THE RATIONALITY PROBLEM FOR SEMISIMPLE GROUP VARIETIES

VLADIMIR I. CHERNOUSOV AND VLADIMIR P. PLATONOV*

1. Introduction

An irreducible algebraic variety X defined over a field K is called K-rational (resp. stably K-rational) if X (resp. $X \times \mathbb{A}_K^m$ for some $m \in \mathbb{N}$) is birationally isomorphic over K to an affine space. The present paper is devoted to the rationality problem for group varieties. This problem has a long history and is especially interesting for semisimple group varieties.

Let G be a semisimple algebraic group defined over a field K. It is well known that the algebraic variety G is unirational over K, i.e. there exists a dominant rational map $\phi : \mathbb{A}_K^m \to G$ over K or equivalently, the field $K(G)$ of K-rational functions on G is contained in a purely transcendental extension of K (see [DG], Exp. XIV).

Rationality problem. *1) Is the variety G rational over K? In other words, is $K(G)$ a purely transcendental extension of K?*
2) If no, is the variety G then stably rational over K?

If G is K-split (i.e. there exists a maximal torus T splitting over K), then G is K-rational. This follows immediately from the Bruhat decomposition: G is birationally isomorphic over K to the product variety $U \times T \times U$ where U is the unipotent radical of a K-defined Borel subgroup B of G containing T. In particular, over an algebraically closed field K, the variety G is always K-rational.

Other examples of rational group varieties are provided by groups of small ranks. It is known that for any field K all algebraic K-tori of dimension at most two are rational over K (see [V]). This fact combined with the Chevalley-Grothendieck theorem and the K-rationality of the variety of maximal tori of G (see [Ch], [DG], Exp. XIV) implies that G is K-rational if rank $G \leq 2$. In particular, varieties of all K-groups of types A_1, A_2, $B_2 = C_2$, G_2 are rational over K. However in general (i.e. for arbitrary G and K), this problem seems to be difficult. For a long time, the following question was open:

Is the variety G rational over K if G is simply connected?

It turned out that this is not the case. First examples of non-rational varieties of simply connected groups were constructed by the second-named author ([P1]):

*Supported in part by an NSERC grant.

For any m which is not squarefree and any l, there exists a division algebra D of index m over $K = \mathbb{Q}(x_1, \dots, x_n)$, or $\mathbb{Q}_p(x_1, \dots, x_n)$, $n \geq 2$, such that the variety $G = \mathbf{SL}_l(D)$ is not stably rational over K.

Here $\mathbb{Q}$ and $\mathbb{Q}_p$ are, respectively, the fields of rational and p-adic numbers and $x_1, \dots, x_n$ are independent variables. In fact, even more was proven: $G(K) = \mathrm{SL}_l(D)$ does not have the weak approximation property.

The following results were inspired by a question posed by P. Deligne. Let f be a non-degenerate quadratic form of rank n over a field K of characteristic $\neq 2$. It is known that the variety $\mathbf{SO}(f)$ is K-rational (see Lemma 5 in §4). In 1978, P. Deligne asked the following question:

Is the spinor variety $\mathbf{Spin}(f)$ rational over K (in particular, over $K = \mathbb{R}$)?

We answered (see [P2], [PC1], [C1]) this question positively in the following cases.

1) If a quadratic form f is K-isotropic, then $\mathbf{Spin}(f)$ is K-rational. In particular, if K is either an algebraic extension of $\mathbb{Q}_p$ or a totally imaginary number field or a global function field, then $\mathbf{Spin}(f)$ is K-rational.
2) $\mathbf{Spin}(f)$ is rational over $K = \mathbb{R}$ or $\mathbb{Q}$.
3) $\mathbf{Spin}(f)$ is rational over any K if f is either a sum of squares or a Pfister form.

For an algebraic number field K, which is not totally imaginary, the question of the rationality of $\mathbf{Spin}(f)$ is still open.

Since all groups of rank ≤ 2 are K-rational, the minimal n for which, a priori, there is a possibility of $\mathbf{Spin}(f)$ being non-rational over K, is $n = 6$. The next result was relatively surprising ([P3]):

Let $n = 4m + 2, n \geq 6$, and let $x_1, \dots, x_{n-2}$ be algebraically independent variables over some field L. If $f(y_1, \dots, y_n) = y_1^2 + x_1 y_2^2 + \dots + x_{n-2} y_{n-1}^2 + x_1 x_2 \cdots x_{n-2} y_n^2$, then the spinor variety $\mathbf{Spin}(f)$ is not stably rational over $K = L(x_1, \dots, x_{n-2})$.

After the discussion above, it becomes clear that it is possible to deal with the question of the rationality of an arbitrary almost simple algebraic K-group only for "good" fields. In the present paper we consider, for instance, the rationality problem for arbitrary almost simple algebraic K-groups over arithmetic fields.

Main Theorem. *Let G be an almost simple algebraic group defined over K, where K is either an algebraic extension of $\mathbb{Q}_p$ or a totally imaginary number field or a global function field. If G is not of type $A_n, n \geq 3$, then G is a rational variety over K.*

This statement cannot be extended to groups of type $A_n, n \geq 3$. In fact, now due to A. Merkurjev and M. Rost (see [M1]) we know that the variety $G = \mathbf{SL}_1(D)$, where D is a division algebra of degree $4m$ over any field K, is not rational over K. In particular, this implies that there exists a $\mathbb{Q}_p$-form G of type D_3 such that G is not rational even over $\mathbb{Q}_p$ (recall that $D_3 = A_3$), in contrast to the case $D_n,\ n \geq 4$.

If $\mathrm{ind}(D)$ is prime we have the following conjecture which is still open:

Conjecture (Platonov [P1]). *Let D be a division algebra of prime index over an arbitrary field K. Then the variety $\mathbf{SL}_1(D)$ is rational over K.*

If $\mathrm{ind}(D)$ is not squarefree, then there exists the following conjecture which is also still open:

Conjecture (Suslin [Su]). *If $\mathrm{ind}\,(D)$ of a simple algebra D is not squarefree, then the generic element χ_D of $\mathrm{SK}_1(D_{K(X)})$, where $X = \mathbf{SL}_1(D)$, $D_{K(X)} = D \otimes_K K(X)$, is not trivial; in particular, the variety $\mathbf{SL}_1(D)$ is not K-rational.*

These conjectures and results both of the present paper (see §§ 3-10) and Merkurjev's paper [M2] on the rationality of adjoint classical groups suggest that the answer to the question whether the variety of a given almost simple algebraic K-group G is rational over K very likely depends only on some discrete parameters such as for example the K-rank, the Tits K-index of G, indices of Tits algebras and so on.

Our main theorem yields immediately

Corollary. *Let K be as in the main theorem and G be a semisimple algebraic group over K which is either simply connected or adjoint without almost simple components of types $A_n, n \geq 3$. Then G is K-rational.*

An intermediate semisimple group G (i.e. one which is neither simply connected nor adjoint) is not necessarily K-rational. There exists even a quasi-split semisimple K-group G which is not K-rational. To produce such an example consider $\alpha_1, \alpha_2, \alpha_3 \in K^*$ which are independent modulo $(K^*)^2$ and any positive number n. Let

$$f_l = \sum_{i=1}^{n} x_i^2 - \sum_{j=1}^{n} y_j^2 + u_l^2 - \alpha_l v_l^2, \quad l = 1, 2, 3,$$

be $(2n+2)$-dimensional quadratic forms over K. Then we have

Proposition. $(\mathbf{SO}(f_1) \times \mathbf{SO}(f_2) \times \mathbf{SO}(f_3))/ < (-1,-1,-1) >$ *is not a K-rational group.*

Another example of a non-rational semisimple group was constructed by Ph. Gille. In fact, the first example of a non-rational semisimple algebraic group which is neither simply connected nor adjoint is due to J-P. Serre. In [S] (P.156-159) he constructed a group G which does not have the weak approximation property; in particular, its variety is not rational (recall that every smooth rational variety has the weak approximation property).

The strategies used to prove the non-rationality of a given variety are very different from those used to prove its rationality. A proof of the non-rationality is usually connected with some new invariants (for instance, $\mathrm{SK}_1(D)$). On the other hand, a proof of the rationality of an algebraic variety involves a direct parametrization of its K-points depending on that particular variety. So it is natural that the proof of our main theorem depends on the distinct types of G. In fact, we prove that *the variety of any almost simple group with a given Tits' index (see* [T]*), which is admissible over the local and global fields under consideration, is rational over an arbitrary field K. In particular, we prove that if the anisotropic semisimple kernel Δ of* *an almost simple group G defined over any field K, $char(K) \neq 2$, has the property that any almost simple K-component of Δ has rank at most 2, then the variety of G is K-rational.* Besides, we also consider some additional Tits indices which do not occur over local and global fields. It seems unlikely to obtain an essential generalization of these results.

A sketch of the proof of our main theorem for p-adic (i.e. finite extensions of Q_p) fields was published in [PC2].

Acknowledgements. This joint work began in 1995 while the first-named author visited the University of Bielefeld as a guest of SFB 343 "Diskrete Strukturen in der Mathematik" and the second-named author visited the Max-Planck-Institute of Mathematics. The final version of this paper was written during our stay at the University of Bielefeld in 1996 supported by the Alexander von Humboldt Foundation. The authors gratefully acknowledge the support of all these institutions and the hospitality of the Max-Planck-Institute of Mathematics and the University of Bielefeld.

2. **Some properties of algebraic groups over arbitrary fields**

Let G be an almost simple algebraic group defined over an arbitrary field K. If S is a maximal K-split torus in G, then the generalized Bruhat decomposition (see [BT]) implies that $G \approx U \times Z_G(S) \times U$ over K, where we use $\approx$ to denote birationally isomorphic varieties over a basic field and U is the unipotent radical of a minimal parabolic K-subgroup P of G. Since U is K-rational, it suffices to show the K-rationality of the variety $Z_G(S)$. But, in turn, the bundle $Z_G(S) \to Z_G(S)/S$ is locally trivial, since S is the K-split torus, and so $Z_G(S) \approx (Z_G(S)/S) \times S$. Thus, if we prove that for a given group G the quotient $Z_G(S)/S$ is K-rational, then G will also be so.

It is known that $Z_G(S)$ is a reductive group. So $Z_G(S) = \Delta{\cdot}S'$, where S' is the central torus of $Z_G(S)$ and $\Delta = [Z_G(S), Z_G(S)]$ is the commutator subgroup called the semisimple K-anisotropic kernel of G. There are two possibilities for S':

(i) $S' = S$ (this is true for any inner type group);
(ii) $S' \neq S$, and so G is an outer type group.

Usually we will reduce our consideration to the case when for any extension F of K one has $H^1(F, S') = 1$, and hence by virtue of Lemma 1 below, we will get

$$Z_G(S) \approx (Z_G(S)/S') \times S' \approx (\Delta/\Delta \cap S') \times S';$$

therefore, the variety G is K-rational if $\Delta/\Delta \cap S'$ and S' are.

