DEGENERATIONS FOR MODULES OVER REPRESENTATION-FINITE BISERIAL ALGEBRAS*

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INTRODUCTION

Main results. Throughout the paper K denotes a fixed algebraically closed field. By an algebra A we mean an associative finite dimensional K-algebra with an identity, and by an A-module a finite-dimensional (unital) left A-module. We shall denote by $\operatorname{mod} A$ the category of A-modules, by Γ_A the Auslander-Reiten quiver of A, and by τ_A the Auslander-Reiten translation $D\mathrm{Tr}$ in Γ_A.

For an algebra A with basis $a_1 = 1, \dots, a_n$, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\operatorname{mod}_A(d)$ of d-dimensional A-modules consists of n-tuples $m = (m_1, \dots, m_n)$ of $d \times d$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum m_k a_{ijk}$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\operatorname{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [16]). We shall agree to identify a d-dimensional A-module M with the point of $\operatorname{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(K)$-orbit of a module M in $\operatorname{mod}_A(d)$. Then one says that a module N in $\operatorname{mod}_A(d)$ is a degeneration of a module M in $\operatorname{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\operatorname{mod}_A(d)$, and we denote this fact by $M \leq_{\mathrm{deg}} N$. Thus $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\leq_{\mathrm{deg}}$ in terms of representation theory .

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [8], [9], [10], Ch. Riedtmann [19], and A. Skowroński and the author [22], [24] connecting $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$ and $\leq$ on the isomorphism classes in $\operatorname{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \leq_{\mathrm{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.

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- $M \leq N$: $\Leftrightarrow$ $[X,M] \leq [X,N]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \mathrm{Hom}_A(X,Y)$ by $[X,Y]$. Then for modules M and N in $\mathrm{mod}\,_A(d)$ the following implications hold:

$$M \leq_{\mathrm{ext}} N \Longrightarrow M \leq_{\mathrm{deg}} N \Longrightarrow M \leq N$$

(see [9], [19]). Unfortunately the reverse implications are not true in general, and it would be interesting to find out when they are. This is the case for the representations of Dynkin quivers and Kronecker modules [9]. It was shown recently in [10] that $\leq_{\mathrm{deg}}$ and $\leq$ coincide for representations of extended Dynkin quivers.

In the representation theory of algebras an important role is played by biserial algebras. Following K. Fuller [13], an algebra A is called biserial if the radical of any indecomposable nonuniserial projective, left or right, A-module is a sum of two uniserial submodules whose intersection is simple or zero. A distinguish class of biserial algebras is formed by the special biserial algebras, introduced in [21]. These are the algebras which are Morita equivalent to bound quiver algebras KQ/I, where (Q,I) satisfies the following conditions

(S1) The numbers of arrows starting, respectively ending, in any fixed vertex of Q are bounded by 2.

(S2) For any arrow α of Q there is at most one arrow β and at most one arrow γ such that $\beta\alpha$ and $\alpha\gamma$ are not in I.

(S3) There is an upper bound for the lengths of paths in Q, which do not belong to I.

Further, a string algebra is a special biserial algebra satisfying the additional condition that I is generated by paths ("monomials").

It has been proved by A. Skowroński and J. Waschbüsh in [21] that every representation-finite biserial algebra is special biserial. Moreover, Thiele showed in [23] that for any compact Riemann surface $\mathcal{S}$ with nonempty boundary, there is a representation-finite string algebra $\Lambda(\mathcal{S})$ such that the geometric realization of its Auslander-Reiten quiver is homeomorphic to $\mathcal{S}$.

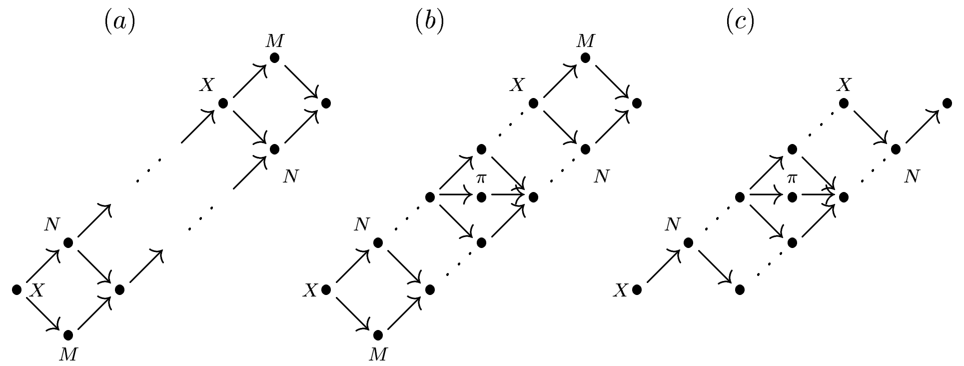
We are interested in degenerations of modules over representation-finite biserial algebras and our first main result is as follows.

THEOREM 1. *Let A be a representation-finite biserial algebra and M, N d-dimensional A-modules. Then $M \leq_{\mathrm{deg}} N$ if and only if $M \leq N$.*

A path $X_0 \to X_1 \cdots \to X_k$ in the Auslander-Reiten quiver Γ_A is called *sectional* if $\tau X_{i+1} \neq X_{i-1}$ for all $0 < i < k$. Our second theorem gives a characterization of representation-finite biserial algebras A for which the orders $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ do not coincide on $\mathrm{mod}\, A$.

Theorem 2. *Let A be a representation-finite biserial algebra. Then the following conditions are equivalent:*

(i) *There exist A-modules M, N such that $M \leq_{\deg} N$ and $M \not\leq_{ext} N$.*
(ii) *There exist A-modules M, N such that $M <_{\deg} N$ and N is indecomposable.*
(iii) *There exists a sectional path in Γ_A of the form $X_0 \to X_1 \to \cdots \to X_k = X_0$ consisting of noninjective modules.*
(iv) *There exists a sectional path in Γ_A of the form $X_0 \to X_1 \to \cdots \to X_k = X_0$ consisting of nonprojective modules.*
(v) *There is in Γ_A a full translation subquiver of one of the form*



where $X \to N \to \cdots \to X$ is a sectional path consisting of noninjective modules.

As an application of the above theorems we get the following theorem.

Theorem 3. *Let A be a block of the group algebra $K[G]$ of a finite group G. If A is representation-finite, then the orders $\leq_{\text{ext}}$, $\leq_{\deg}$ and $\leq$ coincide for A-modules.*

The paper is organized as follows. In Section 1 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules and their degenerations which we apply in our investigations. Sections 2,3 and 4 are devoted to the proofs of Theorems 1,2 and 3, respectively.

For basic background on the topics considered here we refer to [2], [6], [9], [16] and [20]. The results presented in this paper form a part of my doctoral dissertation written under supervision of professor A. Skowroński.

1. PRELIMINARY RESULTS

1.1. Throughout the paper A denotes a fixed finite dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by indecomposable modules, and by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$. By an A-module is meant an object from $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}\, D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. For a module M we denote by $\ell(M)$ the composition length of M in $\operatorname{mod} A$ and by $[M]$ we denote the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same simple composition factors including the multiplicities. Further, for X, Y from $\operatorname{mod} A$ we abbreviate $\dim_K \mathrm{Hom}_A(X, Y)$ by $[X, Y]$.

