On Bicritical Snarks

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Abstract

Bicritical snarks are the irreducible ones with respect to the reductions considered by Nedela and Škoviera in [7]. We give a natural characterization of bicritical snarks and we characterize the elementary extensions of a snark. This solves Problem No. 6 of [7], and yields a method to construct all snarks starting from the irreducible ones.

Then we show that for some $n \geq 10$ and for each even $n \geq 92$ there is a bicritical snark of order n. This solves Problem No. 5 of [7].

1 Introduction

We are using standard graph theoretical terminology and notation in this paper. We define a *snark* to be a cubic graph with chromatic index $\chi' = 4$.

There are two main questions which lead to the study of reduction of snarks. The first one is the question about the intrinsic properties of cubic graphs which force them being a snark. The hope is that these properties can be detected in the irreducible snarks, and, since every snark is reducible to an irreducible one every snark should have this property, too.

The second one is the question about a method to construct all snarks recursively starting from the irreducible snarks. Here the hope is that the reverse operation (of a reduction) can be described without reflecting on the reduction process and hence it would yield such a method.

There are many papers dealing with this topic, cf. [1, 2, 5, 6, 7, 8, 9, 10, 11], and we refer the reader to one these papers for an more extensive introduction.

In this paper we consider the approach of Nedela and Škoviera [7]. They investigate the following reduction:

Let $G = (V(G), E(G))$ be a snark and let $F \subset E(G)$ be an k-edge cut ($k \geq 0$) whose removal divides G into two components H_1 and H_2. If the chromatic index of one of the components is 4, say $\chi'(H_1) = 4$, then H_1 can be extended to a snark $H = (V(H), E(H))$

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with $|V(H)| \leq |V(G)|$, and H is called a *k-reduction* of G. If $|V(H)| < |V(G)|$ then H is called a *proper* k-reduction of G.
A snark is called *k-irreducible* if it has no proper m-reduction for each $m < k$, and it is called *irreducible* if it k-irreducible for each $k \geq 1$.
A snark G is called *bicritical* if $\chi'(G - \{v, w\}) = 3$ for any two vertices $v, w \in V(G)$.

Theorem 1.1 **([7])** *A snark is irreducible if and only if it is bicritical.*

Concerning the second question, i.e. to find a construction method for all snarks starting from the irreducible ones, Nedela and Škoviera introduce the notion of an elementary extension of a snark.
Let G and H be snarks. Then G is called an *elementary extension* of H, if there is a set F of two or three edges in H such that $H - F$ can be extended to G by adding two vertices and edges. Note that $\chi'(H - F) = 4$, since H is a proper k-reduction of G, $k \in \{4, 6\}$.
Clearly, for each snark G there is a sequence of snarks $G_1, G_2, \dots, G_n$ such that G_1 is an irreducible snark, $G_n = G$, and G_{i+1} is an elementary extension of G_i, $i = 1, \dots, n-1$.
The problem is to localize the set F_i in G_i. Nedela and Škoviera formulated the following problem:

Problem 1.2 **([7])** *For a given snark G, characterize all elementary extensions of G.*

In Section 2 we give a very natural characterization of the irreducible snarks which gives an answer to the first question from the beginning with respect to k-reductions. Then we solve Problem 1.2, which leads to a recursive construction method for all snarks starting from the irreducible ones.
Thus the two main questions in the field of snark reductions are answered for the approach of Nedela and Škoviera.
The set of the irreducible snarks is characterized by Theorems 1.1 and 2.1, but it is an infinite set. In [7] it is proved that the flower snark J_{2k+1} is irreducible for each $k \geq 2$, where the flower snark J_{2k+1} is the graph with vertex set $VJ_{2k+1} = \{a_i, b_i, c_i, d_i | i = 0, 1, \dots, 2k\}$ and edge set $EJ_{2k+1} = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} | i = 0, 1, \dots, 2k\}$.
Thus there are irreducible snarks of $8k + 4$ for each $k \geq 2$. In [7] it is also stated that there are no irreducible of orders 12, 14, 16 and 24, but there are such snarks of orders 10, 18, and 22. In [1] these results are obtained independently, in fact there is shown that there are exactly 2 of order 18, 1 of order 20, 2 of order 22, 111 of order 26 and that there are 33 irreducible snarks of order 28.
Nedela and Škoviera asked the following questions:

Problem 1.3 **([7])** *For which even number $n \geq 10$ does there exist an irreducible snark of order n? In particular, does there exist an irreducible snark of each sufficiently large order?*

In Section 3 we will give a partial solution of the first and a positive answer to the second question.