For our considerations it is essential to know the intersection $\Delta \cap S'$ in an explicit form. Let T be a maximal torus of G defined over K and containing S. Let $\Sigma = R(T, G)$ be the root system of G relative to T, and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots. If $\{X_\alpha, \alpha \in \Sigma; H_{\alpha_1}, \dots, H_{\alpha_n}\}$ is a Chevalley basis of the Lie algebra, then G is generated by the corresponding root subgroups $G_\alpha = < x_{\pm\alpha}(t) \mid t \in \overline{K} >$, and the torus T is generated by $T_\alpha = T \cap G_\alpha = < h_\alpha(t) >$, where $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$ and $w_\alpha(t) =$

$x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ (see [St]). Furthermore, if G is simply connected then $T = < h_{\alpha_1}(t_1) > \times \cdots \times < h_{\alpha_n}(t_n) >$ and for all roots $\alpha, \beta \in \Sigma$ one has

$$h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta(t^{<\beta,\alpha>}u), \tag{1}$$

where $<\beta, \alpha> = 2(\beta,\alpha)/(\alpha,\alpha)$ (see [St]).

In this notation our torus S is the identity component of the subgroup defined by the following system of equations:

$$\left\{ \begin{array}{rcl} \beta(t) & = & \sigma^*(\beta)(t) \qquad \text{for all } \beta \in \Pi \text{ and all } \sigma \in \Gamma = Gal(\overline{K}/K) \\ \alpha_i(t) & = & 1 \quad \text{for all non-distinguished vertices} \\ & & \qquad \alpha_i \text{ of the Tits } K\text{-index} \end{array} \right. \tag{2}$$

where $t \in T$ and σ^* denotes the natural action of Γ on Π (see [T]). If G is simply connected and $t = \prod_{j=1}^n h_{\alpha_j}(t_j)$ then it follows from (1) that the second equation of the system (2) is equivalent to

$$\prod_{j=1}^{n} t_j^{2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j)} = 1 \tag{3}$$

Recall also that for groups G of inner type, we have $\sigma^*(\beta) = \beta$ for all $\beta \in \Pi$ and all $\sigma \in \Gamma$. If $X \subset \Sigma$ then the subgroup generated by $G_{\pm\alpha}$ for all $\alpha \in X$ will be denoted by G_X.

We also need to consider more general bundles, for which we use the following

Lemma 1. *Let G_1 be a connected algebraic group defined over an arbitrary field K. If G_2 is a connected subgroup of G_1 defined over K such that for any extension E/K the kernel of the natural morphism $H^1(E, G_2) \to H^1(E, G_1)$ is trivial, then the variety G_1 is birationally isomorphic to $(G_1/G_2) \times G_2$ over K.*

We will assume from now on that $char K \neq 2$ (unless otherwise stated).

3. Groups of type $\mathbf{A_n}$

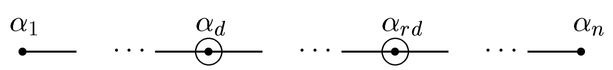
3.1. Isotropic simply connected groups of inner type $\mathbf{A_n}$

Let G be an isotropic simply connected K-group of type ${}^1A_n$. Denote its K-rank by r. Then G is isomorphic over K (see [T]) to the special linear group $\mathbf{SL}_{r+1}(D)$, where D is a central division algebra of degree d over K and $(r+1)d = n+1$.

Proposition 1. *The variety G is birationally isomorphic over K to the product $\mathbf{SL}_1(D) \times \mathbb{A}_K^t$, where $t = (n+1)^2 - d^2$.*

Remark 1. From what has been said in §2 and the direct matrix realization of G it is easy to give a simple proof consisting of a few lines only. However we prefer to present a more complicated proof since it can be applied to groups of other types, too.

Proof. The Tits K-index of G is as follows:



So the system (2) can be written in the form

$$\begin{cases} \alpha_1(t) &= 1, & \dots, & \alpha_{d-1}(t) &= 1 \\ \alpha_{d+1}(t) &= 1, & \dots, & \alpha_{2d-1}(t) &= 1 \\ \dots & & \dots & & \dots \\ \alpha_{n-d+1}(t) &= 1, & \dots, & \alpha_n(t) &= 1 \end{cases}$$

Let $t = \prod_{i=1}^{n} h_{\alpha_i}(t_i)$. Then taking into account (3) one can rewrite the latter in the form

$$(4) \qquad \begin{cases} t_1^2 = 1, & t_2^2/t_1t_3 = 1, & \dots, & t_{d-1}^2/t_{d-2}t_d = 1 \\ \dots & & & \dots \\ t_{n-d+1}^2/t_{n-d}t_{n-d+2} = 1, & \dots & \dots, & t_n^2/t_{n-1} = 1 \end{cases}$$

The semisimple K-anisotropic kernel Δ of G coincides with the product of the simply connected groups $H_1 = G_{X_1}, \dots, H_{r+1} = G_{X_{r+1}}$, where $X_1 = \{\alpha_1, \dots, \alpha_{d-1}\}, \dots, X_{r+1} = \{\alpha_{n-d+1}, \dots, \alpha_n\}$, each of them being isomorphic over K to $\mathbf{SL}_1(D)$. Considering the connected component of the subgroup defined by (4) and adding equations $t_d = 1$, $t_{2d} = 1$, $\dots$, $t_{rd} = 1$ one can easily obtain that the intersection $\Delta \cap S'$ ($= \Delta \cap S$) is generated by all the products of the form $a_1 a_2 \cdots a_{r+1}$, where

$$(5) \qquad \begin{aligned} a_1 &= h_{\alpha_1}(\zeta_1) h_{\alpha_2}(\zeta_1^2) \cdots h_{\alpha_{d-1}}(\zeta_1^{d-1}) \\ &\dots \\ a_r &= h_{\alpha_{n-2d+1}}(\zeta_r) h_{\alpha_{n-2d+2}}(\zeta_r^2) \cdots h_{\alpha_{n-d-1}}(\zeta_r^{d-1}) \\ a_{r+1} &= h_{\alpha_{n-d+1}}(\zeta_1^{-1} \cdots \zeta_r^{-1}) h_{\alpha_{n-d+2}}(\zeta_1^{-2} \cdots \zeta_r^{-2}) \cdots \\ & \quad h_{\alpha_n}(\zeta_1^{-(d-1)} \cdots \zeta_r^{-(d-1)}) \end{aligned}$$

and $\zeta_i^d = 1$ for all $i = 1, \dots, r$.

Let $Z_1, \dots, Z_{r+1}$ be the centers of the simply connected groups $H_1, \dots, H_{r+1}$ defined above. It is clear that Z_i can be identified with μ_d, where μ_d is the $\mathrm{Gal}(\overline{K}/K)$-module d-th roots of unity. Consider the following exact sequence

$$1 \longrightarrow (\mu_d)^r \xrightarrow{\varphi} H_1 \times \cdots \times H_{r+1} \xrightarrow{\psi} \Delta/\Delta \cap S \longrightarrow 1,$$

where $\varphi(\zeta_1, \dots, \zeta_r) = (a_1, \dots, a_{r+1})$ and the elements $a_1, \dots, a_{r+1}$ are given by (5). Our goal is to prove that the variety $\overline{\Delta} = \Delta/\Delta \cap S$ is birationally isomorphic over K to the product

$$\psi(1 \times \cdots \times 1 \times H_{r+1}) \times \left(\overline{\Delta}/\psi(1 \times \cdots \times 1 \times H_{r+1})\right).$$

By Lemma 1, we need only to check that for any extension E/K one has

$$\mathrm{Ker}\left[\tau : H^1\left(E, \psi(1 \times \cdots \times 1 \times H_{r+1})\right) \longrightarrow H^1(E, \overline{\Delta})\right] = 1.$$

Let $\lambda \in \mathrm{Ker}\,\tau$. Since the restriction $\psi|_{1 \times \dots \times 1 \times H_{r+1}}$ is injective there exists a unique $\lambda' \in H^1(E, 1 \times \cdots \times 1 \times H_{r+1})$ such that $\lambda = \theta(\lambda')$, where θ is the natural bijection

$$\theta : H^1(E, 1 \times \cdots \times 1 \times H_{r+1}) \longrightarrow H^1\left(E, \psi(1 \times \cdots \times 1 \times H_{r+1})\right).$$

By construction, $\theta(\lambda')$ is trivial in $H^1(E, \overline{\Delta})$, hence

$$\lambda' \in \operatorname{Im}\left(H^1(E,(\mu_d)^r) \longrightarrow H^1(E,H_1\times\cdots\times H_{r+1})\right). \tag{6}$$

Identify $H^1(E,\mu_d)$, $H^1(E,H_i)$ with $E^*/(E^*)^d$, $E^*/\mathrm{Nrd}\left((D\otimes_K E)^*\right)$, respectively. Then $\lambda' = (1,\dots,1,a\,\mathrm{Nrd}(D_E))$ for some $a\in E^*$ and condition (6) implies that there exist $b_1,\dots,b_r\in E^*$ such that

$$\begin{array}{rcl} b_1\mathrm{Nrd}(D_E) & = & \mathrm{Nrd}(D_E) \\ \cdots & & \cdots \\ b_r\mathrm{Nrd}(D_E) & = & \mathrm{Nrd}(D_E) \\ \left(b_1^{-1}\cdots b_r^{-1}\right)\mathrm{Nrd}(D_E) & = & a\,\mathrm{Nrd}(D_E) \end{array}$$

It follows immediately that $a\in\mathrm{Nrd}(D_E)$, i.e. $\lambda'=1$ and hence $\lambda=1$. Thus, we proved that

$$\overline{\Delta}\approx\psi(1\times\cdots\times1\times H_{r+1})\times\left(\overline{\Delta}/\psi(1\times\cdots\times1\times H_{r+1})\right).$$

Now to complete the proof it remains to observe that the variety

$$\begin{gathered}\overline{\Delta}/\psi(1\times\cdots\times1\times H_{r+1})\approx\\ (H_1\times\cdots\times H_{r+1})/(1\times\cdots\times1\times H_{r+1})\varphi\left((\mu_d)^r\right)\approx\\ (H_1/Z_1)\times\cdots\times(H_r/Z_r)\approx\mathbf{PSL}_1(D)\times\cdots\times\mathbf{PSL}_1(D)\end{gathered}$$

is clearly rational over K, yielding the required result. □

Corollary 1. *For any central subgroup $\mu\subset G$ the variety of the group G/μ is rational over K if $d\le 3$.*

Proof. Keep the notation above. It suffices to prove the K-rationality of the variety $X=\Delta/\Delta\cap<S,\mu>$. If $\Delta\cap<S,\mu>=\Delta\cap S$ then $X=\overline{\Delta}$ and hence X is rational over K since $\mathbf{SL}_1(D)$ has rank at most 2. Otherwise the same arguments as above show that X is birationally isomorphic over K to the variety

$$(H_1/Z_1)\times\cdots\times(H_{r+1}/Z_{r+1})\approx\mathbf{PSL}_1(D)\times\cdots\times\mathbf{PSL}_1(D)$$

and hence is K-rational. □

Remark 2. If $d=4$, then G is not K-rational. This was proved in [P4] for special fields, and in [M1] for arbitrary fields.

3.2. Simply connected groups of outer type $\mathbf{A_n}$

Let L/K be a separable quadratic field extension and D be a central division algebra of finite dimension over L with an involution of the second kind on D, trivial on K. Consider a right vector space V over D of finite dimension and a non-degenerate hermitian form h on V. Then the *unitary group* $\mathbf{U}(V,h)$ of isometries of V with respect to h coincides with $\mathbf{U}(B,\tau)$, where $B=\mathrm{End}_D(V)$ and τ is the adjoint involution of h (see [Sch]), and the *special unitary group* $\mathbf{SU}(V,h)$ equals $\mathbf{SU}(B,\tau)$ and has type ${}^2A_n$. It follows from [CM] that $\mathbf{SU}(B,\tau)$ is stably birationally isomorphic over K to $\mathbf{SU}(D,\sigma)$ for some involution σ of D of the second type, and this fact will be crucial when we consider groups of type E_6. So for the sake of expository completeness we include here the proof of this statement given in [CM] and inspired by the proof of Proposition 1 in [PC1].

