# On the Hajós number of graphs

G. Gutin*
Odense University
Odense, Denmark
and
Brunel University of West London
Uxbridge, Middlesex, UK
z.g.gutin@brunel.ac.uk

A. V. Kostochka†
Novosibirsk State University
Novosibirsk, Russia
sasha@math.nsc.ru

B. Toft‡
Odense University
Odense, Denmark
btoft@imada.ou.dk

January 31, 1997

**Abstract**

A graph $G$ is said to have property $P_m$ if it contains no subdivision of $K_{m+1}$ and no subdivision of $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$. Chartrand, Geller and Hedetniemi [3] (see also Problem 6.3 in [11]) conjectured that the set of vertices (respectively, edges) of any graph with property $P_m$ can be partitioned into $m-n+1$ subsets such that each of these subsets induces a graph with property $P_n$, provided $m \geq n \geq 1$ (respectively, $m \geq n \geq 2$). We prove that both conjectures fail when $m > cn^2$ for some positive constant $c$. In fact, we prove that under the condition $m > cn^2$, there exists a graph $G$ with property $P_m$ such that in every colouring of its vertices or edges with $m$ colours there is a monochromatic subgraph $H$ with Hajós number $h(H) > n$, that is, with a subdivision of $K_{n+1}$. In addition, we prove bounds of Nordhaus-Gaddum type for the Hajós number.

# 1 Introduction

For a graph $G$, let *the Hajós number* $h(G)$ denote the maximum $k$ such that $G$ contains a subdivision of $K_k$.

*This work was partially supported by the grant for discrete mathematics of the Danish Natural Science Research Council.

†This author finished his part of work during his stay at SFB343 "Diskrete Strukturen in der Mathematik" of Bielefeld University. His work was also partially supported by the grant 96-01-01614 of the Russian Foundation for Fundamental Research.

‡This work was partially supported by the grant for discrete mathematics of the Danish Natural Science Research Council.

For a positive integer $m$, Chartrand, Geller and Hedetniemi [3] denote by $P_m$ the property of a graph $G$ to have $h(G) \leq m$ and not to contain a subdivision of the complete bipartite graph $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$ either. Let $f_1(m,n)$ (respectively, $f_2(m,n)$) denote the minimum $k$ such that for any graph $G$ with property $P_m$, there exists a partition of its vertices (respectively, edges) into $k$ sets such that each set induces a subgraph with property $P_n$. The conjecture in [3] (see also Problem 6.3 in [11]) that $f_1(m,n) \leq m-n+1$ for each $m \geq n \geq 1$ is sometimes referred to as the $(m,n)$-conjecture, and the conjecture that $f_2(m,n) \leq m-n+1$ for each $m \geq n \geq 2$ as the $[m,n]$-conjecture. Chartrand, Geller and Hedetniemi proved the $(m,n)$-conjecture for all $n$ and $m$ satisfying $1 \leq n \leq m \leq 4$ except for the case $(4,1)$ which is the 4CT. They also proved the $[m,n]$-conjecture for all $n$ and $m$ satisfying $2 \leq n \leq m \leq 4$, except the case $[4,3]$. The $[4,3]$-conjecture was proved by Heath [10]. The $(m,n)$- and $[m,n]$-conjectures were mentioned in the excellent survey [19]. Woodall [19] remarked that perhaps the corresponding conjectures in terms of contractions, rather than subdivisions, are more likely to be true.

Indeed, Jørgensen [12], and Hanson and Toft [9] proved that for each $n$ the $(m,n)$-conjecture fails for almost all graphs. They used the ideas of Erdős and Fajtlowicz [5] who showed that for almost all graphs on $p$ vertices, the Hajós number is at most $(2+\epsilon)\sqrt{p}$. The first aim of this paper is to show that both the $(m,n)$-conjecture and the $[m,n]$-conjecture are not true when $m > cn^2$, and that also even the contraction version of the $[m,n]$-conjecture is false for $m$ large enough with respect to $n$.