2 Characterization of Elementary Extensions

A graph G with $\Delta(G) = k$ is called *k-critical* if $\chi'(G) = k + 1$ and $\chi'(G - e) < \chi'(G)$ for every edge e of G. It follows from a theorem of Vizing, see e.g. [4] (Theorem 10.3), that each snark contains a 3-critical subgraph.
Considering irreducible snarks it is obvious that they cannot contain small 3-critical subgraphs. Therefore the following characterization is very natural.

Theorem 2.1 *For each snark $G = (V, E)$ the following statements are equivalent.*

i) G is irreducible.

ii) For all $v \in V$ the graph $G - v$ contains a (connected) spanning 3-critical subgraph and it contains no 3-critical subgraph of order $\leq |V| - 2$.

Proof. Theorem 1.1 allows to replace assertion i by: G is bicritical.
($i) \Rightarrow ii)$): Let $v \in V$. Then $G - v$ is not 3-colorable and therefore it has a 3-critical subgraph of order $\leq |V| - 1$. Suppose it has a 3-critical subgraph H of smaller order than $|V| - 1$. Then there is a vertex $x \in G - v$ which is not a vertex of H and therefore $G - \{v, x\}$ is not 3-colorable. This contradicts the fact that G is bicritical.
($ii) \Rightarrow i)$): Suppose there are $v, w \in V$ such that $G - \{v, w\}$ is not 3-colorable. Then $G - v$ contains a 3-critical subgraph which is of order $\leq |V| - 2$.

□

This shows that with respect to this reduction the intrinsic property which is responsible for being a snark is that snarks contain 3-critical subgraphs.
The following theorem solves Problem 1.2.

Theorem 2.2 *Let H be a snark and let G be a cubic graph of order $|H| + 2$. Then G is an elementary extension of H if and only if there are two vertices v and w in G such that*

1. *$G - \{v, w\}$ is subgraph of H, and*
2. *there is a 3-critical graph which is subgraph of H and of G.*

Proof. ($\Rightarrow$) Let G be an elementary extension of H. That is, H contains a set F of two or three edges such that $\chi'(H - F) = 4$, and $H - F$ can be extended to G by adding two vertices v and w and edges. Thus $G - \{v, w\} = H - F$ which is subgraph of H.
Since $\chi'(H - F) = 4$ this graph has a 3-critical subgraph which is subgraph of G, too.

($\Leftarrow$) Let G be a cubic graph which has two vertices v and w such that $G - \{v, w\}$ is subgraph of H. Since G contains a 3-critical subgraph it follows that G is a snark.
We have to show that there is a set $F \subset E(H)$ of two or three edges such that $\chi'(H - F) = 4$ and $H - F$ can be extended to G.

Let N be the set neighbors of v and w in G. Define $F = E(H[N]) - E(G[N])$. Then follows $|F| \in \{2, 3\}$, since H is cubic.
Assume $\chi'(H - F) = 3$. But H is a snark and hence every 3-critical subgraph of H contains at least one edge of F. Therefore, and since $F \cap E(G) = \emptyset$, it follows that every 3-critical subgraph of H is different from each 3-critical subgraph of G. This contradicts the second condition and the statement is proved.

□

Since $\chi'(G - v) = 4$ for each vertex v of a snark G there is always a set of two or three edges in G which can be removed to construct an elementary extension of G. The results above imply the following method to find all appropriate sets F in G such that $G - F$ can be extended to an elementary extension of G. We state

Corollary 2.3 *Let G be a snark. A set F of two or three edges in G can be removed from G and $G - F$ can be extended to a snark by adding two vertices and edges if and only if there is a 3-critical subgraph of G which does not contain an edge of F.*

3 Families of Bicritical Snarks

In this section we show that there are irreducible snarks of order n for some $n \geq 10$ and for each even $n \geq 92$. This is a consequences of results of [3] and [9]. Fiorini proved in [3] a sufficient condition under which the dot product preserves the property of being hypohamiltonian. To state Fiorini's results we give the definitions.
A graph G is *hypohamiltonian* if it is not hamiltonian but $G - v$ is hamiltonian for every vertex v of G.

