ON THE MODALITY OF PARABOLIC SUBGROUPS OF LINEAR ALGEBRAIC GROUPS

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Abstract. For a linear algebraic group G over an algebraically closed field k and a parabolic subgroup P of G the *modality of* P is defined to be the maximal number of parameters upon which a family of P-orbits on $\operatorname{Lie} P_u$ depends and it is denoted by $\operatorname{mod} P$, where P_u is the unipotent radical of P. The principal aim of this note is a generalization of two basic "monotonicity" results from [19] to positive characteristic: (1) If Θ is a semisimple automorphism of G and P is Θ-stable, then $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$. (2) If G is reductive, $\operatorname{char} k$ is a good prime for G, and H is a closed reductive subgroup of G normalized by a maximal torus $T \subset P$ of G, then $\operatorname{mod}(P \cap H) \leq \operatorname{mod} P$.

1. Introduction

Let G be a linear algebraic group over an algebraically closed field k and let P be a parabolic subgroup of G, i.e., G/P is a complete variety. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$ is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u$ depends and is denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$; likewise for the action of P on P_u. The *modality of* P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$. The principal aim of this paper is to generalize two "monotonicity" results on the modality from [19] to positive characteristic:

Theorem 1.1. *Let G be a linear algebraic group and let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. Then $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$.*

Theorem 1.2. *Let G be a reductive algebraic group and suppose that* $\operatorname{char} k$ *is a good prime for G. Let H be a closed reductive subgroup of G normalized by a maximal torus T of G and P a parabolic subgroup of G containing T. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Analogues of Theorems 1.1 and 1.2 were proved in [19, Theorem 2.13] under the assumption that $\operatorname{char} k$ is zero. The proofs in *loc. cit.* do not generalize to positive characteristic, as they make use of the separability of the orbit maps for the action of P on $\mathfrak{p}_u$.

These "monotonicity" theorems from [19] provide helpful inductive tools for comparing the modality of parabolic subgroups of various groups and, apart from [19] itself, have been used frequently in subsequent papers on the subject, such as [11], [18], [23], [24], and [25]. Theorems 1.1 and 1.2 allow us to extend the principal results from all of these papers to positive characteristic. See also [11] and [12].

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These results are proved in Section 3. In the remaining parts of the paper we present some complements to the main theme. We briefly indicate the results.

In characteristic zero the exponential mapping is a P-equivariant isomorphism between $\mathfrak{p}_u$ and P_u, and thus $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. In positive characteristic a similar result can be proved using Springer's map [26] between the variety of unipotent elements of G and the variety of nilpotent elements of $\mathfrak{g}$.

Theorem 1.3. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and suppose that* $\operatorname{char} k$ *is a good prime for G. Then* $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$.

In view of Kirillov's orbit method for the classification of unitary representations of nilpotent Lie groups and in connection with primitive ideals of the enveloping algebra of a complex nilpotent Lie algebra, the coadjoint representation of a Lie group is of particular interest [15, §15], [5].

We consider an algebraic group G and a rational G-module V. Modifying an argument from [21] we show

Theorem 1.4. *Let G be an algebraic group and let V be a rational G-module. Then* $\operatorname{mod}(G : V^*) = \operatorname{mod}(G : V)$*. Moreover, if* $\operatorname{mod}(G : V) = 0$*, then there is a bijection between the sets of G-orbits on V and V^*.*

For a proof see also [17]. As a special case of Theorem 1.4 we have

Corollary 1.5. *Let G be a linear algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod}(P : \mathfrak{p}_u^*) = \operatorname{mod}(P : \mathfrak{p}_u)$*. Moreover, if* $\operatorname{mod} P = 0$*, then there is a bijection between the sets of P-orbits on $\mathfrak{p}_u$ and $\mathfrak{p}_u^*$.*

Let $\mathcal{T}$ and $\mathcal{T}^*$ be the tangent and cotangent bundle of the flag manifold G/P. As a consequence of Corollary 1.5 we obtain