Let $w \in V$ be an anisotropic vector and let $W = w^{\perp}$ be the orthogonal complement to w. A non-trivial isometry $\psi \in \mathrm{U}(V,h)$, which is the identity on W, is called a *reflection* on V. The image of $\psi - \mathrm{id}$ equals wD. Hence the space W is determined by ψ as the orthogonal complement to $\psi(v) - v$ for any $v \in V$, such that $\psi(v) \neq v$ (see [Sch], p.268).

Lemma 2. *Let v and v' be anisotropic vectors in V, such that $h(v,v) = h(v',v')$ and $v' - v$ is anisotropic. Then there is a unique reflection $\psi \in \mathrm{U}(V,h)$, such that $\psi(v) = v'$.*

Proof. Denote $v' - v$ by w and $w^{\perp}$ by W. Since

$$h(v,v) = h(v',v') = h(v,v) + h(v,w) + h(w,v) + h(w,w),$$

it follows that $v \notin W$. Hence the linear map ψ, defined by the conditions $\psi(v) = v'$ and $\psi|_W = \mathrm{id}|_W$, is clearly a reflection.

Conversely, let ψ be a reflection, such that $\psi(v) = v'$. Since $w = \psi(v) - v$ is non-trivial, ψ must be identical on W and hence is uniquely determined. □

Let $v \in V$ be an anisotropic vector. Denote by V' the orthogonal complement of v in V and by h' the restriction of h to V'. Then h' is a non-degenerate hermitian form on V' and $\mathbf{U}(V',h')$ can be identified with the subgroup in $\mathbf{U}(V,h)$ consisting of all φ, such that $\varphi(v) = v$. Denote the quotient $\mathbf{U}(V,h)/\mathbf{U}(V',h')$ by X and the subvariety of reflections by $I \subset \mathbf{U}(V,h)$.

Lemma 3. *The restriction of the natural surjection $\pi : \mathbf{U}(V,h) \to X$ to I is a birational isomorphism.*

Proof. Denote by I' the open subvariety in I consisting of all reflections ψ, such that $\psi(v) \neq v$, i.e. $\psi(v) - v$ is anisotropic, and by U' the open subvariety in $\mathbf{U}(V,h)$ consisting of all φ, such that $\varphi(v) - v$ is anisotropic. By Lemma 2,

$$U' = I' \cdot \mathbf{U}(V',h') \simeq I' \times \mathbf{U}(V',h'),$$

i.e. the restriction of π to I' yields an isomorphism between I' and the image of U' in X. □

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{SU}(V',h') & \longrightarrow & \mathbf{SU}(V,h) & \xrightarrow{\rho} & X & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathbf{U}(V',h') & \longrightarrow & \mathbf{U}(V,h) & \xrightarrow{\pi} & X & \to & 1.
\end{array}$$

By Lemma 3, the map π splits rationally. We want to show that ρ is also rationally split. Let E be the function field of X. Since π splits, there is $\varphi \in \mathrm{U}(V_E, h_E)$ such that $\pi_E(\varphi)$ is the generic point in $X(E)$. Denote by T the norm 1 torus of the extension L/K. More precisely,

$$T = \ker\Big(R_{L/K}(\mathrm{G}_{m,L}) \overset{N_{L/K}}{\longrightarrow} \mathrm{G}_{m,K}\Big).$$

The reduced norm gives the homomorphism

$$\mathrm{Nrd} : \mathbf{U}(B,\tau) \to T.$$

Consider also the group homomorphism

$$\alpha = \alpha_B : R_{L/K}(\mathrm{GL}_{1,B}) \to T,$$

defined by the formula $\alpha(b) = \mathrm{Nrd}(b)/\tau\big(\mathrm{Nrd(b)}\big)$.

Proposition 2. ([M3], Prop. 6.1) *For any field extension E/K the images of Nrd_E and α_E coincide.* □

By Proposition 2, the image of the reduced norm homomorphism

$$\mathrm{Nrd} : \mathrm{U}(V_E, h_E) \to (L \otimes_K E)^*$$

equals $(1-\varepsilon)\mathrm{Nrd}(B_E) = (1-\varepsilon)\mathrm{Nrd}(D_E)$ (here ε is the generator of the Galois group of $L \otimes_K E/E$) and hence does not depend on the hermitian form h. In particular, the reduced norm homomorphism

$$\mathrm{Nrd} : \mathrm{U}(V'_E, h'_E) \to (L \otimes_K E)^*$$

has the same image. Therefore, we can modify φ by an element from $\mathrm{U}(V'_E, h'_E)$ such that $\mathrm{Nrd}(\varphi) = 1$, i.e. $\varphi \in \mathrm{SU}(V_E, h_E)$ and hence ρ splits rationally. In particular, $\mathbf{SU}(V,h)$ is birationally isomorphic to $\mathbf{SU}(V',h') \times X$.

Lemma 4. *The variety $\mathbf{SU}(V,h)$ is birationally isomorphic over K to the product $\mathbf{SU}(V',h') \times \mathbb{A}^s_K$ for some $s \in \mathbb{N}$.*

Proof. By Lemma 3, it suffices to show that I is a rational variety. Denote by $\mathrm{P}_D(V)$ the subvariety of 1-dimensional D-subspaces in V. This is a rational projective homogeneous variety. Consider the dominant morphism $\mu : I \to \mathrm{P}_D(V)$, taking a reflection ψ to the image of $\psi - \mathrm{id}$. The fiber over a given anisotropic 1-dimensional D-subspace $V_1 \subset V$ is isomorphic to the unitary group $\mathbf{U}(V_1, h_1)$, where h_1 is the restriction of h on V_1. Since the variety of the unitary group is rational (see Lemma 5 below), so are the generic fiber of μ and hence also I. □

Proposition 3. *There is an involution σ of the second kind on D such that $\mathbf{SU}(V,h)$ is birationally isomorphic over K to $\mathbf{SU}(D,\sigma) \times \mathbb{A}^r_K$ for some $r \in \mathbb{N}$.*

Proof. By Lemma 4 and induction, it suffices to consider the case $\dim(V) = 1$. But in this case $\mathbf{SU}(V,h) = \mathbf{SU}(D,\sigma)$ for a certain involution σ on D of the second kind. □

Corollary 2. *Let B be a central simple L-algebra with an involution τ of the second kind, trivial on K. If $\mathrm{ind}(B) \le 3$, then for any central subgroup $\mu \subset \mathbf{SU}(B,\tau)$ the variety of the group $G = \mathbf{SU}(B,\tau)/\mu$ is rational over K.*

Proof. By the Wedderburn theorem, $B \simeq \mathrm{End}_D(V)$, where D is a central division algebra over L and V is a vector space over D. Then the involution τ is isomorphic to the adjoint involution with respect to a certain non-degenerate hermitian form h on V. Repeating verbatim the same arguments as in Corollary 1 and using Proposition 3, we obtain that the variety G is birationally isomorphic over K to the product of either $\mathbb{A}^t_K \times \mathbf{SU}(D,\sigma)$ or $\mathbb{A}^t_K \times \mathbf{PSU}(D,\sigma)$. In both cases the second factor is a semisimple group of rank at most 2, and hence is rational. □

Remark 3. Since $\mathbf{SU}(B,\tau)_L \simeq \mathbf{SL}_1(B)$, it follows from [M1] that the group $\mathbf{SU}(B,\tau)$ is not rational if $\mathrm{ind}(B)$ is divisible by 4.

Corollary 3. *Let (B,τ) and (B',τ') be central simple L-algebras with involutions of the second kind. If B and B' are Brauer equivalent algebras, then the groups $\mathbf{SU}(B,\tau)$ and $\mathbf{SU}(B',\tau')$ are stably birationally isomorphic.*

Proof. By Proposition 3, it suffices to show that groups $\mathbf{SU}(D,\sigma)$ and $\mathbf{SU}(D,\sigma')$, where σ and σ' are involutions of the second kind on a division algebra D, are stably birationally isomorphic. We have $\sigma' = \mathrm{Int}(d)\circ\sigma$ for some σ-symmetric invertible element $d \in D$. Consider the 2-dimensional D-space $V = D \oplus D$ and the hermitian diagonal form $h = \langle 1, d\rangle$ on V. By Lemma 4, the groups $\mathbf{SU}(D,\sigma)$ and $\mathbf{SU}(D,\sigma')$, being subgroups in $\mathbf{SU}(V,h)$, are stably birationally isomorphic to this group. □

Remark 4. Corollary 3 shows that the stable birational type of the group variety $\mathbf{SU}(B,\tau)$ depends only on the Brauer class of B and does not depend on the involution τ.

4. Isotropic groups of type $\mathbf{B_n}$

Proposition 4. *Let G be a K-isotropic group of type B_n. Then the variety G is rational over K.*

Proof. There are two possibilities for G to be either $G \simeq \mathbf{SO}_{2n+1}(q)$, where q is a K-isotropic quadratic form of dimension $2n+1$, or $G \simeq \mathbf{Spin}(q)$. In the first case one can apply

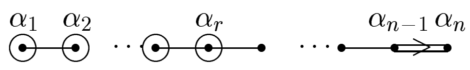
Lemma 5. *Let L be either the field K or a quadratic extension of K and B any central simple algebra over L with an involution τ of either the first or the second kind. Then the connected component of the unitary group $\mathbf{U}(B,\tau)$ is rational over K.*

Proof. The map

$$\varphi : \mathfrak{U} \longrightarrow (\mathbf{U})^o, \qquad X \longrightarrow (1-X)/(1+X),$$

where $\mathfrak{U}$ is the corresponding Lie algebra, consisting of skew elements in B with respect to τ, is clearly a birational K-isomorphism. □

In the simply connected case we first consider the Tits K-index of G :



It follows that the system (1) is equivalent to

$$\alpha_{r+1}(t) = 1, \ldots, \alpha_{n-1}(t) = 1,\ \alpha_n(t) = 1.$$

Putting $t = \prod_{i=1}^n h_{\alpha_i}(t_i)$ and taking into account (3), from the above we obtain an equivalent system

$$\begin{cases} t_{r+1}^2/t_r t_{r+2} = 1, \ldots, t_{n-2}^2/t_{n-3}t_{n-1} = 1, \\ \\ t_{n-1}^2/t_{n-2}t_n^2 = 1,\ t_n^2/t_{n-1} = 1 \end{cases}$$

Again, considering the connected component of the subgroup defined by the latter system and adding equations $t_1 = \ldots = t_r = 1$, we finally get that $\Delta \cap S' = \Delta \cap S = < h_{\alpha_n}(-1) >$ is the center of G. This means that

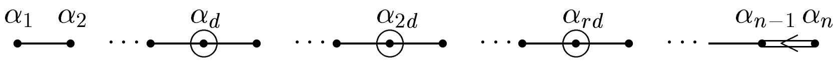
$$\Delta/\Delta \cap S = G_{\{\alpha_{r+1},\ldots,\alpha_n\}}/ < h_{\alpha_n}(-1) > \simeq \mathbf{SO}(q_{an}),$$

where q_{an} is the anisotropic part of q, and it remains to apply Lemma 5. $\square$

5. **Isotropic groups of type $\mathbf{C_n}$**

Any almost simple simply connected K-group G of type C_n can be realized (see [T]) as a special unitary group $\mathbf{SU}_{2n/d}(D, h)$, where h is a non-degenerate hermitian form on the $2n/d$-dimensional right vector space $V = D^{2n/d}$, D being a central division algebra over K, provided with a proper involution τ of the first kind. Hence it follows from Lemma 5 that the variety G is always rational over K.

