{1}{y^2}(dx^2 + dy^2)$. One can also consider H as a solvable Lie group identifying (x, y) with the matrix

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$

Let H^n be a direct product of n copies of H. We will use the coordinates $(y, x) = (y_1, \ldots, y_n, x_1, \ldots, x_n)$ on H^n and where appropriate the coordinates $t_i = \ln y_i$. A left-invariant metric on H^n is given by $ds^2 = \sum_{i=1}^n \frac{1}{y_i^2}(dx_i^2 + dy_i^2) = \sum_{i=1}^n dt_i^2 + e^{-2t_i} dx_i^2$.

For any real r define the *horosphere*

$$S_\infty(r) = \{(t, x) \in H^n : \sum_1^n t_i = r\}, \tag{1}$$

and the *horoball*

$$B_\infty(r) = \{z \in H^n : \sum_1^n t_i > r\}.$$

$S_\infty(0)$ is a Lie subgroup of H^n and $S_\infty(r) = g(r/n)S_\infty(0)$, where $g(s) = (s, \ldots, s; 0, \ldots, 0)$, $s \in \mathbb{R}$ is a 1-parameter subgroup of H^n written in (t, x)-coordinates. Thus all the $S_\infty(r)$ are isometric (as Riemannian submanifolds of H^n).

2.2 The Epstein-Thurston construction

Proposition. *For any natural $n \geq 2$ and for any real r there is a sequence of Lipschitz $(n-1)$-cycles b_m in $S_\infty(r)$ satisfying the following conditions:*

1) The mass of b_m and the diameter of b_m are bounded above by a polynomial in m.

2) Let c_m be chains with $\partial c_m = b_m$ and $c_m \subset H^n - B_\infty(r)$, where $B_\infty(r) = \{(t,x) \in H^n : \sum t_i > r\}$. Then the mass of c_m increases exponentially with m.

Proof of the proposition

Constructing $b = b_m$ in $S_\infty(0)$.

Cycle $b = b_m$ is combinatorially equivalent to the outward pointing normal bundle Z to the boundary of the n-cube $[-1,1]^n$ in $\mathbb{R}^n$. The outward pointing normal vector at a point in the interior of an $(n-1)$-dimensional face of the cube is the standard unit vector. The faces of the cube $[-1,1]^n$ are in one-one correspondence with the partitions $[1,n] = I \cup J \cup L$ with nonempty $I \cup J$, namely the face f_{IJ} is defined by

$$f_{IJ} = \{(x_1, \ldots, x_n) : x_i = 1, x_j = -1, x_l \in [-1,1], i \in I, i \in J, l \in L\}.$$

Any convex linear combination of the various basis vectors e_i for $i \in I$ and $-e_j$ for $j \in J$ is defined to be a normal vector at $x \in f_{IJ}$. The normal vectors based at x form a simplex with vertices the points e_i and $-e_j$. The corresponding cell Z_{IJ} of Z is a product of a simplex of dimension $|I| + |J| - 1$ and a cube of dimension $n - |I| - |J|$, so it has dimension $n-1$. We now embed Z in $S_\infty(0)$ to obtain the cycle $b = b_m$. We do this by embedding each face separately. Recall from (1) that $S_\infty(0) = S_0 \times \mathbb{R}^n$, where $S_0 = \{t \in \mathbb{R}^n; \sum t_i = 0\}$. The image b_{IJ} of the face Z_{IJ} of Z in $S_\infty(0)$ corresponds to the product of a simplex $\triangle_{IJ} \subset S_0$ with a cube $\square_{IJ} \subset \mathbb{R}^n$. The simplex $\triangle_{IJ}$ is defined by

$$\triangle_{IJ} = \text{convex hull } \{v_k | k \in I \cup J\},$$

where

$$v_k = (m, \ldots, m, \underbrace{-m(n-1)}_{k}, m, \ldots, m)$$

and the cube $\Box_{IJ}$ is defined by $\Box_{IJ} = e^m f_{IJ}$. Finally we define $b_{IJ} = \triangle_{IJ} \times \Box_{IJ}$ and $b = \bigcup b_{IJ}$. The correspondence $Z_{IJ} \to b_{IJ}$ preserves the adjacency relations between the faces Z_{IJ}, thus defining a homeomorphism between Z and b which is Lipschitzian (even smooth on each Z_{IJ}).

We assert that b is the cycle in question.