Corollary 1.6. *Let G be a linear algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod}(G : \mathcal{T}) = \operatorname{mod}(G : \mathcal{T}^*) = \operatorname{mod} P$*. Moreover, if* $\operatorname{mod} P = 0$*, then there is a bijection between the sets of G-orbits on the two G-bundles.*

2. Notation and Recollection

For an algebraic group G we denote its Lie algebra by $\operatorname{Lie} G$ or by $\mathfrak{g}$, the identity component of G by G^0, and its unipotent radical by G_u. For the Lie algebra of G_u we write $\mathfrak{g}_u$.

Throughout, the identity element of an algebraic group is denoted by e. For $x \in G$ let $\operatorname{Int}(x)$ denote the inner automorphism of G given by conjugation by x. For a subset $S \subset G$ we denote by $C_G(S) := \{g \in G \mid \operatorname{Int}(s)g = g \text{ for all } s \in S\}$ the centralizer of S in G, and likewise by $C_{\mathfrak{g}}(S) := \{Y \in \mathfrak{g} \mid \operatorname{Ad}(s)Y = Y \text{ for all } s \in S\}$ the centralizer of S in $\mathfrak{g}$.

Suppose that G acts morphically on an algebraic variety X. For x in X the G-orbit in X through x is denoted by $G \cdot x$ and for $Z \subset X$ a subset $\cup_{z \in Z} G \cdot z$ by $G \cdot Z$. Recall from [19] that the *modality* of the action of G on X is defined as $\operatorname{mod}(G : X) := \max_Z \min_{z \in Z} \operatorname{codim}_Z G^0 \cdot z$, where Z runs through all irreducible G^0-invariant subvarieties of X. In case X is an irreducible variety let $k(X)^G$ denote the field of G-invariant rational functions on X. As $\min_{x \in X} \operatorname{codim}_X G \cdot x =$

$\operatorname{trdeg} k(X)^G$ (e.g. see [20, 2.3]), $\operatorname{mod}(G : X)$ measures the maximal number of parameters upon which a family of G-orbits on X depends. The modality of the action of G on X is 0 precisely when G admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [1]. Our definition here is due to Vinberg [32], see also [20, 5.2].

Let G be a linear algebraic group and P a parabolic subgroup of G. We call $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$ the *modality of* P. In earlier papers on this subject the modality of P is defined as $\operatorname{mod}(P : P_u)$. When using results from [19], these papers were only concerned with fields of characteristic zero. In that case the exponential mapping is a P-equivariant isomorphism between $\mathfrak{p}_u$ and P_u, and thus $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. So the present definition is compatible with the one introduced in [19]. Nonetheless, if G is reductive Theorem 1.3 asserts that this equality still holds provided $\operatorname{char} k$ is a good prime for G.

Suppose that G is simple (over its center). Let ϱ be the highest root of the root system of G. A prime is said to be *bad* for G if it divides the coefficient of a simple root in ϱ, else it is called *good* for G, see [28, §I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the Chevalley commutator relations for G. Thus, if $\operatorname{char} k$ is very bad for G, there are degeneracies in these relations. The same notions apply to reductive groups by means of simple components [30, 3.6].

Let G be reductive and let H be a closed subgroup of G normalized by a maximal torus T of G, i.e., H is a *regular* subgroup of G, cf. [10]; likewise for subalgebras of $\mathfrak{g}$. In that case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H)$, is a subset of $\Psi = \Psi(G, T)$, the root system of G relative to T. If G is reductive and $\operatorname{char} k$ is zero or a prime which is not very bad for G, then $\Psi(H)$ is closed under addition in Ψ. Furthermore, if H is reductive and regular, $\Psi(H)$ is a symmetric subsystem of Ψ. For α a root of G we denote by U_α the corresponding one-parameter unipotent subgroup of G.

