Growth of polynilpotent varieties of Lie algebras and fast increasing entire functions

Petrogradsky V.M. *

February 17, 1997

Faculty of Mathematics
Ulyanovsk State University
Lev Tolstoy 42, Ulyanovsk, 432700 Russia
e-mail: vmp@mmf.univ.simbirsk.su

current address:
Faculty of Mathematics, University of Bielefeld
Postfach 100131, 33501 Bielefeld, Germany
e-mail: petrogra@Mathematik.Uni-Bielefeld.de

Abstract

A growth function $c_n(\mathrm{V})$ for varieties of Lie algebras is studied; where $c_n(\mathrm{V})$ is the dimension of a linear span of all multilinear words on n distinct letters in a free algebra $F(\mathrm{V}, X)$ of the variety V. To any nontrivial variety of Lie algebras V corresponds the complexity function $\mathcal{C}(\mathrm{V}, z)$, which is an entire function of a complex variable. In case of a polynilpotent variety of Lie algebras V estimates for complexity function are studied, in most cases it is of infinite order. A connection between a growth of fast increasing entire function and an asymptotic of its Taylor coefficients is studied. The main result is an asymptotic for a function $c_n(\mathrm{V})$ in case of the polynilpotent variety of Lie algebras V. Also we prove an analog of Regev's theorem on an upper bound for an arbitrary variety of Lie algebras. As a result we obtain a better bound than an estimate, obtained by the author earlier.

1 Introduction: growth of varieties of Lie algebras

Let H be a Lie algebra Lie over a field K. Let $F = F(X)$ be a free Lie algebra with a countable set of free generators $X = \{x_i | i \in \mathbb{N}\}$. Then $0 \neq f \in F$ is said an *identity* (or *identical relation*) in H, iff for any $a_i \in H$, $i \in \mathbb{N}$ and any homomorphism $\phi : F \to H$, $\phi(x_i) = a_i$, $i \in \mathbb{N}$ we have $\phi(f) = 0$; also in this case we say that H is a *PI-algebra*. Suppose that $I \subset F$, then a class of all Lie algebras V, satisfying identities I, is said a *variety*. A set of all identities $\mathrm{V}(X)$, which are true in a given variety V, is an ideal in $F(X)$, so we can consider $F(\mathrm{V}, X) = F(X)/\mathrm{V}(X)$, which is called *free algebra of the variety* V. It has a property that $\forall H \in \mathrm{V}$, $\forall a_i \in H$, $i \in \mathbb{N}$, $\exists! \phi : F(\mathrm{V}, X) \to H : \phi(x_i) = a_i$, $i \in \mathbb{N}$. A variety is called nontrivial if it satisfies some nonzero identity. For the theory of varieties of Lie algebras see a monograph [1].

Let $P_n(\mathrm{V}) \subset F(\mathrm{V}, X)$ be a set of multilinear elements on letters $x_1, \dots, x_n$. Namely, $P_n(\mathrm{V})$ is a linear span of all monomials on $x_1, \dots, x_n$ (more exactly, their images in $F(\mathrm{V}, X)$) such that each

*partially supported by grant RFFI 96-01-00146; the author is gratefull to the University of Bielefeld for hospitality, where he was DAAD-fellow

letter $x_i,\ i=1,\dots,n$ enters monomials exactly one time. For a given variety V appears a function of *codimension growth*

$$c_n(\mathrm{V}) = c_n(F(\mathrm{V},X),X) = \dim_K P_n(\mathrm{V}), \quad n\in\mathbb{N}.$$

It is evident that for other variables $x_{i_1},\dots,x_{i_n}\in X$ we will get the same number. A behaviour of the sequence $c_n(\mathrm{V})$ is an important characteristic of the variety V.

The same observations hold also for varieties of associative algebras. About the codimension growth of associative algebras the following fact is known.

Theorem 1.1 (Regev, [2], see also [1]) *Let* V *be a variety of associative algebras which satisfies some nontrivial identity of degree d. Then*

$$c_n(\mathrm{V}) \le C^n, \quad n\in\mathbb{N}; \qquad \textit{where} \quad C=(d-1)^2.$$

Recall the definition of a polynilpotent variety of Lie algebras $\mathrm{N}_{s_q}\cdots\mathrm{N}_{s_2}\mathrm{N}_{s_1}$, corresponding to the tuple $(s_q,\dots,s_2,s_1)$. If $q=1$, then one has a variety of nilpotent Lie algebras N_{s_1}, defined by an identity

$$T_1(x_1,\dots,x_{s_1}) = [x_1,\dots,x_{s_1}] \equiv 0.$$

(Lie brackets we consider to be leftnormed, that is $[a_1,\dots,a_n]=[\dots[a_1,a_2],\dots,a_n]$). Suppose that we have constructed an identity $T_{q-1}(x_1,\dots,x_t)$, which defines $\mathrm{N}_{s_{q-1}}\cdots\mathrm{N}_{s_1}$, then $\mathrm{N}_{s_q}\cdots\mathrm{N}_{s_1}$ is defined by an identity

$$T_q(x_{11},\dots,x_{ts_q}) = [T_{q-1}(x_{11},\dots,x_{t1}), T_{q-1}(x_{12},\dots,x_{t2}),\dots,T_{q-1}(x_{1s_q},\dots,x_{ts_q})] \equiv 0,$$

where $x_{ij},\ 1\le i\le t,\ 1\le j\le s_q$ are distinct elements from X. It is easy to see that $\mathrm{N}_{s_q}\cdots\mathrm{N}_{s_1}$ consists of all Lie algebras H, such that there exists a chain of ideals $0{=}H_{q+1}{\subset}H_q\subset\dots\subset H_1=H$, $H_i/H_{i+1}\in\mathrm{N}_{s_i}$. If $s_q{=}\dots{=}s_2{=}s_1$ then one has a variety of solvable Lie algebras A^q.

A growth of Lie algebras not exceeding an exponential growth has been studied extensively, see review [3]. Unlike associative case, (see Theorem 1.1) a growth of a rather small variety AN_2 is *overexponential* [4] (i.e. it cannot be bounded by any exponent). Moreover, in some sense, this situation is typical for Lie algebras. So, we have a vast area of the overexponential growth for Lie algebras.

In [5] there was suggested a scale to measure the overexponential growth of Lie algebras. In terms of this scale a growth of polynilpotent varieties of Lie algebras was described (see Theorem 2.1). The goal of the present paper is to give more precise description of the growth of polynilpotent varieties (Theorem 2.2, see also Lemma 3.3). As a result appears more fine scale for the overexponential growth (Section 2). Our main instrument is a *complexity function* of the variety, we study estimates on its growth (Section 3). Of independent interest is an asymptotic for the complexity function for the polynilpotent Lie variety, obtained in Theorem 3.2. In Section 4 we study an asymptotics for coefficients of fast increasing entire function, this is an important step in a proof of Theorem 2.2. In Section 5 we directly prove the lower estimates for Theorem 2.2. In Section 6 we prove an analog of Theorem 1.1 for Lie algebras (Theorem 2.3).

2 More fine scale for growth of Lie PI-algebras and growth of polynilpotent varieties

In [5] appeared the following types of overexponential growths. Denote

$$\ln^{(1)}x=\ln x,\ \ln^{(s+1)}x=\ln(\ln^{(s)}x); \quad \exp^{(1)}x=\exp x,\ \exp^{(s+1)}x=\exp(\exp^{(s)}x), \quad s\in\mathbb{N}.$$

Consider a series of functions of a natural argument $\Psi^q_\alpha(n)$, $q = 2, 3, \dots$ with a real parameter α:

$$\Psi^q_\alpha(n) = \begin{cases} (n!)^{\frac{\alpha-1}{\alpha}}, & \alpha > 1, \quad q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha > 0, \quad q = 3, 4 \dots \end{cases}$$

Let $f(n)$ be a function of a natural argument, we will compare it with our etalon functions in the following way. Suppose that $\tau_\gamma(n)$ is a function of a natural argument increasing with respect to a real parameter γ. Let $c \in \mathbb{R}$ be fixed. Then we denote

$$\begin{aligned} f(n) \sim^c \tau_c(n) &\iff \inf\{\gamma > 0 \mid f(n) \overset{\text{a}}{\leq} \tau_\gamma(n)\} = c, \\ f(n) \approx^c \tau_c(n) &\iff \inf\{\gamma > 0 \mid f(n) \overset{\text{a}}{\leq} \tau_\gamma(n)\} = \sup\{\gamma > 0 \mid \tau_\gamma(n) \overset{\text{a}}{\leq} f(n)\} = c, \end{aligned}$$

where we write $f(n) \overset{\text{a}}{\leq} g(n)$ for two functions of a natural argument iff $\exists N\, \forall n \geq N,\ f(n) \leq g(n)$. This notation we also will use for functions of a real argument. Two notions $\sim^c$ and $\approx^c$ differ so that in the second case we approximate our function $f(n)$ by etalon function $\tau_\gamma(n)$ from both sides:

$$\begin{aligned} f(n) \approx^c \tau_c(n) &\iff \forall \varepsilon > 0\ \exists N:\ \tau_{c-\varepsilon}(n) \leq f(n) \leq \tau_{c+\varepsilon}(n), \quad n \geq N. \\ &\iff \exists \varepsilon(n) : \mathbb{N} \to \mathbb{R};\ f(n) = \tau_{c+\varepsilon(n)}(n);\ \lim_{n\to\infty} \varepsilon(n) = 0. \end{aligned}$$

(An equivalence of these conditions is easily verified). But in the first case we approximate $f(n)$ by etalon functions only from above. The main result of [5] is as follows.

