Derived equivalence and stable equivalence of repetitions of algebras of finite global dimension

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Abstract. For repetitions (= repetitive "algebras") of algebras of finite global dimension we will show that there exists a triangle equivalence between their bounded derived categories of finitely presented modules iff there exists a triangle equivalence between their stable categories of finitely presented modules.

Introduction

Throughout this paper k denotes an algebraically closed field. The word algebra always means a finite-dimensional k-algebra with identity, and all categories and functors are assumed to be k-linear. Basic algebras are regarded as *spectroids* (= locally finite-dimensional categories) which are *finite* (=having only finitely many objects)([**2**]).

Consider the following sentences for *locally bounded spectroids* (= locally bounded categories) R and S:

- $\mathrm{D}(R, S)$ There is a triangle equivalence between the derived categories of the categories of modules over R and S (in this case R and S are called *derived equivalent*);
- $\mathrm{Db}(R, S)$ There is a triangle equivalence between the bounded derived categories of the categories of finitely presented modules over R and S (in this case R and S are called *derivatively equivalent* in [**2**, 12.5]); and
- $\mathrm{St}(R, S)$ (In the case that both R and S are selfinjective,) there is a triangle equivalence between the stable categories of the categories of finitely presented modules over R and S (in this case R and S are called *stably equivalent*).

Then first, by Rickard [**8**, Corollary 8.3] $\mathrm{D}(R, S)$ and $\mathrm{Db}(R, S)$ are equivalent if both R and S are finite. But if they have infinitely many objects, then this equivalence is not clear because Rickard's argument used in [**8**, Proposition 8.1] does not work in this case (see Example 2.6).

Second, assume that both R and S are selfinjective. Then by Keller-Vossieck [**6**, 2.3 Exemple] (or by the categorical version of Rickard [**9**, Theorem 2.1]) we see that $\mathrm{Db}(R, S)$ implies $\mathrm{St}(R, S)$. Sometimes the converse holds; e.g., in the case that R is a representation-finite selfinjective algebra of class A_n ([**1**, Theorem 5.8]).

1991 *Mathematics Subject Classification.* 16G30, 18E30.

In this paper we show that the converse statement holds for repetitions (= repetitive "algebras") $A^{\mathbb{Z}}$ of algebras A of finite global dimension. Note that repetitions of (basic) algebras are selfinjective locally bounded spectroids and that locally bounded spectroids of this form are not very special (see Pogorzały [**7**, Theorem]).

Now let A and B be algebras. Then by Happel [**3**] or by Keller-Vossieck [**6**, 3.1] $\mathrm{St}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ implies $\mathrm{Db}(A, B)$ when both A and B are of finite global dimension. As explained above $\mathrm{Db}(A, B)$ and $\mathrm{D}(A, B)$ are equivalent. Further by [**1**, Theorem 1.5] $\mathrm{D}(A, B)$ implies $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$. Hence if $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ implies $\mathrm{Db}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$, then we are done. But as explained above this is not clear. In section 2 we then investigate when $\mathrm{D}(R, S)$ implies $\mathrm{Db}(R, S)$ for arbitrary spectroids R, S. We consider an intermediate condition $\mathrm{H}(R, S)$ that there is a triangle equivalence from the homotopy category of right bounded complexes of finitely generated projective modules over R to that over S. We show that $\mathrm{D}(R, S)$ implies $\mathrm{H}(R, S)$ if and only if the triangle equivalence in $\mathrm{D}(R, S)$ is *of right bounded type* (Lemma 2.3, Definition 2.4); and that if this is the case, then $\mathrm{D}(R, S)$ implies $\mathrm{Db}(R, S)$ (Lemma 2.5). In section 3 we show that the triangle equivalence in $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ constructed from that in $\mathrm{D}(A, B)$ as in [**1**] is of right bounded type (Proposition 3.1), which completes the proof of the assertion.

1. Preliminaries

For an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$, $\mathcal{C}^{\mathrm{b}}(\mathcal{A})$ and $\mathcal{C}^{-}(\mathcal{A})$ the category of differential complexes, the category of bounded differential complexes and the category of right bounded differential complexes in $\mathcal{A}$, respectively; and by $\mathcal{H}(\mathcal{A})$, $\mathcal{H}^{\mathrm{b}}(\mathcal{A})$ and $\mathcal{H}^{-}(\mathcal{A})$ the corresponding homotopy categories. When $\mathcal{A}$ is an abelian category, we denote by $\mathcal{D}(\mathcal{A})$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the corresponding derived categories. For each morphism f in $\mathcal{C}(\mathcal{A})$, $C(f)$ denotes the *mapping cone* of f; and $\overline{f}$ denotes the homotopy class of f in $\mathcal{H}(\mathcal{A})$.