**Theorem 1** *There exists a constant $c$ such that if $m > cn^2$, then there exists a graph $G$ with property $P_m$ such that for all partitions of its vertices or of its edges into $m$ colour classes, there is a monochromatic subgraph $H$ with Hajós number $h(H) > n$.*

**Theorem 2** *For each $n > 1$, there exists an $m_0$ such that for every $m > m_0$, there exists a graph $G$ having neither $K_{m+1}$ nor $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$ as a minor such that for all partitions of its edges into $m$ colour classes at least one monochromatic subgraph $H$ has $K_{n+1}$ as a minor. (It follows that the $[m,n]$-conjecture is false, even in terms of contractions).*

The second aim of the paper is to find bounds of Nordhaus-Gaddum type for the Hajós number. Zelinka [20] conjectured that $\eta(G) + \eta(\bar{G}) \leq n+1$ for each graph $G$ on $n$ vertices, where $\eta(G)$ is the Hadwiger number of $G$. This is not true (see [15]), but for the Hajós number instead of the Hadwiger number the bound is true.

**Theorem 3** *For each positive integer $n$,*

$$\max\{h(G) + h(\bar{G}) \mid |V(G)| = n\} = n+1, \tag{1}$$

$$\max\{h(G) \times h(\bar{G}) \mid |V(G)| = n\} = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor. \tag{2}$$

Then we discuss $\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\}$ whose order of magnitude is $\sqrt{n}$, as it follows from the results of Bollobás and Thomason [2] and Komlós and Szemerédi [13, 14]. The exact value is likely to have a complicated formula. We shall give a simple proof that

**Theorem 4** *For all $n$*

$$\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\} > 0.4\sqrt{n}. \tag{3}$$

We suspect that $h(G)\times h(\bar{G}) \geq n$ for all graphs $G$ on $n$ vertices, but have no proof of this in general. Let $T$ be the set of positive integers $t$ such that any graph $G$ with $h(G)=t$ and $n$ vertices has at most $t^2n/2$ edges. Thomassen's result in [18] implies that $4 \in T$. Komlós and Szemerédi [14] proved that for some $t_0$ every $t > t_0$ is in $T$.

**Theorem 5** *Let $G$ be any graph with $h(G)=t$ and $n$ vertices. If $t \in T$ and $n \geq 4t^3$ or if $t \leq 3$, then $h(G)\times h(\bar{G}) \geq n$.*

Moreover, we can give a better bound when $h(G) \leq 2$:

**Theorem 6** *Let $G$ be a graph on $n$ vertices with $h(G) \leq 2$. Then $h(\bar{G}) \geq \lfloor \frac{2n+1}{3} \rfloor$ with two exceptions:*
*(a) $G$ is the path $P_4$ on four vertices;*
*(b) $G$ is the tree $T_7$ on 7 vertices, obtained from $K_{1,3}$ by subdividing each of its edges by one vertex.*

The bound in Theorem 6 is best possible.

## 2 Bounds of Nordhaus-Gaddum type

**Proof of Theorem 3** The only part of Theorem 3 needed to be proved is the statement that for each graph $G$ on $n$ vertices,

$$h(G)+h(\bar{G}) \leq n+1. \tag{4}$$

Indeed, the inequality $h(G)\times h(\bar{G}) \leq \lfloor (\frac{n+1}{2})^2 \rfloor$, follows from (4) and the well known inequality on arithmetic and geometric means. Both inequalities are equalities for $G$ being the disjoint union of $K_{\lceil \frac{n+1}{2} \rceil}$ and $\overline{K}_{\lfloor \frac{n-1}{2} \rfloor}$.

Suppose that for some $r$ and $b$, $K_n$ contains edge disjoint subdivisions $R$ and $B$ of $K_r$ and $K_b$, respectively. We shall refer to the edges of $R$ as "red" edges and to those of $B$ as "blue" ones. Let $VR$ and $VB$ denote the set of branching vertices in $R$ and $B$, respectively, $VRB = VR \cap VB$, $W = V(K_n) \setminus (VR \cup VB)$. We may assume that $|VRB| \geq 2$, since otherwise $r+b \leq n+1$.

If $x \in VRB$, $y \in VR$ and the path $P$ in $R$, representing the edge $xy$ is not $xy$, then the other end $z$ of the first edge $xz$ in $P$ belongs to $(VB - VRB) \cup W$. Thus, either $P$ meets $W$ or $P$ "wastes" a red edge between a vertex in $VRB$ and a vertex $z$ in $VB - VRB$, and we have a new situation in $B$ of the same type. Let $xz'$ be the first edge on the blue path $P'$ from $x$ to $z$. Then $z' \in (VR - VRB) \cup W$. Thus either $P'$ meets $W$ or $P'$ "wastes" a blue edge between $x$ and a vertex in $VR - VRB$. Continuing like this, we must end up with a vertex in $W$. Observe that no intermediate vertex $z$ belongs to $VRB$. If $y \in VRB$, we correspond the final vertex in $W$ to the pair $x, y$. Since each $w \in W$ lies on at most one red path and at most one blue path, it can be used at most twice. But for any two vertices $x$ and $y$ in $VRB$, either the red or the blue path connecting them is not an edge. So, $2|W| \geq \binom{|VRB|}{2}$, which is possible only if $|W| \geq |VRB| - 1$. This is equivalent to the inequality $r + b \leq n + 1$.