By a Levi subgroup of a reductive group G we simply mean a Levi subgroup of some parabolic subgroup of G.

An automorphism of an algebraic group G is called *semisimple* if it can be achieved by conjugation by a semisimple element of some (linear) algebraic group containing G [29].

As general references for algebraic groups we cite the books by Borel [2] and Springer [27]. For information on root systems the reader should consult Bourbaki [6]. Concerning semisimple automorphisms of linear algebraic groups we refer to Steinberg's monograph [29] and for general results on subgroups of reductive groups normalized by a maximal torus to [4].

3. Monotonicity results for the Modality in arbitrary Characteristic

The following is the principal theorem in [22, Theorem A]. Its advantage for our purpose is that it leads to the desired finiteness results avoiding separability considerations of the orbit maps involved.

Theorem 3.1. *Let A be an affine algebraic group over an algebraically closed field and let C and D be closed subgroups of A. Assume that C is a connected* *normal subgroup of A and that D is diagonalizable. Let V be a rational A-module and let λ be a weight of D on V. Then the fixed point subgroup C^D has only a finite number of orbits on $C \cdot v \cap V_\lambda$ for $v \in V_\lambda$.*

Suppose that G is a linear algebraic group defined over an algebraically closed field K. For Θ an automorphism of G let $\theta = d\Theta_e$ be the corresponding automorphism of $\mathfrak{g}$. Let G^Θ denote the Θ-fixed point subgroup of G and likewise $\mathfrak{g}^\theta$ the θ-fixed point subalgebra of $\mathfrak{g}$; similarly for Θ-stable subgroups of G.

Lemma 3.2. *Let G be a linear algebraic group and let Θ be a semisimple automorphism of G. Then $(G_u)^\Theta = (G^\Theta)_u$.*

Proof. As G_u is a characteristic subgroup of G, it is fixed by Θ. By definition G/G_u is reductive and Θ induces a semisimple automorphism of this quotient which we also denote by Θ. Then $(G/G_u)^\Theta$ is again reductive by [29, 8.1]. Since Θ is semisimple, we have $\mathrm{Lie}(G^\Theta) = \mathfrak{g}^\theta$ by [2, 9.1] and likewise for G_u and G/G_u. As θ is semisimple and leaves $\mathfrak{g}_u$ invariant, there is a θ-stable k-subspace of $\mathfrak{g}$, say $\mathfrak{r}$, complementing $\mathfrak{g}_u$. Thus $\mathfrak{g}/\mathfrak{g}_u \cong \mathfrak{r}$ as $\langle\theta\rangle$-modules, and so $\dim(\mathfrak{g}/\mathfrak{g}_u)^\theta = \dim \mathfrak{r}^\theta$. Furthermore, since $\mathfrak{g}_u$ and $\mathfrak{r}$ are θ-stable, we also have $\mathfrak{g}^\theta = (\mathfrak{g}_u)^\theta \oplus \mathfrak{r}^\theta$, and so $\mathfrak{g}^\theta/(\mathfrak{g}_u)^\theta \cong \mathfrak{r}^\theta$ as $\langle\theta\rangle$-modules. In particular, $\dim \mathfrak{g}^\theta/(\mathfrak{g}_u)^\theta = \dim(\mathfrak{g}/\mathfrak{g}_u)^\theta$. Since $\mathrm{Lie}(G/G_u)^\Theta \cong (\mathfrak{g}/\mathfrak{g}_u)^\theta$ and $\mathrm{Lie}\, G^\Theta/(G_u)^\Theta \cong \mathfrak{g}^\theta/(\mathfrak{g}_u)^\theta$, we conclude that $\dim G^\Theta/(G_u)^\Theta = \dim(G/G_u)^\Theta$. Observe that the image of G^Θ under the canonical epimorphism $G \to G/G_u$ is a subgroup of $(G/G_u)^\Theta$ isomorphic to $G^\Theta/(G_u)^\Theta$. Then, since these two groups have the same dimension, their identity components are isomorphic. A linear algebraic group is reductive precisely when its identity component has this property. Thus $G^\Theta/(G_u)^\Theta$ is reductive. As $G^\Theta/(G^\Theta)_u$ is the largest reductive quotient of G^Θ, we infer that $\dim(G^\Theta)_u \leq \dim(G_u)^\Theta$. However, since Θ is semisimple and G_u is connected and unipotent, $(G_u)^\Theta$ is connected by [2, 9.4], and therefore, $(G_u)^\Theta \subseteq (G^\Theta)_u$. Finally, comparing dimensions, we conclude that $(G^\Theta)_u = (G_u)^\Theta$, as desired.