1.2. Following [19], for M, N from $\operatorname{mod} A$, we set $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all A-modules X. The fact that $\leq$ is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander (see [4], [8]). Observe that, if M and N have the same dimension and $M \leq N$, then $[M] = [N]$. Moreover, M. Auslander and I. Reiten have shown in [5] that, if $[M] = [N]$ for modules M and N, then for all nonprojective indecomposable A-modules X and all noninjective indecomposable modules Y the following formulas hold (see [18]):

$$[X, M] - [M, \tau X] = [X, N] - [N, \tau X]$$
$$[M, Y] - [\tau^- Y, M] = [N, Y] - [\tau^- Y, N]$$

Hence, if $[M] = [N]$, then $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all A-modules X.

1.3. Let M and N be A-modules with $[M] = [N]$ and

$$\Sigma : \ 0 \to D \to E \to F \to 0$$

an exact sequence in $\operatorname{mod} A$. Following [19] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$ and δ_Σ on A-modules X as follows

$$\begin{aligned} \delta_{M,N}(X) &= [N, X] - [M, X] \\ \delta'_{M,N}(X) &= [X, N] - [X, M] \\ \delta_\Sigma(X) &= \delta_{E, D\oplus F}(X) = [D \oplus F, X] - [E, X]. \end{aligned}$$

From the Auslander-Reiten formulas (1.2) we get the following very useful equalities

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \qquad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, we get that the following conditions are equivalent:

(1) $M \leq N$,
(2) $\delta_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$,
(3) $\delta'_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$.

1.4. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M,Z)$ the multiplicity of Z as a direct summand of M. For a nonprojective indecomposable A-module U we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U): \ 0 \to \tau U \to E(U) \to U \to 0,$$

and for an injective indecomposable A-module I we set $E(I) = I/\mathrm{soc}(I)$, $\tau I = 0$. Moreover, for a noninjective indecomposable A-module W, we define $\pi(W)$ to be the unique indecomposable projective-injective direct summand of $E(\tau^- W)$ if such a summand exists, or 0 otherwise.

We shall need the following lemma.

LEMMA 1.5. *Let M, N be A-modules with $[M] = [N]$ and U an indecomposable A-module. Then*

(i) $\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U)$
$= \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta'_{M,N}(U)$.

(ii) *If $Y \to U \to Z$ is a sectional path in Γ_A, then*
$\mu(N,U) - \mu(M,U) \le \delta_{M,N}(U) - \delta_{M,N}(Y) + \delta'_{M,N}(U) - \delta'_{M,N}(Z)$.

(iii) *If U is noninjective and $\pi(U)$ is nonzero, then*
$\mu(N,\pi(U)) - \mu(M,\pi(U)) = -\delta_{M,N}(U)$.

Proof. (i) If U is nonprojective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \mathrm{Hom}_A(M,\tau U) \to \mathrm{Hom}_A(M,E(U)) \to \mathrm{rad}(M,U) \to 0,$$

and hence we get

$$[M,\tau U \oplus U] - [M,E(U)] = [M,U] - \dim_K \mathrm{rad}(M,U) = \mu(M,U).$$

Similarly, we have

$$[N,\tau U \oplus U] - [N,E(U)] = \mu(N,U).$$

Then we obtain the equalities

$$\begin{aligned} \mu(N,U) - \mu(M,U) &= ([N,\tau U \oplus U] - [M,\tau U \oplus U]) - (N,[E(U)] - [M,E(U)]) \\ &= \delta_{M,N}(\tau U) + \delta_{M,N}(U) - \delta_{M,N}(E(U)) \\ &= \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta'_{M,N}(U). \end{aligned}$$

Assume now that U is projective. Then $\mathrm{Hom}_A(M,\mathrm{rad}\,U) \simeq \mathrm{rad}(M,U)$, and so

$$[M,U] - [M,\mathrm{rad}\,U] = \mu(M,U).$$

Similarly, we have

$$[N, U] - [N, \operatorname{rad} U] = \mu(N, U).$$

Therefore, we get

$$\begin{aligned}
\mu(N,U) - \mu(M,U) &= ([N,U] - [M,U]) - ([N, \operatorname{rad} U] - [M, \operatorname{rad} U]) \\
&= \delta_{M,N}(U) - \delta_{M,N}(\operatorname{rad} U) \\
&= \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U) \\
&= \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta'_{M,N}(U),
\end{aligned}$$

because $\delta'_{M,N}(U) = 0$.

(ii) This follows from (i), since $Y \oplus \tau Z$ is a direct summand of $E(U)$ and $\delta_{M,N}(\tau Z) = \delta'_{M,N}(Z)$.

(iii) This also follows from (i), since $\tau\pi(U) = 0$, $\delta_{M,N}(\pi(U)) = 0$ and $E(\pi(U)) = U$.

We shall need also the following Lemma $(3+3+2)$ from [3(2.1)] and its direct consequence.

Lemma 1.6. *Let*

$$\Sigma_1 : 0 \longrightarrow M_1 \xrightarrow{\left[\begin{smallmatrix} u_1 \\ f_1 \end{smallmatrix}\right]} M_2 \oplus N_1 \xrightarrow{[f_2, u_2]} N_2 \longrightarrow 0,$$

$$\Sigma_2 : 0 \longrightarrow M_2 \xrightarrow{\left[\begin{smallmatrix} v_1 \\ f_2 \end{smallmatrix}\right]} M_3 \oplus N_2 \xrightarrow{[f_3, v_2]} N_3 \longrightarrow 0.$$

be short exact sequences in mod A. *Then the sequence*

$$\Sigma_3 : \ 0 \longrightarrow M_1 \xrightarrow{\left[\begin{smallmatrix} v_1 u_1 \\ f_1 \end{smallmatrix}\right]} M_3 \oplus N_1 \xrightarrow{[\, f_3,\, -v_2 u_2 \,]} N_3 \to 0$$

is exact. Moreover, we have $\delta_{\Sigma_3} = \delta_{\Sigma_1} + \delta_{\Sigma_2}$.

Applying the above lemma and Lemma 2.5 in [22] we get the following result (see Corollary 2.2 [3]).

Corollary 1.7. *Let $X_{i,j}$, $Y_{i,j}$ be A-modules for all $0 \le i \le t$, $0 \le j \le s$, such that for each pair (i,j) with $0 \le i < t$, $0 \le j < s$, we have Auslander-Reiten sequence*

$$0 \to X_{i,j} \to X_{i+1,j} \oplus X_{i,j+1} \oplus Y_{i,j} \to X_{i+1,j+1} \to 0,$$

where all $X_{i,j}$ are nonzero with the possible exeptions of $X_{t,0}$ and $X_{0,s}$. Then there exists a short exact sequence

$$\Sigma : 0 \to X_{0,0} \to \left(\bigoplus_{0 \le i < t} \bigoplus_{0 \le j < s} Y_{i,j} \right) \oplus X_{t,0} \oplus X_{0,s} \to X_{t,s} \to 0$$

such that

$$\delta_\Sigma(W) = \sum_{0\le i<t}\sum_{0\le j<s} \mu(W, X_{i,j}).$$

1.8. Here and later on A denotes a representation-finite biserial algebra. For any two A-modules M and N with $M < N$ we define $\mathcal{F}(M,N)$ to be a full translation subquiver of Γ_A consisting all modules $X \in \Gamma_A$ such that $\delta_{M,N}(X) > 0$. Of course, $\mathcal{F}(M,N)$ is nonempty and does not contain injective modules. By [21] for any Auslander-Reiten sequence of A-modules, the middle term has at most two noninjective indecomposable summands. Hence, every arrow in $\mathcal{F}(M,N)$ belongs to an unique maximal sectional path in $\mathcal{F}(M,N)$, and every vertex in $\mathcal{F}(M,N)$ belongs to at most two maximal sectional path in $\mathcal{F}(M,N)$.