2.2.1. Lemma. *The mass of b_m grows polynomially with m.*

Proof. Recall the definition of the mass. To count the mass of a singular (Lipshitz) simplex $f : \Delta^k \to M$ one should integrate over the Δ^k the *k − dimensional Jacobian* $V(x)$ of f, which can be defined as follows. The derivative D_x exists almost everywhere and if this is the case then D_x sends an orthonormal basis at x to a k-tuple of vectors tangent to M at $f(x)$. This k-tuple defines a parallelopiped in the tangent space $T_x M$, which has a k-volume $V(x) \geq 0$. The mass of a chain is defined by linearity. In our case we parametrize $b_{IJ} = \triangle_{IJ} \times \Box_{IJ}$ "by itself" and the volume $V(t,x)$ at the point $(t,x) \in \triangle_{IJ} \times \Box_{IJ}$ is equal to $e^{-2\sum_{l \in L} t_l} = e^{-2|L|m}$. Consequently,

$$\text{mass } b_{IJ} = e^{-2|L|m} \times (\text{euclidean volume of } b_{IJ}) =$$

$$e^{-2|L|m} \times (\text{euclidean volume of } \Box_{IJ}) \times (\text{euclidean volume of } \triangle_{IJ}) =$$

$$(\text{euclidean volume of } \triangle_{IJ}).$$

and the last volume grows as $m^{dim\triangle_{IJ}}$.

2.2.2. Lemma. diam b_{IJ} *grows polynomially with* m.

Proof. Any two points of b_{IJ} can be joined by a piecewise linear path such that for each linear part either the t-coordinates or the x-coordinates are constant. If the x-coordinates are constant then the length of the path coincides with the euclidean length and bounded above by the euclidean diameter of $\triangle_{IJ}$. If the t-coordinates are constant then the path is parallel to some x_l-axis for $l \in L$, so the t_l-coordinate of any point of the path is equal to m. Then the length of the path is bounded above by $\int\limits_{-e^m}^{e^m} \frac{dx_l}{e^m} = 2$. □

Finding c with $\partial c = b$

Let $\triangle^0_{IJ}$ be the convex hull of the origin and $\triangle_{IJ}$ and let $c_{IJ} = \triangle^0_{IJ} \times \square_{IJ}$. The boundary of $\bigcup_{I,J} c_{IJ}$ consists of b and the cube $c_0 = [-e^m, e^m]^n$. To complete the n-chain $\bigcup_{I,J} c_{IJ}$ to a chain c whose boundary is b we fill in the cube c_0.

Other c_m in $S_\infty(0)$ with $\partial c_m = b_m$

2.2.3. Lemma. *The (closed) n–form $\omega = dx_1 \wedge \ldots \wedge dx_n$ on H^n (written in (t,x)-coordinates) is bounded on $S_\infty(r)$.*

Proof. Recall that the norm of the form ω on the manifold M is

$$||\omega|| = \sup\{|\omega(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n)| \ ; \ \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n \in T_xM, |\boldsymbol{\xi}_1|, \ldots, |\boldsymbol{\xi}_n| \leq 1, x \in M\}.$$

Let

$$\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{in}; \eta_{i1}, \ldots, \eta_{in}), i = 1, \ldots, n,$$

be tangent vectors at $(t, x) \in S_\infty(r)$ of norm ≤ 1. This means that

$$\boldsymbol{\xi}_i^2 = \xi_{i1}^2 + \ldots + \xi_{in}^2 + e^{-2t_1}\eta_{i1}^2 + \ldots + e^{-2t_n}\eta_{in}^2 \leq 1 \tag{2}$$

and

$$t_1 + \ldots + t_n = r.$$

Now

$$\omega(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) = (dx_1 \wedge \ldots \wedge dx_n)(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) = \det_{1 \leq i,j \leq n} |\eta_{ij}| =$$

$$e^r \det_{1 \leq i,j \leq n} |e^{-t_i}\eta_{ij}|.$$

But the rows of the determinant $\det_{1 \leq i,j \leq n} |e^{-t_i}\eta_{ij}|$ are of euclidean norm ≤ 1 in view of (2), hence the whole expression is not larger than e^r. □

2.2.4. Lemma. *ωc_m grows exponentially with m.*

Proof. First note that $\omega c_{IJ} = 0$. Indeed, recall that $c_{IJ} = \triangle_{IJ}^0 \times \square_{IJ}$ and that $\square_{IJ}$ lies in a hyperplane $x_i = const$ for $i \in I \cup J$, thus the integration with respect to x_i gives zero. It remains to integrate ω over the cube c_0 which obviously gives just the euclidean volume of this cube, that is $2^n e^{mn}$. □