Let G_1 and G_2 be snarks. The *dot product* $G_1 \cdot G_2$ is a snark (see [6]) and it is defined as follows:
1. take $G_1' = G_1 - \{ab, cd\}$, where ab, cd are two nonadjacent edges of G_1;
2. take $G_2' = G_2 - \{x, y\}$, where x, y are adjacent vertices in G_2;
3. let w, z and u, v be the other neighbors of x and y in G_2, respectively. Then $G_1 \cdot G_2$ is the graph $G = (V, E)$ with $V = VG_1' \cup VG_2'$ and $E = EG_1' \cup EG_2' \cup \{aw, bz, cu, dv\}$.

Let $G = (V, E)$ be a graph. A pair of vertices (v, w) is *good in G* if there is a hamiltonian path with terminal vertices v, w.
Two pairs of vertices $((v, w), (x, y))$ are *good in G* if there are two disjoint spanning paths in G with terminal ends v, w and x, y, respectively.

Theorem 3.1 ([3]) *Let G be a hypohamiltonian snark having two independent edges $e = uv$, $f = xy$ for which*

1. *each of (u, x), (u, y), (v, x), (v, y), $((u, v), (x, y))$ is good in G and*
2. *for each vertex w, one of (u, v), (x, y) is good in $G - w$,*

and let H be an hypohamiltonian snark. Then $G \cdot H$ is a hypohamiltonian snark.

Theorem 3.2 ([3]) *The flower snark J_k is hypohamiltonian for each $k \geq 2$.*

Fiorini mentioned without proof that the two Blanuša snarks and the double-star snark D (cf. [11]) are hypohamiltonian, and that J_9 satisfies the conditions of Theorem 3.1. We verified these results and extend the latter.

Lemma 3.3 *The flower snarks J_9, J_{11} and J_{13} satisfy the conditions of Theorem 3.1.*

A proof of this Lemma is given in the appendix.

Theorem 3.4 ([9]) *Each hypohamiltonian snark is bicritical.*

The converse is not true. The Goldberg snarks [5] on 22 vertices are bicritical and they are not hypohamiltonian, cf. [3]. The following theorem solves Problem 1.3.

Theorem 3.5 *There is an irreducible snark of order n*

1. *for each $n \in \{m | m \geq 64$ and $m \equiv 0 \bmod 8\}$.*
2. *for each $n \in \{10, 18, 26\} \cup \{m | m \geq 98$ and $m \equiv 2 \bmod 8\}$.*
3. *for each $n \in \{m | m \geq 20$ and $m \equiv 4 \bmod 8\}$.*
4. *for each $n \in \{22, 30\} \cup \{m | m \geq 54$ and $m \equiv 6 \bmod 8\}$.*
5. *for each even $n \geq 92$.*

Proof. Theorem 1.1 allows to argue with bicritical graphs. In the introduction we already state that there are bicritical graphs of orders 10, 18, 22, 26 and of orders $8k + 4$ for all $k \geq 2$.
By Theorem 3.4 it suffices to show that there are hypohamiltonian snarks of the orders in question.
Since the flower snarks are hypohamiltonian and J_9 satisfies the conditions of Theorem 3.1 it follows that there are hypohamiltonian snarks of order n for each $n = 8k + 4 + 34m = 8(4m + k) + 2(m + 2)$, $k \geq 2$ and $m \geq 0$.
For $m = 1$ we obtain numbers $8n' + 6$ for all $n' \geq 6$, and the double-star snark of order 30 is hypohamiltonian. Thus 4 holds true.
For $m = 2$ we obtain numbers $8n'$ for all $n' \geq 11$. By Theorem 3.1 and Lemma 3.3 the graphs $D \cdot J_9$, $D \cdot J_{11}$ and $D \cdot J_{13}$ are hypohamiltonian and they have orders 64, 72, and 80, respectively. Hence 1 holds true.
For $m = 3$ we obtain numbers $8n' + 2$ for all $n' \geq 15$. Again by Theorem 3.1 and Lemma 3.3 the graphs $D \cdot J_9 \cdot J_9$, $D \cdot J_9 \cdot J_{11}$ and $D \cdot J_{11} \cdot J_{11}$, are hypohamiltonian and they have orders 98, 106, and 114, respectively. Thus 2 holds true.