Theorem 2.1 ([5]) *Let* $\mathrm{V} = \mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$, $q \geq 2$ *be a polynilpotent variety of Lie algebras. Then there exists an infinitesimal such that*

$$c_n(\mathrm{V}) = \Psi^q_{s_1+o(1)}(n) = \begin{cases} (n!)^{\frac{s_1+o(1)-1}{s_1}}, & q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/(s_1+o(1))}}, & q = 3, 4 \dots \end{cases}$$

If an iteration of a logarithm enters some formula, then we consider that this formula is valid from the point where this logarithm is defined and positive. In [5] the assertion of this theorem was written in other way $c_n(\mathrm{V}) \approx^{s_1} \Psi^q_{s_1}(n)$. For a more precise description of a growth we consider a series of functions with two real parameters α, β:

$$\Psi^q_{\alpha,\beta}(n) = \begin{cases} (n!)^{\frac{\alpha-1}{\alpha}} \beta^{n/\alpha}, & \alpha \geq 1,\ \beta > 0; \quad q = 2; \\ \dfrac{n! \cdot (\beta/\alpha)^{n/\alpha}}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha > 0,\ \beta > 0; \quad q = 3, 4 \dots \end{cases}$$

Here is the main result of the present paper.

Theorem 2.2 *Let* $\mathrm{V} = \mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$, $q \geq 2$ *be a polynilpotent variety. Then there exists an infinitesimal such that*

$$c_n(\mathrm{V}) = \Psi^q_{s_1, s_2+o(1)}(n) = \begin{cases} (n!)^{\frac{s_1-1}{s_1}} (s_2 + o(1))^{n/s_1}; & q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/s_1}} \Big(\dfrac{s_2 + o(1)}{s_1}\Big)^{n/s_1}; & q = 3, 4 \dots \end{cases}$$

In [5] Theorem 2.1 the result was obtained by direct evaluations of some sums of coefficients. But in Theorem 2.2 another important instrument is the *complexity function* of a variety M (see below). For $q = 2$ the upper bound for $c_n(\mathrm{V}) \sim^{s_2} \Psi^q_{s_1,s_2}(n)$ was also obtained earlier [5].

By the following result we see that the scale above is rather complete (also, this is a better estimate than that in [5]).

Theorem 2.3 *Let* V *be a variety of Lie algebras satisfying some nontrivial identity of degree* $m > 3$. *Then there exists an infinitesimal (depending only on* m*), such that*

$$c_n(\mathrm{V}) \le \frac{n!}{(\ln^{(m-3)} n)^n}(1+o(1))^n.$$

Remark. For a codimension growth not exceeding exponent it is typical to use the Joung diagrams [3]. But even for a rather small variety $\mathrm{V} = \mathrm{AN}_2$ by Theorem 2.2 we have $c_n(\mathrm{V}) = \sqrt{n!}(1+o(1))^n$. It is easy to see that for an irreducible representation π of the symmetric group S_n one has $\dim_K \pi < \sqrt{n!}$; the number of all irreducible representations is rather small: $\rho(n) \approx \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$. Hence, from a dimension viewpoint, S_n-module $P_n(\mathrm{V})$ may contain all irreducible S_n-modules. So, for Lie algebras we need other technique, than the Joung diagrams.

Remark that other interesting object is a *growth of finitely generated algebras.* Here for Lie PI-algebras also appears a series of types of growths between polynom and exponent [6]. Also, in [7] is found an asymptotic for growth of a free algebra $L = F(\mathrm{V}, k)$ of a rank k in case of a polynilpotent variety V.

3 Complexity functions for some varieties and their growth

Recall that for the free Lie algebra $F = F(X)$ one has $c_n(F) = (n-1)!$ For a given variety V a *complexity function* is defined as:

$$\mathcal{C}(\mathrm{V}, z) = \sum_{n=1}^{\infty} \frac{c_n(\mathrm{V})}{n!} z^n, \quad z \in \mathbb{C}.$$

An upper bound for an arbitrary nontrivial variety of Lie algebras [8] can be formulated as

Theorem 3.1 ([9]) *Let* V *be a nontrivial variety of Lie algebras. Then* $\mathcal{C}(\mathrm{V}, z)$ *is an entire function of a complex variable.*

For some varieties it is possible to compute the complexity function explicitly.

Lemma 3.1 ([5]) *The variety of solvable Lie algebras* A^q *has the following complexity function*

$$\mathcal{C}(\mathrm{A}^q, z) = \sigma_0(z) + \ldots + \sigma_{q-1}(z), \quad \textit{where}$$
$$\sigma_0(z) = z, \quad \sigma_{i+1} = 1 + (\sigma_i(z) - 1)\exp(\sigma_i(z)), \ i = 0, 1, 2, \ldots$$

Lemma 3.2 ([5]) *For a variety* $\mathrm{N}_c\mathrm{A}$ *we have*

$$\mathcal{C}(\mathrm{N}_c\mathrm{A}, z) = z + \sum_{i=1}^{c} \frac{1}{i}\Big(1 + \exp(z)(z-1)\Big)^i. \tag{1}$$

Complexity functions are useful for computations of a codimension growth. We can illustrate this by an example. We write $f(n) \approx g(n)$, $n \to \infty$ iff $\lim_{n\to\infty} f(n)/g(n) = 1$.

Lemma 3.3 *For* $\mathrm{N}_c\mathrm{A}$ *one has* $c_n(\mathrm{N}_c\mathrm{A}, z) \approx c^{n-c-1} n^c$.

Proof. Remark that for a series expansion $f(z) = z^k \exp(\mu z)$, $k \in \mathbb{N}$ of type $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ one has $a_n \approx \mu^{n-k} n^k$. Indeed, computation shows that for $n \geq k$ we obtain $a_n = \mu^{n-k} \frac{n!}{(n-k)!} \approx \mu^{n-k} n^k$.

So, the maximal possible growth in (1) gives the term $\frac{1}{c} z^c \exp(cz)$, thus yielding the result. $\square$

We will consider complexity functions in a more general context. Suppose that we are given an algebra (groupoid), which is generated by a set $X = \{x_i \mid i \in \mathbb{N}\}$. Let A be a subspace in the algebra (subset in the groupoid).

For any set of distinct elements $Y = \{x_{i_1}, \ldots, x_{i_n}\} \subset X$ by $P_n(A, Y)$ we denote a linear span (subset) of all multilinear elements of degree n on Y in A. Suppose that the dimension of this linear span (number of elements) $c_n(A, Y)$ does not depend on the chose of Y, but depends only on n. In this case we write $c_n = c_n(A, Y)$; also in this case we say that for A is *defined a complexity function with respect to* X; the latter being written as

$$\mathcal{C}(A, z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n.$$

(where the sum is taken from $n = 0$, $c_0 = 1$ for associative algebras and groupoids with unity). Remark that A need not necessarily consist of multilinear elements. The complexity function is one of examples of *exponential generating functions* [10].

Lemma 3.4 *Let* Γ *be a groupoid (an algebra) generated by a set* $X = \{x_i | i \in \mathbb{N}\}$. *Suppose that for subsets (subspaces)* $A \subset \Gamma$, $B \subset \Gamma$ *complexity functions are defined. Suppose also that all elements* $a * b$, $a \in A$, $b \in B$ *of the set* $A * B \subset \Gamma$ *are distinct (linearly independent); where* $*$ *denotes the multiplication in the groupoid (algebra). Then* $A * B$ *has a complexity function, and* $\mathcal{C}(A * B, z) = \mathcal{C}(A, z)\mathcal{C}(B, z)$.