Remark 3.3. If the group G in Lemma 3.2 admits a Levi complement to G_u and if all such complements are conjugate in G, then one can show that there always exists a Θ-stable Levi subgroup H of G. Thus $G^\Theta = (G_u)^\Theta H^\Theta$ is a Levi decomposition of G^Θ by Lemma 3.2.

Such a Levi decomposition exists for instance, when $\mathrm{char}\, k = 0$ [16], when G is a parabolic subgroup of some connected reductive group [4, §4], or more generally, when G is a connected subgroup of maximal rank of some connected reductive group [4, §3].

We briefly give an argument in the second case. Suppose P is a Θ-stable parabolic subgroup of a connected reductive group G. Since Θ is semisimple, it fixes a maximal torus of G in P by [29, 7.5]. Thus Θ acts on the root system $\Psi = \Psi(G, T)$ of G relative to T. Let $\Psi(P)$ and $\Psi(P_u)$ be the set of roots of P, respectively P_u, relative to T. Define $L := \langle T, U_\alpha \mid \alpha \in \Psi(P) \backslash \Psi(P_u)\rangle$. Then L is a Levi complement in P, i.e., $P = LP_u$ and $L \cap P_u = \{e\}$, and $L \cong P/P_u$ is reductive, cf. [4]. Since Θ leaves P and P_u invariant, it acts on $\Psi(P)$ and $\Psi(P_u)$ and thus it permutes $\Psi(P) \setminus \Psi(P_u)$. Because it also fixes T, it stabilizes L; whence, $P^\Theta = L^\Theta(P_u)^\Theta$. Since Θ acts semisimply on L and since L is reductive, so is L^Θ by [29, 8.1].

We return to the general situation. Let Θ be a semisimple automorphism of the linear group G and let P be a Θ-stable parabolic subgroup of G. By [29, 7.2] there is a Θ-stable Borel subgroup B of G in P, and from [2, 11.14] it follows that $(B^\Theta)^0$ is a Borel subgroup of $(G^\Theta)^0$, whence of G^Θ. Since $(B^\Theta)^0 \subseteq P^\Theta$, the latter is again a parabolic subgroup of G^Θ.

With this setup we formulate a first consequence of Theorem 3.1.

Corollary 3.4. *Let G be a linear algebraic group and let Θ be a semisimple automorphism of G. Let P be a parabolic subgroup of G and N a closed normal subgroup of P, both fixed by Θ. Then* $\operatorname{mod}(P^\Theta : \mathfrak{n}^\theta) \leq \operatorname{mod}(P : \mathfrak{n})$.

Proof. First suppose that G is connected. If we define $A := P \rtimes \langle\Theta\rangle$, $C := P$, $D := \langle\Theta\rangle$ and $V := \mathfrak{n}$, then the hypotheses of Theorem 3.1 are satisfied: A is affine, $\mathfrak{n}$ is a rational A-module, P is connected by [27, 7.3.8], because G is, and D is diagonalizable. Note that $\mathfrak{n}^D = \mathfrak{n}^\theta$ is the 0-weight space of D on $\mathfrak{n}$. Hence for each x in $\mathfrak{n}^\theta$ the intersection $P \cdot x \cap \mathfrak{n}^\theta$ is a finite union of orbits under $P^D = P^\Theta$ by Theorem 3.1. Consequently, $\operatorname{mod}(P^\Theta : \mathfrak{n}^\theta) = \operatorname{mod}(P : P \cdot \mathfrak{n}^\theta)$. Finally, the statement follows, as $\operatorname{mod}(P : \mathfrak{n}) \geq \operatorname{mod}(P : P \cdot \mathfrak{n}^\theta)$.