2. PROOF OF THEOREM 1

2.1. Let (Q,I) be a bound quiver satisfying the conditions (S1), (S2) and (S3), such that the algebra A is Morita equivalent to KQ/I. Let J be a minimal ideal in KQ generated by paths (monomials) and containing I. Then KQ/J is a string algebra (see [11]). For an arrow $\alpha : i \to j$ in Q we denote by $\alpha^{-1} : j \to i$ a formal inverse of α. Given such a formal inverse $l = \alpha^{-1}$, one writes $l^{-1} = \alpha$. Let $\overline{Q_1}$ be the set of all arrows of Q and their formal inverses. A *word* is a sequence of the form $\omega = l_1 l_2 \dots l_n$ such that $l_i \in \overline{Q_1}$ for all $1 \le i \le n$, the starting vertex of l_i is the ending vertex of l_{i+1} and $l_i^{-1} \ne l_{i+1}$ for all $1 \le i < n$. An inverse of ω is a word $\omega^{-1} = l_n^{-1} l_{n-1}^{-1} \dots l_1^{-1}$. Following [23] we define a *proper string of length* $n \ge 1$ to be a word $\omega = l_1 l_2 \dots l_n$ such that no proper subsequence of ω or its inverse belongs to J. An inverse of a proper string is still a proper string.

For each vertex i of Q we introduce the set of zero strings (nonproper strings) $\mathcal{H}_i$ as follows

$$\mathcal{H}_i = \{s(l), e(l^{-1});\ l : i \to j \in \overline{Q_1}, j \in Q\}.$$

If $l_1 l_2$ is a word such that $l_1 l_2 \notin I$ and $l_2^{-1} l_1^{-1} \notin I$, then we identify the zero strings $e(l_2)$ and $s(l_1)$. Since (Q,I) satisfies the conditions (S1) and (S2), $\mathcal{H}_i$ consists of 2 or 4 zero strings. Further, $\mathcal{H}_i$ has 4 elements if and only if there exist arrows $\alpha \in Q$ starting at x, $\beta \in Q$ ending at x, and for any arrows $\gamma \in Q$ starting at x and $\sigma \in Q$ ending at x we have $\gamma\sigma \in I$ (compare with Lemma 4 in [23]). For any proper string $\omega = l_1 l_2 \dots l_n$ and any zero string z, we define $s(\omega) = s(l_n)$, $\vec{s}(\omega) = l_n$, $e(\omega) = e(l_1)$, $\vec{e}(\omega) = l_1$ and $s(z) = e(z) = z$.

Since A is representation-finite algebra, there is no proper string ω such that ω^n is a proper string for all $n \ge 1$, by [21].

2.2. To every string ω, one knows how to attach an indecomposable string A-module $M(\omega)$ (see [11]). Of course, for any zero string z, the module $M(z)$ is simple, and for any proper strings ω, u we have $M(\omega) = M(u)$ if and only if

$\omega = u$ or $\omega = u^{-1}$. A first extensive study of such string modules was done by I. M. Gelfand and V. A. Ponomarev, see [14].

It follows from [21] that for any indecomposable A-module N which is not projective-injective, there is a string ω such that $N = M(\omega)$. Similarly as in [23], we introduce the following order. We fix a zero string z. In any string ω such that $e(\omega) = z$, we replace every arrow by the latin letter c and every inverse of an arrow by the latin letter a, and we add at the right end of the string the latin letter b (we replace the zero string z by b). Then the resulting words of the latin alphabet may be compared by the usual lexicographical order. From the description of Auslander-Reiten quivers of special biserial algebras given in [21] we infer that for any sectional path $X_0 \to X_1 \to \cdots \to X_k$ in Γ_A consisting of noninjective A-modules, there exist a zero string z and a lexicographicaly ordered sequence of strings $(\omega_0, \omega_1, \ldots, \omega_k)$ such that $X_i = M(\omega_i)$ and $e(\omega_i) = z$ for all $0 \leq i \leq k$. Hence, we get a following result.

Lemma. *Let $X_0 \to X_1 \to \cdots \to X_k = X_0$ be a sectional path in Γ_A consisting of noninjective modules. Then there exists a proper string γ such that $X_0 = M(\gamma)$ and $e(\gamma) = e(\gamma^{-1})$.*

Observe that every proper string γ with $e(\gamma) = e(\gamma^{-1})$ uniquely determines a string v and a proper string ω such that $\gamma = v^{-1}\omega v$ and $\vec{s}(\omega)$, $\vec{e}(\omega)$ are both arrows.

On the other hand, for any zero string z and a lexicographicaly ordered sequence $(\omega_1, \omega_2, \ldots, \omega_k)$ of strings with $e(\omega_i) = z$ for all $1 \leq i \leq k$, there are numbers $1 = j_1 < j_2 < \cdots < j_k = s$ and a sectional path $X_1 \to X_2 \to \cdots \to X_s$ in Γ_A such that $X_{j_i} = M(\omega_i)$ for all $1 \leq i \leq k$.

Proposition 2.3. *Let $\mathcal{F}$ be a full translation subquiver of Γ_A consisting of noninjective modules and assume that each vertex of $\mathcal{F}$ belongs to two maximal sectional paths in $\mathcal{F}$. Then there exists a maximal sectional path in $\mathcal{F}$ consisting pairwise nonisomorphic modules.*

Proof. Assume that for every maximal sectional path $X_0 \to X_1 \to \cdots \to X_k$ in $\mathcal{F}$ there exist numbers $0 \leq i < j \leq k$ such that $X_i = X_j$. Let $\mathcal{B}$ be the set of all modules Z in $\mathcal{F}$ for which there exists a sectional path in $\mathcal{F}$ starting and ending at Z. Applying Lemma 2.2 we get for any $Z \in \mathcal{B}$ a string v and a proper string ω such that $Z = M(v^{-1}\omega v)$, $e(\omega) = e(\omega^{-1})$ and $\vec{s}(\omega)$, $\vec{e}(\omega)$ are both arrows.

Assume now, that for any $Z = M(v^{-1}\omega v) \in \mathcal{B}$ there is a module $Z' = M(t^{-1}ut)$ in $\mathcal{B}$ with $s(t) = e(\omega)$. Since $\mathcal{B}$ is finite, there exist number $k \geq 2$ and modules $Z_i = M(v_i^{-1}\omega_i v_i) \in \mathcal{B}$ such that $s(v_{i+1}) = e(\omega_i) = e(\omega_i^{-1})$ and $\vec{s}(\omega_i)$, $\vec{e}(\omega_i)$ are arrows for all $1 \leq i \leq k$ (we set $v_{k+1} = v_1$, $\omega_{k+1} = \omega_1$). Let $\xi_i = \omega_i^{-1}v_i$ provided $\vec{s}(\omega_{i+1}v_{i+1})$ is an arrow in Q and $\xi_i = \omega_i v_i$ otherwise, for all $1 \leq i \leq k$. Then $(\xi_k\xi_{k-1}\ldots\xi_1)^n$ is a proper string for all $n \geq 1$. This leads to contradiction, since A is representation-finite algebra.