Proof. We observe that $c_n(A * B) = \sum_{i=0}^{n} \binom{n}{i} c_i(A) c_{n-i}(B)$, where binomial coefficients correspond to all divisions of the set $\{x_1, \ldots, x_n\}$ into two parts. This yields an assertion for the series. $\square$

Lemma 3.5 ([9]) *Let* $L = F(\mathrm{V}, X)$ *be a free Lie algebra of some variety (or Lie algebra for which a complexity function is defined), generated by a set* $X = \{x_i | i \in \mathbb{N}\}$. *Then for its universal enveloping algebra* $U(L)$ *a complexity function is defined (also with respect to* X*). Moreover,* $\mathcal{C}(U(L), z) = \exp(\mathcal{C}(L, z))$.

Proof. Considerations in [9] are based on computations. We supply an easy clear proof. Let us make use of the standard construction of the universal enveloping algebra:

$$U(L) \cong T(L)/\operatorname{Id}\{x \otimes y - y \otimes x - [x, y] | x, y \in L\}, \quad T(L) = \mathop{\oplus}_{m=0}^{\infty} T^m, \quad T^m = \underbrace{L \otimes \cdots \otimes L}_{m}.$$

By Lemma 3.4 we obtain $\mathcal{C}(T^m, z) = \mathcal{C}(L, z)^m$. We count multilinear elements only, hence a number of elements $y_{j_1} \otimes \cdots \otimes y_{j_m} \in T^m$, $y_{j_i} \in L$ is $m!$ times bigger than a number of corresponding ordered elements in $U(L)$. As a result one has

$$\mathcal{C}(U(L), z) = \sum_{m=0}^{\infty} \frac{\mathcal{C}(T^m, z)}{m!} = \sum_{m=0}^{\infty} \frac{\mathcal{C}(L, z)^m}{m!} = \exp(\mathcal{C}(L, z)).$$

$\square$

Our next constructions will happen inside a free Lie algebra $F(X)$ with a countable free generating set $X = \{x_i | i \in \mathbb{N}\}$. Suppose that in this algebra there exists a subalgebra $\widetilde{F} = \widetilde{F}(\widetilde{X})$ with a free generating set $\widetilde{X} = \{\widetilde{x}_i | i \in \mathbb{N}\}$. Suppose that for $\widetilde{X}$ a complexity function is defined $\mathcal{C}(\widetilde{X}, z) = \phi(z)$ (next, all complexity functions are considered with respect to X).

Lemma 3.6 *Let $\widetilde{F} = \mathop{\oplus}\limits_{i=1}^{\infty} \widetilde{F}_i$ be a natural grading given by $\widetilde{X}$. Then $\mathcal{C}(\widetilde{F}_i, z) = \phi^i(z)/i$.*

Proof. By Lemma 3.4 a set of all associative $\widetilde{X}$-words of length i has a complexity function $\phi^i(z)$. Recall that an associative word is said *regular* (or a *Lyndon word*) iff for any decomposition $x = ab$ one has $x > ba$ (an order is lexicographical from the left). There is a one-to-one correspondence between regular words and basis elements of a free Lie algebra [11]. From any multilinear $\widetilde{X}$-word we obtain a regular word by cyclic shifts such that a maximal letter will be the first. This yields the assertion. $\square$

Denote $\sum_{n=0}^{\infty} a_n z^n \prec \sum_{n=0}^{\infty} b_n z^n \iff a_n \le b_n,\ n \ge 0$. Let us introduce functions

$$A_d(z) = \frac{z^d}{d}, \qquad B_d(z) = \sum_{i=1}^{d} \frac{z^i}{i}, \qquad d \ge 1;$$
$$G(z) = \exp(z) - z - 1, \qquad H(z) = z\ \exp(z).$$

By Shirshov's theorem (see [1]) $(\widetilde{F})^{d+1}$ is a free Lie algebra.

Proposition 3.1 *For $(\widetilde{F})^{d+1}$, $d \ge 1$ there exist a free generating set Y with*

$$G(A_d(\phi(z))) \prec \mathcal{C}(Y, z) \prec H(B_d(\phi(z))). \tag{2}$$

Proof. Let $\Gamma = \Gamma(\overline{X})$ be a free groupoid of non-associative monomials in the set $\overline{X} = \{\overline{x_i} | i \in \mathbb{N}\}$ and $*$ be an operation in it. For convenience let us identify elements of Γ with their canonical images in $\widetilde{F}$ under an extension of a mapping $\overline{x}_i \to \widetilde{x}_i$, $i \in \mathbb{N}$. For $\overline{x} \in \Gamma$ by $\deg \overline{x}$ we denote its degree with respect to $\overline{X}$.

Remark that $S = \Gamma \cap (\widetilde{F})^{d+1}$ is a two-sided ideal in Γ. Suppose that R is the Hall basis for $\widetilde{F}$, i.e. $u < v$ if a degree of u is less than that of v; also suppose that $\widetilde{x}_i < \widetilde{x}_j$ for $i < j$. Then $u \in R \backslash S$, $v \in R \cap S$ implies that $u < v$. An element $w \in S$ is called S-reducible iff $w = u{*}v$, $u, v \in S$; otherwise it is called S-irreducible.

A set Y of all S-irreducible elements from $R \cap S$ is a free generating system in $(\widetilde{F})^{d+1}$ [1, 2.4.2]. Denote $Z = R \backslash S = \{w \in R | \deg w \le d\}$. If $w \in Y$, $w = u * v$, $u, v \in R$, then by definition of the Hall basis $u > v$, and therefore $v \in Z$. They are two possibilities. Either $u \in Z$ and our process stops or $u \notin Z$; in which case $u = u_1 * u_2$, $u_1 > u_2$, $u_2 \le v$, thus $u_2 \in Z$. In the last case we repeat our process to u_1. Finally, $w \in Y$ can be uniquely presented as

$$w = [z_{i_1}, z_{i_2}, \ldots, z_{i_n}], \qquad z_{i_j} \in Z, \quad n \ge 2; \tag{3}$$
$$z_{i_1} > z_{i_2} \le \ldots \le z_{i_n}, \quad [z_{i_1}, z_{i_2}] \notin Z,\ [z_{i_1}, z_{i_2}] \in R. \tag{4}$$

Conversely, one easily verifies that any such element belongs to Y. Let us evaluate $\mathcal{C}(Y, z)$. By Lemma 3.6 $\mathcal{C}(Z, z) = \sum_{i=1}^{d} \phi^i(z)/i = B_d(\phi(z))$. Let Y_1 be a set of multilinear elements of type (3) where the last two conditions in (4) are omitted. Then

$$Y_1 \subset Y_2 = \{[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] \,|\, \forall z_{i_1},\ z_{i_2} \le z_{i_2} \le \ldots \le z_{i_n}, \quad z_{i_j} \in Z, \quad n \ge 1\}.$$

Ordered sequences of z_{i_j} correspond to the basis of $U(\widetilde{L})$. Applying Lemmas 3.4, 3.5, we obtain

$$\mathcal{C}(Y, z) \prec \mathcal{C}(Y_2, z) = \mathcal{C}(Z, z)(\exp(\mathcal{C}(Z, z))) = H(B_d(\phi(z))). \tag{5}$$

In order to get a lower bound for $\mathcal{C}(Y, z)$ we consider $\overline{Z} = \{x \in Z \mid \deg x = d\}$, then $\mathcal{C}(\overline{Z}, z) = \phi^d(z)/d = A_d(\phi(z))$. Remark that for $z_{i_j} \in \overline{Z}$ the last two conditions in (4) are satisfied. Then

$$\overline{Y} = \{[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] \mid z_{i_j} \in \overline{Z},\ n \ge 2,\ z_{i_1} > z_{i_2} \le \ldots \le z_{i_n}\} \subset Y.$$

Remark that all inequalities are strict, by changing of places of elements z_{i_1}, z_{i_2}, we obtain a set with lesser complexity function; for which, as above

$$\begin{aligned}
&\overline{\overline{Y}} = \{[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] \mid z_{i_j} \in \overline{Z},\ n \geq 2,\ z_{i_1} \leq z_{i_2} \leq \ldots \leq z_{i_n}\};\\
&\mathcal{C}(Y, z) \succ \mathcal{C}(\overline{\overline{Y}}, z) = \exp(\mathcal{C}(\overline{Z}, z)) - \mathcal{C}(\overline{Z}, z) - 1 = G(A_d(\phi(z))). \qquad (6)
\end{aligned}$$

Now estimates (5),(6) are as required. □

Proposition 3.2 *Let* $\mathrm{V} = \mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$, $q \geq 2$ *be a polynilpotent variety. Then for its complexity function the following estimates hold*

$$\sum_{i=1}^{q} A_{s_i}(\alpha_i(z)) \prec \mathcal{C}(\mathrm{V}, z) \prec \sum_{i=1}^{q} B_{s_i}(\beta_i(z)),$$

where $\alpha_1(z) = \beta_1(z) = z$ *and* $\alpha_{i+1}(z) = G(A_{s_i}(\alpha_i(z)))$, $\beta_{i+1}(z) = H(B_{s_i}(\beta_i(z)))$, $i = 1, \ldots, q$.