$\square$

The result of Erdős and Fajtlowicz [5] mentioned in the introduction implies that $\min\{h(G) + h(\bar{G}) \mid |V(G)| = n\} \leq 5\sqrt{n}$ for large $n$. On the other hand, Komlós and Szemerédi [13, 14] proved that $h(G) \geq \sqrt{\frac{2e}{n}}$ when $\frac{2e}{n}$ is large enough ($e = |E(G)|$). Either $G$ or $\bar{G}$ has $e \geq \frac{n(n-1)}{4}$, hence for $n$ large enough either $h(G)$ or $h(\bar{G})$ is at least $\sqrt{\frac{n-1}{2}}$. Thus the order of magnitude of $\min\{h(G) + h(\bar{G}) \mid |V(G)| = n\}$ is $\sqrt{n}$. We shall not use the results of Komlós and Szemerédi in this connection. Instead, with the help of the following simple lemma, which will also be used in the next section, we shall see that in fact for *all* $n$, $\min\{h(G) + h(\bar{G}) \mid |V(G)| = n\}$ is not much less than $\sqrt{n}$.

**Lemma 1** *Let $|V(G)| = n$ and $|E(G)| \geq 0.24(n-1)n$. Then $h(G) \geq 0.4\sqrt{n} - 1.5$.*

**Proof:** Let $A = \{v \in V(G) \mid deg_G(v) \geq 0.43(n-1)\}$ and $a = |A|$. Then $0.43(n-1)(n-a) + (n-1)a > 0.48(n-1)n$, and hence $a > n/12$. Let $F$ be the graph with the vertex-set $A$ such that a pair $ab$ is in $E(F)$ iff either $ab$ is in $E(G)$ or the number of paths of length two in $G$ connecting $a$ and $b$ is at least $n/11$. Assume that $A' = \{a_1, a_2, a_3\}$ is an independent set in $F$. Then $A'$ is an independent set in $G$, and $|N_G(a_i) \cap N_G(a_j)| \leq (n-1)/11$ for $j \neq i$. Since $A' \subset A$, $|N_G(a_1) \cup N_G(a_2) \cup N_G(a_3)| \geq 3 \cdot 0.43(n-1) - 3 \cdot (n-1)/11 > n - 1.5 > |V(G)| - |A'|$, a contradiction. Thus, the independence number of $F$ is at most two.

According to an old bound of Erdős and Szekeres [6], $F$ contains some clique $Q$ of size $q$, where $q$ is the largest integer such that

$$\binom{q+1}{2} \leq n/12. \tag{5}$$

We find in $G$ a subdivision of $K_q$ with $Q$ as the set of branching vertices in the following simple way. First we use all the edges of $G(Q)$. Then for each $x$ and $y$ in $Q$ with no edge $xy$, we assign to the pair $xy$ a path of length two. Assume that we can't do it for some $x$ and $y$ in $Q$. Since $xy \in E(F)$, there are $m = \lceil n/11 \rceil$ paths of length 2 connecting $x$ and $y$. If, say, $r$ of these paths have vertices in $Q$ as intermediate vertices, then $G(Q)$ has at least $2r$ edges and, since each of the remaining $m-r$ vertices also are already used in some path of length two, we have $2r+(m-r) \leq \binom{q}{2}-1$. Taking (5) into account, we get $n/11 < n/12$, a contradiction. Thus, $h(G) \geq q$.

Since $q$ is the largest integer satisfying (5), we have $\binom{q+2}{2} > n/12$, and so, $(h(G)+1)(h(G)+2) > n/6$, which implies the statement of the lemma. $\square$

Lemma 1 immediately implies Theorem 4.