2.2.5. Lemma. *Let c'_m be some sequence of chains in $S_\infty(0)$ with $\partial c'_m = b_m$, then* mass c'_m *grows exponentially with m.*

Proof. Since ω is closed and $S_\infty(0)$ is contractible, the value of ω on each n-chain with boundary b_m is the same, in particular $\omega c'_m = \omega c_m$. Next, ω is of norm ≤ 1, so $\omega c'_m \leq$ mass c'_m and the lemma follows from the exponential growth of ωc_m just proven. □

Completion of the proof of the proposition.

First notice that left invariance gives us the construction of b_m in any horosphere $S_\infty(r)$ with the properties proven above. Fix a real r and consider the corresponding sequence of cycles b_m in $S_\infty(r)$. Let $M_0 = H^n - B_\infty(r)$ be the submanifold of H^n obtained by removing the horoball $B_\infty(r)$ from H^n. Let c'_m be some sequence of chains in M_0 with $\partial c'_m = b_m$. We have to prove that mass c'_m grows exponentially with m. We need the following

2.2.6. Lemma. *The projection $\pi : M_0 \to S_\infty(r)$ along geodesics orthogonal to the horosphere $S_\infty(r)$ does not increase distances and consequently does not increase the mass.*

Proof. Note first that the curve $g(s) = (s, \dots, s; x_1, \dots, x_n), s \in \mathbb{R}$ is a geodesic in H^n. Indeed, in the hyperbolic plane H with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) = dt^2 + e^{-2t}dx^2,\ t = \ln y,$$

the vertical lines $t \mapsto (t, x)$ are geodesic. Since the metric on H^n is the euclidean sum of metrics on the components we conclude that the diagonal curve $g(s) = (s, \dots, s; x_1, \dots, x_n), s \in \mathbb{R}$ is geodesic in H^n.

The projection $\pi(t, x)$ of the point

$$(t, x) = (t_1, \dots, t_n; x_1, \dots, x_n),\ \sum_{i=1}^{n} t_i = t < r$$

onto the horosphere $S_\infty(r)$ along the geodesic $g(s)$ is

$$(t_1 + s, \dots, t_n + s; x_1, \dots, x_n),\ s = \frac{r - t}{n}.$$

The metric in (t, x) is

$$ds^2 = \sum_{i=1}^{n} dt_i^2 + e^{-2t_i}dx_i^2,$$

and in $\pi(t,x)$ it is

$$ds^2 = \sum_{i=1}^{n} dt_i^2 + e^{-2t_i - 2s} dx_i^2,$$

thus the projection does not increase the metric. $\square$

It follows from the lemma 2.2.4 that the mass of $\pi c'_m$ grows exponentially with m and we conclude that the mass of c'_m also grows exponentially with m. This finishes the proof of the proposition.

3 Proof of the theorem

We will find a cocompact proper action of $\Gamma = \mathrm{SL}_2(\mathcal{O})$ on some submanifold $M \subset H^n$ and apply the Epstein-Thurston criterion to prove that Γ is not combable. We need some standard facts from reduction theory [vdG].

3.1 Action of Γ on H^n and reduction theory

The general reference here is [vdG]. Let K be a totally real number field of degree $n = [K : \mathbb{Q}]$ and let $\mathcal{O}$ be its ring of integers. The n distinct embeddings $\sigma_i : K \to \mathbb{R}$ of K into $\mathbb{R}$ induce an embedding $\sigma : K \to \mathbb{R}^n$ and an embedding of Γ into $\mathrm{SL}_2(\mathcal{O})^n$. The image of $\mathcal{O}$ under σ in $\mathbb{R}^n$ is a lattice of rank n and the image of Γ is a discrete subgroup in $\mathrm{SL}_2(\mathbb{R})^n$ and therefore Γ acts properly and isometrically on H^n.

The action of Γ on H^n is not cocompact and reduction theory permits us to find a Γ-invariant contractible submanifold M with boundary, obtained by removing from H^n a disjoint union of open horoballs such that the action of Γ on M is cocompact and proper.