Let μ be the centre of G. To study the variety of the adjoint group $\overline{G} = G/\mu$ we will first describe the maximal K-split torus S in G. The Tits K-index of the group under consideration is as follows:



Hence the semisimple anisotropic kernel Δ is clearly the direct product of groups $G_{X_1}, \ldots, G_{X_{r-1}}$, where $X_1 = \{\alpha_1, \ldots, \alpha_{d-1}\}, \ldots, X_{r-1} = \{\alpha_{(r-1)d+1}, \ldots, \alpha_{rd-1}\}$, each of them being isomorphic to $\mathbf{SL_1}(D)$, and G_{X_r}, where $X_r = \{\alpha_{rd+1}, \ldots, \alpha_n\}$. Furthermore, S is described by the system

$$\begin{cases} \alpha_1(t) = 1, & \ldots, \quad \alpha_{d-1}(t) = 1 \\ \cdots & \qquad\qquad \cdots \\ \alpha_{(r-1)d+1}(t) = 1, & \ldots, \quad \alpha_{rd-1}(t) = 1 \\ \alpha_{rd+1}(t) = 1, & \ldots, \quad \alpha_n(t) = 1 \end{cases}$$

Adding equations $t_d = t_{2d} = \cdots = t_{rd} = 1$, we conclude that $\Delta \cap S$ coincides with the product of the centers of the groups $G_{X_1}, \ldots, G_{X_{r-1}}$.

Consider the canonical map

$$\varphi : C_G(S) \longrightarrow \overline{C_G(S)} = C_G(S)/\mu$$

Since $\varphi(S)$ is a K-split torus we have

$$\overline{C_G(S)} \simeq \varphi(S) \times \left(\overline{C_G(S)}/\varphi(S)\right) \simeq$$
$$\varphi(S) \times (C_G(S)/ < S, \mu >) \simeq \varphi(S) \times \Delta/[\Delta \cap < S, \mu >]$$

But μ is generated by the element $\prod_{i=1}^{[(n+1)/2]} h_{\alpha_{2i-1}}(-1)$, where the square bracket means the integer part of a number, and hence the intersection $\Delta \cap < S, \mu >$ is exactly the center of Δ. Therefore,

$$\Delta/\Delta \cap < S, \mu > \simeq \overline{G}_{X_1} \times \cdots \times \overline{G}_{X_{r-1}} \times \overline{G}_{X_r} \simeq$$
$$\mathbb{A}^{r(d^2-1)} \times \mathbf{PSU}(D, h_{an})$$

Thus, we proved

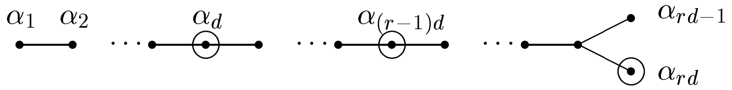
Proposition 5. *Let G be as above. Then $\overline{G} \approx \mathbf{PSU}(D, h_{an}) \times \mathbb{A}_K^t$, where $t = \dim G - \dim \mathbf{PSU}(D, h_{an})$.*

Corollary 4. *$\overline{G}$ is rational over K, if $n - rd \leq 2$.*

6. Isotropic groups of type $^1\mathbf{D}_\mathbf{n}$

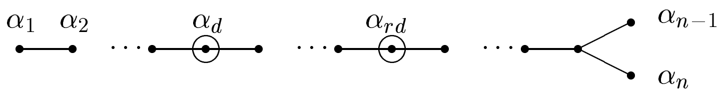
Any almost simple simply connected group G of type ^1D_n can be realized as a spinor group $\mathbf{Spin}(D, h)$ associated to a non-degenerate hermitian form h of discriminant one over a central division algebra D of degree d with an involution τ of the first kind and of orthogonal type. There are two different cases to be considered: the Witt index r of the hermitian form h is either maximal possible, i.e. $r = n/d$, or not. In the first case the Tits K-index of G is of the form

(7)



and in the second case

(8)



6.1. The case $\mathbf{rd = n}$ and $\mathbf{G}$ is simply connected

The assumption $rd = n$ implies (see (7)) that the semisimple anisotropic kernel Δ of G is isomorphic over K to the direct product of r copies of the group $\mathbf{SL}_1(D)$. Denote the simple components of Δ by

$$H_1 = G_{\{\alpha_1, \dots, \alpha_{d-1}\}}, \dots, H_r = G_{\{\alpha_{(r-1)d+1}, \dots, \alpha_{rd-1}\}}$$

and their centers by $Z_1, \dots, Z_r$. It is clear that Z_i can be identified with μ_d for all $i = 1, \dots, r$. As in the previous sections a straightforward computation shows that $\Delta \cap S$ is generated by the following elements:

$$(\zeta, 1, \dots, 1, \zeta), (1, \zeta, \dots, 1, \zeta), \cdots, (1, \dots, 1, \zeta, \zeta), (1, \dots, 1, \lambda)$$

where ζ (resp. λ) is a primitive root of unity of degree d (resp. $d/2$).

Put $\overline{H}_r = H_r/\mu_{d/2} \simeq \mathbf{SL}_1(D)/\mu_{d/2}$ and consider the exact sequence

$$1 \longrightarrow \mu_d^{r-1} \xrightarrow{\varphi} H_1 \times \cdots \times H_{r-1} \times \overline{H}_r \xrightarrow{\psi} \overline{\Delta} = \Delta/\Delta \cap S \longrightarrow 1,$$

where $\varphi(\zeta_1, \dots, \zeta_{r-1}) = (\zeta_1, \dots, \zeta_{r-1}, \overline{\zeta}_1 \cdot \ldots \cdot \overline{\zeta}_{r-1})$ and $\overline{\zeta}_i$ denotes the image of ζ_i under the canonical map $H_r \longrightarrow \overline{H}_r$.

As in §3 (see Proposition 1) one can easily prove that for all extensions E/K

$$\operatorname{Ker}\left[H^1(E, \psi(1 \times \cdots \times 1 \times \overline{H}_r)) \longrightarrow H^1(E, \overline{\Delta})\right] = 1.$$

It follows immediately that $\overline{\Delta}$ is birationally isomorphic over K to the direct product of the variety $\overline{H}_r$ and the variety

$$\overline{\Delta}/\psi(1 \times \cdots \times 1 \times \overline{H}_r) \simeq H_1/Z_1 \times \cdots \times H_{r-1}/Z_{r-1}.$$

But $H_i/Z_i \simeq \mathbf{PSL}_1(D)$ is rational over K and therefore, we have

Proposition 6. *Let G be as above. Then the variety G is birationally isomorphic over K to the product $\mathbf{SL}_1(D)/\mu_{d/2} \times \mathbb{A}_K^t$, where $t = \dim G - (d^2 - 1)$.*

Corollary 5. *Let G be as above and $d \leq 4$. Then G is K-rational.*

Proof. If $d = 1$, then G is split and there is nothing to prove. If $d = 2$, then clearly $\mu_{d/2} = 1$ and $\mathbf{SL}_1(D)/\mu_{d/2} = \mathbf{SL}_1(D)$ is rational over K. Finally, if $d = 4$, then the group $\mathbf{SL}_1(D)/\mu_2$ is of type $D_3 = A_3$ and so one can apply Lemma 5. ☐

6.2. The case $\mathbf{rd = n}$ and G is not simply connected

Consider the corresponding exact sequence $1 \longrightarrow \mu \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$, where $\widetilde{G}$ is the simply connected covering. Let $\widetilde{S}$ be a maximal K-split torus in $\widetilde{G}$, S its image in G. If $d = 1$, then G is split and hence K-rational. So we may assume in the sequel that $d \geq 2$. The center Z of $\widetilde{G}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, there are four possibilities for μ to be considered:

$$\begin{array}{rcl} \text{(i)} \quad \mu & = & < h_{\alpha_1}(-1)h_{\alpha_3}(-1)\cdots h_{\alpha_{rd-1}}(-1) > \\ \text{(ii)} \quad \mu & = & < h_{\alpha_1}(-1)h_{\alpha_3}(-1)\cdots h_{\alpha_{rd}}(-1) > \\ \text{(iii)} \quad \mu & = & < h_{\alpha_{rd-1}}(-1)h_{\alpha_{rd}}(-1) > \\ \text{(iv)} \quad \mu & = & < h_{\alpha_1}(-1)h_{\alpha_3}(-1)\cdots h_{\alpha_{rd-1}}(-1),\ h_{\alpha_{rd-1}}(-1)h_{\alpha_{rd}}(-1) > \end{array}$$

From the system (2) one can easily obtain that in case (i) one has $\mu \subset \widetilde{S}$; therefore, we have

$$C_G(S) \simeq S \times (C_G(S)/S) \simeq S \times (C_{\widetilde{G}}(\widetilde{S})/\widetilde{S}),$$

which implies that G is birationally K-isomorphic to the product of an affine space and the variety $\mathbf{SL}_1(D)/\mu_{d/2}$ (see Proposition 6); in particular, the variety G is K-rational, if $d \leq 4$.

In cases (ii), (iii) one can make sure that the non-trivial element of μ can be written as the product of an element belonging to the torus $\widetilde{S}$ and a generator ζ of the center of $H_r = \widetilde{G}_{\{\alpha_{(r-1)d+1},\dots,\alpha_{rd-1}\}}$. It follows that

$$C_G(S) \simeq S \times (C_G(S)/S) \simeq S \times \left(C_{\widetilde{G}}(\widetilde{S})/(\widetilde{S},\zeta)\right) \simeq$$
$$S \times (H_1/Z_1 \times \cdots \times H_r/Z_r) \simeq S \times \mathbf{PSL}_1(D) \times \cdots \times \mathbf{PSL}_1(D),$$

hence in these cases G is K-rational for any d.

Finally, in case (iv) arguing as above we obtain that

$$C_G(S) \simeq S \times (H_1/Z_1 \times \cdots \times H_r/Z_r)$$

is also a K-rational variety. Thus, summarizing we have

Proposition 7. *Let G be as above. If $d \leq 4$, then G is rational over K.*

6.3. The case $\mathbf{n - rd > 0}$ and G is simply connected

As above, put $\Delta = H_1 \times \cdots \times H_{r+1}$, where

$$H_1 = G_{\{\alpha_1,\dots,\alpha_{d-1}\}}, \dots, H_r = G_{\{\alpha_{(r-1)d-1},\dots,\alpha_{rd-1}\}}, H_{r+1} = G_{\{\alpha_{rd+1},\dots,\alpha_n\}}.$$

Identify the centers $Z_1, \dots, Z_r$ of $H_1, \dots, H_r$ with μ_d. As in the previous sections it is easy to show that the intersection $\Delta \cap S$ consists of all elements of the form $(\zeta_1, \dots, \zeta_r, \zeta_1^{d/2} \cdots \zeta_r^{d/2})$, where $\zeta_i^d = 1$ for all $i = 1, \dots, r$, and $\zeta_1^{d/2} \cdots \zeta_r^{d/2}$ is considered as an element of the group

$$Z_{r+1} =< \pm 1 >=< h_{\alpha_{n-1}}(-1) h_{\alpha_n}(-1) > .$$

Put $\overline{H}_i = H_i/\mu_{d/2} \simeq \mathbf{SL}_1(D)/\mu_{d/2}$, $i = 1, \dots, r$, and consider the exact sequence

$$1 \longrightarrow \mu_2^r \xrightarrow{\varphi} \overline{H}_1 \times \cdots \times \overline{H}_r \times H_{r+1} \xrightarrow{\psi} \overline{\Delta} = \Delta/\Delta \cap S \longrightarrow 1,$$

where $\varphi(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r, \alpha_1 \cdots \alpha_r)$ and as above $\alpha_1, \dots, \alpha_r$, $\alpha_1 \cdots \alpha_r$ are considered as central elements of the groups $\overline{H}_1, \dots, \overline{H}_r$, H_{r+1}.