Let R be a spectroid. We denote by $\mathrm{Mod}\,R$ the category of all (right) R-modules; by $\mathrm{mod}\,R$ the full subcategory of $\mathrm{Mod}\,R$ consisting of finitely presented objects; by $\mathrm{Pro}\,R$ the full subcategory of $\mathrm{Mod}\,R$ consisting of projective objects; by $\mathrm{pro}\,R$ the full subcategory of $\mathrm{Pro}\,R$ consisting of finitely generated projective objects; by $\mathcal{X}_R$ the full subcategory of $\mathrm{pro}\,R$ consisting of representable functors $R(-, x)$ for some $x \in R$; and by $\underline{\mathrm{mod}}\,R$ the (projectively) stable category of $\mathrm{mod}\,R$. We set $D := \mathrm{Hom}_k(-, k)$. By $\eta = \eta_R : R \to \mathcal{X}_R$ we denote the Yoneda isomorphism defined by $\eta(x) = R(-, x)$ for all $x \in R$. As in the notation $R(-, x)$, by (-) or (?) we denote a place for a variable. When we want to stress the difference of variables, we use different symbols like $R(-, ?)$.

Further $\mathcal{H}_{\mathrm{p}}(R)$ denotes the full subcategory of $\mathcal{H}(\mathrm{Mod}\,R)$ consisting of *homotopically projective* objects, i.e., formed by complexes P of R-modules such that $\mathcal{H}(\mathrm{Mod}\,R)(P, N) = 0$ for all acyclic complexes N of R-modules. (In Keller [**5**] homotopically projective complexes are called DG modules having property (P).) The inclusion $\mathcal{H}_{\mathrm{p}}(R) \to \mathcal{H}(\mathrm{Mod}\,R)$ admits the right S-adjoint ([**6**]) $\mathbf{p} : \mathcal{H}(\mathrm{Mod}\,R) \to \mathcal{H}_{\mathrm{p}}(R)$ and the canonical functor $\mathcal{H}(\mathrm{Mod}\,R) \to \mathcal{D}(\mathrm{Mod}\,R)$ induces a triangle equivalence $\mathcal{H}_{\mathrm{p}}(R) \to \mathcal{D}(\mathrm{Mod}\,R)$. By $\mathbf{p}$ we also denote a quasi-inverse of this equivalence. (See [**5**] for details.)

We recall some definitions needed later.

DEFINITION 1.1. We say that spectroids A and B are *derived equivalent* if one of the following equivalent conditions holds ([**1**], [**5**], [**8**]):

(1) There is a triangle equivalence $\varphi : \mathcal{D}(\operatorname{Mod} B) \to \mathcal{D}(\operatorname{Mod} A)$; and
(2) There is a full subcategory E of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)$ with the following properties:
   (a) For any $T, U \in E$ and for any $n \neq 0$, $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)(T, U[n]) = 0$;
   (b) $\mathcal{X}_A$ is contained in the smallest full triangulated subcategory of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)$ closed under direct summands and isomorphisms that contains E; and
   (c) E is isomorphic to B.

In this case φ is called a *derived equivalence* from B to A; the triple (A, E, B) is called a *tilting triple*, and E is called a *tilting spectroid* for A.

DEFINITION 1.2. Let A be a spectroid.

(1) We define a spectroid, the *repetition* $A^{\mathbb{Z}}$ of A as follows (Cf. [**2**]). Objects are the pairs $x^{[n]} := (x, n)$ with $x \in A$ and $n \in \mathbb{Z}$.

$$A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) := \begin{cases} \{f^{[n]} := (f, n) | f \in A(x, y)\} & \text{if } m = n; \\ \{\varphi^{[n]} := (\varphi, n) | \varphi \in DA(y, x)\} & \text{if } m = n + 1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The composition

$$A^{\mathbb{Z}}(y^{[m]}, z^{[l]}) \times A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) \to A^{\mathbb{Z}}(x^{[n]}, z^{[l]})$$

is given as follows:

   (i) If $m = n, l = m$, then this is the composition of A:

$$A(y, z) \times A(x, y) \to A(x, z).$$

   (ii) If $m = n, l = m+1$, then this is given by the right A-module structure of $DA(-, ?)$:

$$DA(z, y) \times A(x, y) \to DA(z, x).$$

   (iii) If $m = n + 1, l = m$, then this is given by the left A-module structure of $DA(-, ?)$:

$$A(y, z) \times DA(y, x) \to DA(z, x).$$

   (iv) Otherwise the composition is zero.

(2) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $A^{\mathbb{Z}}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow A^{\mathbb{Z}}, x \mapsto x^{[n]}$, the embedding functor.
(3) For each $n \in \mathbb{Z}$ and for each $P \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)$, we set $P^{[n]} := P \otimes_A A^{\mathbb{Z}}(-, \mathbb{1}^{[n]}?) \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} A^{\mathbb{Z}})$. For each full subcategory E of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)$ we denote by $\hat{E}$ the full subcategory of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} A^{\mathbb{Z}})$ consisting of $P^{[n]}$ with $P \in E$, $n \in \mathbb{Z}$.

Note that if a spectroid A is locally bounded, then so is $A^{\mathbb{Z}}$.

2. From derived equivalence to derivative equivalence

DEFINITION 2.1. Let R be a spectroid.