Combining the results 1, 2, 3 and 4 yields that there is a hypohamiltonian snark G_i of order $92 + 2i$ for each $0 \leq i \leq 16$. Thus for each even $n \geq 126$ there is $0 \leq i \leq 16$ such that the iterated dot product $G_i \cdot J_9 \cdot \ldots \cdot J_9$ yields a hypohamiltonian snark of order n. Hence there is a bicritical snark of order n for each even $n \geq 92$.

$\square$

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Appendix

Proof of Lemma 3.3

Let the vertices and edges be denoted as in the definition of J_{2k+1}, $k \geq 2$.
We show that edges b_0c_0 and b_4c_4 satisfy the conditions of Theorem 3.1 in each of J_9, J_{11} and J_{13}.

Claim 1: Vertices (c_0, c_4) are good in J_9, J_{11} and J_{13}.
Proof. The Hamilton-paths are for
J_9: $c_0, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.
J_{11}: $c_0, d_{10}, c_9, b_9, a_9, a_{10}, b_{10}, c_{10}, d_9, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, d_5, c_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.
J_{13}: $c_0, d_{12}, c_{11}, b_{11}, a_{11}, a_{12}, b_{12}, c_{12}, d_{11}, c_{10}d_9, b_9, a_9, a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.
□

Claim 2: Vertices (c_0, b_4) are good in J_9, J_{11} and J_{13}.
Proof. The Hamilton-paths are for
J_9: $c_0, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.
J_{11}: $c_0, d_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_9, a_{10}, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.
J_{13}: $c_0, d_{12}, b_{12}, c_{12}, d_{11}, b_{11}, c_{11}, d_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.
□

Claim 3: Vertices (b_0, c_4) are good in J_9, J_{11} and J_{13}.
Proof. The Hamilton-paths are for
J_9: $b_0, a_0, a_1, b_1, d_1, c_0, d_8, b_8, a_8, a_7, b_7, c_7, d_6, b_6, a_6a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, a_8, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$
J_{11}: $b_0, a_0, a_1, b_1, d_1, c_0, d_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, c_7, d_6, b_6, a_6a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, c_8, b_8, d_8, c_9, b_9, d_9, c_{10}, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$.
J_{13}: $b_0, a_0, a_1, b_1, d_1, c_0, d_{12}, b_{12}, a_{12}, a_{11}, a_{10}, a_9, a_8a_7, b_7, c_7, d_6, b_6, a_6a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, c_8, b_8, d_8, c_9, b_9, d_9, c_{10}, b_{10}, d_{10}c_{11}, b_{11}, d_{11}, c_{12}, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$.
□

Claim 4: Vertices (b_0, b_4) are good in J_9, J_{11} and J_{13}.
Proof. The Hamilton-paths are for
J_9: $b_0, d_0, c_8, b_8, d_8, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.
J_{11}:$b_0, d_0, c_{10}, a_{10}, d_{10}, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, a_7, a_6, b_6, d_6, c_7, d_8, c_9, b_9, d_9, c_8, b_8, a_8, a_9, a_{10}, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.

J_{13}:$b_0, d_0, c_{12}, a_{12}, d_{12}, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, d_8, b_8, c_8, d_9, c_{10}, d_{11}, b_{11}, c_{11}, d_{10}, b_{10}, a_{10}, a_{11}, a_{12}, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.

□

Claim 5: The two pairs of vertices $((b_0, c_0), (b_4, c_4))$ are good in J_9, J_{11} and J_{13}.
Proof. The cycles are for
J_9: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, b_7, c_7, d_8, b_8, a_8, a_0, b_0$ and $c_4, d_5, c_6, d_7, c_8, d_0, c_1, d_2, c_3, d_4, b_4$.
J_{11}: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, d_{10}, b_{10}, a_{10}, a_0, b_0$ and $c_4, d_5, c_6, d_7, b_7, c_7, d_8, b_8, c_8, d_9, c_{10}, d_0, c_1, d_2, c_3, d_4, b_4$.
J_{13}: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, a_8, a_9, a_{10}, a_{11}, b_{11}, c_{11}, d_{12}, b_{12}, a_{12}, a_0, b_0$ and $c_4, d_5, c_6, d_7, b_7, c_7, d_8, b_8, c_8, d_9, b_9, c_9, d_{10}, b_{10}, c_{10}, d_{11}, c_{12}, d_0, c_1, d_2, c_3, d_4, b_4$.