Lemma 6. *For any extension E/K one has*

$$\mathrm{Ker}\left[H^1(E, \psi(\overline{H}_1 \times 1 \times \cdots \times 1)) \longrightarrow H^1(E, \overline{\Delta})\right] = 1$$

Proof. Pick any λ' in the kernel of the morphism above. The restriction $\psi|_{\overline{H}_1 \times 1 \times \cdots \times 1}$ is clearly injective and so λ is the image of the unique element $\lambda' \in H^1(E, \overline{H}_1 \times 1 \times \cdots \times 1)$. By assumption, $\theta(\lambda')$ is trivial in $H^1(E, \overline{\Delta})$, where

$$\theta : H^1(E, \overline{H}_1 \times 1 \times \cdots \times 1) \longrightarrow H^1(E, \overline{\Delta})$$

is the canonical map. This means that

$$\lambda' \in \mathrm{Im}\left[H^1(E, \mu_2^r) \longrightarrow H^1(E, \overline{H}_1 \times \cdots \times \overline{H}_r \times H_{r+1})\right] \tag{9}$$

Identify $H^1(E, \mu_2^r)$ with $\left(E^*/(E^*)^2\right)^r$. Then in view of (9) we conclude that there are $b_1, \dots, b_r \in E^*$ such that

$$(\lambda', 1, \dots, 1) = (\varphi_1(b_1(E^*)^2), \dots, \varphi_r(b_r(E^*)^2), \varphi_{r+1}(b_1 \cdots b_r(E^*)^2)), \tag{10}$$

where $\varphi_i : H^1(E, \mu_2) \to H^1(E, \overline{H}_i)$ is the canonical map.

First, we are going to show that $b_2, \dots, b_r$ are squares in E^* modulo $\mathrm{Nrd}(D \otimes_K E)^*$. To this end, consider the commutative diagram

$$\begin{array}{ccccccccc}
1 \longrightarrow & \mu_{d/2} & \longrightarrow & H_i \simeq \mathbf{SL}_1(D) & \longrightarrow & \overline{H}_i & \longrightarrow 1 \\
& \| & & \uparrow & & \uparrow & \\
1 \longrightarrow & \mu_{d/2} & \longrightarrow & \mu_d & \longrightarrow & (\pm 1) & \longrightarrow 1
\end{array}$$

It induces the commutative diagram with exact rows

$$\begin{array}{ccccc}
H^1(E, \mu_{d/2}) & \xrightarrow{\gamma_6} & H^1(E, \mathbf{SL}_1(D)) & \xrightarrow{\gamma_2} & H^1(E, \overline{H}_i) \\
\| & & \uparrow \gamma_1 & & \uparrow \gamma_4 \\
H^1(E, \mu_{d/2}) & \xrightarrow{\gamma_5} & H^1(E, \mu_d) & \xrightarrow{\gamma_3} & H^1(E, (\pm 1))
\end{array} \tag{11}$$

As above, identify $H^1(E, \mu_d)$, $H^1(E, \mu_{d/2})$, $H^1(E, (\pm 1))$ respectively with $E^*/(E^*)^d$, $E^*/(E^*)^{d/2}$ and $E^*/(E^*)^2$. Then one can easily make sure that $\gamma_3\left(x(E^*)^d\right) = x(E^*)^2$ and $\gamma_5\left(x(E^*)^{d/2}\right) = x^2(E^*)^d$. So in view of (10) we have

$$(\gamma_4 \circ \gamma_3)\left(b_i(E^*)^d\right) = \gamma_4\left(b_i(E^*)^2\right) = 1.$$

On the other hand, it follows from diagram (11) that

$$(\gamma_4 \circ \gamma_3)\left(b_i(E^*)^d\right) = (\gamma_2 \circ \gamma_1)\left(b_i(E^*)^d\right),$$

therefore, $\gamma_1\big(b_i(E^*)^d\big) \in \operatorname{Ker}\gamma_2 = \operatorname{Im}\gamma_6$, forcing b_i, $i = 2, \dots, r$, to be squares in E^* modulo $\mathrm{Nrd}(D \otimes_K E)^*$, as claimed.

From (10) it also follows that $b_1 \cdots b_r (E^*)^2$ lies in the kernel of the natural map $H^1(E, \mu_2) \longrightarrow H^1(E, H_{r+1})$ consisting of spinor norms. However, any spinor norm can be written as the product of a square and an element, which is a reduced norm in $D \otimes_K E$ (see [Z]). So without loss of generality we may assume that $b_1(E^*)^2 = \mathrm{Nrd}\,(b)(E^*)^2$ for some $b \in D \otimes_K E$. Then, by construction and from diagram (11), we have

$$\begin{gathered}\lambda' = \gamma_4\left(\mathrm{Nrd}\,(b)(E^*)^2\right) = (\gamma_4 \circ \gamma_3)\left(\mathrm{Nrd}\,(b)(E^*)^d\right) = \\ (\gamma_2 \circ \gamma_1)\left(\mathrm{Nrd}(b)\,(E^*)^d\right) = \gamma_2(1) = 1,\end{gathered}$$

obtaining the desired result. □

Proposition 8. *Let G be a simply connected group of type ${}^1D_n$ with the Tits index* (8)*. Then $G \approx \mathbb{A}^t_K \times \mathbf{SL}_1(D)/\mu_{d/2}$ for some $t \in \mathbb{N}$.*

Proof. Lemma 6 combined with Lemma 1 implies that the variety $\overline{\Delta}$ is birationally isomorphic over K to the product of $\overline{H}_1 \simeq \mathbf{SL}_1(D)/\mu_{d/2}$ and a K-rational variety $X = \overline{\Delta}/\psi(\overline{H}_1 \times 1 \times \cdots \times 1)$ for

$$\begin{aligned} X \;\simeq\; & \left[\overline{H}_1 \times \cdots \times \overline{H}_r \times H_{r+1}\right] / \left[(\overline{H}_1 \times 1 \times \cdots \times 1)\varphi(\mu_2^r)\right] \;\simeq \\ & \overline{H}_2/(\pm 1) \times \cdots \times \overline{H}_r/(\pm 1) \times H_{r+1}/(\pm 1) \;\simeq \\ & \mathbf{PSL}_1(D) \times \cdots \times \mathbf{PSL}_1(D) \times \mathbf{SU}(D, h_{an}) \end{aligned}$$

is clearly rational over K. □

Corollary 6. *Let G be as above. Then the variety G is K-rational, if $d \leq 4$.*

6.4. The case $\mathbf{n - rd > 0}$ and $\mathbf{G}$ is not simply connected

This case can be handled just as the case $rd = n$. Namely, consider the corresponding exact sequence $1 \longrightarrow \mu \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$, where $\widetilde{G}$ is simply connected covering. Let $\widetilde{S}$ be a maximal K-split torus in $\widetilde{G}$, S its image in G. There are four possibilities for μ to be considered:

(i)	μ	$=$	$< h_{\alpha_{n-1}}(-1) h_{\alpha_n}(-1) >$,	
(ii)	μ	$=$	$< h_{\alpha_1}(-1) h_{\alpha_3}(-1) \cdots h_{\alpha_{n-1}}(-1) >$	(n is even),
(iii)	μ	$=$	$< h_{\alpha_1}(-1) h_{\alpha_3}(-1) \cdots h_{\alpha_n}(-1) >$	(n is even),
(iv)	μ		coincides with the center of $\widetilde{G}$.	

In the first case $G \simeq \mathbf{SU}(D, h)$ and by Lemma 5, the variety G is K-rational. In cases (ii), (iii) the non-trivial element of μ can be written as product of an element belonging to $\widetilde{S}$ and the element

$$\zeta = h_{\alpha_{rd+1}}(-1) h_{\alpha_{rd+3}}(-1) \cdots h_{\alpha_{n-1}}(-1)$$

belonging to the center of $H_{r+1} = G_{\{\alpha_{rd+1},\dots,\alpha_n\}}$. So keeping the notation of the section 6.3 we have

$$C_G(S) \simeq S \times [C_G(S)/S] \simeq S \times \left[C_{\widetilde{G}}(\widetilde{S})/ < \widetilde{S}, \zeta >\right] \simeq$$
$$S \times \left[(\overline{H}_1 \times \cdots \times \overline{H}_r \times H_{r+1})/ < \varphi(\mu_2^r), \zeta >\right].$$

Again, arguing as in 6.3 one can finally show that

$$(\overline{H}_1 \times \cdots \times \overline{H}_r \times H_{r+1})/ < \varphi(\mu_2^r), \zeta > \simeq$$
$$\overline{H}_1 \times \mathbf{PSL}_1(D) \times \cdots \times \mathbf{PSL}_1(D) \times H_{r+1}/Z_{r+1},$$

where Z_{r+1} denotes the center of H_{r+1}. Finally, in case (iv) analogous arguments show that

$$C_G(S) \simeq S \times \mathbf{PSL}_1(D) \times \cdots \times \mathbf{PSL}_1(D) \times H_{r+1}/Z_{r+1}.$$

Summarizing we have

Proposition 9. *1) In case* (i) *G is rational over K.*
2) In cases (ii), (iii) *we have $G \approx \mathbb{A}_K^t \times \mathbf{PSU}(D, h_{an}) \times \mathbf{SL}_1(D)/\mu_{d/2}$*
3) In case (iv) *we have $G \approx \mathbb{A}_K^t \times \mathbf{PSU}(D, h_{an})$*

Corollary 7. *G is rational over K if $d \le 4$ and $n - rd \le 3$.*

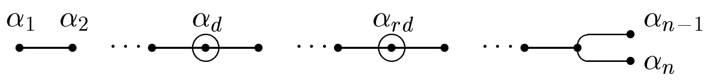
Proof. It is enough to observe that all K-anisotropic adjoint groups of type ${}^1D_l$ are K-rational if $l \le 3$. Indeed, the case $l \le 2$ is trivial. If $l = 3$ then the group under consideration is K-isomorphic to $\mathbf{PSL}_1(T)$ for some central simple algebra T and hence is rational. □

7. Isotropic groups of type $^2\mathbf{D_n}$

As in the previous section any simply connected group $\widetilde{G}$ of the type under consideration is the spinor group $\mathbf{Spin}(D, h)$ associated to a non-degenerate hermitian form h of discriminant $\neq 1$ over a division algebra D of degree d with an involution τ of the first kind and of orthogonal type. Let r be its K-rank, μ any central subgroup and $G = \widetilde{G}/\mu$. Let us also label simple roots as in § 6. Then there are three possibilities to be considered: μ coincides either with 1 or with the center of $\widetilde{G}$ or $\mu = < h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1) >$. Denote the quotient $\widetilde{G}/\mu$ by G.

