A category of pseudo-tangles with classifying space $\Omega^\infty S^\infty$

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1 Introduction

The category $\mathcal{F}_{Sym} = \coprod_{n \geq 0} \Sigma_n$ is the free symmetric monoidal category on one generating object, and it is well known that the group completion of its classifying space $B\mathcal{F}_{Sym}$ is $\Omega^\infty S^\infty$, i. e. stable homotopy of the 0-sphere.
If a monoidal category S satisfies certain properties, one obtains by Quillen's "S-inverse-construction" [1] a new category $S^{-1}S$ and a canonical arrow $S \longrightarrow S^{-1}S$ which yields a group completion $BS \longrightarrow B(S^{-1}S)$ on the level of classifying spaces.
$\mathcal{F}_{Sym}$ has a geometric interpretation: Morphisms $n \longrightarrow n$ are given by "braids with unconsidered crossings" (pseudo-braids) on n strings, composition by putting one picture on the top of the other and the tensor product by juxtaposition.
In this paper it is shown that $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ is equivalent to a category $\mathcal{G}$ with an analogous geometric description and hence this category has classifying space $\Omega^\infty S^\infty$.
Section 2 of this article contains basic definitions, a short review of the S-inverse-construction and states the main theorem. Section 3 is devoted to the proof of the main theorem.

2 Definitions and theorems

2.1 Definition: Let $\mathcal{C}$ be a small category. Its nerve is the simplicial set whose k–simplices are diagrams in $\mathcal{C}$ of the form $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k$. The i-th face (resp. degeneracy) of this simplex is obtained by deleting the object X_i (resp. replacing X_i by $id : X_i \longrightarrow X_i$) in the evident way. The classifying space $B\mathcal{C}$ of $\mathcal{C}$ is the toplogical realization of its nerve .—

Some general facts about classifying spaces may be found in [4].

2.2 Convention: All terms concerning monoidal categories (sometimes also called tensor categories) are used in the sense of [2] .—

2.3 Remarks: The associativity isomorphisms of a small monoidal category describe a natural transformation from the functor $\otimes \circ (\otimes \times id)$ to the functor $\otimes \circ (id \times \otimes)$. Hence one obtains a homotopy associative multiplication on its classifying space which is in fact an A_∞-structure because of the pentagon rule. In case of a strict monoidal category the classifying space $B\mathcal{C}$ is a topological monoid.
This paper only deals with small strict monoidal categories, so the terms "small" and "strict" are omitted in the sequel.
If a monoidal category is even braided (cf. [2, p. 315]), the braid isomorphisms describe a natural transformation from the functor $\otimes$ to the functor $\otimes \circ T$, where $T : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ is the switch functor $(A, B) \longmapsto (B, A)$. Hence the classifying space of a braided monoidal category, in particular $B\mathcal{F}_{Sym}$, is a homotopy commutative monoid.—

2.4 Definition: An H-space X is called admissible if it is homotopy associative and if left translation by any given element is homotopic to right translation by the same element. X is called grouplike if $\pi_0 X$ is a group under the induced multiplication .—

Obviously the classifying space of a braided monoidal category is admissible.

2.5 Definition: Let $f : X \longrightarrow Y$ be an H-map between admissible H-spaces, where Y is grouplike. f is called a group completion if $f_* : H_*(X; R) \longrightarrow H_*(Y; R)$ is a localization of the ring $H_*(X; R)$ at its multiplicative submonoid $\pi_0 X$ for every commutative coefficient ring R .—