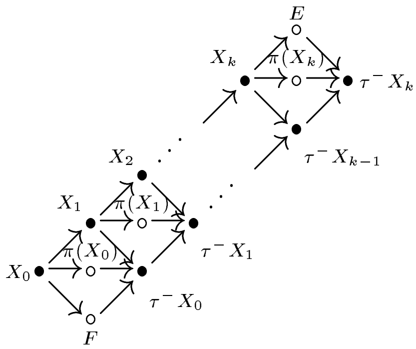
Hence, there exists a module $Z = M(v^{-1}\omega v) \in \mathcal{B}$ such that for any $Z' = M(t^{-1}ut) \in \mathcal{B}$ we have $s(t) \neq e(\omega)$, what is equivalent to $e(t^{-1}) \neq e(v)$. Since $(v^{-1}\omega^{-1}v, v^{-1}, v^{-1}\omega v)$ is a lexicographicaly ordered sequence of strings such that $e(v^{-1}\omega^{-1}v) = e(v^{-1}) = e(v^{-1}\omega v)$, then we have a maximal sectional path in $\mathcal{F}$ of the form

$$X \cdots \to Z = M(v^{-1}\omega^{-1}v) \to \cdots \to M(v^{-1}) \to \cdots \to Z = M(v^{-1}\omega v) \to \cdots X'.$$

By our assumption, $M(v^{-1}) = M(v)$ belongs to another maximal sectional path in $\mathcal{F}$: $Y_0 \to Y_1 \to \cdots \to Y_l$, such that $Y_i = Y_j = M(t^{-1}ut)$ with $e(t^{-1}) = e(v)$, what leads to contradiction and finishes the proof.

2.4. Proof of Theorem 1. Let M, N be modules with $[M] = [N]$ and $M \leq N$. We shall show that $M \leq_{\deg} N$. We proceed by induction on $\sum_{X \in \Gamma_A} \delta_{M,N}(X) \geq 0$. If this sum equals zero then $[M, X] = [N, X]$ for all A-modules X and by [4], M is isomorphic to N. So, we may assume that $\sum_{X \in \Gamma_A} \delta_{M,N}(X) > 0$.

Let X_0 be a module in $\mathcal{F}(M, N)$ which does not belong to two maximal sectional paths in $\mathcal{F}(M, N)$ if such module exists or by Proposition 2.3, $X_0 \to X_1 \to \cdots \to X_k$, $(k \geq 1)$ be a maximal sectional path in $\mathcal{F}(M, N)$ consisting pairwise nonisomorhic modules, otherwise. In the first case we set $k = 0$. Then we have a mesh-complete subquiver of the form



where E, F, $\pi(X_0), \ldots, \pi(X_k)$ are indecomposable or zero. Observe that in both cases ($k \geq 1$ and $k = 0$), E is either zero or $E \notin \mathcal{F}(M, N)$, and F is either zero or projective, or $\tau F \notin \mathcal{F}(M, N)$. This implies that $\delta_{M,N}(E) = 0$ and $\delta'_{M,N}(F) = \delta_{M,N}(\tau F) = 0$. Applying Corollary 1.7 to the meshes starting at X_i for all $0 \leq i \leq k$, we get a short exact sequence

$$\Sigma : 0 \to X_0 \to E \oplus F \oplus \pi \to \tau^- X_k \to 0,$$

such that $\pi = \bigoplus_{i=0}^{k} \pi(X_i)$ and

$$\delta_\Sigma(Y) = \delta_{E \oplus F \oplus \pi, X_0 \oplus \tau^- X_k}(Y) = \sum_{i=0}^{k} \mu(Y, X_i)$$

for any A-module Y. Since X_i belongs to $\mathcal{F}(M,N)$ for any $0 \le i \le k$, then

$$\delta_{M,N}(Y) \ge \delta_{E\oplus F\oplus\pi, X_0\oplus\tau^- X_k}(Y)$$

for any $Y \in \Gamma_A$. This implies that

$$M \oplus X_0 \oplus \tau^- X_k \le N \oplus E \oplus F \oplus \pi.$$

Moreover,

$$\sum_{Y\in\Gamma_A} \delta_{M\oplus X_0\oplus\tau^- X_k, N\oplus E\oplus F\oplus\pi}(Y) = \sum_{Y\in\Gamma_A} (\delta_{M,N}(Y) - \delta_\Sigma(Y)) < \sum_{Y\in\Gamma_A} \delta_{M,N}(Y).$$

Therefore, by our inductive assumption,

$$M \oplus X_0 \oplus \tau^- X_k \le_{\rm deg} N \oplus E \oplus F \oplus \pi.$$

Since

$$M \oplus E \oplus F \oplus \pi \le_{\rm ext} M \oplus X_0 \oplus \tau^- X_k,$$

then

$$M \oplus E \oplus F \oplus \pi \le_{\rm deg} N \oplus E \oplus F \oplus \pi.$$

The equality $\delta'_{M,N}(F) = 0$ implies $[F \oplus \pi, M \oplus E] = [F \oplus \pi, N \oplus E]$, because π is projective and $[M \oplus E] = [N \oplus E]$. Then by the cancellation theorem for degenerations proved in [9, Corollary 2.5], we have $M \oplus E \le_{\rm deg} N \oplus E$. The equality $\delta_{M,N}(E) = 0$ implies $[M,E] = [N,E]$. Applying now the dual cancellation theorem for degenerations, we obtain the required relation $M \le_{\rm deg} N$, what finishes our proof.

3. PROOF OF THEOREM 2

3.1. We fix two numbers $r, s \ge 1$. A translation subquiver Δ of Γ_A with vertices $\Delta_{i,j}$, $0 \le i \le r$, $0 \le j \le s$, arrows

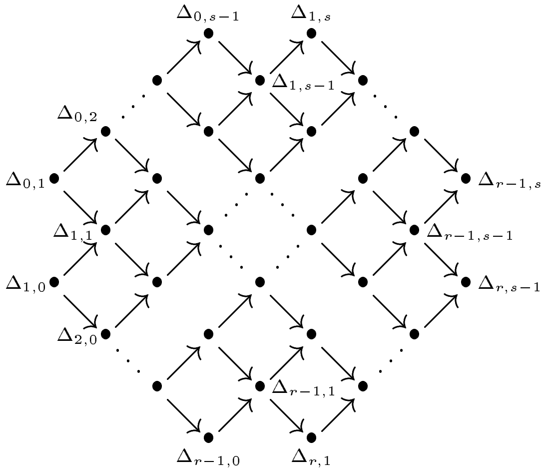
$$\begin{aligned} &\Delta_{i,j} \to \Delta_{i+1,j} \qquad \text{for any } 0 \le i < r, 0 \le j \le s, \\ &\Delta_{i,j} \to \Delta_{i,j+1} \qquad \text{for any } 0 \le i \le r, 0 \le j < s, \end{aligned}$$

and translations

$$\tau\Delta_{i+1,j+1} = \Delta_{i,j} \qquad \text{for any } 0 \le i < r, 0 \le j < s,$$

is called a *rectangle of size* (r,s). Moreover, the vertices $\Delta_{0,0}$, $\Delta_{r,0}$, $\Delta_{0,s}$ and $\Delta_{r,s}$ are called the *corners* of Δ.

If $r, s \geq 2$, then a full translation subquiver $\overline{\Delta}$ of a rectangle Δ of size (r, s), obtained from Δ by removing its corners ($\Delta_{0,0}$, $\Delta_{r,0}$, $\Delta_{0,s}$ and $\Delta_{r,s}$) is called a *rectangle without corners of size* (r, s).