(1) For each $T \in \mathcal{D}(\operatorname{Mod} R)$, we set $r(T) := \sup\{i \in \mathbb{Z} | T^i \neq 0\}$ and $\tilde{r}(T) := \sup\{i \in \mathbb{Z} | H^i(T) \neq 0\}$. In particular, we define $r(0) := -\infty$.
(2) A subcategory E of $\mathcal{D}(\operatorname{Mod} R)$ is said to be *right bounded* in case there is some $b \in \mathbb{Z}$ such that for any $T \in E$ we have $\tilde{r}(T) \leq b$.

The following are easily verified.

REMARK 2.2. Let R be a spectroid and $\mathcal{H}$ one of the categories $\mathcal{H}^-(\operatorname{Pro} R)$, $\mathcal{H}^-(\operatorname{pro} R)$ and $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$.

(1) If $X \cong Y$ in $\mathcal{D}(\operatorname{Mod} R)$, then $\tilde{r}(X) = \tilde{r}(Y)$.
(2) Any $P \in \mathcal{H}$ has a decomposition $P = P' \oplus P''$ with $P', P'' \in \mathcal{H}$ as complexes such that $P \cong P'$, $P'' \cong 0$ in $\mathcal{H}$ and $\tilde{r}(P) = r(P') = \tilde{r}(P')$.
(3) Let E be a subcategory of $\mathcal{D}(\operatorname{Mod} R)$ all the objects of which are isomorphic to objects of $\mathcal{H}$. Then E is right bounded iff there is some $b \in \mathbb{Z}$ such that for any $T \in E$ there exists some $T' \cong T$ in $\mathcal{H}$ with $r(T') \leq b$.
(4) Let $X \to Y \to Z \to X[1]$ be a triangle in $\mathcal{D}(\operatorname{Mod} R)$. Then $\tilde{r}(Y) \leq \max\{\tilde{r}(X), \tilde{r}(Z)\}$.

LEMMA 2.3. *Let R, S be spectroids, and $\varphi : \mathcal{D}(\operatorname{Mod} R) \to \mathcal{D}(\operatorname{Mod} S)$ a triangle equivalence with a quasi-inverse φ'. Then the following are equivalent.*

(1) *φ induces a triangle equivalence $\mathcal{H}^-(\operatorname{pro} R) \to \mathcal{H}^-(\operatorname{pro} S)$; and*
(2) *Both $\varphi(\mathcal{X}_R)$ and $\varphi'(\mathcal{X}_S)$ are right bounded.*

PROOF. (2) $\Rightarrow$(1). By symmetry it is enough to show the following.

CLAIM. *If $\varphi(\mathcal{X}_R)$ is right bounded, then for every $P \in \mathcal{H}^-(\operatorname{pro} R)$ there exists some $C \in \mathcal{H}^-(\operatorname{pro} S)$ such that $\varphi(P) \cong C$ in $\mathcal{D}(\operatorname{Mod} S)$.*

Since $X \cong \mathbf{p}X$ in $\mathcal{D}(\operatorname{Mod} S)$ for all $X \in \mathcal{D}(\operatorname{Mod} S)$, it is enough to show the claim for $\mathbf{p}\varphi$ instead of φ. Hence we may regard φ as a triangle equivalence $\mathcal{H}_{\mathrm{p}}(R) \to \mathcal{H}_{\mathrm{p}}(S)$. Put $P = (P^i, d^i)$. We may assume that $P^i = 0$ for all $i > 0$. Put for each $i \geq 0$,

$$F_i := (\cdots \to 0 \to P^{-i} \xrightarrow{d^{-i}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \dots) \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R),$$

and define morphisms $f_i := (f_i^j) : F_i \to F_{i+1}$ in $\mathcal{C}(\operatorname{Mod} R)$ for all $i \geq 0$ by $f_i^j := \mathbb{1}_{P^j}$ if $-i \leq j \leq 0$ and $f_i^j := 0$ otherwise. Then we have triangles

$$F_i \xrightarrow{\overline{f_i}} F_{i+1} \longrightarrow P^{-(i+1)}[i+1] \longrightarrow F_i[1]$$

for all $i \geq 0$ and a triangle

$$\bigoplus_{p\geq 0} F_p \xrightarrow{\overline{\Phi}} \bigoplus_{q\geq 0} F_q \longrightarrow P \longrightarrow \big(\bigoplus_{p\geq 0} F_p\big)[1]$$

in $\mathcal{H}_{\mathrm{p}}(R)$, where

$$\Phi := \begin{pmatrix} \mathbb{1}_{F_0} & & & \\ -f_0 & \mathbb{1}_{F_1} & & \\ & -f_1 & \mathbb{1}_{F_2} & \\ & & \ddots & \ddots \end{pmatrix}.$$