2.6 S^{-1}-Construction: Let S be a monoidal category. Define a category $S^{-1}S$ in the following way: $S^{-1}S$ has the same objects as $S \times S$, and a morphism from (A_-, A_+) to (B_-, B_+) is an isomorphism class of triples

$$(K,\ \ f_- : K \otimes A_- \longrightarrow B_-,\ \ f_+ : K \otimes A_+ \longrightarrow B_+)$$

where K is an object and f_-, f_+ are morphisms of S. An isomorphism between triples (K, f_-, f_+) and (K', f'_-, f'_+) is an isomorphism $K \xrightarrow{\cong} K'$ in S which makes the following diagrams commute:

$$\begin{tikzcd} K \otimes A_- \arrow[rr, "\cong"] \arrow[dr, "f_-"'] & & K' \otimes A_- \arrow[dl, "f'_-"] \\ & B_- & \end{tikzcd} \qquad \begin{tikzcd} K \otimes A_+ \arrow[rr, "\cong"] \arrow[dr, "f_+"'] & & K' \otimes A_+ \arrow[dl, "f'_+"] \\ & B_+ & \end{tikzcd}$$

$S^{-1}S$ has a canonical monoidal structure: On objects the tensor product is given by $(A_-, A_+) \otimes (B_-, B_+) := (A_- \otimes B_-, A_+ \otimes B_+)$. If F and G are morphisms, represented by triples (K, f_-, f_+) and (L, g_-, g_+) respectively, then $F \otimes G$ is represented by

$$(K \otimes L,\ \ (f_- \otimes g_-) \circ (id \otimes \tau \otimes id),\ \ (f_+ \otimes g_+) \circ (id \otimes \tau \otimes id)),$$

where τ denotes the switch map. The canonical functor $S \longrightarrow S^{-1}S$, given by $A \longmapsto (0, A)$, preserves the monoidal structure.

Theorem: Suppose S satisfies the following conditions:

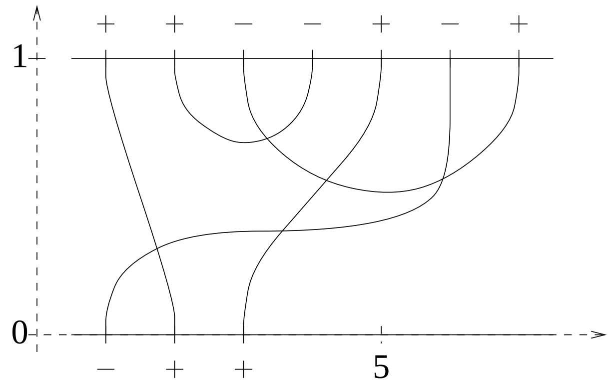
1. S is commutative up to natural isomorphism
2. all morphisms of S are isomorphisms
3. all translations $S \longrightarrow S$ are faithful

Then the canonical functor $S \longrightarrow S^{-1}S$ induces a group completion on the level of classifying spaces .— (See [1])

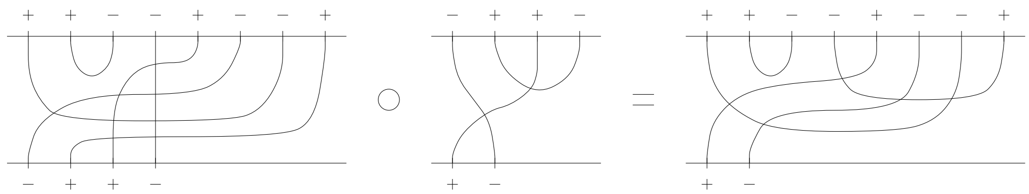
2.7 Definition: Let $\mathcal{G}$ be the following category: The set of objects is the free monoid on the alphabet $\{+, -\}$. We think of an object with k letters as the subset $\{1, \ldots, k\} \subset \mathbb{R}$ with an assigned $+$ or $-$ to each i.
A morphism of a word w with k letters to another word w' with k' letters is represented by an unordered set $\{f_\alpha\}$ of $\frac{1}{2}(k+k')$ paths $f_\alpha : [0,1] \longrightarrow \mathbb{R} \times [0,1]$ subject to the following conditions:

1. Each f_α connects two different points of the subset $\{1, \ldots, k\} \times \{0\} \cup \{1, \ldots, k'\} \times \{1\}$
2. Each point of $\{1, \ldots, k\} \times \{0\} \cup \{1, \ldots, k'\} \times \{1\}$ is in the image of exactly one f_α.
3. Each f_α connects either an element from $\{1, \ldots, k\} \times \{0\}$ with an element from $\{1, \ldots, k'\} \times \{1\}$ having the same sign (called "strings") or two elements from $\{1, \ldots, k'\} \times \{1\}$ having different signs (called "arcs").