LEMMA 3.2. *There is no translation subquiver $\overline{\Delta}$ of Γ_A which is a rectangle without corners of size (r, s) and such that $\Delta_{1,1}$ is projective and $\Delta_{r-1,s-1}$ is injective.*

Proof. Assume that there exists such a rectangle without corners. We set

$$J = \{(i,j); 0 \leq i < r, 0 \leq j < s\} \setminus \{(0,0), (r-1,0), (0,s-1), (r-1,s-1)\}.$$

Then

$$\ell(\Delta_{i,j}) + \ell(\Delta_{i+1,j+1}) \geq \ell(\Delta_{i+1,j}) + \ell(\Delta_{i,j+1})$$

for any $(i,j) \in J$ and

$$\begin{aligned}
\ell(\Delta_{r-1,0}) + \ell(\Delta_{r,1}) &\geq \ell(\Delta_{r-1,1}), \\
\ell(\Delta_{0,s-1}) + \ell(\Delta_{1,s}) &\geq \ell(\Delta_{1,s-1}), \\
\ell(\Delta_{1,1}) &> \ell(\Delta_{0,1}) + \ell(\Delta_{1,0}), \\
\ell(\Delta_{r-1,s-1}) &> \ell(\Delta_{r,s-1}) + \ell(\Delta_{r-1,s}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\ell(\Delta_{r-1,0}) + \ell(\Delta_{r,1}) + \ell(\Delta_{0,s-1}) + \ell(\Delta_{1,s}) + \ell(\Delta_{1,1}) + \ell(\Delta_{r-1,s-1}) \\
&+ \sum_{(i,j)\in J} (\ell(\Delta_{i,j}) + \ell(\Delta_{i+1,j+1})) > \sum_{(i,j)\in J} (\ell(\Delta_{i+1,j}) + \ell(\Delta_{i,j+1})) \\
&+ \ell(\Delta_{r-1,1}) + \ell(\Delta_{1,s-1}) + \ell(\Delta_{0,1}) + \ell(\Delta_{1,0}) + \ell(\Delta_{r,s-1}) + \ell(\Delta_{r-1,s}).
\end{aligned}$$

This leads to $0 > 0$ and we obtain a contradiction.

LEMMA 3.3. *Let Δ be a translation subquiver of Γ_A which is a rectangle of size (r,s) with $\Delta_{r,0}=\Delta_{0,s}$ for some $r,s\geq 1$. Then $\Delta_{r-1,0}=\Delta_{0,s-1}$.*

Proof. Assume that there exists a rectangle Δ of size (r,s) such that $\Delta_{r,0}=\Delta_{0,s}$ and $\Delta_{r-1,0}\neq\Delta_{0,s-1}$. We claim that we can choose such a rectangle of size (r,s) with indecomposable projective A-module $\Delta_{0,h}$ for some $1\leq h\leq s$. For, if this is not the case, then by induction on the number $k\geq 0$, we define vertices $\Delta_{-k,j}$ of Γ_A for $0\leq j\leq s$ for which the vertices $\Delta_{i,j}$ for $-k\leq i\leq r-k$, $0\leq j\leq s$ form a rectangle Δ^k of size (r,s) with $\Delta_{r-k,0}=\Delta_{-k,s}$ and $\Delta_{r-k-1,0}\neq\Delta_{-k,s-1}$. Assume that a rectangle Δ^k is defined for some $k\geq 0$. Since $\Delta_{-k,j}$ is nonprojective for any $1\leq j\leq s$, we may set $\Delta_{-(k+1),s}=\Delta_{r-(k+1),0}$ and $\Delta_{-(k+1),j}=\tau\Delta_{-k,j+1}$ for all $0\leq j\leq s-1$. Of course, $\Delta_{-(k+1),s}\neq\Delta_{-k,s-1}$ and therefore, the vertices $\delta_{i,j}$ for $-(k+1)\leq i\leq r-(k+1)$, $0\leq j\leq s$ form a rectangle Δ^{k+1} of size (r,s) with $\Delta_{r-(k+1),0}=\Delta_{-(k+1),s}$. But if the vertices $\Delta_{r-(k+1)-1,0}$ and $\Delta_{-(k+1),s-1}$ coincide then $\Delta_{r-k,0}=\Delta_{-k,s}=\tau^-\Delta_{-(k+1),s-1}=\tau^-\Delta_{r-(k+1)-1,0}=\Delta_{r-k-1,1}$ what yields to contradiction, because Γ_A has no multiple arrows ($\Delta_{r-k-1,0}\to\Delta_{r-k,0}=\Delta_{r-k-1,1}$). Hence, we obtain an infinite sectional path in Γ_A of the form $\cdots\to\Delta_{r-2,0}\to\Delta_{r-1,0}\to\Delta_{r,0}$ what is impossible, because A is of representation-finite. This proves our claim.

So, we may assume that there is $1\leq h\leq s$ such that $\Delta_{0,h}$ is projective and $\Delta_{0,j}$ is non-projective for all $h<j\leq s$. In a similar way we get a number $l\geq s$ and a rectangle Δ' of size (r,l) consisting of vertices $\Delta_{i,j}$, for all $0\leq i\leq r$, $0\leq j\leq l$ and with corners $\Delta_{0,0}$, $\Delta_{r,0}$, $\Delta_{0,l}$ and $\Delta_{r,l}$, such that $\Delta_{g,l}$ is injective for some $0\leq g<r$. Then $\Delta_{0,j}=\Delta_{r,j-s}$ for any $s\leq j\leq l$. Without loss of generality, we may assume that g is minimal ($\Delta_{i,l}$ is not injective for $0\leq i<g$). We set

$$
\begin{array}{lcll}
\nabla_{i,j} & = & \Delta_{i-1,j+h-1}, & \text{for any} \quad 1\leq i\leq g+2, 0\leq j\leq (l-h)+1,\\
\nabla_{0,j} & = & \tau\Delta_{0,j+h}, & \text{for any} \quad 1\leq j\leq s-h,\\
\nabla_{0,j} & = & \Delta_{r-1,j-s+h-1}, & \text{for any} \quad s-h<j\leq (l-h)+1,\\
\nabla_{1,(l-h)+2} & = & \Delta_{r,l-s+1}, & \\
\nabla_{i,(l-h)+2} & = & \tau^-\Delta_{i-2,l}, & \text{for any} \quad 1<i\leq g+1.
\end{array}
$$

Then we obtain a rectangle without corners ∇ of size $(g+2,l-h+2)$ in Γ_A. By Lemma 3.2, we get a contradiction, since the module $\nabla_{1,1}=\Delta_{0,h}$ is projective and the module $\nabla_{g+1,l-h+1}=\Delta_{g,l}$ is injective.

LEMMA 3.4. *Let M and N be two A-modules such that $[M]=[N]$ and $M<N$. Then there is an Auslander-Reiten sequence*

$$0\to Z\to Y\oplus Y'\to\tau^-Z\to 0,$$

such that $\delta_{M,N}(Z)>0$, Y is a direct summand of M and Y' is indecomposable.

Proof. Assume first that there is no arrow in $\mathcal{F}(M,N)$. Let Z be any vertex of $\mathcal{F}(M,N)$ and U be any direct summand of the middle term of Auslander-Reiten sequence

$$0 \to Z \to Y \oplus Y' \to \tau^- Z \to 0,$$

where Y' is indecomposable. Of course, there is an arrow $Z \to U$ in Γ_A and an arrow $\tau U \to Z$ in Γ_A provided U is nonprojective. Hence, $\delta_{M,N}(U) = 0$ and $\delta'_{M,N}(U) = \delta_{M,N}(\tau U) = 0$. Applying Lemma 1.5(i) we get

$$\mu(N,U) - \mu(M,U) = -\delta_{M,N}(E(U)) \le -\delta_{M,N}(Z) < 0,$$

and consequently U is a direct summand of M. Since $Y \oplus Y'$ has no multiple indecomposable direct summand, $Y \oplus Y'$ is a direct summand of M too.