7.1. The case $\mathbf{n - rd > 1}$

The Tits K-index is as follows:



Proposition 10. *1) $\widetilde{G} \approx \mathbb{A}_K^t \times \mathbf{SL}_1(D)/\mu_{d/2}$ for some $t \in \mathbb{N}$; in particular, $\widetilde{G}$ is K-rational, if $d \le 4$.*
2) If $\mu = < h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1) >$, then G is rational over K.
3) If μ coincides with the centre of G, then $G \approx \mathbb{A}_K^t \times \mathbf{PSU}(D, h_{an})$ for some $t \in \mathbb{N}$.

For the proof repeat verbatim the same arguments as in 6.3, 6.4 and so we omit it. $\square$

7.2. The case $\mathbf{n - rd = 1}$

The Tits K-index is as follows:

(12)

Proposition 11. *The variety G is rational over K.*

Proof. If $\mu = < h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1) >$, then $G \simeq \mathbf{SU}(D,h)$ and hence by Lemma 5, G is K-rational. Thus, we have to consider the simply connected and adjoint cases only. To do this we need the following claim.

Lemma 7. *Let $T = (\alpha, \beta)$ be a quaternion algebra over K and $\Delta = H_1 \times \cdots \times H_t$, where $H_i(K) \simeq \mathbf{SL}_1(T)$ for all $i = 1, \dots, t$. Then for any central subgroup $A \subset Z(\Delta)$ the quotient Δ/A is a rational variety over K.*

Proof. Let $\overline{H}_i = H_i/(\pm 1)$, $i = 1, \dots, t$, and $\overline{\Delta} = \overline{H}_1 \times \cdots \times \overline{H}_t$. It is clear that $\overline{H}_i$ is a K-rational variety and furthermore, its K-points can be parametrized in such a way that $K(\overline{H}_i) = K(x_{1,i}, x_{2,i}, x_{3,i})$ and

$$K(H_i) = K(\overline{H}_i)\left(\sqrt{1 - \alpha x_{1,i}^2 - \beta x_{2,i}^2 + \alpha\beta x_{3,i}^2}\,\right),$$

where $x_{1,i}$, $x_{2,i}$, $x_{3,i}$ are algebraically independent variables over K. Put

$$f_1 = 1 - \alpha x_{1,1}^2 - \beta x_{2,1}^2 + \alpha\beta x_{3,1}^2,\ \dots,\ f_t = 1 - \alpha x_{1,t}^2 - \beta x_{2,t}^2 + \alpha\beta x_{3,t}^2.$$

Denote the field of rational functions $K(\Delta)$ by E and consider any intermediate subfield

$$K' = K(x_{1,1}, x_{2,1}, x_{3,1}, \dots, x_{1,t}, x_{2,t}, x_{3,t}) \subset F \subset E$$

Obviously, F can be obtained by adding to K' some square roots of the form $\sqrt{f_{i_1} \cdots f_{i_s}}$. We are going to show that F is a purely transcendental extension of K by induction on the number of these square roots.

To begin with, let us consider first the case $F = K'(\sqrt{f_{i_1} \cdots f_{i_s}}\,)$. Then F is clearly the field of K-rational functions on the variety given by the equation $z^2 = f_{i_1} \cdots f_{i_s}$ or equivalently, $z^2 f_{i_1} = f_{i_2} \cdots f_{i_s}$. Its left part is a 4-dimensional Pfister form

$$g = z^2 - \alpha(x_{1,i_1}z)^2 - \beta(x_{2,i_1}z)^2 + \alpha\beta(x_{3,i_1}z)^2.$$

and obviously it represents over the field

$$\widetilde{K} = K(x_{1,i_2}, x_{2,i_2}, x_{3,i_2}, \dots, x_{1,i_s}, x_{2,i_s}, x_{3,i_s})$$

the elements $f_{i_2}, \dots, f_{i_s}$. This means that the $\widetilde{K}$-defined quadric $g = f_{i_2} \cdots f_{i_s}$ has $\widetilde{K}$-rational points and hence it defines a $\widetilde{K}$-rational variety, as claimed.

In the general case, suppose that F is obtained by adjoining to the field K' the square roots $\sqrt{f_{i_1} \cdots f_{i_\alpha}}, \dots, \sqrt{f_{j_1} \cdots f_{j_\beta}}$. Without loss of generality we may assume that the function f_{j_β} occurs in one square root only. Then our field F is the field of K-rational functions on the variety V given by the equations

$$\begin{cases} z_1^2 & = & f_{i_1}\cdots f_{i_\alpha} \\ & \cdots & \\ z_{s-1}^2 & = & f_{l_1}\cdots f_{l_\gamma} \\ z_s^2 f_{j_\beta} & = & f_{j_1}\cdots f_{j_{\beta-1}} \end{cases}$$

Arguing as above we may look at the last equation as a $\widetilde{K}$-defined quadric in the variables

$$z_s,\ \tilde{x}_1 = z_s x_{1,j_\beta},\ \tilde{x}_2 = z_s x_{2,j_\beta},\ \tilde{x}_3 = z_s x_{3,j_\beta}.$$

As above, this defines a $\widetilde{K}$-rational variety and furthermore, the variables $z_s,\ \tilde{x}_1,\ \tilde{x}_{,}\ \tilde{x}_3$ do not occur in the other equations of the system under consideration. Therefore, to complete the proof it remains to apply the induction step. $\square$

Let us proceed with the proof of Proposition 11 and first consider the case of the simply connected group $\widetilde{G}$. Here we have (see (12)) $\dim S' - \dim S = 1$, where as usually S' denotes the center of $C_G(S)$. Then the quotient $C_G(S)/S$ is an almost direct product of the 1-dimensional central torus S'/S and the semisimple group $\overline{\Delta} = \Delta/\Delta\cap S$, where $\Delta = H_1\times\cdots\times H_{(n-1)/2}$ and $H_1 = G_{\alpha_1}$, $H_2 = G_{\alpha_3},\ldots, H_{(n-1)/2} = G_{\alpha_{n-2}}$. Since $H_i(K)\simeq \mathbf{SL}_1(D)$ for all $i = 1,2,\ldots,(n-1)/2$, Lemma 7 means that the variety $\overline{\Delta}$ is rational over K. A straightforward check also shows that $\Delta\cap S$ is generated by the elements

$$h_{\alpha_1}(-1)h_{\alpha_{n-2}}(-1),\ h_{\alpha_3}(-1)h_{\alpha_{n-2}}(-1),\ldots,\ h_{\alpha_{n-4}}(-1)h_{\alpha_{n-2}}(-1).$$

If $(S'/S)\cap\overline{\Delta} = 1$, then $C_G(S) = S'/S\times\overline{\Delta}$ is K-rational since all 1-dimensional K-tori are rational over K. Otherwise, the intersection $(S'/S)\cap\overline{\Delta}$ must coincide with the center of $\overline{\Delta}$ for it has order 2. Then the group $C_G(S)/S$ is part of an exact sequence

$$1\longrightarrow \mu_2^{(n-1)/2}\xrightarrow{\varphi}(S'/S)\times H_1\times\cdots\times H_{(n-1)/2}\xrightarrow{\psi} C_G(S)/S\longrightarrow 1$$

where

$$\begin{gathered}\varphi(-1,1,\ldots,1) = (1,-1,1,\ldots,1,-1),\\ \varphi(1,-1,1,\ldots,1) = (1,1,-1,1,\ldots,1,-1)\\ \cdots\\ \varphi(1,\ldots,1,-1,1) = (1,\ldots,1,-1,-1),\\ \varphi(1,\ldots,1,-1) = (-1,1,\ldots,1,-1)\end{gathered}$$

By construction, $S'/S\simeq R^{(1)}_{L/K}(G_{m,L})$, where L is a quadratic extension of K such that G has inner type over L. Obviously, L splits D ; therefore, each H_i is also split over L. Then arguing just as in § 3 one can easily see that

$$\operatorname{Ker}\left[H^1(E,\psi(1\times\cdots\times 1\times H_{(n-1)/2}))\longrightarrow H^1(E,C_G(S)/S)\right] = 1$$

for any extension E/K. This means that $C_G(S)/S$ is birationally isomorphic to

$$\psi(1 \times \cdots \times 1 \times H_{(n-1)/2}) \times \left[(C_G(S)/S)/\psi(1 \times \cdots \times 1 \times H_{(n-1)/2})\right] \simeq$$
$$H_{(n-1)/2} \times H_1/\mu_2 \times \cdots \times H_{(n-3)/2}/\mu_2 \times (S_1/S)/\mu_2$$

and so is a K-rational variety, yielding the required result.

The adjoint case is completely analogous to the previous one and so we omit it. Proposition 11 is proved. $\square$

7.3. General theorem

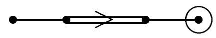
Combining previous results we get the following

Theorem 1. *Let G be a K-isotropic almost simple group of classical type ${}^{1,2}D_n$. If G is not K-rational, then in the notation above either $G \approx \mathbf{SL}_1(D)/\mu_{d/2} \times \mathbb{A}_K^m$ or $G \approx \mathbf{PSU}(D, h_{an}) \times \mathbb{A}_K^m$ or $G \approx \mathrm{SL}_1(D)/\mu_{d/2} \times \mathbf{PSU}(D, h_{an}) \times \mathbb{A}_K^m$ for some $m \in \mathbb{N}$. In particular, G is K-rational if $d \leq 4$ and either $n - rd \leq 3$ and G has inner type or $n - rd \leq 2$ and G has outer type.*

8. Isotropic groups of types $\mathbf{F_4}$, ${}^{3,6}\mathbf{D_4}$

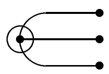
Proposition 12. *The variety of any K-isotropic group G of type F_4 is rational over K.*

Proof. For a K-split group the statement is trivial. So we may assume that G is K-isotropic, but not K-split. Then the Tits K-index (see [T]) is of the form



It follows that the semisimple K-anisotropic kernel Δ has type B_3. Solving the system (2) one gets that $\Delta \cap S$ coincides with the centre of Δ. This means that $\overline{\Delta} = \Delta/\Delta \cap S \simeq \mathbf{SO}(f)$ for some 7-dimensional quadratic form f and it remains only to apply Lemma 5. $\square$

Now we turn to type ${}^{3,6}D_4$. As above we may assume that G is not quasi-split over K. Then the only possibility (see [T]) for the Tits K-index is to be of the form

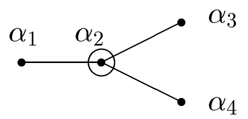


Let L be any cubic extension of K such that G has type ${}^{1,2}D_4$ over L. Then the semisimple K-anisotropic kernel Δ of the corresponding simply connected group is isomorphic to $\mathrm{R}_{\mathrm{L/K}}(\mathbf{SL}_1(D))$, where D is a quaternion algebra over L.

Lemma 8. *D has a maximal subfield E of the form $E = L(\sqrt{\alpha})$ for some $\alpha \in K$.*

Proof. By [KMRT], Proposition 43.7, it suffices to show that the algebra $T = \mathrm{cor}_{L/K}(D)$ is trivial over K or equivalently, $T_L = T \otimes_K L$ is trivial over L. Denote all K-embeddings of L into an algebraic closure $\overline{K}$ by τ_1, τ_2, τ_3 and let $D_1 = {}^{\tau_1}D$, $D_2 = {}^{\tau_2}D$, $D_3 = {}^{\tau_3}D$ be the corresponding twisting algebras. Then we have $T_L = D_1 \otimes_L D_2 \otimes_L D_3$. To compute T_L recall that the Tits L-index of G is of the form

(13)



and by construction, the quaternion algebras D_1, D_2, D_3 correspond to the simple components $G_{\alpha_1}\, G_{\alpha_3}\, G_{\alpha_4}$ of the semisimple L-anisotropic kernel Δ_L of G.