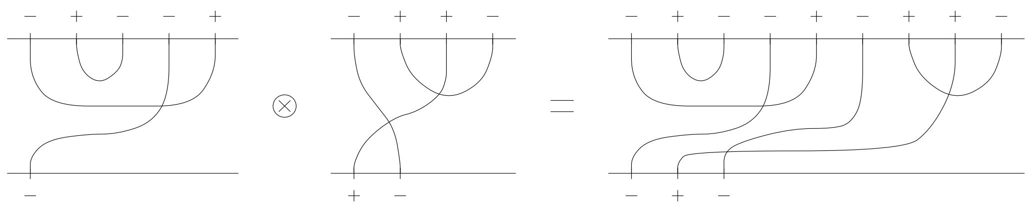
Two such systems $\{f_\alpha\}$ and $\{f'_\alpha\}$ represent the same morphism if there is a family of homotopies $\{H_\alpha : f_\alpha \simeq f'_\alpha\}$ which keep boundary points fixed. Morphisms can be illustrated by pictures like this:



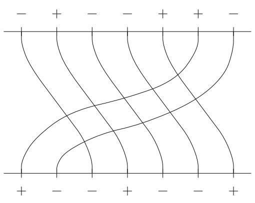
We will call such a picture "pseudo-tangle" because it looks like a tangle with unconsidered crossings.
To find the picture for the composition of two pseudo-tangles $w \longrightarrow w'$ and $w' \longrightarrow w''$, one stacks the latter picture onto the first one, compresses the second coordinate into the unit intervall and forgets the intermediate state:



$\mathcal{G}$ has a natural monoidal structure: On objects the tensor product is given by juxtaposition of words (with the empty word as unit) and on morphisms by putting the appropriate pseudo-tangles side by side and making the necessary adjustments to the endpoints:



Further, $\mathcal{G}$ has a natural braiding, hence its classifying space is homotopy commutative and admissible. The braid isomorphisms $c_{w,w'} : w \otimes w' \longrightarrow w' \otimes w$ are given by pictures of the following pattern:

$\mathbf{c}_{+-,-+--+} =$ 

2.8 Remark: There are two canonical embeddings $i_+, i_- : \mathcal{F}_{Sym} \longrightarrow \mathcal{G}$ which are induced by $+ \longmapsto +$ and $+ \longmapsto -$ respectively and which preserve the monoidal structure strictly.
$\mathcal{F}_{Sym} = \coprod_{n\geq 0} \Sigma_n$ will be identified with its image $i_+(\mathcal{F}_{Sym})$, i.e. the full subcategory of $\mathcal{G}$ which consists of those words that are given by the free monoid on the letter "+". Its morphisms will be called pseudo-braids .—

Now we are able to formulate the main result:

2.9 Main Theorem: There is a monoidal functor $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym} \longrightarrow \mathcal{G}$ which induces a homotopy equivalence of H-spaces on classifying spaces. Hence $B\mathcal{G}$ has the homotopy type of $\Omega^\infty S^\infty$.—

The first statement implies the latter one because the canonical functor $\mathcal{F}_{Sym} \longrightarrow \mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ is a group completion on classifying spaces ($\mathcal{F}_{Sym}$ satisfies the necessary conditions given in the S^{-1}-construction). The rest follows from the following well-known theorem:

2.10 Theorem: The canonical map $\coprod_{n\geq 0} B\Sigma_n \longrightarrow \Omega^\infty S^\infty$ is a group completion .— (See [3] or [5])

3 Proof of the main theorem

The required functor $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym} \longrightarrow \mathcal{G}$ is given by a composition

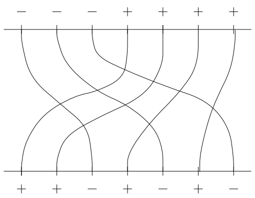
$$\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym} \xrightarrow{J} \mathcal{R} \xrightarrow{\mathcal{I}} \mathcal{G}$$

where $\mathcal{R}$ is a subcategory of $\mathcal{G}$, the inclusion I an equivalence of categories and J an isomorphism.

An object of $\mathcal{G}$ is called ordered, if it is of the pattern $w = --\cdots-++\cdots+$. Let $\mathcal{R}$ be the full subcategory of $\mathcal{G}$ which consists of all ordered words, and let $I : \mathcal{R} \longrightarrow \mathcal{G}$ be the inclusion.

Consider the following functor $R : \mathcal{G} \longrightarrow \mathcal{R}$:
For a given word w let $\hat{w} := R(w)$ be the word which is obtained by ordering its letters, i.e. $R(++-+-) = --+++$ etc.

Let $s_w : w \longrightarrow \hat{w}$ be the isomorphism which orders the letters in that way that the ordering among the letters "+" and "−" is preserved seperately.

$$s_{++-+-+-} =$$


Define R on morphisms $\alpha : w_1 \longrightarrow w_2$ by $R(\alpha) := s_{w_2} \circ \alpha \circ s_{w_1}^{-1}$.
R is obviously well-defined, and the functorial properties are easily checked:

$$R(id_w) = s_w \circ s_w^{-1} = id_{\hat{w}} \qquad \text{for all } \ w,$$

and for any given composition $w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\beta} w_3$ we have

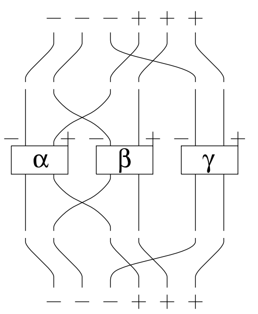
$$R(\beta \circ \alpha) = s_{w_3} \circ \beta \circ \alpha \circ s_{w_1}^{-1} = s_{w_3} \circ \beta \circ s_{w_2}^{-1} \circ s_{w_2} \circ \alpha \circ s_{w_1}^{-1} = R(\beta) \circ R(\alpha)$$

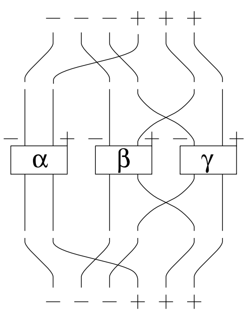
3.1 Lemma: $\mathcal{R}$ is a monoidal category with tensor product $\hat{\otimes} := R \circ \otimes$ and is tensor equivalent to $\mathcal{G}$ by the functors R and I. In particular, the classifying spaces of $\mathcal{R}$ and $\mathcal{G}$ are homotopy equivalent as H-spaces .—

Proof: The following identities are immediate for any objects w_1, w_2, w_3:

$$\begin{array}{lrcl} (1) & s_{(w_1 \hat{\otimes} w_2) \otimes w_3} \circ (s_{w_1 \otimes w_2} \otimes id_{w_3}) & = & s_{w_1 \otimes (w_2 \hat{\otimes} w_3)} \circ (id_{w_1} \otimes s_{w_2 \otimes w_3}) \\ (2) & s_{w_1 \otimes w_2} & = & s_{\hat{w_1} \otimes \hat{w_2}} \circ (s_{w_1} \otimes s_{w_2}) \end{array}$$