**Proof of Theorem 5 :** If $h(G) \leq 3$, then $G$ is $h(G)$-colourable (Hadwiger [8]; Dirac[4]). In such a colouring one of the colour classes has size at least $|V(G)|/h(G)$. Then $\bar{G}$ contains a complete graph of this size, and hence $h(G) \times h(\bar{G}) \geq |V(G)|$.

Let $t \in T$, $t \geq 4$ and

$$n \geq 4\,t^3. \tag{6}$$

For a graph $H = (V, E)$ the *edge-density* $ed(H)$ is defined as $\frac{2|E|}{|V|\cdot(|V|-1)}$. Note that if $x \in V$ and $d(x) \geq \frac{2|E|}{|V|}$ then $ed(H-x) \leq ed(H)$. By the definition of $T$,

$$ed(G) \leq t^2/(n-1).$$

Observe also that, for each $k \leq |V(H)|$, the average edge-density over all induced subgraphs of $H$ with $k$ vertices equals $ed(H)$. Thus there is a $k$-subset $W$ such that $ed(h[W]) \leq ed(H)$.

Denote $f = \left\lceil \sqrt{3nt/10} \right\rceil$. By (6), $f > t^2$, which by the definition of $T$, is at least the average degree of $G$. Thus, successively removing from $G$ vertices of degree at least $f$, we come up with a graph $G-X$ of maximum degree at most $f-1$ and such that $ed(G-X) \leq ed(G)$. Since $f \cdot |X| \leq |E(G)|$, the number $|X|$ of deleted vertices is at most $t^2n/(2f) \leq \sqrt{5t^3n/6}$.

In $G-X$, let $W$ be a vertex-subset of size $\lceil n/t \rceil$ with minimum edge-density. By the above remarks,

$$ed(G[W]) \leq ed(G-X) \leq ed(G) \leq t^2/(n-1).$$

Since $\lceil n/t \rceil \leq \frac{n-1}{t}+1$, it follows that

$$\begin{aligned} e(G[W]) &\leq \frac{t^2}{n-1}\,0.5\,\lceil n/t \rceil(\lceil n/t \rceil - 1) \\ &\leq 0.5\,t\lceil n/t \rceil \leq 0.5(n+t-1). \end{aligned}$$

For each edge $xy$ of $G[W]$ we want to find a vertex $z$ of $G-X-W$ such that $xz$ and $yz$ are not edges in $G$. Furthermore, the vertices $z$ corresponding to different $xy$ should be different. If this can be accomplished, then there is a subdivision of the complete graph on $\lceil n/t \rceil$ vertices in $\bar{G}$ with the vertices of $W$ as branch vertices. Hence $h(G) \times h(\bar{G}) \geq n$.

If the number of possible $z$ for each $xy$ is at least $0.5(n+t-1)$, then this can indeed be accomplished. Remembering that each vertex in $G-X$ has degree at most $f-1 < \sqrt{3nt/10}$ we need only to have the following inequality fulfilled:

$$n - |X| - |W| - 2(f-1) \geq 0.5(n+t-1).$$

But

$$n - |X| - |W| - 2(f-1) \geq n - \sqrt{5t^3 n/6} - \lceil n/t \rceil - 2\sqrt{3nt/10},$$

which exceeds $0.5(n+t-1)$ provided $t \geq 4$.

Indeed,

$$n - \sqrt{5t^3 n/6} - n/t - 1 - 2\sqrt{3nt/10} > 0.5(n+t-1)$$

as it is equivalent to

$$\left(\frac{1}{2} - \frac{1}{t}\right)n + \left(t\sqrt{\frac{5t}{6}} - \sqrt{\frac{6t}{5}}\right)\sqrt{n} - \frac{1+t}{2} > 0,$$

whose left hand side is a quadratic polynomial of $\sqrt{n}$ which is always positive since its first coefficient and discriminant are positive ($t \geq 4$). $\square$

To support further our conjecture on the minimum possible product of the Hajós number of a graph and its complement, we find the minimum possible Hajós number of the complement of a forest $T$ (i.e., graph $T$ with $h(T) \leq 2$) on $n$ vertices. It is straightforward to check that for the exceptional graphs in the statement of Theorem 6, we have $h(\bar{P}_4) = h(P_4) = 2$, $h(\bar{T}_7) = 4$. Before proving the theorem, we show that in all other cases its bound is attained on paths.