The functor φ gives us triangles

$$\varphi(F_i) \xrightarrow{\varphi(\overline{f_i})} \varphi(F_{i+1}) \longrightarrow \varphi(P^{-(i+1)})[i+1] \longrightarrow \varphi(F_i)[1]$$

for all $i \geq 0$ and a triangle

$$\bigoplus_{p\geq 0} \varphi(F_p) \xrightarrow{\Phi_*} \bigoplus_{q\geq 0} \varphi(F_q) \longrightarrow \varphi(P) \longrightarrow \big(\bigoplus_{p\geq 0} \varphi(F_p)\big)[1]$$

in $\mathcal{H}_{\mathrm{p}}(S)$, where

$$\Phi_* := \begin{pmatrix} \overline{\mathbb{1}}_{\varphi(F_0)} & & & \\ -\varphi(\overline{f_0}) & \overline{\mathbb{1}}_{\varphi(F_1)} & & \\ & -\varphi(\overline{f_1}) & \overline{\mathbb{1}}_{\varphi(F_2)} & \\ & & \ddots & \ddots \end{pmatrix}.$$

Now since $\varphi(\mathcal{X}_R)$ is right bounded, $\varphi(\operatorname{pro} R)$ is also right bounded. Thus by Remark 2.2 there is some $b \in \mathbb{Z}$ such that for each $T \in \varphi(\operatorname{pro} R)$ we have some $T' \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} S)$ with $T' \cong T$ in $\mathcal{H}_{\mathrm{p}}(S)$ and with $r(T') \le b$. In particular, for each $i \ge 0$, there exists some $Y_i \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} S)$ such that $Y_i \cong \varphi(P^{-i})[i]$ in $\mathcal{H}_{\mathrm{p}}(S)$ and that $r(Y_i) \le b-i$. Put $X_i := \varphi(F_i)$ and $\overline{u_i} := \varphi(\overline{f_i})$ for all $i \ge 0$. Then we have triangles

$$Y_{i+1}[-1] \xrightarrow{\overline{w_i}} X_i \xrightarrow{\overline{u_i}} X_{i+1} \xrightarrow{\overline{v_i}} Y_{i+1}$$

for all $i \ge 0$. Choose an isomorphism $\overline{t_0} : X_0 \xrightarrow{\sim} Y_0$ in $\mathcal{H}_{\mathrm{p}}(S)$. From this $\overline{t_0}$ construct inductively isomorphisms $\overline{t_i} : X_i \xrightarrow{\sim} C(t_{i-1}w_{i-1})$ for all $i \ge 1$ so that we have a commutative diagram

$$\begin{tikzcd} Y_0 \arrow[r, "\overline{u'_0}"] & C(t_0w_0) \arrow[r] & \cdots \arrow[r] & C(t_{i-1}w_{i-1}) \arrow[r, "\overline{u'_i}"] & C(t_iw_i) \arrow[r] & \cdots \\ X_0 \arrow[u, "\overline{t_0}"] \arrow[r, "\overline{u_0}"] & X_1 \arrow[u, "\overline{t_1}"] \arrow[r] & \cdots \arrow[r] & X_i \arrow[u, "\overline{t_i}"] \arrow[r, "\overline{u_i}"] & X_{i+1} \arrow[u, "\overline{t_{i+1}}"] \arrow[r] & \cdots \end{tikzcd}$$

by using commutative diagrams with triangle rows:

$$\begin{tikzcd} Y_1[-1] \arrow[r, "\overline{t_0w_0}"] & Y_0 \arrow[r, "\overline{u'_0}"] & C(t_0w_0) \arrow[r] & Y_1 \\ Y_1[-1] \arrow[u, "\mathbf{1}"] \arrow[r, "\overline{w_0}"'] & X_0 \arrow[u, "\overline{t_0}"] \arrow[r, "\overline{u_0}"'] & X_1 \arrow[u, "\overline{t_1}"] \arrow[r, "\overline{v_0}"'] & Y_1 \arrow[u, "\mathbf{1}"] \end{tikzcd}, \dots,$$

$$\begin{tikzcd} Y_{i+1}[-1] \arrow[r, "\overline{t_iw_i}"] & C(t_{i-1}w_{i-1}) \arrow[r, "\overline{u'_i}"] & C(t_iw_i) \arrow[r] & Y_{i+1} \\ Y_{i+1}[-1] \arrow[u, "\mathbf{1}"] \arrow[r, "\overline{w_i}"'] & X_i \arrow[u, "\overline{t_i}"] \arrow[r, "\overline{u_i}"'] & X_{i+1} \arrow[u, "\overline{t_{i+1}}"] \arrow[r, "\overline{v_i}"'] & Y_{i+1} \arrow[u, "\mathbf{1}"] \end{tikzcd}, \dots$$

where u'_i are the canonical injections in $\mathcal{C}(\operatorname{Mod} S)$. Let C be the union of the $C(t_pw_p)$. Then we have a commutative diagram with triangle rows:

$$\begin{tikzcd} Y_0 \oplus \bigoplus_{p\ge 0} C(t_pw_p) \arrow[r, "\overline{\Phi'}"] & Y_0 \oplus \bigoplus_{q\ge 0} C(t_qw_q) \arrow[r] & C \arrow[r] & (Y_0 \oplus \bigoplus_{p\ge 0} C(t_pw_p))[1] \\ \bigoplus_{p\ge 0} X_p \arrow[u, "\bigoplus_{p\ge 0}\overline{t_p}"] \arrow[r, "\Phi_*"'] & \bigoplus_{q\ge 0} X_q \arrow[u, "\bigoplus_{q\ge 0}\overline{t_q}"] \arrow[r] & \varphi(P) \arrow[r] & (\bigoplus_{p\ge 0} X_p)[1], \end{tikzcd}$$

in $\mathcal{H}_{\mathrm{p}}(S)$, where

$$\Phi' := \begin{pmatrix} \mathbb{1}_{Y_0} & & & \\ -u_0' & \mathbb{1}_{C(t_0 w_0)} & & \\ & -u_1' & \mathbb{1}_{C(t_{-1} w_{-1})} & \\ & & \ddots & \ddots \end{pmatrix}.$$