If G is not connected, we replace P by P^0 in the argument above. This is still a normal subgroup of $P \rtimes \langle\Theta\rangle$. Since $\operatorname{mod}(P : \mathfrak{n}) = \operatorname{mod}(P^0 : \mathfrak{n})$, the desired result follows.

Remark 3.5. In the statement of Corollary 3.4 we may replace $\mathfrak{n}$ by a quotient of the form $\mathfrak{n}/\mathfrak{m}$, where $M \subset N$ are closed normal subgroups of P both fixed by Θ.

As a special case of Corollary 3.4 we obtain Theorem 1.1.

Corollary 3.6. *Let G be a linear algebraic group and let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. Then* $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$.

Proof. Since P_u is a characteristic subgroup of P, it is also Θ-stable. It follows from Lemma 3.2 above that $(P^\Theta)_u = (P_u)^\Theta$. Therefore, $\operatorname{Lie}(P^\Theta)_u = \operatorname{Lie}(P_u)^\Theta$. Since Θ is semisimple, $\operatorname{Lie}(P_u)^\Theta = \mathfrak{p}_u^\theta$ by [2, 9.1], and thus $\operatorname{mod} P^\Theta = \operatorname{mod}(P^\Theta : \mathfrak{p}_u^\theta)$. The statement now follows from Corollary 3.4.

We proceed to some particular instances of Corollary 3.6.

Corollary 3.7. *Let G be a linear algebraic group and P a parabolic subgroup of G. Let s be a semisimple element in P and let $H = C_G(s)$. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Proof. Note that $\Theta = \operatorname{Int}(s)$ is a semisimple automorphism of G, and $H = G^\Theta$.

Corollary 3.8. *Let G be a linear algebraic group and P a parabolic subgroup of G. Let T be a maximal torus of G in P, S a subtorus of T, and $H = C_G(S)$. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Proof. By [2, 8.18] there exists an element $s \in S$ such that $C_G(s) = C_G(S)$. Thus Corollary 3.7 applies and the desired result follows. Note that H is connected whenever G is, by [2, 11.12].

For the remainder of this section we assume that G is reductive.

Remark 3.9. Let P be a parabolic subgroup of G. For studying $\operatorname{mod} P$ we may assume without loss that G is connected, simply connected, and semisimple. For, as $P \cap G^0$ has finite index in P, we have $\operatorname{mod} P \cap G^0 = \operatorname{mod} P$. So, we may suppose that G is connected. Then $G = G'Z(G)^0$, where G' is semisimple. Since P and $P \cap G'$ only differ by central elements, we have $\operatorname{mod} P \cap G' = \operatorname{mod} P$. So we may further assume that G is semisimple. If G is simply connected and $\overline{G}$ is in the same isogeny class as G, then $\operatorname{mod} P \cap \overline{G} = \operatorname{mod} P$, as $P \to P \cap \overline{G}$ has finite central kernel. Hence we may assume that G is a connected simply connected semisimple algebraic group.

The same reductions also apply if we consider $\operatorname{mod}(P : \mathfrak{n}/\mathfrak{m})$ or $\operatorname{mod}(P : N/M)$, where $M \subset N$ are closed normal subgroups of P.