So, we may assume that there is an arrow in $\mathcal{F}(M,N)$. Let $X_0 \to X_1 \to \cdots \to X_k$ be a maximal sectional path in $\mathcal{F}(M,N)$ for some $k \ge 1$. Let $Y_l \to Y_{l+1} \to \cdots \to Y_0 = X_0 \to \cdots \to Y_r$ be another maximal sectional path in $\mathcal{F}(M,N)$ containing X_0 for some $l \le 0 \le r$, $l \ne r$, if such path exists, or $l = r = 0$, $Y_0 = X_0$ otherwise.

Assume now that for any arrow $Y_r \to W$ in Γ_A, either $W = \pi(Y_r)$ or $W = \tau^- Y_{r-1}$ provided $r \ge 1$, or $W = X_1$ provided $r = 0$. We set $Z = Y_r$, $Y = \pi(Y_r)$ and $Y' = \tau^- Y_{r-1}$ provided $r \ge 1$ or $Y' = X_1$ provided $r = 0$. Then we have an Auslander-Reiten sequence

$$0 \to Z \to Y \oplus Y' \to \tau^- Z \to 0,$$

where Y' is indecomposable A-module and by Lemma 1.5(iii), $Y = \pi(Y_r)$ is a direct summand of M.

Thus we may assume that there is a non-projective-injective indecomposable module Y_{r+1} such that $Y_0 \to \cdots \to Y_{r+1}$ is a sectional path in Γ_A, $\delta_{M,N}(Y_{r+1}) = 0$ and Y_1 is not isomorphic to X_1. If $\delta'_{M,N}(Y_1) > 0$, then Y_1 is nonprojective, $\delta_{M,N}(\tau Y_1) = \delta'_{M,N}(Y_1) > 0$, consequently τY_1 belongs to $\mathcal{F}(M,N)$ and $\tau Y_1 \to X_0 \to X_1 \to \cdots \to X_k$ is a sectional path in $\mathcal{F}(M,N)$ what yields to the contradiction with maximality of the sectional path $X_0 \to X_1 \to \cdots \to X_k$. This implies that $\delta'_{M,N}(Y_1) = 0$. Applying Lemma 1.5 we get

$$\mu(N,Y_i) - \mu(M,Y_i) \le \delta_{M,N}(Y_i) - \delta_{M,N}(Y_{i-1}) + \delta'_{M,N}(Y_i) - \delta'_{M,N}(Y_{i+1}),$$

for any $1 \le i \le r$, and

$$\begin{aligned}\mu(N,Y_{r+1}) - \mu(M,Y_{r+1}) &\le \delta_{M,N}(Y_{r+1}) - \delta_{M,N}(Y_r) + \delta'_{M,N}(Y_{r+1}) \\ &= -\delta_{M,N}(Y_r) + \delta'_{M,N}(Y_{r+1}).\end{aligned}$$

Hence,

$$\sum_{i=1}^{r+1} \mu(N,Y_i) - \mu(M,Y_i) \le -\delta_{M,N}(Y_0) + \delta'_{M,N}(Y_1) = -\delta_{M,N}(Y_0) < 0,$$

because Y_0 belongs to $\mathcal{F}(M,N)$. This implies that there is a number $1 \leq j \leq r+1$ such that Y_j is a direct summand of M. We set $Z = Y_{j-1}$, $Y = Y_j \oplus \pi(Y_{j-1})$, $Y' = \tau^- Y_{j-2}$ provided $j \geq 2$, or $Y' = X_1$ provided $j = 1$. Then we have an Auslander-Reiten sequence

$$0 \to Z \to Y \oplus Y' \to \tau^- Z \to 0,$$

where Y' is indecomposable A-module and $\delta_{M,N}(Z) > 0$. Moreover, by Lemma 1.5(iii), $\pi(Y_{j-1})$ is a direct summand of M, what implies that Y is a direct summand of M. This completes the proof.

3.5. Proof of Theorem 2. Clearly, (v) implies (iii) and (ii) implies (i). That (iii) is equivalent to (iv) follows from the fact that $X_0 \to X_1 \to \cdots \to X_k = X_0$ is a sectional path in Γ_A consisting of nonprojective modules if and only if $\tau X_0 \to \tau X_1 \to \cdots \to \tau X_k = \tau X_0$ is a sectional path in Γ_A consisting of noninjective modules.

(iii) $\Rightarrow$ *(v)*. Let $X_0 \to X_1 \to \cdots \to X_k = X_0$ be a sectional path in Γ_A consisting of noninjective modules. By the result of R. Bautista and S. O. Smalø[7] $X_{k-1} = \tau X_1$. Since the middle term of any Auslander-Reiten sequence of A-modules has at most two nonprojective-injective indecomposable direct summand, we have Auslander-Reiten sequences:

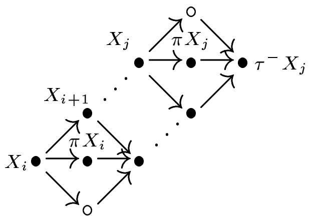
$$0 \to X_i \to \tau^- X_{i-1} \oplus X_{i+1} \oplus \pi(X_i) \to \tau^- X_i \to 0,$$

for any $0 < i < k$ and

$$0 \to X_0 \to X_1 \oplus \pi(X_0) \oplus M' \to \tau^- X_0 \to 0,$$

where $\pi(X_i)$ and M' are indecomposable or zero.

Assume that there exist numbers $0 \leq i < j \leq k$ such that $\pi(X_i) \neq 0$ and $\pi(X_j) \neq 0$. Then we have a subquiver of the form



Applying Corollary 1.7 to the meshes starting at X_s, for $i \leq s \leq j$, we get an exact sequence

$$0 \to X_i \to \pi(X_i) \oplus \pi(X_j) \oplus W \to \tau^- X_j \to 0$$

for some A-module W, what implies that

$$\ell(X_i) + \ell(\tau^- X_j) \geq \ell(\pi(X_i)) + \ell(\pi(X_j)).$$

On the other hand $X_i = \mathrm{rad}(\pi(X_i))$ and $\tau^- X_j = \pi(X_j)/\mathrm{soc}(\pi(X_j))$, what implies $\ell(X_i) < \ell(\pi(X_i))$ and $\ell(\tau^- X_j) < \ell(\pi(X_j))$. This leads to contradiction and hence, $\pi(X_0) = \pi(X_k) = 0$ and there is at most one number $0 < i < k$ such that $\pi(X_i) \neq 0$. Hence, if we set $X = X_0 = X_k$ and $N = X_1$, we get in Γ_A a subquiver of one of the form (a), (b), or (c).

$(v) \Rightarrow (ii)$. Applying Corollary 1.7 to the meshes starting at $X_0 = X$, $X_1 = N$,..., X_{k-1}, we get an exact sequence

(a) $0 \to X \to X \oplus M' \to N \to 0$,

(b) $0 \to X \to X \oplus M' \oplus \pi \to N \to 0$,

or

(c) $0 \to X \to X \oplus \pi \to N \to 0$.

Thus in all cases we have an exact sequence $0 \to X \to X \oplus M \to N \to 0$ with N indecomposable, and finally, by Proposition 3.4 in [19], we infer that $M <_{\deg} N$.