Using the axiom of triangulated categories and the five lemma we see that $C \cong \varphi(P)$ in $\mathcal{H}_{\mathrm{p}}(S)$. Since as graded modules $C(t_i w_i) = C(t_{i-1} w_{i-1}) \oplus Y_{i+1} = \cdots = Y_0 \oplus Y_1 \oplus \cdots Y_i \oplus Y_{i+1}$, we have $C = \bigoplus_{i \geq 0} Y_i$ as graded modules. It follows from $r(Y_i) \leq b - i$ that $C^t = 0$ for all $t > b$ and that $C^t \in \operatorname{pro} S$ for all $t \in \mathbb{Z}$. Therefore $C \in \mathcal{H}^-(\operatorname{pro} S)$, as desired.

(1) $\Rightarrow$(2). Assume that (1) holds. Suppose that $\varphi(\mathcal{X}_R)$ is not right bounded. Then there exist a sequence $(P_i)_{i \geq 0}$ of objects of $\mathcal{X}_R$ such that $\tilde{r}(\varphi(P_i)[i]) < \tilde{r}(\varphi(P_{i+1})[i+1])$ for all $i \geq 0$. Put

$$P := \bigoplus_{i \geq 0} P_i[i] = (\cdots \to P_2 \xrightarrow{\ 0\ } P_1 \xrightarrow{\ 0\ } P_0 \to 0 \to 0 \to \cdots).$$

Then P is in $\mathcal{H}^-(\operatorname{pro} R)$ but by construction, $\varphi(P) = \bigoplus_{i \geq 0} \varphi(P_i)[i] \not\cong C$ for all $C \in \mathcal{H}^-(\operatorname{pro} S)$, a contradiction. Hence $\varphi(\mathcal{X}_R)$ must be right bounded. Similarly $\varphi'(\mathcal{X}_S)$ must be right bounded. $\square$

Definition 2.4. A derived equivalence φ between spectroids R and S is said to be *of right bounded type* if φ satisfies the condition (1) or (2) in Lemma 2.3.

Lemma 2.5. *Let R, S be spectroids, and φ a derived equivalence from R to S. If φ is of right bounded type, then φ induces a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} R) \to \mathcal{D}^{\mathrm{b}}(\operatorname{mod} S)$.*

Proof. We denote by $\mathcal{H}^-_{\mathrm{b}}(\operatorname{pro} R)$ and by $\mathcal{H}^{\mathrm{b}}_{\leq N}(\operatorname{pro} R)$ for all $N \in \mathbb{Z}$, the full subcategory of $\mathcal{H}^-(\operatorname{pro} R)$ formed by those objects X with $H^n(X) = 0$ for almost all n, and the full subcategory of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$ formed by those objects Y with $H^n(Y) = 0$ for all $n > N$, respectively. First we show the following.

Claim 1. *For each $N \in \mathbb{Z}$, there exists an $N' \in \mathbb{Z}$ such that $\varphi(\mathcal{H}^{\mathrm{b}}_{\leq N}(\operatorname{pro} R)) \subseteq \mathcal{H}^{\mathrm{b}}_{\leq N'}(\operatorname{pro} S)$.*

In fact, by Lemma 2.3 it follows from the assumption that we have some $b \in \mathbb{Z}$ with $\varphi(\mathcal{X}_R) \subseteq \mathcal{H}^{\mathrm{b}}_{\leq b}(\operatorname{pro} S)$. Thus $\varphi(\operatorname{pro} R) \subseteq \mathcal{H}^{\mathrm{b}}_{\leq b}(\operatorname{pro} S)$. Let $N \in \mathbb{Z}$ and $T := (T^i, d^i)_{i \in \mathbb{Z}} \in \mathcal{H}^{\mathrm{b}}_{\leq N}(\operatorname{pro} R)$. Then $\tilde{r}(T) \leq N$. It is enough to show that $\varphi(T) \in \mathcal{H}^{\mathrm{b}}_{\leq N+b}(\operatorname{pro} S)$. This is trivial for $T = 0$. So assume that $T \neq 0$. Put $p := \min\{i \in \mathbb{Z} | T^i \neq 0\}$, $q := r(T)$ and $l(T) := q - p + 1 \geq 1$. By Remark 2.2 we may assume that $q = \tilde{r}(T)$. Then $q \leq N$. We show by induction on $l(T)$ that $\varphi(T) \in \mathcal{H}^{\mathrm{b}}_{\leq N+b}(\operatorname{pro} S)$.