If G is a connected simply connected semisimple algebraic group and Θ is a semisimple automorphism of G, then G^Θ is reductive and connected by [29, 8.1]. In particular, this applies to $C_G(s)$, where s is a semisimple element of G [29, 8.5], whence to H in Corollaries 3.7 and 3.8. Thus by Remark 3.9 we may replace H by H' in these statements. In Corollary 3.8, the centralizers of subtori of maximal tori of G are precisely the Levi subgroups of G [4, 4.15]. Thus we may reformulate Corollary 3.8 in this case as follows.

Corollary 3.10. *Let G be a reductive algebraic group and P a parabolic subgroup of G containing a maximal torus T of G. Let H be a Levi subgroup of G normalized by T, or the derived subgroup thereof. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Let G be a connected, simply connected simple algebraic group. The following criterion of Deriziotis [9, 2.3] gives precise information as to when a reductive subgroup H of G is the centralizer of a semisimple element of G.

Let T be a maximal torus in G, Ψ the set of roots of G with respect to T, and W the associated Weyl group of G. Fix a set of simple roots Π of Ψ, and let $\widetilde{\Pi} := \Pi \cup \{-\varrho\}$.

Theorem 3.11. *Let G be a connected, simply connected simple algebraic group. Assume that* $\operatorname{char} k$ *is a good prime for G. Let H be a connected reductive subgroup of G of maximal rank. Fix a maximal torus T of G in H. Let Π be a basis of the root system of G with respect to T. Then H is the centralizer of some semisimple element of G if and only if $\Psi(H)$ has a basis which is W-conjugate to a proper subset of* $\widetilde{\Pi}$.

See [8, Proposition 11] for a criterion which is equivalent to the one of Theorem 3.11 in positive characteristic, but also applies in characteristic zero.

We now consider the case of Theorem 1.2, namely a closed reductive subgroup of G normalized by a maximal torus T of G.

Theorem 3.12. *Let G be a reductive algebraic group and suppose that* $\operatorname{char} k$ *is a good prime for G. Let H be a closed reductive subgroup of G normalized by a maximal torus T of G and P a parabolic subgroup of G containing T. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Proof. By Remark 3.9 we may suppose that G is connected, simply connected, and semisimple and likewise that H is connected and semisimple. Let $\Psi(H)$ be the set of roots of H relative to T which is a closed symmetric subsystem of $\Psi = \Psi(G,T)$, the root system of G relative to T. These can be determined by means of the algorithm of Borel-de Siebenthal [3] applied to each of the simple components of G, see also [6, Exc. Ch. VI §4.4]. According to that, H is the derived subgroup of a Levi subgroup of some connected semisimple subgroup of G of maximal rank. By Corollary 3.10 we may assume that H is a connected semisimple subgroup of G of maximal rank. Such an H can be obtained by successive applications of the Borel-de Siebenthal algorithm applied first to the simple components of G and then successively to further irreducible components. If H is maximal among such groups, it is then the centralizer of some semisimple element of G according to Theorem 3.11 applied to each of the simple components of G. Inductively we obtain a chain of connected semisimple subgroups of G of maximal rank each of which is the centralizer of some semisimple element of the next group with H at the beginning and G at the end of the chain. If $\operatorname{char} k$ is a good prime for G, it is also a good prime for each connected semisimple subgroup of G of maximal rank [28, 4.7]. Since each of these simple components is also simply connected, the result then follows from Corollary 3.7 and Theorem 3.11.

Remark 3.13. Observe that the proofs of Corollaries 3.7, 3.8, 3.10 and of Theorem 3.12 equally apply if we consider the P-action on a closed (connected) normal subgroup N of P in P_u instead of P_u itself and the Q-action on $Q_u \cap N$ instead of Q_u; likewise for quotients of the form N/M where $M \subseteq N$ are closed normal subgroups of P in P_u.

We close this section with a discussion of characteristic restrictions for some applications of the results above.