On the other hand, it follows from (13) that up to isogeny G can be realized over L as a spinor group $\mathbf{Spin}(D', h)$ associated to a non-degenerate 4-dimensional hermitian form h over a quaternion L-algebra D' with an involution σ of the first type and of orthogonal type. Since h is L-isotropic it can be written as the direct sum $h = h_1 \oplus h_2$, where h_1 is a 2-dimensional L-isotropic subform. Then a straightforward computation shows that for a L-split torus $S \subset \mathbf{Spin}(D', h_1)$ the commutator of $Z_G(S)$ is the direct product of three groups of type A_1, corresponding to D' and the components of the Clifford algebra $C(D', h_2)$. Furthermore, it is known that the simple components of the Clifford algebras $C(D', h)$, $C(D', h_2)$ of the form h and its anisotropic part h_2 coincide in $\mathrm{Br}(L)$ and the tensor product of the components of $C(D', h)$ equals D' in $\mathrm{Br}(L)$ (see [KMRT]). Therefore, the tensor product of D' and the components of $C(D', h_2)$ is a matrix algebra over L, as claimed. $\square$

Proposition 13. *The variety of any K-isotropic group G of type ${}^{3,6}D_4$ is rational over K.*

The proof is purely computational. Let first G be the simply connected group. Since $\Delta \cap S = (\pm 1)$, where $(\pm 1) \longrightarrow \mathrm{R_{L/K}}(\mathbf{SL}_1(D))$ is the diagonal embedding, it suffices to prove the K-rationality of the variety $\mathrm{R_{L/K}}(\mathbf{SL}_1(D))/(\pm 1)$.

Let $D = (\alpha, \beta)_L$, $L = K(\theta)$ for some $\beta, \theta \in L$. The group $\mathrm{R_{L/K}}(\mathbf{SL}_1(D))$ is described by the following system of equations:

$$\begin{gathered}(x_1 + x_2\theta + x_3\theta^2)^2 - \alpha(y_1 + y_2\theta + y_3\theta^2)^2 - \\ \beta(u_1 + u_2\theta + u_3\theta^2)^2 + \alpha\beta(v_1 + v_2\theta + v_3\theta^2)^2 = 1\end{gathered} \tag{14}$$

Then the field of K-rational functions $P = K\left(\mathrm{R_{L/K}}(\mathbf{SL}_1(D))\right)$ is generated over K by the coordinate functions $x_1, x_2, x_3, y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2, v_3$, and the field $K\left(\mathrm{R_{L/K}}(\mathbf{SL}_1(D))/(\pm 1)\right)$ can be identified with the subfield of invariants P^σ of the automorphism σ mapping x_i, y_i, u_i, v_i respectively to $-x_i, -y_i, -u_i, -v_i$ for all $i = 1, 2, 3$.

Making a substitution $\overline{v}_2 = v_2/v_1, \overline{v}_3 = v_3/v_1, \overline{v}_1 = 1/v_1$ and $\overline{x}_i = x_i/v_1, \overline{y}_i = y_i/v_1, \overline{u}_i = u_i/v_1,\ i = 1,2,3$, we reduce (14) to the form

$$\begin{aligned}(\overline{x}_1 + \overline{x}_2\theta + \overline{x}_3\theta^2)^2 - \alpha(\overline{y}_1 + \overline{y}_2\theta + \overline{y}_3\theta^2)^2 - \\ \beta(\overline{u}_1 + \overline{u}_2\theta + \overline{u}_3\theta^2)^2 + \alpha\beta(1 + \overline{v}_2\theta + \overline{v}_3\theta^2)^2 = \overline{v}_1^2,\end{aligned} \tag{15}$$

and $\sigma(\overline{x}_i) = \overline{x}_i,\ \sigma(\overline{y}_i) = \overline{y}_i,\ \sigma(\overline{u}_i) = \overline{u}_i,\ i = 1,2,3,\ \sigma(\overline{v}_i) = \overline{v}_i,\ i = 2,3,\ \sigma(\overline{v}_1) = -\overline{v}_1$. Expanding the left part of (15) along the basis $1, \theta, \theta^2$ we arrive at a system of the form

$$\begin{cases} f_1 = \overline{v}_1^2 \\ f_2 = 0 \\ f_3 = 0 \end{cases}$$

where f_1, f_2, f_3 are quadratic forms which do not depend on the variable $\overline{v}_1$. Furthermore, the variables $\overline{x}_1, \overline{y}_1$ occur in the quadratic forms f_2, f_3 in the first degree only. This means that the rational functions $\overline{x}_1, \overline{y}_1, w = \overline{v}_1^2$ can be expressed as rational functions of the variables $\overline{x}_i, \overline{y}_i, \overline{v}_i,\ i = 2,3$ and $\overline{u}_j,\ j = 1,2,3$, implying $P^\sigma = K(\overline{x}_2, \overline{x}_3, \overline{y}_2, \overline{y}_3, \overline{v}_2, \overline{v}_3, \overline{u}_1, \overline{u}_2, \overline{u}_3)$ is a purely transcendental extension of K, as claimed.

Since the center μ of G does not contain K-defined subgroups of order two it remains only to consider the case of the adjoint group $\overline{G} = G/\mu$. In this case we have to prove the K-rationality of the quotient $C_G(S)/ < S, \mu >$. But $C_G(S)/ < S, \mu > \simeq \Delta/\Delta \cap < S, \mu >$ and it is easy to see that the subgroup $\Delta \cap < S, \mu >$ coincides with the center of Δ. Therefore, keeping the notation of the simply connected case we have

$$\begin{aligned}\Delta/\Delta \cap < S, \mu > \simeq \mathrm{R}_{\mathrm{L/K}}\,(\mathbf{SL}_1(D))/\mathrm{R}_{\mathrm{L/K}}(\pm 1) \simeq \\ \mathrm{R}_{\mathrm{L/K}}\,(\mathbf{SL}_1(D)/(\pm 1)) \simeq \mathrm{R}_{\mathrm{L/K}}(\mathbf{PSL}_1(D)),\end{aligned}$$

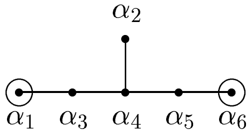
whence $C_G(S)/ < S, \mu >$ is clearly a K-rational variety. The proposition is completely proved. ☐

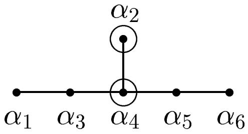
9. Isotropic groups of type $\mathbf{E_6}$

9.1. The case of inner type groups

Proposition 14. *Let G be any almost simple K-isotropic group of type ${}^1E_6$. Then G is rational over K.*

Proof. Keep the notation of the previous sections. In particular, let G be a simply connected group, S its maximal K-split torus, Δ a semisimple K-anisotropic kernel, μ the center of G and $\overline{G} = G/\mu$ the corresponding adjoint group. The non-split Tits K-index of G is one of the following:

a) 

b) 

In case a) a straightforward computation shows that both the center μ_1 of $\Delta = G_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}$ and μ lie in S. So both G and $\overline{G}$ are birationally isomorphic over K to the product $\mathbb{A}_K^{50} \times G_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}/\mu_1$. Futhermore, it is obvious that $G_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}} \simeq \mathbf{Spin}\,(f)$ for some 8-dimensional quadratic form f and

f posseses the following us property important for: if f is isotropic over some extension E/K then it is split over E. This fact implies immediately that f is a Pfister form and hence the K-rationality of G and $\overline{G}$ follows from the following

Proposition 15. *If f is a K-defined Pfister form, then the variety* $\mathbf{PSO}(f)$ *is rational over K.*

To the best of our knowledge this statement was not formulated before in such a form. To prove it one can use, for example, the same arguments as in either [C2] or [M2] and so we omit the proof. ◻

Let us proceed with the proof of the last case b). First we consider the case of a simply connected group G. It follows from b) that $\Delta = H_1 \times H_2$, where $H_1 = G_{\{\alpha_1,\alpha_3\}}$ and $H_2 = G_{\{\alpha_5,\alpha_6\}}$. Since H_i is the group of inner type A_2 we have $H_i \simeq \mathbf{SL}_1(D_i)$ for some division algebra D_i of degree 3. As before, we have to prove the rationality of the variety $\overline{\Delta} = \Delta/\Delta \cap S$. Solving the system (2) one gets $\Delta \cap S$ has order 3 and so $\overline{\Delta}$ is a part of the exact sequence

$$1 \longrightarrow \mu_3 \longrightarrow H_1 \times H_2 \xrightarrow{\psi} \overline{\Delta} \longrightarrow 1$$

Clearly, any extension E/K splitting H_1 splits G and hence H_2 too. So the division algebras D_1, D_2 have the same set of splitting fields and hence we have either $D_1 \simeq D_2$ or $D_1 \simeq D_2^{-1}$; in particular, the division algebras D_1, D_2 have the same reduced norms. Using this fact it is easy to show that for any extension E/K

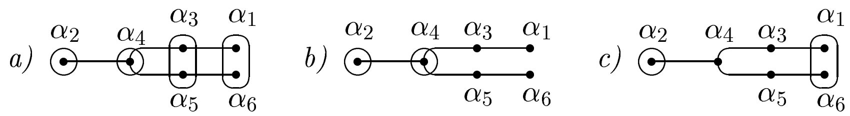
$$\mathrm{Ker}\left[H^1(E, \psi(1 \times H_2)) \longrightarrow H^1(E, \overline{\Delta})\right] = 1,$$

implying $\overline{\Delta} \simeq H_1 \times \overline{H}_2$, where $\overline{H}_2$ is the corresponding adjoint group. It remains to observe that both H_1 and $\overline{H}_2$ have rank 2 and hence are K-rational.

Finally, one can make sure that $\mu \subset \Delta$ and $\Delta \cap S \neq \mu$. So the K-rationality of $\overline{G}$ follows from the K-rationality of $\Delta/\Delta \cap < S, \mu > \simeq \overline{H}_1 \times \overline{H}_2$. Proposition 14 is completely proved. ◻

9.2. The case of outer type groups

Proposition 16. *Let G be a K-group with Tits K-index of the form*



Then G is rational over K.

Proof. If G is adjoint then one can use exactly the same arguments as above. Since every 2-dimensional torus is K-rational, the variety G is K-rational in the quasi-split case a).

Consider a simply connected group G corresponding to diagram b). For such a group G we are going to compare the K-anisotropic kernels of G and its subgroup $G_1 = G_X$ where $X = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. It follows from picture b) that G_1 is a simply connected K-isotropic group of type ${}^2A_5$. Let S (resp. S_1) be a maximal K-split torus in G (resp. G_1).