Case $l(T) = 1$. In this case $p = q$. $T = T^q[-q]$. Hence $\varphi(T) = \varphi(T^q)[-q] \in \mathcal{H}^{\mathrm{b}}_{\leq b+q}(\operatorname{pro} S) \subseteq \mathcal{H}^{\mathrm{b}}_{\leq N+b}(\operatorname{pro} S)$.

Case $l(T) > 1$. Put

$$T' := (\cdots \to 0 \to T^{p+1} \xrightarrow{d^{p+1}} \cdots \xrightarrow{d^{q-1}} T^q \to 0 \to \cdots)\cdot$$

Then we have a triangle of the form $T' \to T \to T^p[-p] \to T'[1]$ in $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, R)$, which yields a triangle $\varphi(T') \to \varphi(T) \to \varphi(T^p[-p]) \to \varphi(T')[1]$ in $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, S)$. Since $q \le N$, we have $T^p[-p], T' \in \mathcal{H}^{\mathrm{b}}_{\le N}(\mathrm{pro}\, R)$ and by construction $l(T^p[-p]), l(T') < l$. Hence by the induction hypothesis, we have $\varphi(T^p)[-p], \varphi(T') \in \mathcal{H}^{\mathrm{b}}_{N+b}(\mathrm{pro}\, S)$. Thus $\varphi(T) \in \mathcal{H}^{\mathrm{b}}_{\le N+b}(\mathrm{pro}\, S)$ by Remark 2.2. This proves the claim.

Next we show the following.

CLAIM 2. *The following are equivalent for all $X \in \mathcal{H}^-(\mathrm{pro}\, R)$:*

(1) *$X \in \mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, R)$; and*

(2) *For each $N \in \mathbb{Z}$, there exists an $n_0 \in \mathbb{Z}$ such that for any $Y \in \mathcal{H}^{\mathrm{b}}_{\le N}(\mathrm{pro}\, R)$ and for any $n \ge n_0$, we have $\mathcal{H}^-(\mathrm{pro}\, R)(Y[n], X) = 0$.*

(1) $\Rightarrow$(2): let X be in $\mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, R)$ and $\{i \in \mathbb{Z} | H^i(X) \neq 0\} \subseteq [p,q]$. Then (2) holds if we take $n_0 > N - p$ for all $N \in \mathbb{Z}$.

(2) $\Rightarrow$(1): Assume that (2) holds. Then in particular there exists an $n_0 \in \mathbb{Z}$ such that for any $Y \in \mathrm{pro}\, R$ and for any $n \ge n_0$ we have $\mathcal{H}^-(\mathrm{pro}\, R)(Y[n], X) = 0$. Thus $H^{-n}(X) = 0$. This proves the claim.

Now we show that $\varphi(\mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, R)) \subseteq \mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, S)$. Let φ' be a quasi-inverse of φ. Applying Claim 1 to φ', we obtain that for each $N \in \mathbb{Z}$, there exists an $N' \in \mathbb{Z}$ with $\varphi'(\mathcal{H}^{\mathrm{b}}_{\le N}(\mathrm{pro}\, S)) \subseteq \mathcal{H}^{\mathrm{b}}_{\le N'}(\mathrm{pro}\, R)$. Let $X \in \mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, R)$. By Claim 2, there exists an $n_0 \in \mathbb{Z}$ such that for any $Z \in \mathcal{H}^{\mathrm{b}}_{\le N'}(\mathrm{pro}\, R)$ and for any $n \ge n_0$, we have $\mathcal{H}^-(\mathrm{pro}\, R)(Z[n], X) = 0$. Then for any $Y \in \mathcal{H}^{\mathrm{b}}_{\le N}(\mathrm{pro}\, S)$ and for any $n \ge n_0$ we have $\mathcal{H}^-(\mathrm{pro}\, S)(\varphi(\varphi'(Y)[n]), \varphi(X)) = 0$, thus $\mathcal{H}^-(\mathrm{pro}\, S)(Y[n], \varphi(X)) = 0$. Therefore by applying Claim 2 to S, we see that $\varphi(X) \in \mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, S)$. Similarly we have $\varphi'(\mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, S)) \subseteq \mathcal{H}^-_{\mathrm{b}}(\mathrm{pro}\, R)$. Hence the assertion follows. $\square$

EXAMPLE 2.6. There exist locally bounded spectroids R and S and a triangle equivalence $\varphi : \mathcal{D}(\mathrm{Mod}\, R) \to \mathcal{D}(\mathrm{Mod}\, S)$ which does not induce a triangle equivalence $\mathcal{H}^-(\mathrm{pro}\, R) \to \mathcal{H}^-(\mathrm{pro}\, S)$ but induces a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, R) \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, S)$. Hence in particular, the converse of the previous lemma does not hold in general. For instance, let Q be the quiver whose vertex set is $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and whose arrow set is empty, and set $R = S$ to be the locally bounded spectroid kQ. Let E be the full subcategory of $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, S)$ formed by $S(-,n)[-n]$ with $n \in \mathbb{N}_0$. Then it is easy to check that (S, E, R) is a tilting triple. Construct a triangle equivalence $\varphi : \mathcal{D}(\mathrm{Mod}\, R) \to \mathcal{D}(\mathrm{Mod}\, S)$ as in [**5**, 9.2] using this triple. Then $\varphi(\mathcal{X}_R) = E$, which is not right bounded by construction. Hence by Lemma 2.3, φ does not induce a triangle equivalence $\mathcal{H}^-(\mathrm{pro}\, R) \to \mathcal{H}^-(\mathrm{pro}\, S)$. But as well known φ induces a triangle equivalence $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, R) \to \mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, S)$, thus it induces a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, R) \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, S)$ because $R = S$ are of finite global dimension.