**Lemma 2** *For a path $P_n$ on $n$ vertices, we have*

$$h(\bar{P}_n) \leq \lfloor \frac{2n+1}{3} \rfloor. \tag{7}$$

**Proof:** Let $P = v_1, \ldots, v_n$ be a path, and $M$ be the set of branching vertices in a subdivision of $K_m$ in $\bar{P}$. For each edge $xy$ in $P[M]$, we need a path in $\bar{P}$ connecting $x$ and $y$, and containing a vertex in $V(P) \setminus M$. Thus,

$$|E(P[M])| \leq n - m. \tag{8}$$

Among sets $M$ of cardinality $m$ with the minimum possible $|E(P[M])|$, consider an $M'$ with the minimum sum of numbers of its elements. Observe that $M'$ possesses the following properties:

(a) if $v_{i+1} \notin M'$, $v_{i+2} \in M'$, then $v_i \in M'$ (otherwise we move $v_{i+2}$ to $v_{i+1}$);
(b) if $v_{i+1} \in M'$, $v_{i+2} \in M'$, then $v_i \in M'$ (otherwise we move $v_{i+1}$ to $v_i$):
(c) $v_1 \in M'$.

It follows that, for some $k$, $M' = \{v_1, \ldots, v_k, v_{k+2}, v_{k+4}, \ldots\}$. Thus, $k = |E(P[M'])| + 1$, and

$$m \leq (n+k)/2. \tag{9}$$

Substituting (8) into (9), we get $m \leq (2n+1)/3$. $\square$

**Proof of Theorem 6:** We use induction on $n$. For $n \leq 4$ , the statement is easy (including exception (a)).

Let $G$ be a counter-example with the smallest number $n$ ($n \geq 5$) of vertices. Then $G$ possesses the following properties.

**(i)** *$G$ is connected.*

Assume first that $G$ has an isolated vertex $v$. By the minimality of $G$, $h(\bar{G} - v) \geq \lfloor 2(n-1)/3 \rfloor$ (even if $G-v$ is $P_4$ or $T_7$). Hence $h(\bar{G}) \geq 1 + \lfloor 2(n-1)/3 \rfloor = \lfloor (2n+1)/3 \rfloor$, a contradiction. If $G$ is not connected and has no isolated vertices, choose pendant vertices, say $x$ and $y$, in distinct components of $G$. We may assume that neighbours of $x$ and $y$ are $a$ and $b$, respectively, and that the component containing $x$ is of maximum order in $G$. Let $H = G - \{x, a, y\}$. By the minimality of $G$, $h(\bar{H}) \geq \lfloor (2(n-3)+1)/3 \rfloor$. Now we prove that $h(\bar{G}) \geq 2 + h(\bar{H})$, which easily implies $h(\bar{G}) \geq \lfloor (2n+1)/3 \rfloor$. Consider a subdivision $F$ of $K_{h(\bar{H})}$ in $\bar{H}$. Adding branch vertices $x$ and $y$, and the path $yab$ if $b$ is a branch vertex in $F$, we obtain a subdivision of $K_{2+h(\bar{H})}$ in $\bar{G}$.

**(ii)** *If $n \neq 7, 10$, then no two pendant vertices in $G$ share a neighbour.*

Indeed, if pendant vertices $x$ and $y$ both are adjacent to a vertex $a$, the graph $H = G - \{x, a, y\}$ is neither $P_4$ nor $T_7$. So, $h(\bar{H}) \geq \lfloor (2(n-3)+1)/3 \rfloor$, and, since $x$ and $y$ are adjacent to all vertices in $\bar{G} - a$, we have $h(\bar{G}) \geq 2 + h(\bar{H})$.

**(iii)** *The diameter of $G$ is at least four.*

Otherwise, $G$ has at least $n-2$ pendant vertices and so $h(\bar{G}) \geq n-2$.

Choose two pendant vertices $x$ and $y$ on distance at least four. Let their neighbours be $a$ and $b$, respectively. We construct $H$ as we did in the proof of (i). If $h(\bar{H}) \geq \lfloor (2(n-3)+1)/3 \rfloor$, then we are done as in (i). Thus the only situations we have to deal with are that $H = P_4$ or $H = T_7$. Let $H' = G - \{x, b, y\}$. Because of the symmetry between $H$ and $H'$, we assume further that $H'$ also is either $P_4$ or $T_7$. Since $G$ has no cycles, $b$ and $a$ are pendant vertices in $H$ and $H'$, respectively.