$(i) \Rightarrow (iii)$. Assume that there is no sectional path in Γ_A of the form $X_0 \to X_1 \to \cdots \to X_k = X_0$ consisting of noninjective modules. We shall show that $M \leq_{\deg} N$ implies $M \leq_{\mathrm{ext}} N$. We proceed by induction on $[N, N] - [M, M] \geq 0$. If $[N, N] - [M, M] = 0$ then $M \simeq N$ by [9]. So we may assume that M is not isomorphic to N and moreover, M and N have no common indecomposable direct summand.

We claim that for any rectangle Δ of size (r, s), $r, s \geq 1$, we have $\Delta_{r,0} \neq \Delta_{0,s}$. For, if this is not the case, then by Lemma 3.3 $\Delta_{r-1,0} = \Delta_{0,s-1}$. We can assume that $r \geq s$. Since the arrows $\Delta_{r-1,0} \to \Delta_{r,0}$ and $\Delta_{0,s-1} \to \Delta_{0,s}$ coincide, we get that $\Delta_{r-i,0} = \Delta_{0,s-i}$ for all $0 \leq i \leq s$. Since A is a representation-finite algebra an over algebraically closed field, there is no multiply arrows in Γ_A. Therefore, $\Delta_{1,0} \neq \Delta_{0,1}$ and $r \neq s$. Then we get a contradiction, because $\Delta_{0,0} \to \Delta_{1,0} \to \cdots \to \Delta_{r-s,0} = \Delta_{0,0}$ is a sectional path in Γ_A consisting of noninjective modules. This proves our claim.

Applying Lemma 3.4 we get a mesh

$$\begin{tikzcd}
& M_1 \arrow[dr] & \\
Z \arrow[ur] \arrow[r] \arrow[dr] & \pi(Z) \arrow[r] & \tau^- Z \\
& Y' \arrow[ur] &
\end{tikzcd}$$

where Z belongs to $\mathcal{F}(M, N)$, M_1 and $\pi(Z)$ are indecomposable or zero, and $M_1 \oplus \pi(Z)$ is a direct summand of M.

Let $Y_{0,0} \to Y_{1,0} \to \cdots \to Y_{t,0} = Z$ be a maximal sectional path in $\mathcal{F}(M,N)$ for some $t \geq 0$, such that $Y_{t-1,0} = \tau Y'$ provided $t \geq 1$. Since there is no infinite sectional path in $\mathcal{F}(M,N)$, there exist a number $u \geq 0$ and a rectangle $\mathcal{Y}$ of size (t,u) consisting of vertices $Y_{i,j} \in \mathcal{F}(M,N)$ for all $0 \leq i \leq t$, $0 \leq j \leq u$, with corners $Y_{0,0}, Y_{t,0}, Y_{0,u}, Y_{t,u}$, and such that $Y_{0,u} \to Y_{1,u}$ is the only arrow in $\mathcal{F}(M,N)$ starting at $Y_{0,u}$ or $\tau^{-}Y_{i,u}$ does not belong to $\mathcal{F}(M,N)$ for some $0 \leq i < t$. We set $Y_{t+1,j} = \tau^{-}Y_{t,j-1}$ for any $0 < j \leq u+1$, $Y_{t+1,0} = M_1$, $Y_{i,u+1} = \tau^{-}Y_{i-1,u}$ for any $0 < i < t$, $Y_{0,u+1} \neq Y_{1,u}$ is a nonprojective and noninjective indecomposable A-module such that $Y_{0,u} \to Y_{0,u+1}$ is an arrow in Γ_A provided such an arrow exists, or $Y_{0,u+1} = 0$ otherwise. If $Y_{0,u+1} = 0$ then we set $v = 0$, $g = 0$, $h = u$.

Now we assume that $Y_{0,u+1} \neq 0$. Take minimal v such that $0 \leq v \leq t$ and $Y_{v,u+1}$ does not belong to $\mathcal{F}(M,N)$. Then $\delta_{M,N}(Y_{v,u+1}) = 0$. By Lemma 1.5 we have

$$\mu(N,Y_{i,j}) - \mu(M,Y_{i,j}) \leq \delta_{M,N}(Y_{i,j}) - \delta_{M,N}(Y_{i,j-1}) + \delta'_{M,N}(Y_{i,j}) - \delta'_{M,N}(Y_{i,j+1})$$

for any $0 \leq i \leq t$, $1 \leq j \leq u$,

$$\begin{aligned}\mu(N,Y_{i,u+1}) - \mu(M,Y_{i,u+1}) \leq & \delta_{M,N}(Y_{i,u+1}) - \delta_{M,N}(Y_{i-1,u+1}) \\ & + \delta_{M,N}(Y_{i-1,u}) - \delta_{M,N}(Y_{i,u})\end{aligned}$$

for any $1 \leq i \leq t$, and

$$\mu(N,Y_{0,u+1}) - \mu(M,Y_{0,u+1}) \leq \delta_{M,N}(Y_{0,u+1}) - \delta_{M,N}(Y_{0,u}) + \delta'_{M,N}(Y_{0,u+1})$$

Then

$$\begin{aligned}\sum_{i=0}^{v}\sum_{j=1}^{u} & \mu(N,Y_{i,j}) - \mu(M,Y_{i,j}) \leq \\ & \leq \sum_{i=0}^{v}\left(\delta_{M,N}(Y_{i,u}) - \delta_{M,N}(Y_{i,0})\right) + \sum_{i=0}^{v}\left(\delta'_{M,N}(Y_{i,1}) - \delta'_{M,N}(Y_{i,u+1})\right) \\ & = \delta_{M,N}(Y_{v,u}) - \delta_{M,N}(Y_{v,0}) + \sum_{i=0}^{v-1}\left(\delta_{M,N}(Y_{i,u}) - \delta_{M,N}(Y_{i,0})\right) \\ & \quad + \delta'_{M,N}(Y_{0,1}) - \delta'_{M,N}(Y_{0,u+1}) + \sum_{i=0}^{v-1}\left(-\delta'_{M,N}(Y_{i+1,u+1}) + \delta'_{M,N}(Y_{i+1,1})\right) \\ & = \delta'_{M,N}(Y_{0,1}) - \delta_{M,N}(Y_{v,0}) + \delta_{M,N}(Y_{v,u}) - \delta'_{M,N}(Y_{0,u+1}),\end{aligned}$$

because $\delta'_{M,N}(Y_{k+1,l+1}) = \delta_{M,N}(\tau Y_{k+1,l+1}) = \delta_{M,N}(Y_{k,l})$ for any $0 \leq k \leq t$ and $0 \leq l \leq u$. Moreover, we obtain

$$\begin{aligned}\sum_{i=0}^{v}\mu(N,Y_{i,u+1}) - \mu(M,Y_{i,u+1}) & \leq \delta_{M,N}(Y_{v,u+1}) - \delta_{M,N}(Y_{v,u}) + \delta'_{M,N}(Y_{0,u+1}) \\ & = \delta'_{M,N}(Y_{0,u+1}) - \delta_{M,N}(Y_{v,u})\end{aligned}$$

and
$$\sum_{i=0}^{v}\sum_{j=1}^{u+1}\mu(N,Y_{i,j})-\mu(M,Y_{i,j})\leq\delta'_{M,N}(Y_{0,1})-\delta_{M,N}(Y_{v,0})<0,$$
since $Y_{v,0}$ belongs to $\mathcal{F}(M,N)$ and $\delta'_{M,N}(Y_{0,1})\neq 0$ implies that $Y_{0,1}$ is not projective and consequently we would get a sectional path in $\mathcal{F}(M,N)$
$$\tau Y_{0,1}\to Y_{0,0}\to Y_{1,0}\to\cdots\to Y_{t,0}=Z,$$
which contradicts the maximality of the path
$$Y_{0,0}\to Y_{1,0}\to\cdots\to Y_{t,0}=Z.$$
Hence, there are numbers g,h such that $0\leq g\leq v$, $0\leq h\leq u$ and $\mu(M,Y_{g,h+1})>0$.