3. Derivative and stable equivalence of repetitions

PROPOSITION 3.1. *Let A, B be algebras. If A and B are derived equivalent, then there is a derived equivalence between $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ of right bounded type.*

PROOF. We may assume that both A and B are basic and we can regard them as finite spectroids. Let (A, E, B) be a tilting triple with an isomorphism $\psi : E \to B$. Then by [**1**, Theorem 1.5], $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is a tilting triple with an isomorphism $\psi^{\mathbb{Z}} : \hat{E} \to B^{\mathbb{Z}}$. Recall that $\varphi^{\mathbb{Z}}(T^{[n]}) := \varphi(T)^{[n]}$ for all $T^{[n]} \in \hat{E}$. As in Keller [**5**, 9.2] construct triangle equivalences $\varphi : \mathcal{D}(\operatorname{Mod} B) \to \mathcal{D}(\operatorname{Mod} A)$ and $\hat{\varphi} : \mathcal{D}(\operatorname{Mod} B^{\mathbb{Z}}) \to \mathcal{D}(\operatorname{Mod} A^{\mathbb{Z}})$ from the tilting triples (A, E, B) and $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$, respectively. Then we have the following commutative diagrams.

$$
\begin{tikzcd}
\mathcal{D}(\operatorname{Mod} A) & \mathcal{D}(\operatorname{Mod} B) \arrow[l, "\varphi"'] \\
E \arrow[u, "\text{incl.}"] \arrow[d, equal] & \mathcal{X}_B \arrow[l, "\varphi|\mathcal{X}_B"'] \arrow[u, "\text{incl.}"'] \\
E \arrow[r, "\sim", "\psi"'] & B \arrow[u, "\eta_B"', "\sim"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(\operatorname{Mod} A^{\mathbb{Z}}) & \mathcal{D}(\operatorname{Mod} B^{\mathbb{Z}}) \arrow[l, "\hat{\varphi}"'] \\
\hat{E} \arrow[u, "\text{incl.}"] \arrow[d, equal] & \mathcal{X}_{B^{\mathbb{Z}}} \arrow[l, "\hat{\varphi}|\mathcal{X}_{B^{\mathbb{Z}}}"'] \arrow[u, "\text{incl.}"'] \\
\hat{E} \arrow[r, "\sim", "\psi^{\mathbb{Z}}"'] & B^{\mathbb{Z}} \arrow[u, "\eta_{B^{\mathbb{Z}}}"', "\sim"]
\end{tikzcd}
$$

In particular, $\hat{\varphi}(\mathcal{X}_{B^{\mathbb{Z}}}) = \hat{E}$. Since B is a finite spectroid, we can put $\operatorname{obj}(E) = \{T_1, \dots, T_n\}$ with n the number of isoclasses of simple B-modules. Then $\operatorname{obj}(\hat{E}) = \{T_i^{[j]} | i = 1, \dots, n, j \in \mathbb{Z}\}$. Since $T_i^{[j]} := T_i \otimes_A A^{\mathbb{Z}}(-, \mathbb{1}^{[j]}?)$ by definition, we have $r(T_i^{[j]}) = r(T_i)$ for all $i = 1, \dots, n$ and all $j \in \mathbb{Z}$. Hence $\hat{\varphi}(\mathcal{X}_{B^{\mathbb{Z}}})$ is right bounded. Let φ' and $\hat{\varphi}'$ be quasi-inverses of φ and $\hat{\varphi}$, respectively. By Lemma 2.3 it only remains to show that $\hat{\varphi}'(\mathcal{X}_{A^{\mathbb{Z}}})$ is right bounded. First we show the following.

Claim 1. *For any $P \in \operatorname{pro} B$ we have $\hat{\varphi}(P^{[m]}) \cong \varphi(P)^{[m]}$.*

(Here we do not need an argument whether this isomorphism exists for any $P \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} B)$.) It is enough to show this claim for any $P \in \mathcal{X}_B$. Let $P = B(-, x)$ for some $x \in B$. Put $T_x := \varphi(B(-, x)) = \varphi(P)$. Then by definition of $\psi^{\mathbb{Z}}$ it follows from the first commutative diagram above that $\psi^{\mathbb{Z}}(T_x^{[m]}) = \psi(T_x)^{[m]} = x^{[m]}$. From the second commutative diagram above we get $\hat{\varphi}(B^{\mathbb{Z}}(-, x^{[m]})) = T_x^{[m]}$. This proves the claim because $P^{[m]} \cong B^{\mathbb{Z}}(-, x^{[m]})$.