CASE 1. $H = H' = P_4$. This is possible only if $G$ is either the path $P_7$ (and we know that $h(\bar{P}_7) = 5$) or $T_7$.

CASE 2. $H = H' = T_7$. Then $G$ is the tree obtained from two disjoint copies of the path $P_5$ by joining their central vertices with an edge, and $b$ and $a$ are vertices of degree two and of distance two in this tree. Taking instead of $y$ a pendant vertex $z$ of distance five from $x$, we obtain that the graph $G - \{x, a, z\}$ is not isomorphic to $T_7$.

$\square$

# 3 On subdivisions of graphs

Jørgensen [12], and Hanson and Toft [9] observed the following fact:

**Lemma 3** *Almost all graphs on $p$ vertices have property $P_{3\lceil\sqrt{p}\rceil}$.*

The next fact is well known and follows from standard calculations (cf. e.g. [1]).

**Lemma 4** *Almost all graphs on $p$ vertices have the property that for every $k \geq (\log p)^2$ each subgraph on $k$ vertices has at least $0.24k(k-1)$ edges.*

**Proof of Theorem 1:**

We may assume that $n > 12$ since for smaller $n$ the theorem would follow from the case $n = 13$. We choose $c > 100$ so that for each $p > c$, according to Lemmas 3 and 4, there exists a graph $G_p$ on $p$ vertices satisfying $P_{3\lceil\sqrt{p}\rceil}$ and such that every subgraph of $G_p$ with $k \geq (\log p)^2$ vertices has at least $0.24k(k-1)$ edges.

Let $m > cn^2$ and $p = \lfloor (m/3)^2 \rfloor$. Define $k = \max\{\lceil (\log p)^2 \rceil, 9n^2\}$. Consider the graph $G_p$. By the definition, it has the property $P_m$ and each of its subgraphs on $k$ vertices has at least $0.24k(k-1)$ edges. We show that for each partition $\{V_1, \ldots, V_s\}$ of $V(G_p)$ into $s = \lfloor p/k \rfloor$ parts, at least one of $V_i$-s induces a subgraph with the Hajós number at least $n+1$. In a partition $\{V_1, \ldots V_s\}$ of $V(G_p)$, at least one set, say $V_1$, has cardinality at least $k$. Let $G'$ be the subgraph of $G_p$ induced by some subset of $V_1$ of cardinality $k$. By the properties of $G_p$, $G'$ has at least $0.24k(k-1)$ edges and thus, by Lemma 1, $h(G') \geq 0.4\sqrt{k} - 1.5 \geq 0.4 \cdot 3n - 1.5$. For $n \geq 13$, the last expression is at least $n+1$. Thus, the number of parts needed to partition $V(G_p)$ into subgraphs with Hajós number at most $n$ is at least $1 + \lfloor p/k \rfloor$. If $k = \lceil (\log p)^2 \rceil$, this is at least

$$0.5 \left( \frac{m}{6 \log m} \right)^2 > m.$$

If $k = 9n^2$, then $1 + \lfloor p/k \rfloor > \frac{m^2}{81n^2} > m$, as $c > 100$. The theorem for vertex partitions is proved.

For edge partitions, let $G_p$ be as above for a large $c$. Since all subgraphs of size $k$ have edge density at least $0.48$, $G_p$ itself has edge density at least $0.48$ (see the remark in the proof of Theorem 5 b)). Therefore, $|E(G_p)| \geq 0.24 \cdot p(p-1)$. In a partition of the edge-set of $G_p$ into $m$ parts at least one part $F$ has $|E(F)| \geq \frac{0.24\,p(p-1)}{m}$. For this $F$ the average degree is at least $\frac{0.48(p-1)}{m} \geq \frac{2p}{5n} \geq \frac{2m^2}{45m} = \frac{2m}{45} > \frac{2cn^2}{45}$, which is sufficiently large for $c$ sufficiently large. By the theorem of Komlós and Szemerédi [14], $h(F_1) \geq \sqrt{\frac{2cn^2}{45}} > n$ for $c$ sufficiently large.

Thus $G_p$ cannot be partitioned into $m$ spanning subgraphs each with Hajós number at most $n$. $\square$

Note that for any fixed $n$ and $m$ large enough, we have $k = \lceil (\log p)^2 \rceil$ and hence need at least $0.5(\frac{m}{6 \log m})^2$ parts to partition $V(G_p)$ into subgraphs of Hajós number at most $n$.