□

Claim 6: For each $v \in VJ_9$ one of (b_0, c_0), (b_4, c_4) is good in $J_9 - v$.
Proof. A Hamilton-cycle H in $J_9 - a_0$ is: $c_0, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, a_1, a_2, b_2, d_2, c_1, d_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0, 3, 4, 7, 8\}$. Thus shifting the indices of the vertices of the cycle from i to $i + j$ yields that (b_0, c_0) is good in $J_9 - a_j$ for all $j \in \{0, 1, 2, 5, 6\}$, and that (b_4, c_4) is good in $J_9 - a_4$.

The following cycle is a Hamilton-cycle in $J_9 - a_8$: $c_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, a_7, a_6, b_6, d_6, c_7, d_8, b_8, c_8, d_0, c_1, b_1, a_1, a_0, b_0$.
This cycle contains edges b_0c_0, b_1c_1 and b_5c_5. Shifting the indices of the vertices of this cycle from i to $i + j$ yields that (b_0, c_0) is good in $J_9 - a_{j+8}$ for $j \in \{0, 4, 8\}$.

The following cycle is a Hamilton-cycle in $J_9 - b_1$: $c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, b_4, a_4, a_3, a_2, a_1, a_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0, 2, 3, 5, 6\}$. Thus shifting the indices of the vertices of H from i to $i + j$ yields that (b_0, c_0) is good in $J_9 - b_{j+1}$ for $j \in \{0, 3, 4, 6, 7\}$, and that (b_4, c_4) is good in $J_9 - b_{j+1}$ for $j \in \{1, 2, 8\}$.
The following cycle shows that (b_4, c_4) is good in $J_9 - b_6$: $c_4, d_3, b_3, a_3, a_4, a_5, a_6, a_7, b_7, c_7, d_6, c_5, b_5, d_5, c_6, d_7, c_8, d_0, b_0, a_0, a_8, b_8, d_8, c_0, d_1, c_2, b_2, a_2, a_1, b_1, c_1, d_2, c_3, d_4, b_4$.

The following cycle is a Hamilton-cycle in $J_9 - c_0$: $c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0, d_0, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, d_5, c_6, b_6, a_6, a_5, a_4, a_3, b_3, c_3, d_4, b_4$
This cycle contains edges b_ic_i for $i \in \{1, 3, 4, 5, 6, \}$. Thus shifting the indices of the vertices of H from i to $i + j$ yields that (b_4, c_4) is good in $J_9 - c_j$ for $j \in \{0, 1, 3, 7, 8\}$, and that (b_0, c_0) is good in $J_9 - c_j$ for $j \in \{4, 5, 6\}$.

The following cycle shows that (b_4, c_4) is good in $J_9 - c_2$: $c_4, d_3, b_3, a_3a_4, a_5, a_6, b_6, d_6, c_5, b_5, d_5, c_6, d_7, c_8, b_8, a_8, a_7, b_7, c_7, d_8, c_0, d_1, b_1, c_1, d_0, b_0, a_0, a_1, a_2, b_2, d_2, c_3, d_4, b_4$

The following cycle is a Hamilton-cycle in $J_9 - d_0$: $c_0, d_8, c_7, b_7, d_7, c_8, b_8, a_8, a_7, a_6, a_5, b_5, d_5,$ c_6, b_6, d_6, $c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0, 1, 3, 4, 6, 7, 8\}$. Thus shifting the indices of the vertices of H from i to $i+j$ yields that (b_0, c_0) is good in $J_9 - d_j$ for $j \in \{0, 1, 2, 3, 5, 6, 8\}$, and that (b_4, c_4) is good in $J_9 - d_j$ for $j \in \{4, 7\}$.