Let $M_2=Y_{g,h+1}$ and $\pi=\bigoplus_{i=g}^{t}\bigoplus_{j=0}^{h}\pi(Y_{i,j})$. Observe that if $Y_{0,u+1}=0$ then $M_2=Y_{0,u+1}=0$. Then M_2 is a direct summand of M. Applying now Corollary 1.7 to the meshes starting at $Y_{i,j}$ for all $g\leq i\leq t$, $0\leq j\leq h$ we get an exact sequence
$$\Sigma:0\to Y_{g,0}\to M_1\oplus M_2\oplus\pi\to Y_{t+1,h+1}\to 0,$$
such that
$$\delta_\Sigma(X)=\delta_{M_1\oplus M_2\oplus\pi,Y_{g,0}\oplus Y_{t+1,h+1}}(X)=\sum_{i=g}^{t}\sum_{j=0}^{h}\mu(X,Y_{i,j})$$
for any A-module X.

We claim that the modules $Y_{i,j}$, $g\leq i\leq t$, $0\leq j\leq h$ are pairwise nonisomorphic. For, if this is not the case, $Y_{i,j}$ is isomorphic to $Y_{k,l}$ for some $(i,j)\neq(k,l)$ such that $g\leq i\leq k\leq t$ and $0\leq j,l\leq h$. Of course, $Y_{i+1,j}$ is isomorphic to $Y_{k+1,l}$ or $Y_{k,l+1}$. If $i=k$ then we can assume that $j<l$ and we get a sectional path $Y_{i,j}\to Y_{i,j+1}\to\cdots\to Y_{i,l}=Y_{i,j}$ consisting of noninjective modules, what yields to contradiction with our assumption. Hence, $i<k$ and in a similar way we obtain $j\neq l$.

Assume that $j<l$. If $Y_{i+1,j}=Y_{k,l+1}$, then we get a contradiction, because
$$Y_{k,j}\to Y_{k,j+1}\to\cdots\to Y_{k,l}=Y_{i,j}\to Y_{i+1,j}\to\cdots\to Y_{k,j}$$
is a sectional path consisting of noninjective modules. Thus $Y_{i+1,j}=Y_{k+1,l}$ and consequently
$$Y_{0,0}=Y_{k-i,l-j},\qquad Y_{1,0}=Y_{k-i+1,l-j}.$$
This yields to contradiction, since
$$Y_{k-i-1,l-j}\to Y_{k-i,l-j}=Y_{0,0}\to Y_{1,0}\to\cdots\to Y_{t,0}=Z$$

is a sectional path in $\mathcal{F}(M,N)$. So, $j>l$ and we get a rectangle of size $(k-i,j-l)$ with corners $Y_{i,l}$, $Y_{i,j}$, $Y_{k,l}$ and $Y_{k,j}$, what implies that $Y_{k,l}$ is not isomorphic to $Y_{i,j}$.

Suppose that for some $g \le i \le t$, $0 \le j \le h$, the module $\pi(Y_{i,j})$ is nonzero. By Lemma 1.5(iii), we get

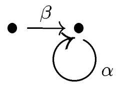
$$\mu(N,\pi(Y_{i,j})) - \mu(M,\pi(Y_{i,j})) = -\delta_{M,N}(Y_{i,j}) < 0.$$

Hence, $\pi(Y_{i,j})$ is a direct summand of M.

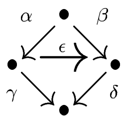
We shall show that $M_1 \oplus M_2 \oplus \pi$ is a direct summand of M. If M_1 and M_2 are nonzero, then we have a rectangle of size $(t-g+1,h+1)$ with corners $Y_{g,0}$, M_1, M_2, $Y_{t+1,h+1}$, and consequently M_1 is not isomorpic to M_2. So, it remains to prove that for any $g \le i,k \le t$, $0 \le j,l \le h$, $(i,j) \ne (k,l)$, if $\pi(Y_{i,j})$ and $\pi(Y_{k,l})$ are nonzero, then they are not isomorphic. But $\pi(Y_{i,j}) \not\simeq \pi(Y_{k,l})$ if and only if $Y_{i,j} \not\simeq Y_{k,l}$, what is proved above.

Hence, $M = M_1 \oplus M_2 \oplus \pi \oplus M'$ for some A-module M'. Let $W = Y_{g,0} \oplus Y_{t+1,h+1} \oplus M'$. Then $\delta_{M,W}(X) = 1$ for any $X \in \mathcal{Z} = \{Y_{i,j}, g \le i \le t, 0 \le j \le h\}$ and $\delta_{M,W}(X) = 0$ for other indecomposable A-modules. Since $\delta_{M,N}(X) \ge 1$ for any $X \in \mathcal{Z}$, we have $\delta_{M,W} \le \delta_{M,N}$ and consequently $M <_{\mathrm{ext}} W \le N$. Since $[N,N]-[W,W] < [N,N]-[M,M]$, then $W \le_{\mathrm{ext}} N$, by our inductive assumption. This implies $M <_{\mathrm{ext}} N$, what finishes the proof.

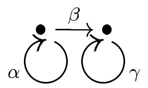
3.6. We give examples of bound quivers (Q_a,I_a), (Q_b,I_b) and (Q_c,I_c) such that KQ_j/I_j is representation-finite biserial algebra and among the quivers (a), (b) and (c) (in Theorem 2) only the quiver (j) is a full translation subquiver of Γ_{KQ_j/I_j}, for $j=a,b,c$.

Q_a :


I_a is generated by α^2,

Q_b :


I_b is generated by $\epsilon\alpha$, $\delta\epsilon$ and $\gamma\alpha-\delta\beta$,

Q_c :


I_c is generated by α^2, γ^2 and $\beta\alpha-\gamma\beta$.

4. PROOF OF THEOREM 3

Let G be a finite group and A be a representation-finite block of $K[G]$. We know that then A is special biserial. Furthermore, if A is not semi-simple, then A

is Morita equivalent to a bound quiver algebra KQ/I, where Q is a Brauer quiver, by the Theorem of Dade-Janusz-Kupisch (see [12], [15], [17]). Furthermore, if we replace each vertex of Q by an edge, and each cycle of Q by a vertex, then we get a finite, connected, undirected graph without loops or cycles, called *Brauer tree* (see [2] for more precise definition). Let γ be a proper string in (Q, I) such that the starting and the ending vertices of γ coincide. Then either $\alpha = \vec{s}(\gamma)$ and $\beta = \vec{e}(\gamma)$ are arrows in Q belonging to the same cycle of Q, or $\alpha = \vec{s}(\gamma^{-1})$ and $\beta = \vec{e}(\gamma^{-1})$ are arrows in Q belonging to the same cycle of Q. This implies that $e(\gamma) \neq e(\gamma^{-1})$. Applying Lemma 2.2, Theorem 1 and Theorem 2(iv) we obtain the thesis of Theorem 3.

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