Proof. Remark, that by the combinatorial interpretation of $G(z), H(z)$ in the proposition above, a relation for complexity functions $f(z) \prec g(z)$ always implies $H(f(z)) \prec H(g(z))$, $G(f(z)) \prec G(g(z))$.

Let $F = F(X)$ be a free Lie algebra, consider a chain of ideals $F = F_1 \supset F_2 \supset \ldots \supset F_{q+1}$, $F_{i+1} = (F_i)^{s_i}$, $i = 1, \ldots, q$. Let us prove by induction, that for F_i there exists a set of free generators X_i such that

$$\alpha_i(z) \prec \mathcal{C}(X_i, z) \prec \beta_i(z). \qquad (7)$$

If $i = 1$, then we set $X_1 = X$, and $\mathcal{C}(X_1, z) = \alpha_1(z) = \beta_1(z) = z$. For $i = k > 1$, by the inductive hypothesis, there exist a free generating set X_k for F_k with $\alpha_k(z) \prec \mathcal{C}(X_k, z) \prec \beta_k(z)$. By application of Proposition 3.1 for $\widetilde{F} = F_k$, $d = s_k$, $\phi(z) = \mathcal{C}(X_k, z)$, we obtain a free generating set $Y = X_{k+1}$ for F_{k+1}, with

$$\mathcal{C}(X_{k+1}, z) \prec H(B_{s_k}(\mathcal{C}(X_k, z))) \prec H(B_{s_k}(\beta_k(z))) = \beta_{k+1}(z)$$

because $H(z)$ is a monotone function as it was remarked earlier. Analogously we obtain a lower estimate.

Remark that F_i/F_{i+1} is the free nilpotent Lie algebra of the class s_i with the free generating set X_i [1]. Also $F(\mathrm{V}) \cong F/F_{q+1}$, and by Lemma 3.6 and (7) we have

$$\mathcal{C}(\mathrm{V}, z) = \sum_{i=1}^{q} \mathcal{C}(F_i/F_{i+1}, z) = \sum_{i=1}^{q} B_{s_i}(\mathcal{C}(X_i, z)) \prec \sum_{i=1}^{q} B_{s_i}(\beta_i(z)).$$

By analogy we obtain a lower bound. □

Theorem 3.2 *Let* $\mathrm{V} = \mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$, $q \geq 2$ *be a polynilpotent variety and* $f(z) = \mathcal{C}(\mathrm{V}, z)$ *be its complexity function. Then*

$$\lim_{r \to \infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{s_1}} = \frac{s_2}{s_1}.$$

Proof. Coefficients of complexity functions are real and nonnegative, therefore by a substitution $z = r \in \mathbb{R},\ r > 0$ a sign $\prec$ turns into an ordinary inequality. If, for simplicity, we suppose that $G(z) = \exp(z)$, then the last proposition gives an estimate

$$\mathrm{M}_f(r) = \mathcal{C}(\mathrm{V}, r) \geq A_{s_q}(\alpha_q(r)) = \frac{1}{s_q} \underbrace{\exp\left(\frac{s_q}{s_{q-1}} \exp\left(\ldots \left(\frac{s_3}{s_2} \exp\left(\frac{s_2}{s_1} r^{s_1}\right)\right) \ldots\right)\right)}_{q-1 \text{ times } \exp}.$$

A correct estimate, as well as an upper estimate with use of $H(z)$, is more complicated. Nevertheless, it is easy to see that they yield the desired assertion

$$\lim_{r\to\infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{s_1}} = \frac{s_2}{s_1}.$$

□

Remark that it will be interesting to obtain an explicit formula for the complexity function of the polynilpotent variety similar to the formulas from Lemmas 3.1, 3.2.

4 Fast increasing entire functions

Let $f(z)$ be an entire function of a complex variable. Denote $\mathrm{M}_f(r) = \max\limits_{|z|=r} |f(z)|$. Then *order* is defined in the following way [12], [13]:

$$\operatorname{ord} f = \inf\{\rho \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp(r^\rho)\} = \overline{\lim_{r\to\infty}} \frac{\ln\ln \mathrm{M}_f(r)}{\ln r}.$$

Also, a *type* is defined, provided that the order $\rho = \operatorname{ord} f$ is known [12], [13]:

$$\operatorname{typ} f = \inf\{\sigma \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp(\sigma r^\rho)\} = \overline{\lim_{r\to\infty}} \frac{\ln \mathrm{M}_f(r)}{r^\rho}.$$

The following Adamar's formula connecting a growth of an entire function and an asymptotic of its Taylor coefficients is very well known.

Theorem 4.1 ([13, 3.2.3]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ and type σ. Then*

$$\overline{\lim_{n\to\infty}} n^{1/\rho} \sqrt[n]{|a_n|} = (\sigma e \rho)^{1/\rho}.$$

By analogy we can define an *order* and a *type* of the *level* q as (the second provided that $\operatorname{ord}_q f = \rho$):

$$\begin{aligned} \operatorname{ord}_q f = &\quad \inf\{\rho \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp^{(q)}(r^\rho)\ \} = \quad \overline{\lim_{r\to\infty}} \frac{\ln^{(q+1)} \mathrm{M}_f(r)}{\ln r}; \\ \operatorname{typ}_q f = &\quad \inf\{\sigma \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp^{(q)}(\sigma r^\rho)\ \} = \quad \overline{\lim_{r\to\infty}} \frac{\ln^{(q)} \mathrm{M}_f(r)}{r^\rho}. \end{aligned}$$

In these notations Theorem 3.2 implies

Corollary 4.1 *Let $\mathrm{V} = \mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$, $q \geq 2$ be a polynilpotent Lie variety and $f(z) = \mathcal{C}(\mathrm{V}, z)$ be its complexity function. Then*

$$\operatorname{ord}_{q-1} f(z) = s_1, \qquad \operatorname{typ}_{q-1} f(z) = s_2/s_1.$$

There exist formulae for fast increasing entire functions, analogous to the Adamar's formula, see review [14], and [15]. In some cited papers a connection between a growth of $\ln^{(s)} \mathrm{M}_f(r)$ and a growth of $\ln^{(t)} r,\ t>1$ is studied. A more common character has Theorem 2 in [16], but our case also does not fit into its conditions (remark also that in its wording there is a misprint: to class L^0 should belong the function $\beta(x)$ itself, but not its derivative). So, let us prove the result that we need.

Theorem 4.2 *Suppose that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is an entire function. Then for any fixed numbers $q\in\mathbb{N},\ q\geq 3;\ \lambda>0$ holds*

$$\overline{\lim_{r\to\infty}} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{\lambda}} = \overline{\lim_{n\to\infty}} \ln^{(q-2)} n\ |a_n|^{\lambda/n}.$$

Proof. Suppose that $\overline{\lim}_{r\to\infty} \ln^{(q-1)} \mathrm{M}_f(r) r^{-\lambda}=\sigma<\infty$. Then for any $\varepsilon>0$ there exists r_0 such that

$$\mathrm{M}_f(r)\leq \exp^{(q-1)}(\overline{\sigma} r^{\lambda}),\ r\geq r_0;\quad \overline{\sigma}=\sigma+\varepsilon.$$

By Caushy formula [12] we obtain an estimate:

$$|a_n|\leq \frac{\exp^{(q-1)}(\overline{\sigma} r^{\lambda})}{r^n}.$$

Let us substitute $r_0=(1/\overline{\sigma}\ln^{(q-2)}(n/\ln n))^{1/\lambda}$, then

$$|a_n|\leq \frac{\exp(n/\ln n)\ \overline{\sigma}^{n/\lambda}}{(\ln^{(q-2)}(n/\ln n))^{n/\lambda}} = \frac{(1+o(1))^n\ \overline{\sigma}^{n/\lambda}}{(((1+o(1))\ \ln^{(q-2)} n)^{n/\lambda}} = \frac{(\overline{\sigma}+o(1))^{n/\lambda}}{(\ln^{(q-2)} n)^{n/\lambda}};$$

Since $\varepsilon>0$ has been taken arbitrarily, we obtain

$$\overline{\lim_{n\to\infty}} \ln^{(q-2)} n\ |a_n|^{\lambda/n}\leq \sigma = \overline{\lim_{r\to\infty}} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{\lambda}}. \tag{8}$$

In case $\sigma=\infty$ the inequality (8) is trivial.