By construction, the semisimple K-anisotropic kernels Δ, Δ_1 of G and G_1 coincide and moreover, using the algorithm in §2 it is easy to show that $\Delta \cap S = \Delta \cap S_1$. Therefore, we have birational K-isomorphisms

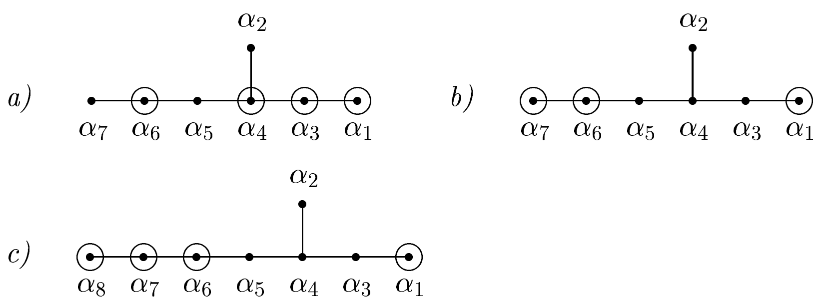
$$G \approx \mathbb{A}_K^{62} \times \overline{\Delta}, \qquad G_1 \approx \mathbb{A}_K^{19} \times \overline{\Delta} \tag{16}$$

where $\overline{\Delta} = \Delta/\Delta \cap S$. However, by Corollary 2 the variety G_1 is rational. So comparing (16) we conclude that G is rational as well.

Now assume that G has diagram c). Then simple computations give that $\Delta \cap S = \Delta \cap S'$, where as usually S' is the center of $C_G(S)$. It follows that the quotient $C_G(S)/S$ is the direct product of the 1-dimensional torus S'/S and the group $\Delta/\Delta \cap S$ of type $A_3 = D_3$ and it remains to observe that by Lemma 5, the variety $\Delta/\Delta \cap S$ is rational since $|\Delta \cap S| = 2$. □

10. Some isotropic groups of types $\mathbf{E_7}$ and $\mathbf{E_8}$

Proposition 17. *Let G be a K-group with Tits K-index of the form*



Then the variety G is K-rational.

Proof. In case a) the assertion follows immediately from Lemma 7. Cases b), c) can be treated similarly as the groups of type E_6. □

11. Groups with low rank semisimple anisotropic kernel

Theorem 2. *Let G be an almost simple group isotropic over K. Suppose the semisimple anisotropic kernel Δ of G has the property that any K-simple component of Δ has rank at most 2. Then the variety G is K-rational.*

The proof follows immediately from the results of the previous sections and the classification of Tits' indices (see [T]). Namely, for classical groups it follows from Corollaries 1,2,4, Proposition 4 and Theorem 1. Any group of type G_2 is K-rational. If G has type F_4 or E_8, then in view of our assumption it is K-split. If G is of type E_6 or E_7, then the rationality follows from Propositions 14, 16, 17. The groups of type ${}^3D_4$ and ${}^6D_4$ are K-quasi-split and hence K-rational. □

12. Proof of the main theorem

Main Theorem. *Let G be an almost simple algebraic group defined over K, where K is either an algebraic extension of $\mathbb{Q}_p$ or a totally imaginary number field or a global function field. If G is not of type A_n, $n \geq 3$, then G is a rational variety over K.*

The proof is based on the following two fundamental statements which are due to Harder in positive characteristic (see [H]) and to Kneser, Harder and Chernousov in characteristic zero (cf. [PR], Ch. 6). Let K be as above. Then one has the following.

1. If G is an almost simple simply connected algebraic K-group, then $H^1(K, G) = 1$.
2. Any almost simple K-group of type other than A_n is isotropic over K.

Let us proceed with the proof.

Types $\mathbf{B_n}$, $\mathbf{C_n}$, $\mathbf{D_n}$. Any central division algebra over K with an involution of the first kind is a quaternion algebra. Therefore, the claim follows from statement 2), Proposition 4, Corollary 4 and Theorem 1.

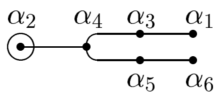
Types $\mathbf{G_2}$, $\mathbf{F_4}$, $\mathbf{E_8}$. By statement 1), G is split over K and so there is nothing to prove.

Type $\mathbf{E_7}$. In view of statement 2) there are only two admissible Tits' K-indices, considered in cases a), b) of Proposition 17 (see [T]).

Type $\mathbf{{}^{3,6}D_4}$. The claim follows from Proposition 13.

Type $\mathbf{E_6}$. Taking into account statement 2), the results of § 9 and comparing with the tables of all admissible Tits' K-indices of groups of type E_6 (cf. [T]) we see that it remains to consider one picture only:

(17)



We are going to show that for the field K under consideration there is no K-defined group G of type ${}^2E_6$ having the Tits K-index (17).

Indeed, it is known that any group of outer type E_6 defined over a local field is quasi-split. Furthermore, in the case of totally imaginary number fields and global fields of positive characteristic G is quasi-split over some cubic extension E/K (cf. [PR], Chapter 6, § 5). Let Δ be its K-anisotropic kernel. It follows from (17) that Δ has type ${}^2A_5$, and hence $\Delta \simeq \mathbf{SU}(A, \tau)$, where A is a central simple algebra over a quadratic extension L of K of degree 6, equipped with an involution τ of the second kind.

Since Δ is quasi-split over the cubic extension E/K the index of algebra A is equal to either 1 or 3. Thus, in both cases the involution τ is the adjoint involution for some non-degenerate hermitian form f over some division algebra D with an involution σ of the second kind.

We claim that the form f is K-isotropic. Really, using the vanishing of the Galois cohomology of the special unitary group associated with D or the Landherr theorem (see [H] and [L]) and repeating the argument given on p. 366 of [PR] one shows that a σ-symmetric element $d \in D$ can be written in the form $g\sigma(g)$ for some $g \in D$ if and only if $\mathrm{Nrd}(d) \in N_{L/K}(L^*)$. Since $\dim f \geq 2$ and $[E : K]$ is odd it follows that f is K-isotropic iff it is E-isotropic. However by construction, Δ is K-anisotropic and quasi-split over E — a condradiction. The proof of the main theorem is complete. □

13. Examples of non-rational quasi-split groups

Consider $\alpha_1, \alpha_2, \alpha_3 \in K^*$ which are independent modulo $(K^*)^2$ and any positive number n. Define the following $(2n+2)$-dimensional quadratic forms over K:

$$f_l = \sum_{i=1}^{n} x_i^2 - \sum_{j=1}^{n} y_j^2 + u_l^2 - \alpha_l v_l^2, \quad l = 1, 2, 3.$$

Then we have

Proposition 18. $H = (\mathbf{SO}(f_1) \times \mathbf{SO}(f_2) \times \mathbf{SO}(f_3)) / < (-1,-1,-1) >$ *is not a K-rational group.*

Proof. Let S be a maximal K-split torus in H and $G = \mathbf{SO}(f_1) \times \mathbf{SO}(f_2) \times \mathbf{SO}(f_3)$. H is a K-quasi-split group and so $C_H(S)$ is a maximal torus in H. Therefore, it is enough to show that the 3-dimensional torus $C_H(S)/S$ is not stably rational over K. To do this let us consider $\widetilde{S} = \varphi^{-1}(S)^\circ$, where $\varphi : G \longrightarrow H$ is the canonical map. It is obvious that $C_G(\widetilde{S})$ is the product of $\widetilde{S}$ and the torus

$$T = R^{(1)}_{L_1/K}(G_{m,L_1}) \times R^{(1)}_{L_2/K}(G_{m,L_2}) \times R^{(1)}_{L_3/K}(G_{m,L_3})$$

where $L_i = K(\sqrt{\alpha_i})$, $i = 1, 2, 3$. Therefore, we have

$$C_H(S)/S \simeq C_G(\widetilde{S}) / < \widetilde{S}, (-1,-1,-1) > \simeq T/(\pm 1)$$

where (-1) embeds into T diagonally. However, by the classification of 3-dimensional tori (see [K]), the torus $T/(\pm 1)$ is not stably K-rational, yielding the required result. $\square$

References

[BT] A. Borel, J. Tits, Groupes reductifs, Publ. Math. IHES **27** (1965), 55-151.

[C1] V. Chernousov, On the rationality of spinor varieties over the field of rational numbers, Dokl. AN BSSR **25** (1981), 293-296.

[C2] V. Chernousov, The group of multipliers of the canonical quadratic form and the stable rationality of the variety PSO, Mat. Zametki **55** (1994), 114-119.

[Ch] C. Chevalley, On algebraic group varieties, J. Math. Soc. Japan **6** (1954), 303—324.

[CM] V. Chernousov, A. Merkurjev, R-equivalence and Special Unitary Groups, Preprint **96—083** 1996, Bielefeld.

[DG] M. Demazure et A. Grothendieck, Schemas en groupes, Springer Lecture Notes **152**, 1970.

[H] G. Harder, Uber die Galoiskohomologie halbeinfacher algebraischer Gruppen, J. reine und angew. Math. **274-275** (1975), 125-138.

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, Book of Involutions, in preparation.

[K] B. Kunyavskii, Selecta Math. Sov. **21** (1990), 1-21.

[L] W. Landherr, Liesche Ringe vom Typus A uber einen algebraischen Zahlkorper und hermitesche Formen uber einem Schiefkorper, Abh. Math. Sem. Hamburg **12** (1938), 200-241.

[M1] A. Merkurjev, Generic element in SK_1 for simple algebras, K-theory **7** (1993), 1-3.

[M2] A. Merkurjev, R-equivalence and rationality problem for semisimple adjoint classical groups, Publ. Math. IHES (to appear).

[M3] A.. Merkurjev, Norm principle for algebraic groups, St. Petersburg Math.J. **7** (1996), 243–264.

[P1] V.P. Platonov, Algebraic groups and reduced K-theory, Proc. Internat. Congress Math., Helsinki **1** (1980), 311-317.

[P2] V.P. Platonov, On the problem of rationality of spinor varieties, Soviet Math. Dokl. **20** (1979), 1027-1031.

[P3] V.P. Platonov, Birational properties of spinor varieties, Proc. Steklov Inst.Math. **3** (1983), 173-182.

[P4] V.P. Platonov, The Tannaka-Artin problem and reduced K-theory, Math. USSR—Izvestiya **10** (1976), 211-243.

[PC1] V.P. Platonov, V.I. Chernousov, On the rationality of canonical spinor varieties, Soviet Math. Dokl. **21** (1980), 830-834.

[PC2] V.P. Platonov, V.I. Chernousov, The rationality problem for group varieties, C.R. Acad. Sci. Paris **322** (1996), 245-250.

[PR] V.P. Platonov, A.S. Rapinchuk, Algebraic groups and number theory, Academic Press, N.Y., 1994, pp.614.

[S] J-P.Serre, Cohomologie Galoisienne, Springer, 1994, pp.181.

[Sch] W. Scharlau, Quadratic and Hermitian Forms, Grundlehren math., Wiss. 270, Springer-Verlag, Berlin, 1985.

[St] R.Steinberg, Lectures on Chevalley groups, Yale University, 1967.

[Su] A.Suslin, SK_1 of division algebras and Galois cohomology, Advances in Soviet Mathematics **4** (1991), 75-101.

[T] J.Tits, Classification of algebraic semisimple groups, Proc. Symp. Pure Math. **9** (1966), 32-62.

[V] V.Voskresenskii, Algebraic tori, Nauka, Moscow, 1977, pp.223.

[Z] H. Zassenhaus, On the spinor norm, Arch. Math. **13** (1962), 434-451.

V. Chernousov, *Current address:* Fakultat fur Mathematik, Universitat Bielefeld, Postfach 10 01 31, 33615 Bielefeld, Germany, *Permanent address:* Institute of Mathematics, Surganova str. 11, Minsk 220072, Belarus

E-mail address: chernous@mathematik.uni-bielefeld.de

V. Platonov, Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario,, N2L3G1 Canada

E-mail address: vplatonov@math.uwaterloo.ca