□

Claim 7: For each $v \in VJ_{11}$ one of (b_0, c_0), (b_4, c_4) is good in $J_{11} - v$, for all v.
Proof. A Hamilton-cycle H in $J_{11} - a_0$ is: $c_0, d_{10}, c_9, b_9, d_9, c_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, c_7, d_8,$ $b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, a_1, a_2, b_2, d_2, c_1, d_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0, 3, 4, 7, 8, 9, 10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_{11} - a_j$ for all $j \in \{0, 1, 2, 3, 4, 7, 8\}$, and that (b_4, c_4) is good in $J_{11} - a_j$ for all $j \in \{5, 6\}$.

The following cycle is a Hamilton-cycle H in $J_{11} - a_{10}$: $c_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2$, a_3, a_4, a_5, b_5, c_5, $d_4, b_4, c_4, d_5c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, d_9, c_8, b_8, d_8, c_9, d_{10}, b_{10}, c_{10}, d_0,$ c_1, b_1, a_1, a_0, b_0.
This cycle contains edge b_1c_1. Shifting the indices of the vertices from i to $i + 10$ yields that (b_0, c_0) is good in $J_{11} - a_9$.

The following cycle is a Hamilton-cycle in $J_{11} - b_1$: $c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_{10}, d_9, b_9, c_9, d_{10},$ b_{10}, a_{10}, a_9, a_8, $a_7, b_7, d_7, c_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, b_4, a_4, a_3, a_2,$ a_1, a_0, b_0.
This cycle contains edges b_ic_i for $i \in \{0, 2, 3, 5, 6, 8, 9\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_{11} - b_{j+1}$ for all $j \in \{0, 2, 3, 5, 6, 8, 9\}$, and that (b_4, c_4) is good in $J_{11} - b_{j+1}$ for all $j \in \{1, 4, 6, 7, 10\}$.

The following cycle is a Hamilton-cycle in $J_{11} - c_0$: $c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0, d_0,$ c_{10}, d_9, b_9, a_9, $a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6a_5, a_5, a_4, a_3$, b_3, c_3, d_4, b_4.
This cycle contains edges b_ic_i for $i \in \{0, 1, 3, 4, 5, 7, 8\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_{11} - c_j$ for all $j \in \{0, 3, 4, 6, 7, 8, 10\}$, and that (b_4, c_4) is good in $J_{11} - c_1$.
The following cycle is a Hamilton-cycle in $J_{11} - c_2$: $c_4, d_3, b_3, a_3, a_4, a_5, a_6, b_6, d_6, c_5, b_5, d_5, c_6,$ $d_7, c_8, b_8, d_8, c_7, b_7, a_7, a_8, a_9, a_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_{10}, c_0, d_1, b_1, c_1, d_0, b_0, a_0, a_1, a_2, b_2, d_2, c_3,$ d_4, b_4.
This cycle contains edges b_1c_1, and b_4c_4, shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_4, c_4) is good in $J_{11} - c_{j+2}$ for all $j \in \{0, 3\}$, and that (b_0, c_0) is good $J_{11} - c_9$.

The following cycle is a Hamilton-cycle in $J_{11} - d_0$: $c_0, d_{10}, c_9, b_9, a_9, a_{10}, b_{10}, c_{10}, d_9, c_8, d_7, b_7,$ c_7, d_8, b_8,$a_8, a_7, a_6, a_5, b_5, d_5, c_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1,$

a_0, b_0.
This cycle contains edges b_ic_i for $i \in \{0,1,3,4,6,7,9,10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{11}-d_j$ for all $j \in \{0,1,2,4,5,7,8,10\}$, and that (b_4,c_4) is good in $J_{11}-d_j$ for all $j \in \{3,6,9\}$.