To obtain an opposite inequality we suppose that $\overline{\lim}_{n\to\infty} \ln^{(q-2)} n\ |a_n|^{\lambda/n}=\sigma<\infty$. Then for any $\varepsilon>0$ there exists n_0 such that

$$|a_n|\leq \frac{\overline{\sigma}^{n/\lambda}}{(\ln^{(q-2)} n)^{n/\lambda}},\ n\geq n_0;\quad \overline{\sigma}=\sigma+\varepsilon.$$

Let us evaluate nth term of the Taylor series

$$\begin{aligned} |a_n r^n| &\leq \frac{\overline{\sigma}^{n/\lambda} r^n}{(\ln^{(q-2)} n)^{n/\lambda}} = \psi_r(n),\quad n\geq n_0; \\ \ln\psi_r(n) &= n\ln(\overline{\sigma}^{1/\lambda} r)-\frac{n}{\lambda}\ln^{(q-1)} n. \end{aligned} \tag{9}$$

In order to find $\max_{n>n_0}\psi_r(n)$ we have an equation

$$(\ln\psi_r(x))'=\ln(\overline{\sigma}^{1/\lambda} r)-\frac{1}{\lambda}\ln^{(q-1)} x-\frac{1}{\lambda\ \ln^{(q-2)} x\cdot\ldots\cdot\ln^{(1)} x}=0. \tag{10}$$

Without loss of generality we consider n_0 big enough that the denominator of the last fraction be greater than 1. We choose $r_1=r_1(n_0)$ so that $(\ln\psi_r(n_0))'>0$ for $r>r_1(n_0)$. It is easy to see that for a point $n_*=n_*(r)=[\exp^{(q-2)}(\overline{\sigma} r^{\lambda})]+1$ by substitution in (10) one has $(\ln\psi_r(n_*))'<0$.

Therefore, for a solution $\xi = \xi(r)$ of the equation (10) we obtain estimates $n_0 < \xi(r) < n_*(r)$. To evaluate a value of the maximum by (9), (10) we derive

$$\begin{aligned}\ln \psi_r(\xi) &= \frac{\xi}{\lambda\ \ln^{(q-2)} \xi \cdot \ldots \cdot \ln^{(1)} \xi} < \xi < n_* = [\exp^{(q-2)}(\overline{\sigma} r^\lambda)] + 1;\\ \psi_r(\xi) &\le \exp^{(q-1)}(\overline{\sigma} r^\lambda) e.\end{aligned}$$

To evaluate a "tail" we consider an inequality

$$\sqrt[n]{|a_n| r^n} \le \frac{\overline{\sigma}^{1/\lambda} r}{(\ln^{(q-2)} n)^{1/\lambda}} < \frac{1}{2},$$

it is valid as soon as $n > n_1(r) = [\exp^{(q-2)}(\overline{\sigma}(2r)^\lambda)] + 1$. Then

$$\sum_{n=n_1+1}^{\infty} |a_n r^n| < \sum_{n=n_1+1}^{\infty} (1/2)^n < 1.$$

As a result

$$\begin{aligned}\mathrm{M}_f(r) &\le \sum_{n=1}^{n_0} |a_n r^n| + \sum_{n=n_0+1}^{n_1} |a_n r^n| + \sum_{n=n_1+1}^{\infty} |a_n r^n| \le\\ &\le C + 1 + n_1(r) * \max_{n_0 < n \le n_1} |a_n r^n| \le C + 1 + n_1(r) * \psi_r(\xi(r)) \le\\ &\le C + 1 + ([\exp^{(q-2)}(\overline{\sigma}(2r)^\lambda)] + 1) \cdot \exp^{(q-1)}(\overline{\sigma} r^\lambda)\ e = \exp^{(q-1)}((\overline{\sigma} + o(1)) r^\lambda).\end{aligned}$$

Since $\varepsilon > 0$ has been taken arbitrarily one has

$$\varlimsup_{r\to\infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^\lambda} \le \sigma = \varlimsup_{n\to\infty} \ln^{(q-2)} n\ |a_n|^{\lambda/n}. \tag{11}$$

In case $\sigma = \infty$ the relation (11) is trivial. Now, the inequalities (8), (11) yield the desired result. □

Later we will make use of a lemma.

Lemma 4.1 *Let $f(z)$ be an entire function. Then $g(z) = \int_0^z f(z)\,dz$ has the same order and type of the same level as the function $f(z)$.*

Proof. In case $q = 1$ this fact is well known. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$; then $b_n = a_{n-1}/n,\ n > 0$. Let us apply Theorem 4.2:

$$\begin{aligned}\varlimsup_{n\to\infty} \ln^{(q-1)} n |b_n|^{\lambda/n} &= \varlimsup_{n\to\infty} \ln^{(q-1)}(n+1)\left|\frac{a_n}{n+1}\right|^{\lambda/(n+1)} =\\ &= \varlimsup_{n\to\infty} \exp\Big(\ln^{(q)}(n+1) + \frac{\lambda \ln|a_n|}{n+1} - \frac{\lambda\ln(n+1)}{n+1}\Big) =\\ &= \varlimsup_{n\to\infty} \exp\Big(\ln^{(q)} n + \frac{\lambda\ln|a_n|}{n+1} + o(1)\Big) =\\ &= \varlimsup_{n\to\infty} \exp\Big(\frac{n}{n+1}(\ln^{(q)} n + \frac{\lambda}{n}\ln|a_n|) + \frac{\ln^{(q)} n}{n+1} + o(1)\Big) = \varlimsup_{n\to\infty} \ln^{(q-1)} n|a_n|^{\lambda/n}.\end{aligned}$$

□

Proof of upper estimates in Theorem 2.2. Recall that in our case $\mathcal{C}(L, z) = \sum_{n=1}^{\infty} a_n z^n$, $a_n = c_n(\mathrm{V})/n!$. If $q \ge 3$, then by Theorems 3.2, and 4.2

$$\varlimsup_{n\to\infty} \ln^{(q-2)} n\ |a_n|^{s_1/n} = \frac{s_2}{s_1};$$

in our notations this can be written as

$$\ln^{(q-2)} n\, |a_n|^{s_1/n} \sim^{s_2} \frac{s_2}{s_1}; \qquad a_n \sim^{s_2} \frac{(s_2/s_1)^{n/s_1}}{(\ln^{(q-2)} n)^{n/s_1}};$$

$$c_n(\mathrm{V}) = a_n n! \sim^{s_2} \frac{n! \cdot (s_2/s_1)^{n/s_1}}{(\ln^{(q-2)} n)^{n/s_1}}.$$

If $q = 2$, then we apply Theorem 4.1:

$$\varlimsup_{n\to\infty} n^{1/s_1} |a_n|^{1/n} = (s_2 e)^{1/s_1};$$

$$c_n(\mathrm{V}) = a_n n! \sim^{s_2} n^{-n/s_1} (s_2 e)^{n/s_1} n!$$

By the Stirling formula $n^n \approx n! e^n / \sqrt{2\pi n}$, this value we substitute in the formula above and obtain the claimed result $c_n(\mathrm{V}) \sim^{s_2} (n!)^{1-1/s_1} (s_2)^{n/s_1}$. $\square$

5 Direct estimates on codimension growth

Theorem 3.2 gives the estimates on the complexity function of the polynilpotent variety from both sides. But Theorem 4.2 allowed us to obtain only the upper estimate for $c_n(\mathrm{V})$. Also, it is hardly possible to obtain the lower estimate making additional use of that $c_n(\mathrm{V})$ is monotone. So, for the lower estimate we will use direct evaluations. Lemma 3.5 in terms of coefficients can be written as [9]:

$$c_n(U(L)) = \sum_{\substack{k_1 n_1 + \ldots + k_t n_t = n \\ n_1 > n_2 > \ldots > n_t}} \frac{n!}{k_1!(n_1!)^{k_1} \ldots k_t!(n_t!)^{k_t}} c_{n_1}^{k_1}(L) \ldots c_{n_t}^{k_t}(L) \tag{12}$$

Proposition 5.1 *Let $\mathrm{V} = \mathrm{N}_c \mathrm{N}_d$. Then there exists an infinitesimal such that*

$$c_n(\mathrm{V}) = \Psi^2_{d,c+o(1)}(n) = (n!)^{\frac{d-1}{d}} (c + o(1))^{n/d}.$$