$\hat{\otimes} : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ is given by a composition of functors and hence a functor. Obviously the empty word yields a strict unit for $\hat{\otimes}$, and the associativity on objects is also clear. The associativity on morphisms can be shown algebraically by using (1) or is to be seen from the following self-explanatory picture:

$$(\alpha \ \hat{\otimes} \ \beta) \ \hat{\otimes} \ \gamma =$$


$$=$$


$$= \alpha \ \hat{\otimes} (\beta \ \hat{\otimes} \ \gamma)$$

R preserves the monoidal structure strictly:
On objects the equation $R(w \otimes w') = R(w)\hat{\otimes} R(w')$ is evident.
Given any two morphisms $w_1 \xrightarrow{\alpha_1} w_1'$ and $w_2 \xrightarrow{\alpha_2} w_2'$ one uses (2):

$$\begin{aligned} R(\alpha_1 \otimes \alpha_2) &= s_{w_1' \otimes w_2'}(\alpha_1 \otimes \alpha_2) s^{-1}_{w_1 \otimes w_2} \\ &= s_{\hat{w_1'} \otimes \hat{w_2'}}(s_{w_1'} \otimes s_{w_2'})(\alpha_1 \otimes \alpha_2)(s_{w_1} \otimes s_{w_2})^{-1} s^{-1}_{\hat{w}_1 \otimes \hat{w}_2} \\ &= s_{\hat{w_1'} \otimes \hat{w_2'}}(R(\alpha_1) \otimes R(\alpha_2)) s^{-1}_{\hat{w}_1 \otimes \hat{w}_2} \\ &= R(\alpha_1) \hat{\otimes} R(\alpha_2) \end{aligned}$$

The inclusion I is monoidal but not strictly. The necessary isomorphisms $\varphi : I(w) \otimes I(w') \longrightarrow I(w \hat{\otimes} w')$ are in fact given by $s_{w \otimes w'} : w \otimes w' \longrightarrow w \hat{\otimes} w'$ and the usual coherence condition (cf. [2, p. 287]) turns out to be formular (1).

Finally, the identity $R \circ I = Id_{\mathcal{R}}$ is obvious, and a natural tensor isomorphism $Id_{\mathcal{G}} \longrightarrow I \circ R$ is given by the isomorphisms s_w:

$$\begin{tikzcd} Id(w_1) \arrow[r, "s_{w_1}"] \arrow[d, "Id(\alpha)"'] & I \circ R(w_1) \arrow[d, "I \circ R(\alpha)"] \\ Id(w_2) \arrow[r, "s_{w_2}"] & I \circ R(w_2) \end{tikzcd} \quad = \quad \begin{tikzcd} w_1 \arrow[r, "s_{w_1}"] \arrow[d, "\alpha"'] & \hat{w}_1 \arrow[d, "s_{w_2} \circ \alpha \circ s^{-1}_{w_1}"] \\ w_2 \arrow[r, "s_{w_2}"] & \hat{w}_2 \end{tikzcd}$$

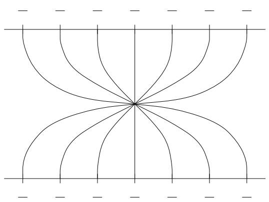
In this case the appropriate coherence condition (cf. [2, p. 288]) is nothing else but formular (2) .—

3.2 Remark:
The proof in fact shows that $B\mathcal{R}$ is a strong deformation retract of $B\mathcal{G}$.—

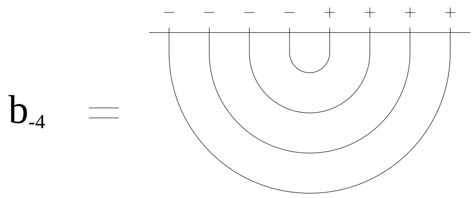
3.3 Lemma:
There is an isomorphism of monoidal categories $J : \mathcal{F}^{-1}_{Sym}\mathcal{F}_{Sym} \longrightarrow \mathcal{R}$.—