The n embeddings of K into the reals induce an embedding of the projective line $\mathbb{P}(K)$ into the product $\mathbb{P}(\mathbb{R})^n$ of n copies of $\mathbb{P}(\mathbb{R})$. The linear action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ induces the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{P}(\mathbb{R})$ by fractional linear transformations and so does $\mathrm{PSL}_2(\mathrm{K})$ on $\mathbb{P}(\mathbb{R})^n$. Associated to each nonzero ideal $A = \mathcal{O}\alpha + \mathcal{O}\beta$ is a *horofunction*

$$\mu_l(z) = \mathrm{Norm}(A) \prod_{i=1}^{n} \frac{\mathrm{Im}\ z_i}{|\sigma_i(\alpha) - \sigma_i(\beta) z_i|^2}$$

which depends only on $l = (\alpha : \beta) \in \mathbb{P}(K)$. Define the *horoball* at l

$$B_l(r) = \{z \in H^n : \ln \mu_l(z) > r\}$$

and *horosphere*

$$S_l(r) = \{z \in H^n : \ln \mu_l(z) = r\}$$

In particular for $l = \infty$ we have $\mu_\infty(z) = \prod_{i=1}^n y_i,\ y_i = \mathrm{Im} z_i$ and the definition coincides with that given in 2.1.

The function μ is Γ-invariant in the sense that $\mu_{gl}(gz) = \mu_l(z)$ for all $g \in \Gamma$. In particular Γ permutes the horoballs of the same radius.

3.1.1. Lemma. *For sufficiently big r the submanifold*

$$M = H^n - \bigcup_{l \in \mathbb{P}(K)} B_l(r)$$

is contractible and $\mathrm{SL}_2(\mathcal{O})$ *acts on it properly and with compact quotient.*

Proof. This is well known fact and we just want to give precise references. The properness was already noticed at the beginning of this subsection. To provide a cocompact action we first note that for sufficiently big r the horoballs $B_l(r)$ and $B_{l'}(r)$ do not intersect for distinct l, l', [vdG], lemma I.2.1. Increasing r we may assume that the closed horoballs of the radius r also do not intersect, hence the set $M = H^n - \bigcup_{l\in\mathbb{P}(K)} B_l(r)$ is a submanifold in H^n. Yet further increasing r enables us to assume that for all cusps $l \in \mathbb{P}(K)$ the horoball $B_l(r)$ is contained in the "ball of influences"

$$F_l = \{(t,x) \in H^n; \mu_l(z) \geq \mu_{l'}(z) \text{ for all } l' \neq l\},$$

see [vdG], p.8. Then the factor space of $H^n - \bigcup_{l\in\mathbb{P}(K)} B_l(r)$ modulo $\mathrm{SL}_2(\mathcal{O})$ is a compact manifold with boundary [vdG], p.10, thus the cocompactness of the action.

Finally to prove contractibility first note that H^n is homeomorphic to $\mathbb{R}^{2n}$ so it is contractible and to prove the contractibility of $M = H^n - \bigcup_{l\in\mathbb{P}(K)} B_l(r)$ it is enough to construct a deformation retraction of H^n onto M. The deformation retraction ϕ_∞ of H^n onto $H^n - B_\infty(r)$ is given by

$$(t,x) \mapsto (1-\tau)t + \tau\frac{rt}{\sum t_i}, 0 \leq \tau \leq 1,$$

for $(t,x) \in B_\infty(r)$ and ϕ_∞ identical on $H^n - B_\infty(r)$. Since $\mathrm{PSL}_2(\mathrm{K})$ acts transitively on $\mathbb{P}(K)$ one can define the deformation retraction ϕ_l of H^n onto $H^n - B_l(r)$ for every $l \in \mathbb{P}(K)$ by equivariance. Now define the deformation retraction of H^n onto $M = H^n - \bigcup_{l\in\mathbb{P}(K)} B_l(r)$ to be identity on $M = H^n - \bigcup_{l\in\mathbb{P}(K)} B_l(r)$ and to be ϕ_l on $B_l(r), l \in \mathbb{P}(K)$. □

3.2 Γ is not combable

We apply the Epstein-Thurston theorem for the action of Γ on M. So suppose that Γ is combable. Take the cycles $b_m \subset S_\infty(r)$ constructed in 2.2. Then the theorem of Epstein-Thurston shows that b_m is the boundary of an n-chain c'_m in M whose mass is bounded above by a polynomial in m. In view of lemma 2.2.6 the mass of the chain $\pi c'_m \subset S_\infty(r)$ also grows polynomially with m. This contradicts lemma 2.2.5. This finishes the proof of the theorem.

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