Remark 3.14. If Θ is a finite automorphism of G, then it is semisimple provided $\operatorname{char} k$ does not divide its order. If G is semisimple, then the group of outer automorphisms of G is finite [29]. So in that case, e.g., if Θ is a graph automorphism of G, Corollaries 3.4 and 3.6 apply provided $\operatorname{char} k$ satisfies this condition. This applies to the extensions of some results using graph automorphisms in [11], [19], [18], and [24].

However, no characteristic restrictions are required for applications of Corollaries 3.7 through 3.10. The proofs of the key theorems from [14], [19], [23], and [25] either use derived subgroups of Levi subgroups of G in inductive arguments, or else can easily be modified to fit that situation. Thus the extensions of these results hold without any restrictions on $\operatorname{char} k$.

4. Global and Infinitesimal Modality

Throughout this section G denotes a reductive algebraic group defined over an algebraically closed field k of positive characteristic and P is a parabolic subgroup of G. We wish to compare $\operatorname{mod}(P : P_u)$ with $\operatorname{mod}(P : \mathfrak{p}_u)$.

Theorem 4.1. *Let G be a connected simply connected simple algebraic group, P a parabolic subgroup of G, and N a closed connected normal subgroup of P contained in P_u. Suppose that* $\operatorname{char} k$ *is a good prime for G. Then there is a P-equivariant bijective morphism of P-varieties $\varphi : N \to \mathfrak{n}$ affording a bijection between the sets of P-orbits on N and on $\mathfrak{n}$.*

Proof. Let $\mathcal{U}$ be the unipotent variety of G and $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$ with the usual G-actions. In [26] Springer showed that, under the hypotheses of the theorem, there is a G-equivariant bijective morphism $\varphi : \mathcal{U} \longrightarrow \mathcal{N}$ which is a homeomorphism of topological spaces; see also [28, Theorem 3.12]. If $\operatorname{char} k$ is a *very good* prime for G (i.e., p is a good prime for G and it does not divide $r+1$ in case G is of type A_r), then there is such a map which is even an isomorphism of varieties, since $\mathcal{N}$ is known to be normal in this instance [31, 6.9].

We choose a maximal torus T of G and a Borel subgroup B of G satisfying $T \subset B \subseteq P$. Let $x \in B_u$ be a regular unipotent element of G. Since φ is G-equivariant, $\varphi(x)$ is regular nilpotent, and thus is contained in the Lie algebra of a unique Borel subgroup $\widetilde{B}$ of G [30, 3.7]. The connected centralizer of a regular unipotent element is contained in a unique Borel subgroup, and likewise for regular nilpotent elements [30, 3.7]. Thus we have $C_G(x)^0 \subset B$ and $C_G(\varphi(x))^0 \subset \widetilde{B}$. By the G-equivariance of φ, we have $C_G(x)^0 = C_G(\varphi(x))^0$, and thus $B = \widetilde{B}$. Since φ is a homeomorphism and $B_u = \overline{B \cdot x}$, it follows that $\varphi(B_u) = \overline{B \cdot \varphi(x)} = \mathfrak{b}_u$.

Let $\alpha \in \Psi^+$. Since $\dim \varphi(U_\alpha) = 1$ and $\varphi(U_\alpha)$ is a T-invariant subvariety of $\mathfrak{b}_u$, there is some root $\beta \in \Psi^+$ such that $\varphi(U_\alpha) \subseteq \mathfrak{g}_\beta$. For, if $\varphi(U_\alpha)$ meets at least two root spaces nontrivially, then, using the action of T, we see that $\dim \varphi(U_\alpha) \geq 2$ (assuming that $\operatorname{rank} G > 1$, else $\varphi(U_\alpha) = \mathfrak{g}_\alpha$ for the single root α in Ψ^+ by the previous paragraph). But then the T-invariance forces the equality $\varphi(U_\alpha) = \mathfrak{g}_\beta$. Since φ is T-equivariant and k is infinite, $\beta = \alpha$. In particular, this shows that any connected T-regular subgroup R of B_u corresponds to its Lie algebra under φ. The desired result now follows.