Proof: To be able to give an explicit formula for J, some special morphisms of $\mathcal{G}$ have to be introduced.
Let $k \in \mathbb{N}$ denote the object $++\ldots++$ of $\mathcal{G}$ which consists of k letters "+" and $-k = --\ldots--$ the word of k letters "−".
Then, let $t_{-k} : -k \longrightarrow -k$ be the morphism of $\mathcal{G}$ the picture of which is given by the permutation $i \longmapsto k - i$ of Σ_k. We will depict it as follows:

t_{-7} =

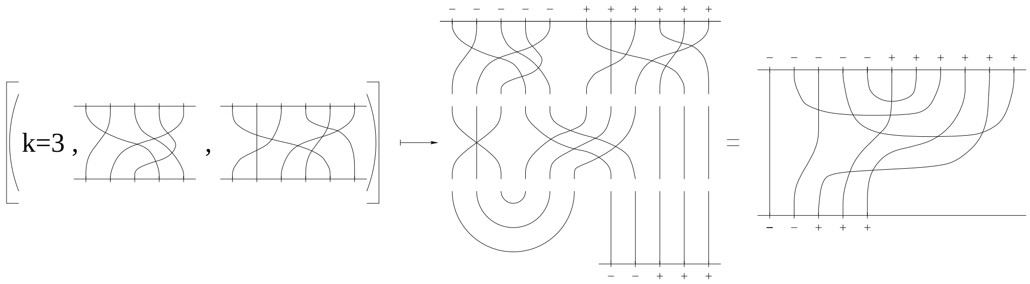


Let $b_{-k} : 0 \longrightarrow -k \otimes k$ denote the morphism which is represented in the following way:



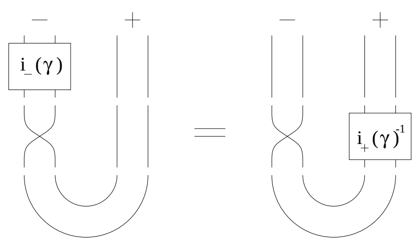
Finally, recall the braid isomorphism $c_{-m,n} : -m \otimes n \longrightarrow n \otimes -m$ of $\mathcal{G}$ and from 2.8 the embeddings $i_+, i_- : \mathcal{F}_{Sym} \longrightarrow \mathcal{G}$.
Now define $J(m,n) := -m \otimes n$ on objects, and if we are given a morphism $\alpha : (m,n) \longrightarrow (k+m, k+n)$, represented by some triple (k, α_-, α_+), let

$$J(\alpha) := (i_-(\alpha_-) \otimes i_+(\alpha_+)) \circ (t_{-k} \otimes c_{k,-n} \otimes id_m) \circ (b_{-k} \otimes id_{-n} \otimes id_m)$$

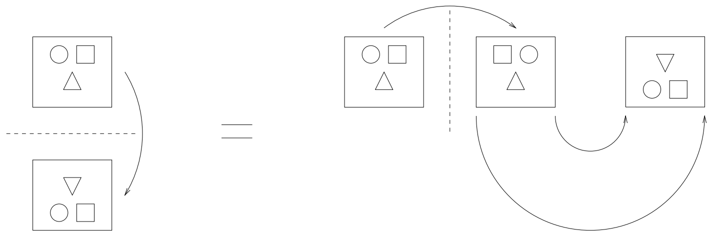


Note: $J(\alpha)$ is indeed a morphism in $\mathcal{R}$ although its constituents are morphisms in $\mathcal{G}$, and the number of arcs in $J(\alpha)$ is equal to k.
The definition of J does not depend on the representing triple (k, α_-, α_+): Let $(k', \alpha'_-, \alpha'_+)$ be another triple which represents the same morphism. Then we have $k = k'$ and there exists an isomorphism γ in $\mathcal{F}_{Sym}$ (i.e. $\gamma \in \Sigma_k$) with $\alpha'_- = \alpha_- \circ (\gamma \otimes id_m)$ and $\alpha'_+ = \alpha_+ \circ (\gamma \otimes id_n)$.
The crucial step in this proof is the observation of the following equation:

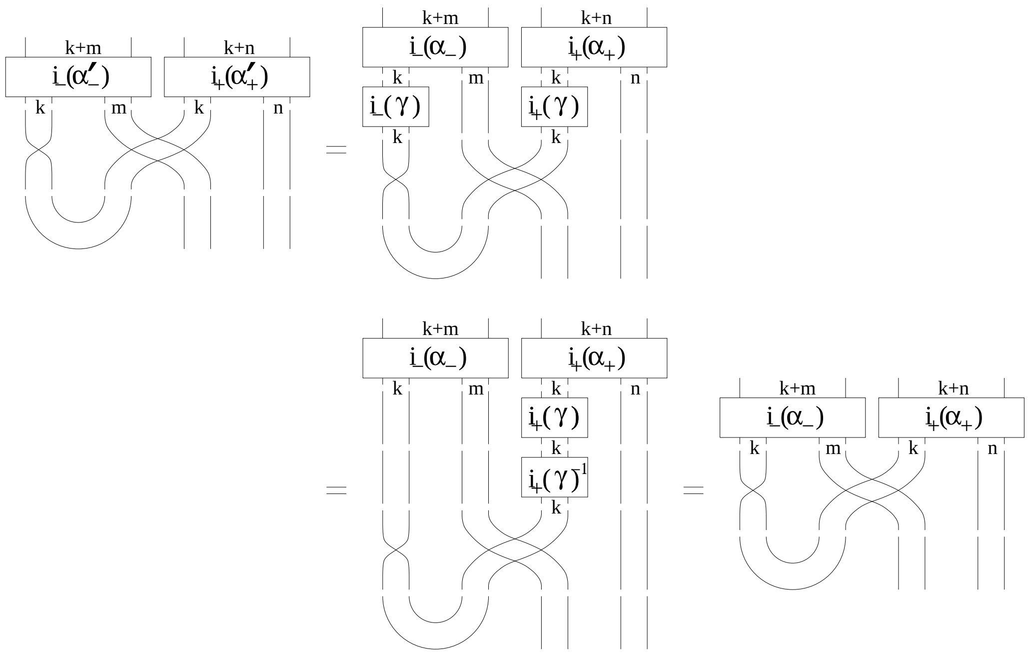
$$((i_-(\gamma) \circ t_{-k}) \otimes id_k) \circ b_{-k} = (t_{-k} \otimes i_+(\gamma)^{-1}) \circ b_{-k} \qquad \text{for all } \gamma \in \Sigma_k$$



It is clear that the inverse of a given pseudo-braid is obtained by reflecting it at its bottom base. But it also can be built by reflecting it at a vertical axis and then rotating it by $180°$ afterwards:

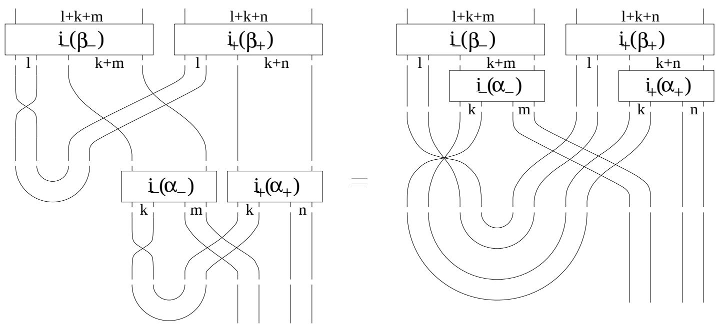


This is exactly what is done by pushing $i_-(\gamma)$ through t_{-k} and b_{-k}. Hence the desired relation is true, and using this we have:



The functoriality of J is checked as follows: The identity of some object (m,n) in $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ has the only representant $(0, id_m, id_n)$. In this case the three morphisms t, c and b are identities and hence the equation $J(id_{m,n}) = id_{-m\otimes n}$ becomes obvious.
For any given composition $(m,n) \xrightarrow{\alpha} (k+m, k+n) \xrightarrow{\beta} (l+k+m, l+k+n)$ the relation $J(\beta \circ \alpha) = J(\beta) \circ J(\alpha)$ is deduced from the following pictures:

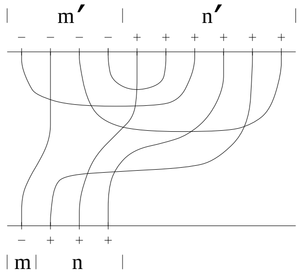
$J(\beta)\circ J(\alpha) =$



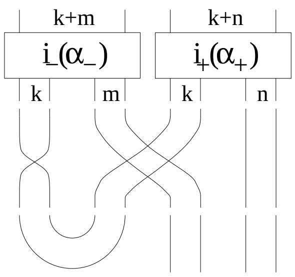
$= J(\beta\circ\alpha)$

The functor J is an isomorphism of categories if it is fully faithful since it is obviously bijective on objects. Suppose we are given a pseudo-tangle $\varphi : -m \otimes n \longrightarrow -m' \otimes n'$ and we are asking for triples (k, α_-, α_+) representing morphisms in $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ which map to φ.

$\varphi =$

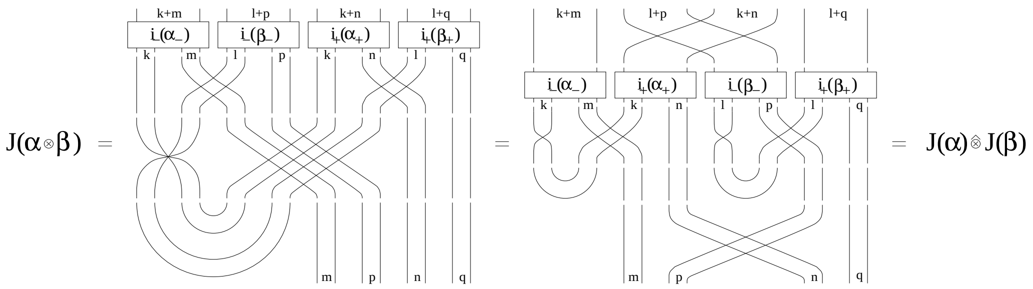


Firstly we observe $m' = k + m, \;\; n' = k + n$ where $k \in \mathbb{N}$ is the number of arcs in φ because every arc adds two letters of different sign to the source object. This number k is inevitably the object needed in the representing triple, and α_- and α_+ are morphisms $k + m \longrightarrow k + m$ and $k + n \longrightarrow k + n$ respectively in $\mathcal{F}_{Sym}$. As the reader has already seen, J applies the following machinery to α_- and α_+:



So, those strings of α_- (resp. α_+) which have their lower end among the last m (resp. n) strings are already completely determined. The other strings contribute to arcs and are not unique. But every concrete choice for the remaining k strings of α_- determines the last k strings of α_+ and vice versa. There are $k!$ such choices and they correspond by the simultaneous action of Σ_k on the first k strings of the lower bound of α_- and α_+ respectively. But this leads precisely to the notion of isomorphic triples in the construction of $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$. Hence all representing triples are isomorphic, and there is exactly one morphism in $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ which maps to φ, i.e. J is in fact an isomorphism of categories.

We finally have to show the preservation of the monoidal structure by J, i.e. the equation $J(\alpha \otimes \beta) = J(\alpha)\hat{\otimes}J(\beta)$ which is directly clear for every two objects. So let (k, α_-, α_+), (l, β_-, β_+) be representants for α and β respectively. Then the desired equation is proved by the following pictures:



This completes the proof of this lemma and of the main theorem .—

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