Proof. The upper estimate has been obtained above. Let $L = F(\mathrm{N}_c\mathrm{N}_d, X)$, $X = \{x_i | i \in \mathrm{N}\}$ and $H = F(\mathrm{N}_d, Y)$, $Y = \{y_i | i \in \mathrm{N}\}$. In order to obtain a lower bound for $c_n(U(H))$ let us consider in (12) a summand corresponding to parameters $t = 2$, $n_1 = d$, $k_1 = [n/d]$; $n_2 = 1$, and k_2 such that equality under the sum (12) be satisfied. Recall that $c_m(H) = (m-1)!$, $1 \le m \le d$; $c_m(H) = 0$, $m > d$. To this summand we apply the Stirling formula

$$\begin{aligned} c_n(U(H)) &\ge \frac{n!}{d^{n/d}(n/d)!d!} = \\ &= \exp(n(\ln n - 1) + o(n) - \tfrac{n}{d}\ln d - \tfrac{n}{d}(\ln n - \ln d - 1) + o(n)) = \\ &= \exp((1-\tfrac{1}{d})(n(\ln n - 1) + \tfrac{1}{2}\ln(2\pi n)) + o(n)) = (n!)^{1-1/d}(1+o(1))^n. \end{aligned} \tag{13}$$

Let us prove the lower bound for a codimension growth of $\mathrm{N}_c\mathrm{N}_d$. We construct a N_c-wreath product $W = Z \operatorname{wr}_{\mathrm{N}_c} H$, where $Z = F(\mathrm{N}_c, Z)$, $Z = \{z_i | i \in \mathbb{N}\}$, $H = F(\mathrm{N}_d, Y)$, $Y = \{y_i | i \in \mathbb{N}\}$ (see [1]). For vector spaces we have an isomorphism $W \cong \overline{Z} \oplus H$, where $\overline{Z}$ is an ideal in W. Also, this ideal is a free algebra in N_c with the following set of free generators: $z_j \circ h_i$; $\{h_i\}$ being an ordered basis of $U(H)$ and $w \circ u$, $w \in \overline{Z}$, $u \in U(H)$ being an extension of a natural action of H on $\overline{Z}$. A mapping $\phi : x_i \mapsto y_i + z_i$, $i \in \mathbb{N}$ can be extended to an embedding $\phi : L \to W$ [1, 4.4.7].

Consider a decomposition of $\phi P_n(L,\{x_1,\dots,x_n\})$ via Y,Z. Monomials, not containing z_i, belong to $P_n(H,\{y_1,\dots,y_n\})$. As for other monomials, by use of anticommutativity and Jacobi identity we present them as linear combinations of elements

$$w=[w_1,\dots,w_s],\quad w_k=[z_{i_k},y_{a(k,1)},\dots,y_{a(k,l_k)}],\quad 1\le s\le c,\ l_k\ge 0,$$
$$s+l_1+\ldots+l_s=n,\quad \{i_1,\dots,i_s\}\cup\{a(k,j)|1\le k\le s,\ 1\le j\le l_k\}=\{1,\dots,n\}.$$

The same arguments as in Poincaré-Birkhof-Witt's theorem prove that this monomials, in turn, are expressed via the following

$$\begin{gathered}w=[w_1,\dots,w_s],\quad w_k=z_{i_k}\circ u_k,\\ u_k\in P_{l_k}(U(H),\{y_{a(k,1)},\dots,y_{a(k,l_k)}\}),\quad 1\le s\le c,\ l_k\ge 0,\\ s+l_1+\ldots+l_s=n,\quad \{i_1,\dots,i_s\}\cup\{a(k,j)|1\le k\le s,\ 1\le j\le l_k\}=\{1,\dots,n\}.\end{gathered}\tag{14}$$

Moreover, if u_k are taken from the basis of $P_{l_k}(\ldots)$ and $i_1=\max\{i_1,\dots,i_s\}$, then the monomials (14) are linearly independent. Let us fix $z_1,\dots,z_c\in Z$. Suppose that $\widetilde{Z}$ is a subspace in $\overline{Z}$, spanned by all monomials, containing $z_1,\dots,z_c$; also let $\pi:\overline{Z}\to\widetilde{Z}$ be a projection parallel to a subspace of monomials of type (14) with other choice of z_{i_j}. Then (14) implies

$$\begin{aligned}&\pi\phi(P_n(L,\{x_1,\dots,x_n\}))=\\ &=\bigoplus_{\substack{c+l_1+\ldots+l_c=n;\\ \cup_{i=1}^{c}Y_i=\{y_{c+1},\dots,y_n\},\ |Y_i|=l_i,\\ \{j_1,\dots,j_c\}=\{1,\dots,c\},\ j_1=c}}[z_{j_1}\circ P_{l_1}(U(H),Y_1),\dots,z_{j_c}\circ P_{l_c}(U(H),Y_c)]\end{aligned}\tag{15}$$

Indeed, by a computation of $\pi\phi([x_{i_1},\dots,x_{i_n}])$, we see that x_{i_j} is substituted by z_{i_j} if $i_j\le c$, and by y_{i_j} otherwise. Therefore, for any monomial from the right hand side of (15), it is not difficult to find a pre-image under $\pi\phi$. Fix $l_i=[n/c]-1$ or $l_i=[n/c]$ such that $c+l_1+\ldots+l_c=n$. The number of distributions of $y_{c+1},\dots,y_n$ into the sets Y_i, $i=1,\dots,c$ we evaluate using the Stirling formula

$$\frac{(n-c)!}{l_1!\ldots l_c!}\ge\frac{(n-c)!}{((n/c)!)^c}=(c+o(1))^n.$$

Applying (13) and (15) we derive

$$\begin{aligned}c_n(L)&\ge(c+o(1))^n c_{l_1}(U(H))\ldots c_{l_c}(U(H))\ge\\ &\ge(c+o(1))^n\left(\left((\frac{n}{c}-2)!\right)^{1-1/d}(1+o(1))^n\right)^c=\\ &=(c+o(1))^n\exp\left(c(1-\frac{1}{d})\left((\frac{n}{c}-2)(\ln n-\ln c-1+o(1))+o(n)\right)\right)=\\ &=(c+o(1))^n\exp\left((1-\frac{1}{d})n(\ln n-1)-(1-\frac{1}{d})n\ln c+o(n)\right)=(n!)^{\frac{d-1}{d}}(c+o(1))^{n/d}.\end{aligned}$$

□

The following estimates will be proved for products of Lie varieties (see [1]).

Proposition 5.2 *Let* V, M=N$_s$V *be varieties of Lie algebras. Suppose that there exists an infinitesimal such that*

$$c_n(\mathrm{V})\ge\Psi^2_{\alpha,\beta+o(1)}(n)=(n!)^{\frac{\alpha-1}{\alpha}}(\beta+o(1))^n;\qquad \textit{where } \alpha\ge 1,\ \beta>0.$$

Then

$$c_n(\mathrm{M})\ge\Psi^3_{\alpha,\beta+o(1)}(n)=\frac{n!}{(\ln n)^{n/\alpha}}(\beta/\alpha+o(1))^n.$$

Proof. Suppose that $L = F(\mathrm{M})$, $H = F(\mathrm{V})$ are free algebras of varieties of countable rank. By the hypothesis for any $0 < \beta' < \beta$ we have $c_n(H) \geq \Psi^2_{\alpha,\beta'}(n)$ from some number $n > N$. If we will prove that beginning from some number $c_n(L) \geq \Psi^3_{\alpha,\beta'+o(1)}(n)$ holds, then the lower estimate will be proved. We omit the prime for convenience, then

$$c_m(H) \geq \Psi^2_{\alpha,\beta}(m) = (m!)^{1-1/\alpha}\beta^{m/\alpha}, \quad m > N.$$

Let us prove a lower estimate for $c_n(U(H))$. Consider in (12) terms with $t = 2$, $n_1 = m \geq N$, $k_1 = [n/m]$; $n_2 = 1$, and k_2 such that equality under the sum be satisfied. Now, using an estimate $a! < (a/e)^a a$, one has

$$c_n(U(H)) \geq \frac{n!\cdot\beta^{n/\alpha}}{(m!)^{\frac{n}{\alpha m}}(n/m)!m!} \geq \frac{n!\cdot\beta^{n/\alpha}}{f(n,m)h(n,m)}; \tag{16}$$

$$h(n,m) = m^{\frac{n}{\alpha m}} n/m\, m!, \quad f(n,m) = \Big(\frac{m}{e}\Big)^{n/\alpha}\Big(\frac{n}{em}\Big)^{n/m};$$

$$\ln f(n,m) = n\Big(\frac{\ln m}{\alpha} - \frac{1}{m} + \Big(\frac{\ln(n/m)}{m} - \frac{1}{\alpha}\Big)\Big). \tag{17}$$

By computation of a derivative, wee see that a minimum of (17), while n being fixed, is achieved in a point

$$m_* = \phi(m_*), \quad \phi(x) = \alpha\ln(n/x). \tag{18}$$

Remark that the mapping $\phi(x)$ is compressing and a solution of (18) can be found by iteration; for us it is enough to take the first two points

$$m_0 = \alpha\ln n, \ m_{i+1} = \phi(m_i), \ i \in \mathbb{N}, \qquad \lim_{i\to\infty} m_i = m_*;$$
$$m_0 = \alpha\ln n > m_* > m_1 = \alpha(\ln n - \ln\ln n - \ln\alpha),$$
$$m_* = \alpha\ln n + o(1). \tag{19}$$