□

Claim 8: For each $v \in VJ_{13}$ one of (b_0,c_0), (b_4,c_4) is good in $J_{13}-v$, for all v.
Proof. A Hamilton-cycle H in $J_{13}-a_0$ is: $c_0, d_{12}, c_{11}, b_{11}, d_{11}, c_{12}, b_{12}, a_{12}, a_{11}, a_{10}, a_9 b_9, d_9$, c_{10}, b_{10}, d_{10}, c_9, d_8, $c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4$, d_3, c_2, d_1, b_1, a_1, $a_2, b_2, d_2, c_1, d_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0,3,4,7,8,10,11,12\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-a_j$ for all $j \in \{0,1,2,3,5,6,9,10\}$, and that (b_4,c_4) is good in $J_{13}-a_j$ for all $j \in \{4,7\}$.
The following cycle is a Hamilton-cycle in $J_{13}-a_0$: $c_0, d_1, c_2, b_2, a_2, a_1, b_1, c_1, d_2, c_3, d_4, b_4$, c_4, d_3, b_3, a_3, $a_4, a_5, a_6, b_6, c_6, d_5, b_5, c_5, d_6, c_7, d_8, b_8, c_8, d_7, b_7, a_7, a_8, a_9, a_{10}, b_{10}, c_{10}, d_9, b_9, c_9$, d_{10}, c_{11}, $d_{12}, b_{12}, a_{12}, a_{11}, b_{11}, d_{11}, c_{12}, d_0, b_0$.
This cycle contains edges b_1c_1, b_2c_2 and b_5c_5. Thus shifting the indices of the vertices of the cycle from i to $i+8$, $i+11$ and $i+12$ yields that (b_0,c_0) is good in $J_{13}-a_8$, $J_{13}-a_{11}$ and $J_{13}-a_{12}$, respectively.

The following cycle is a Hamilton-cycle in $J_{13}-b_1$: $c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_{12}, d_{11}, b_{11}, a_{11}$, $a_{12}, b_{12}, d_{12}, c_{11}, d_{10}, c_9, b_9, d_9, c_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, d_7, c_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6$, d_5, c_4, d_3, $b_3, c_3, d_4, b_4, a_4, a_3, a_2, a_1, a_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0,2,3,5,6,8,9,10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-b_{j+1}$ for all $j \in \{0,3,4,5,7,8,10,11\}$, and that (b_4,c_4) is good in $J_{13}-b_{j+1}$ for all $j \in \{1,2,9,12\}$.
The following Hamilton-cycle shows that (b_4,c_4) is good in $J_{13}-b_7$.
$b_4, a_4, a_5, b_5, c_5, d_4, c_3, d_2, b_2, c_2, d_3, b_3, a_3, a_2, a_1, a_0, b_0, d_0, c_1, b_1, d_1, c_0, d_{12}$, c_{11}, b_{11}, a_{11}, a_{12}, b_{12}, c_{12}, d_{11}, c_{10}, d_9, $b_9, a_9, a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, d_6, b_6, a_6, a_7, a_8, b_8, c_8, d_7, c_6, d_5, c_4$.

The following cycle is a Hamilton-cycle in $J_{13}-c_1$: $c_0, d_{12}, b_{12}, a_{12}, a_0, a_1, b_1, d_1, c_2, d_3, b_3$, c_3, d_2, b_2, a_2, a_3,$a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, d_9, c_8, b_8, d_8, c_9$, d_{10}, c_{11}, b_{11}, a_{11}, $a_{10}, b_{10}, c_{10}, d_{11}, c_{12}, d_0, b_0$.
This cycle contains edges b_ic_i for $i \in \{0,3,4,5,7,8,10,11\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-c_{j+1}$ for all $j \in \{0,2,3,5,6,8,9,10\}$, and that (b_4,c_4) is good in $J_{13}-c_{j+1}$ for all $j \in \{1,4,7,12\}$.
The following cycle shows that (b_0,c_0) is good in $J_{13}-c_{12}$.
$b_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7$, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, $d_8, b_8, c_8, d_9, c_{10}, d_{11}, b_{11}, a_{11}, a_{10}, b_{10}, d_{10}, c_{11}, d_{12}, b_{12}, a_{12}, a_0$, a_1, b_1, c_1, d_0, b_0.

The following cycle is a Hamilton-cycle in $J_{13}-d_0$: $c_0, d_{12}, c_{11}, b_{11}, d_{11}, c_{12}, b_{12}, a_{12}, a_{11}, a_{10}$, a_9, b_9, c_9, d_{10}, b_{10},$c_{10}, d_9, c_8, d_7, b_7, c_7, d_8 b_8, a_8, a_7, a_6, a_5, b_5, d_5, c_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4$,

$b_4,\ c_4,\ d_3,\ c_2,\ d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0$.
The cycle contains edges b_ic_i for $i \in \{0,1,3,4,6,7,9,10,11,12\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_{13} - d_j$ for all $j \in \{0,1,2,3,4,6,7,9,10,12\}$, and that (b_4, c_4) is good in $J_{13} - d_j$ for all $j \in \{5,8,11\}$.

$\square$