**Proof of Theorem 2:**

It was observed by several authors (e.g., in [16, 7] ) that almost every graph on $p$ vertices has Hadwiger number at most $p/\sqrt{\log p}$. Almost repeating any of these arguments, one easily sees that for $k = \lceil p/\sqrt{\log p}\rceil$, almost every graph on $p$ vertices does not contain $K_{k,k}$ as a minor. Thus, for some $w$ and each integer $p > w$, there exists a graph $H_p$ such that
(a) $|V(H_p)| = v$, $|E(H_p)| > p^2/4$;
(b) for $k = \lceil p/\sqrt{\log p}\rceil$, $H_p$ contains neither $K_{k+1}$ nor $K_{k,k}$ as a minor.

By an old result of Mader [17], for each positive integer $n$, there exists a constant $c_n$ such that each graph $G$ with $|E(G)| \geq c_n|V(G)|$ has Hajós number at least $n+1$ (the best known upper bound for $c_n$ is in [14] ).

Consider an arbitrary integer $n > 1$. Let $m_0 = m_0(n)$ be the smallest integer satisfying the properties that
(i) $w \leq m_0\sqrt{\log m_0}/3$;
(ii) $\sqrt{\log m_0} > 12c_n$.

Now, let $m$ be an arbitrary integer greater than $m_0(n)$ and $p = \lceil m\sqrt{\log m}/3\rceil$. By (i), there is a graph $H_v$ satisfying (a) and (b). Since $m > 2p/\sqrt{\log p}$, $H_p$ has neither $K_{m+1}$ nor $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$ as a minor. On the other hand, for each partition $\{E_1, ..., E_m\}$ of the edge-set of $H_p$, at least one $E_i$ contains more than $\frac{p^2}{4m} \geq \frac{p\cdot m\sqrt{\log m/3}}{4m} \geq p\cdot c_n$ edges. Thus, we cannot partition the edge-set of $H_p$ into $m$ spanning subgraphs possessing $P_n$. $\square$

# References


[1] N. Alon and J. Spencer, *The Probabilistic Method*, Wiley, New York, 1992.

[2] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdös and Hajnal on topological complete subgraphs, *to appear.*

[3] G. Chartrand, D. P. Geller and S. T. Hedetniemi, Graphs with forbidden subgraphs, *J. Combin. Theory* **10** (1971), 12–41.

[4] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85–92.

[5] P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *Combinatorica* **1** (1981), 141–143.

[6] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[7] Fernandez de la Véga, On the maximum density of graphs which have no subcontractions to $K^S$, *Discrete Math.* **46** (1983), 109–110.

[8] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljahrsschrift der Naturf. Grsellschaft in Zürich* **88** (1943), 133–142.

[9] D. Hanson and B. Toft, The $(m, n)$-conjecture is false, *Bul. Inst. Combin. Appl.* **11** (1994), 59–66.

[10] L. S. Heath, Edge coloring planar graphs with two outerplanar subgraphs, in: *Proc. Second Annual ACM-SIAM Symp. on Discrete Algorithms* (1991), 195–202.

[11] T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

[12] L. K. Jørgensen, Some probabilistic and extremal results on subdivisions, *J. Graph Theory* **13** (1989), 75–85.

[13] J. Komlós and E. Szemerédi, Topological cliques in graphs, *Combinatorics, Probability & Computing* **3** (1994), 247–256.

[14] J. Komlós and E. Szemerédi, Topological cliques in graphs II, *Combinatorics, Probability & Computing* **5** (1996), 79–90.

[15] A. V. Kostochka, On Hadwiger numbers of a graph and its complement, in: *Finite and Infinite Sets* (ed. A. Hajnal, L. Lovász and V. T. Sós), *Colloquia Math. Soc. János Bolyai* **37** (1984), 537–545.

[16] A. V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, *Metody Discret. Analiz.* **38** (1982), 37-58 (In Russian).

[17] W. Mader, Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind, *Math. Nachr.* **53** (1972), 145-150.

[18] C. Thomassen, Some homeomorphism properties of graphs, *Math. Nachr.* **64** (1974), 119–133.

[19] D. R. Woodall, Improper colouring of graphs, in *Graph Colourings* (R. Nelson and R.J. Wilson, eds.), Pitman Research Notes **218**, Longman 1990, 45-63.

[20] B. Zelinka, Hadwiger number of finite graphs, *Math. Slov.* **26** (1976), 23-30.