After having substituted m_* into (17), the summand in parentheses vanishes by virtue of (18). Using (19), we obtain

$$\ln f(n,m) \geq \ln f(n,m_*) = n\Big(\frac{\ln m_*}{\alpha} - \frac{1}{m_*}\Big) = \frac{n(\ln\ln n + \ln\alpha + o(1))}{\alpha}, \tag{20}$$

$$\ln(f(n,m_*)h(n,m_*)) = n\Big(\frac{\ln m_*}{\alpha} - \frac{1}{m_*}\Big) + \ln h(n,m_*) = \frac{n(\ln\ln n + \ln(\alpha + o(1)))}{\alpha};$$

$$c_n(U(H)) \geq \frac{n!\cdot\beta^{n/\alpha}}{(\ln n(\alpha+o(1)))^{n/\alpha}} = \frac{n!\cdot(\beta/\alpha + o(1))^{n/\alpha}}{(\ln n)^{n/\alpha}} = \Psi^3_{\alpha,\beta+o(1)}(n). \tag{21}$$

Let $L = F(\mathrm{N}_s\mathrm{V})$, $\widetilde{L} = \widetilde{F}(\mathrm{AV})$ be free algebras of countable rank. Let us consider a verbal ideal $I = \mathrm{V}(L)$, then $\widetilde{L} \cong L/I^2$ and $c_n(L) \geq c_n(\widetilde{L})$. Hence, it is enough to prove a lower estimate for a free algebra $\widetilde{L} = F(\mathrm{AV}, X)$, $X = \{x_i | i \in \mathbb{N}\}$. Denote $H = F(\mathrm{V}, Y)$, $Y = \{y_i | i \in \mathbb{N}\}$ and let $\langle Z\rangle = \langle z_i | z_i \in Z\rangle$ be an abelian algebra with a basis $Z = \{z_i | i \in \mathbb{N}\}$. Consider an abelian wreath product $L_1 = Z \operatorname{wr} H = \overline{Z} \oplus H$, $\overline{Z} = Z \otimes U(H)$ [1]. There exists an embedding $\phi : \widetilde{L} \to L_1$, which extends a mapping $\phi : x_i \mapsto z_i + y_i$, $i \in \mathbb{N}$. Let $\pi_i : \overline{Z} \to z_i U(H)$ be a projection parallel to $\oplus_{j\neq i} z_j U(H)$, then it is easy to see that

$$\pi_n\phi(P_n(\widetilde{L}, \{x_1,\ldots,x_n\})) = z_n P_{n-1}(U(H), \{y_1,\ldots,y_{n-1}\});$$
$$c_n(L) \geq c_n(\widetilde{L}) \geq c_{n-1}(U(H)). \tag{22}$$

Computations show that a substitution of $n-1$ into (21) only changes the infinitesimal. Hence, the desired lower estimate is proved. □

Proposition 5.3 *Let* V, M=N_sV *be varieties of Lie algebras. Suppose that*

$$c_n(\mathrm{V}) \geq \Psi^q_{\alpha,\beta+o(1)}(n) = \frac{n!}{(\ln^{(q-2)} n)^{n/\alpha}}(\beta/\alpha+o(1))^{n/\alpha}; \qquad \textit{where } \alpha>0,\ \beta>0.$$

Then

$$c_n(\mathrm{M}) \geq \Psi^{q+1}_{\alpha,\beta+o(1)}(n) = \frac{n!}{(\ln^{(q-1)} n)^{n/\alpha}}(\beta/\alpha+o(1))^{n/\alpha}.$$

Proof. Let $L=F(\mathrm{M})$, $H=F(\mathrm{V})$ be free algebras of countable rank. By assumption for any $\beta'<\beta$ we have $c_n(H)\geq \Psi^q_{\alpha,\beta'}(n)$ beginning from some $n>N$. If we will prove that $c_n(L)\geq \Psi^{q+1}_{\alpha,\beta'+o(1)}(n)$, then the lower bound will be established. For convenience we omit the prime, then

$$c_m(H) \geq \Psi^q_{\alpha,\beta}(m) = \frac{m!\beta^{m/\alpha}}{(\ln^{(q-2)} m)^{m/\alpha}}, \quad m\geq N.$$

Let us prove the lower bound for $c_n(U(H))$. Consider in (12) a term with $t=2$, $n_1=m\geq N$, $k_1=[n/m]$; $n_2=1$, with appropriate k_2. Then, using an estimate $a!<(a/e)^a a$, we obtain

$$c_n(U(H)) \geq \frac{n!\cdot(\beta/\alpha)^{n/\alpha}}{(\ln^{(q-2)} m)^{n/\alpha}(n/m)!m!} \geq \frac{n!\cdot(\beta/\alpha)^{n/\alpha}}{f(n,m)h(n,m)}; \tag{23}$$

$$h(n,m)=n/m\, m!, \quad f(n,m) = (\ln^{(q-2)} m)^{n/\alpha}\Big(\frac{n}{em}\Big)^{n/m};$$

$$\ln f(n,m) = \frac{n}{\alpha}\ln^{(q-1)} m + \frac{n}{m}(\ln(n/m)-1). \tag{24}$$

Computations show that a minimum of (24), while n being fixed, is achieved in a point

$$m_* = \phi(m_*), \quad \phi(x) = \alpha(\ln^{(q-2)} x \ldots \ln^{(1)} x)\ln(n/x). \tag{25}$$

A mapping $\phi(x)$ is compressing and a solution of (25) we find by an iteration, for our estimates we take the first two points

$$m_0=\alpha\ln n,\ m_{i+1}=\phi(m_i),\ i\in\mathbb{N}, \qquad \lim_{i\to\infty} m_i = m_*;$$
$$m_0=\alpha\ln n < m_* < m_1=\phi(m_0);$$
$$\ln m_* = \ln\ln n(1+o(1)). \tag{26}$$

We express from (25) the value of $\ln(n/m_*)$ and substitute in (24); using (26) we have

$$\ln f(n,m) \geq \ln f(n,m_*) = \frac{n}{\alpha}(\ln^{(q-1)} m_* + o(1)) = \frac{n}{\alpha}(\ln^{(q)} n + o(1)); \tag{27}$$

$$\ln(f(n,m_*)h(n,m_*)) = \frac{n}{\alpha}(\ln^{(q)} n+o(1)) + \ln h(n,m_*) = \frac{n}{\alpha}(\ln^{(q)} n + o(1));$$

$$c_n(U(H)) \geq \frac{n!}{(\ln^{(q-1)} n)^{n/\alpha}}(\beta/\alpha+o(1))^{n/\alpha} = \Psi^{q+1}_{\alpha,\beta+o(1)}(n). \tag{28}$$

As above, it is enough to consider $\mathrm{N}_s=\mathrm{A}$, in which case we obtain $c_n(L)\geq c_{n-1}(U(H))$. A substitution of $n-1$ in (28) changes the infinitesimal only. Proposition is proved. □

Proof of theorem 2.1. Let $\mathrm{V}=\mathrm{N}_{s_q}\ldots\mathrm{N}_{s_1}$, $q\geq 2$. The upper estimates have been proved above. Namely, in the previous section the existence of the infinitesimal such that $c_n(\mathrm{V})\leq\Psi^q_{s_1,s_2+o(1)}(n)$ has been established. The lower estimates follow by induction from Propositions 5.1, 5.2, 5.3: $c_n(\mathrm{V})\geq\Psi^q_{s_1,s_2+o(1)}(n)$ for some infinitesimal. Our series of functions is continuous with respect to the parameter, this implies an existence of an infinitesimal such that $c_n(\mathrm{V})=\Psi^q_{s_1,s_2+o(1)}(n)$. □

6 On analogue of Regev's theorem for Lie algebras

The main goal of this section is to prove Theorem 2.3. All words are multilinear. We will use definitions and notations from Section 3; also we preserve notations of [9], where it is possible. Let A be a free associative algebra with a free generating set $X = \{x_i | i \in \mathbb{N}\}$. All words of equal length we order lexicographically from the left. A multilinear word $w \in P_n(A, \{x_1, \dots, x_n\})$ is said *Lie m-decomposable*, iff it can be presented in the form [9]:

$$w = a w_m w_{m-1} \dots w_1 b; \qquad w_t = y_t x_{t1} \dots x_{ts_t},\ s_t \ge 0; \quad y_i, x_{t_j} \in X;$$
$$y_t > x_{t_j},\ 1 \le t \le m, 1 \le j \le s_t; \quad y_m > y_{m-1} > \dots > y_1.$$

If such a presentation does not exist, then w is said *Lie m-indecomposable*. A set of all such words we denote by Q_m. By definition we consider that $1 \in Q_m$. Denote by R_m a subset of all Shirshov-regular words ([11], [1], [9]) in Q_m; also, by definition, $1 \notin R_m$.