Next we show the following.

Claim 2. *For any $T \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} A)$ and for any $m \in \mathbb{Z}$, we have $\tilde{r}(\hat{\varphi}'(T^{[m]})) \leq \tilde{r}(\varphi'(T))$.*

The claim is trivial for $T = 0$. Thus we may assume that $T \neq 0$. Now there is some $P := (P^i, d^i) \in \mathcal{H}^{\mathrm{b}}(\operatorname{pro} B)$ such that $\varphi'(T) \cong P$ in $\mathcal{D}(\operatorname{Mod} B)$. Put $p := \min\{i \in \mathbb{Z} | P^i \neq 0\}, q := r(P), l(P) := q - p + 1 \geq 1$. By Remark 2.2 we may assume that $\tilde{r}(\varphi'(T)) = \tilde{r}(P) = r(P) = q$. We show this claim by induction on $l(P)$.

Assume that $l(P) = 1$. Then $p = q$ and $P = P^q[-q]$. By Claim 1 it follows from $P^q \in \operatorname{pro} B$ that $\hat{\varphi}((P^q)^{[m]}[-q]) \cong \varphi(P^q)^{[m]}[-q] \cong T^{[m]}$, thus $\hat{\varphi}'(T^{[m]}) \cong (P^q)^{[m]}[-q]$. Hence $\tilde{r}(\hat{\varphi}'(T^{[m]})) = q = \tilde{r}(\varphi'(T))$.

Next assume that $l(P) > 1$. Put $P' := (P'^i, d'^i)$ with $P'^i := P^i, d'^i := d^i$ for all $i \neq p$ and $P'^p := 0, d'^p := 0$. Then $\tilde{r}(P') = \tilde{r}(P) = q > p$ and we have a triangle of the form

$$P' \to P \to P^p[-p] \to P'[1]$$

in $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} B)$. This yields a triangle of the form

$$\hat{\varphi}'(\varphi(P')^{[m]}) \to \hat{\varphi}'(T^{[m]}) \to \hat{\varphi}'(\varphi(P^p[-p])^{[m]}) \to \hat{\varphi}'(\varphi(P')^{[m]})[1]$$

in $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} B^{\mathbb{Z}})$ (up to isomorphisms). Applying the induction hypothesis to $\varphi(P')$ and $\varphi(P^p[-p])$, we get $\tilde{r}(\hat{\varphi}'(\varphi(P')^{[m]})) \leq \tilde{r}(P') = q$ and $\tilde{r}(\hat{\varphi}'(\varphi(P^p[-p])^{[m]})) \leq$

$\tilde{r}(P^p[-p]) = p < q$. Therefore $\tilde{r}(\hat{\varphi}'(T^{[m]})) \leq q$ by Remark 2.2. This proves the claim.

In particular by Claim 2 we have $\tilde{r}(\hat{\varphi}'(A^{\mathbb{Z}}(-, x^{[m]}))) \leq \tilde{r}(\varphi'(A(-, x)))$ for all $x \in A$ and all $m \in \mathbb{Z}$. Thus $\tilde{r}(\hat{\varphi}'(T)) \leq \max_{x \in A} \tilde{r}(\varphi'(A(-, x))) \in \mathbb{Z}$ for all $T \in \mathcal{X}_{A^{\mathbb{Z}}}$. Therefore $\hat{\varphi}'(\mathcal{X}_{A^{\mathbb{Z}}})$ is right bounded. □

Summarizing the considerations we get the following.

THEOREM 3.2. *Let A, B be algebras of finite global dimension. Then the following are equivalent.*

(0) *There is a derived equivalence from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ of right bounded type;*
(1) *There is a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathbb{Z}}) \to \mathcal{D}^{\mathrm{b}}(\operatorname{mod} B^{\mathbb{Z}})$;*
(2) *There is a triangle equivalence $\underline{\operatorname{mod}}\, A^{\mathbb{Z}} \to \underline{\operatorname{mod}}\, B^{\mathbb{Z}}$; and*
(3) *There is a derived equivalence from A to B .*

PROOF. (1) ⇒(2). This holds by Keller-Vossieck [**6**, 2.3 Example].
(2) ⇒(3). This holds by Happel [**3**] or by Keller-Vossieck [**6**, 3.1].
(3) ⇒(0). By Proposition 3.1.
(0) ⇒(1). By Lemma 2.5. □

Acknowledgment

I would like to thank Prof. Bernhard Keller for helpful conversations during the ICRA VIII. Especially Lemma 2.5 is due to him; and he showed me how to use his uniqueness theorem [**4**, Theorem] to check that the triangle equivalence constructed as in [**5**] from a tilting spectroid commutes with group actions, which was needed in the first version of the proof (although not used in the present version). Further the work of this paper was done during my stay in Bielefeld in 1996. I would like to thank Prof. Claus M. Ringel and Sonderforschungsbereich 343 for the hospitality and the support.

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