Corollary 4.2. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and N a closed normal subgroup of P contained in P_u. Suppose that* $\operatorname{char} k$ *is a good prime for G. Then* $\operatorname{mod}(P : N) = \operatorname{mod}(P : \mathfrak{n})$.

Proof. By Remark 3.9 we may suppose that G is connected, simply connected, and semisimple. Without loss we may also assume that N is connected. Then, since each simple component of G is also simply connected, and $\operatorname{char} k$ is a good prime for each of these components, and since P is the direct product of its irreducible components and likewise for N, there is a P-equivariant bijective morphism of P-varieties $\varphi : N \to \mathfrak{n}$ affording a bijection between the sets of P-orbits on N and on $\mathfrak{n}$ by Theorem 4.1.

The case $N = P_u$ of Corollary 4.2 yields Theorem 1.3.

Remark 4.3. Having a bijection between the sets of P-orbits on $\mathfrak{p}_u$ and on P_u, as asserted by Theorem 4.1, is considerably stronger than the equality statement of Theorem 1.3, which merely requires that the maximal number of parameters of both actions are the same. It is unknown whether this equality also holds when $\operatorname{char} k$ is a bad prime for G.

Remark 4.4. The map φ in Theorem 4.1 extends to quotients of the form N/M, where $M \subset N$ are closed connected normal subgroups of P; whence Corollary 4.2 extends to such quotients without the connectedness requirement.

5. Modality for the coadjoint action and G-bundles

Let G be a linear algebraic group defined over an algebraically closed field k. In this section we are concerned with the modality of the coadjoint action of a parabolic subgroup P of G on the dual space of $\mathfrak{p}_u$ and some associated G-bundles. We first prove Theorem 1.4. We modify an argument from the proof of Theorem 1 in [21]. Let $\dim V = n$. Consider $M := \{(v, f) \in V \times V^* \mid f(gv) = 0 \text{ for all } g \in G\}$. For $j \in \mathbb{N}$ we define $V^j := \{v \in V \mid \dim G \cdot v = j\}$. Then $V = \bigcup_{j\geq 0} V^j$. Let $\pi : V \times V^* \to V$ be the projection onto the first factor. Define $M^j := \pi^{-1}(V^j) \cap M$. Then $M = \bigcup_{j\geq 0} M^j$ and $\dim M = \max_{j\geq 0} \dim M^j$. By duality, each fiber of the map $\pi : M^j \to V^j$ has dimension $n - j$. Thus $\dim M^j = \dim V^j + n - j$. Since $\mathrm{mod}(G : V) = \max_{j\geq 0}(\dim V^j - j)$, e.g., see [13, 1.9], we have

$$\dim M = \max_{j\geq 0} \dim M^j = \max_{j\geq 0}(\dim V^j - j) + n = \mathrm{mod}(G : V) + n.$$

By dual considerations we also have $\dim M = \mathrm{mod}(G : V^*) + n$, and thus the desired equality follows; see also [17, §2].

The second statement of Theorem 1.4 follows from the proof of [21, Cor. 2], which is also valid in positive characteristic.

For an application of Corollary 1.5 consider the tangent and cotangent bundle of the flag manifold G/P, i.e., $G \times_P \mathfrak{p}_u$ and $G \times_P \mathfrak{p}_u^*$. By construction, the G-orbit of a point in $G \times_P \mathfrak{p}_u$ meets the fiber over some point in G/P (which is isomorphic to $\mathfrak{p}_u$) in a single P-orbit on $\mathfrak{p}_u$. So, there is a canonical bijection between the G-orbits on $G \times_P \mathfrak{p}_u$ and the P-orbits on $\mathfrak{p}_u$; likewise for $G \times_P \mathfrak{p}_u^*$. This observation and Corollary 1.5 imply Corollary 1.6.

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