Lemma 6.1 ([9], 1.2.1) *Let $w = w' x_n w'' \in P_n(A, \{x_1, \dots, x_n\})$. Then $w \in Q_m \iff w' \in Q_m$ and $w'' \in Q_{m-1}$.*

Lemma 6.2 *Let $w \in P_n(A, \{x_1, \dots, x_n\})$. Then*

1. *$w \in R_m \iff w = x_n u,\ u \in Q_{m-1}$;*
2. *$w \in Q_m \iff w$ can be presented in the form*

$$w = v_s \dots v_1, \quad v_i \in R_m,\ i = 1, \dots, s, \quad v_s < \dots < v_1,\ s \ge 1.$$

Proof. 1) Remark that a word $w \in P_n(A, \{x_1, \dots, x_n\})$ is regular iff it begins with the maximal letter x_n.

2) Suppose that $1 \ne w \in Q_m$. By Lemma 6.1 we have $w = w' x_n w''$, $w' \in Q_m$ and $w'' \in Q_{m-1}$. If w' is nontrivial, then we again apply Lemma 6.1 to w', etc. We obtain a presentation

$$\begin{aligned} &w = x_{i_s} u_s \dots x_{i_2} u_2 x_{i_1} u_1, \quad u_j \in Q_{m-1},\ j = 1, \dots, s; \quad s \ge 1; \\ &i_s < \dots < i_2 < i_1 = n; \quad x_{i_t} \text{ is a maximal letter in } x_{i_s} u_s \dots x_{i_t} u_t. \end{aligned} \tag{29}$$

We have $v_t = x_{i_t} u_t \in R_m$ and $w = v_s \dots v_1$, $v_s < \dots < v_1$. Conversely, it is easy to see that any word of such type can be presented as (29) and, by Lemma 6.1, it lies in Q_m. □

Let us consider corresponding complexity functions (the first of which was introduced in [9]):

$$q_m(z) = \mathcal{C}(Q_m, z) = \sum_{n=0}^{\infty} \frac{a_m(n)}{n!} z^n, \quad r_m(z) = \mathcal{C}(R_m, z) = \sum_{n=1}^{\infty} \frac{b_m(n)}{n!} z^n; \quad m \in \mathbb{N}.$$

Lemma 6.3 *The following initial conditions and recursive relations hold*

1. *$r_1(z) = 0$, $q_1(z) = 1$;*
2. *$r_m(z) = \int_0^z q_{m-1}(z)\, dz$; $\quad m > 1$;*
3. *$q_m(z) = \exp(r_m(z)) = \exp(\int_0^z q_{m-1}(z)\, dz)$, $\quad m > 1$.*

Proof. 1) By definition, every nonempty word is 1-decomposable, hence $a_1(n) = 0$, $n > 0$; and by definition $a_1(0) = 1$. Thus, we obtain $r_1(z) = 0$, $q_1(z) = 1$.

2) By Lemma 6.2, every word $w \in R_m \cap P_n(A, \{x_1, \dots, x_n\})$ can be uniquely written as $w = x_n u$, $u \in Q_{m-1} \cap P_{n-1}(A, \{x_1, \dots, x_{n-1}\})$, this means that $c_n(R_m) = c_{n-1}(Q_{m-1})$. Taking into account $r_m(0) = b_m(0) = 0$, for complexity functions this implies $r_m(z) = \int_0^z q_{m-1}(z)\, dz$.

3) By Lemma 6.2, we have an order, as that for universal enveloping algebra and it is easy to see that considerations of Lemma 3.5 remain valid, thus $\mathcal{C}(Q_m, z) = \exp(\mathcal{C}(R_m, z)) = \exp(r_m(z))$. □

Lemma 6.4 *1.* $\mathrm{ord}_{m-2}\, r_m(z) = \mathrm{typ}_{m-2}\, r_m(z) = 1$*;*

2. $\mathrm{ord}_{m-1}\, q_m(z) = \mathrm{typ}_{m-1}\, q_m(z) = 1$.

Proof. We have $q_2(z) = \exp(z)$; $r_3(z) = \exp(z) - 1$. These two functions have the ordinary order and type, both equal to 1. By Lemma 4.1, the integral in Lemma 6.3 does not change the oder and type, while the exponent increases the level by one. □

Proof of theorem 2.3. Arguments, similar to those of [9] prove that $P_n(\mathrm{V})$ is a linear span of all regular m-indecomposable words. Therefore, $c_n(\mathrm{V}) \le b_m(n)$, i.e. $\mathcal{C}(\mathrm{V}, z) \prec r_m(z)$. According to Lemma 6.4, $\mathrm{ord}_{m-2}\, r_m(z) = \mathrm{typ}_{m-2}\, r_m(z) = 1$. By application of Theorem 4.2 for $q = m-1$ we conclude that

$$\overline{\lim_{n\to\infty}}\, \ln^{(m-3)} n \left| \frac{c_n(\mathrm{V})}{n!} \right|^{1/n} = 1;$$

$$c_n(\mathrm{V}) \le \frac{n!}{(\ln^{(m-3)} n)^n} (1 + o(1))^n.$$

□

The estimate $\mathcal{C}(\mathrm{V}, z) \prec q_m(z)$ for varieties of Lie pairs was obtained in [9] as a generalization of results of [8]. In [5] this result was presented in terms of coefficients. In the present paper we gave a clear prove of a formula for $q_m(z)$ (Lemma 6.3). Moreover, an introduction of the function $r_m(z)$ and use of Theorem 4.2 enabled as to obtain the better upper estimate for the growth of an arbitrary nontrivial variety of Lie algebras than that in [5].

References

[1] Yu. A. Bahturin, *Identical relations in Lie algebras*, VNV Science Press, Utrext, 1987.

[2] A. Regev, *Existence of polynomial identities in* $A \otimes B$, Bull. Amer. Math. Soc. **77**, No. 6, (1971), 1067–1069.

[3] S. P. Mishchenko, *Growth of varieties of Lie algebras*, Uspekhi Mat. Nauk **45**, No. 6, (1990), 25–45; Engl. transl., Russian Math. Surveys, **45**, No. 6, (1990), 27–52.

[4] I. B. Volichenko, *On variety* AN_2 *over field of zero characteristic*, Dokl. Akad. Nauk Belarusi XXV, No. 12, (1981), 1063–1066.

[5] V. M. Petrogradsky, *On types of overexponential growth of identities in Lie PI-algebras*, Fundamentalnaya i Prikladnaya Matematika **1**, No. 4, (1995), 989–1007.

[6] V. M. Petrogradsky, *On some types of intermediate growth in Lie algebras*, Uspekhi Mat. Nauk **48**, No. 5, (1993), 181–182; Engl. transl., Russian Math. Surveys, **48**, No. 5, (1993), 181–182.

[7] V. M. Petrogradsky, *Intermediate growth in Lie algebras and their enveloping algebras*, J. Algebra **179**, (1996), 459–482.

[8] A. N. Grishkov, *On growth of varieties of Lie algebras*, Mat. Zametki **44**, No. 1, (1988), 51–54.

[9] Yu. P. Razmyslov, *Identities of algebras and their representations*, Transl. Math. Monogr. vol. 138., Amer. Math. Soc., Providence, RI, 1994.

[10] I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*, Wiley, New York, 1983.

[11] Yu. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, and M. V. Zaicev, *Infinite dimensional Lie superalgebras*, de Gruyter Exp. Math. vol. 7, de Gruyter, Berlin, 1992.

[12] B. V. Shabat, *Introduction in complex analysis*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1992.

[13] M. A. Evgrafov, *Asymptotic estimates and entire functions*, Gordon and Breach., New York, 1961.

[14] A. A. Goldberg, B. Ya. Levin, and I. W. Ostrovski, *Entire and meromorphic functions*, Itogi Nauki i Tekhniki, Sovrem. Probl. Mat. Fund. Naprav. vol. **85**, Komplex analysis, one variable–**1**, Moscow, 5–185.

[15] G. A. Fridman, *Slowly increasing functions and their applications*, Sibirsk. Math. Zh. **7**, No. 5, (1966), 1139–1160.

[16] M. N. Sheremeta, *On connection between growth of maximum module of entire function and module of its Taylor coefficients*, Izv. Vyssh. Uchebn. Zaved. Mat. **57**, No. 2, (1967), 100–108.