THE DERIVED EQUIVALENCE CLASSIFICATION OF REPRESENTATION-FINITE SELFINJECTIVE STANDARD ALGEBRAS

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Introduction

Throughout this paper k denotes an algebraically closed field and all algebras, categories and functors are assumed to be k-linear; in addition all algebras are assumed to be connected, basic, finite-dimensional and with identity.

In [15, Theorem 4.2] Rickard classified Brauer tree algebras up to derived equivalence, in connection with Broué's problem [5]. In [1, Theorem 5.8] we extended this classification to representation-finite selfinjective algebras ($\neq k$) of class A_n using a covering technique for derived equivalence developed there (Cf. Membrillo-Hernandez [13]). In this paper we will further extend this classification to the class RFSS of representation-finite selfinjective standard algebras ($\neq k$). Moreover we will show that in the class of representation-finite selfinjective algebras, any standard algebra is not stably equivalent to any non-standard algebra. It immediately follows from this that the class RFSS is closed under stable equivalence. Note that derived equivalent selfinjective algebras are always stably equivalent by Keller-Vossieck [12, 2.3 Example] or Rickard [15, Theorem 2.1]. As a corollary of our results we see that the converse of this statement holds if one of the algebras is in the class RFSS. Namely, for any $\Lambda \in$ RFSS, a selfinjective algebra is derived equivalent to Λ iff it is stably equivalent to Λ.

Section 1 is devoted to a preparation. In particular we gave the precise form of induced display-functors of repetitions for a necessary class of algebras for later use. In section 2 we state our main results, which also give "complete invariants" under derived equivalence for RFSS and a complete set of representatives of the derived equivalence classes in RFSS. In section 3 we introduce computations in a formal additive hull of a spectroid to make the whole computation easy. In section 4 a covering technique developed in [1] is rephrased and a new tool is given to deal with those algebras whose Auslander-Reiten quivers are "twisted". In section 5 we give the proof of the classification theorem, and in section 6 we give the proof of the statement about standard and non-standard algebras.

1. Preliminaries

Dynkin graphs considered in this paper are only A_n ($n \geq 1$), D_n ($n \geq 4$) or E_n ($n = 6, 7, 8$); and all quivers are assumed to be *locally finite*, i.e., for each vertex x, there are only finitely many arrows which start or stop at x.

We excluded the 1-dimensional algebra k from RFSS because (i) the stable Auslander-Reiten quiver of k is empty, which is different from any algebra in RFSS; and (ii) as easily seen the isoclass of k in the class of selfinjective algebras is closed under derived equivalence; namely if Λ is a selfinjective algebra then $\Lambda \cong k$ iff Λ is derived equivalent to k.

For an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ and by $\mathcal{C}^b(\mathcal{A})$ the category of differential complexes and the category of bounded differential complexes in $\mathcal{A}$, respectively; and by $\mathcal{H}(\mathcal{A})$ and $\mathcal{H}^b(\mathcal{A})$ the corresponding homotopy categories. When $\mathcal{A}$ is an exact category, we denote by $\mathcal{D}(\mathcal{A})$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the corresponding derived categories. For a triangulated category $\mathcal{T}$, and a class of objects E in $\mathcal{T}$, $\mathrm{tri}(E)$ denotes the smallest full triangulated subcategory of $\mathcal{T}$ closed under direct summands and isomorphisms that contains E.

1.1. **Spectroids.** As in [1] we regard every algebra as a finite spectroid. Recall from Gabriel-Roiter [7] that a small category A is called a *spectroid* (= a *locally finite-dimensional category* in [6]) if the following three conditions are satisfied:

(i) Distinct objects of A are not isomorphic;
(ii) Every object of A has a local endomorphism algebra; and
(iii) The space $A(x, y)$ is finite-dimensional for every $x, y \in A$.

For a spectroid A, A is called *finite* if A has only a finite number of objects; and A is called *locally bounded* if for every $x \in A$, there are only finitely many $y \in A$ such that $A(x, y) \neq 0$ or $A(y, x) \neq 0$.

For a spectroid A, we denote by $\operatorname{Mod} A$ the category of all (right) A-modules; by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely presented objects; and by $\operatorname{pro} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely generated projective objects. In addition, $\underline{\operatorname{mod}} A$ denotes the (projectively) stable category of $\operatorname{mod} A$, and we set $D := \operatorname{Hom}_k(-, k)$.

1.2. **Repetitions.** We recall the definition of the repetition of a spectroid. Let A be a spectroid.

(1) We define a spectroid, the *repetition* $A^{\mathbb{Z}}$ of A as follows (Cf. [10], [8], [7]). Objects are the pairs $x^{[n]} := (x, n)$ with $x \in A$ and $n \in \mathbb{Z}$.

$$A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) := \begin{cases} \{f^{[n]} := (f, n) | f \in A(x, y)\} & \text{if } m = n; \\ \{\varphi^{[n]} := (\varphi, n) | \varphi \in DA(y, x)\} & \text{if } m = n + 1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The composition

$$A^{\mathbb{Z}}(y^{[m]}, z^{[l]}) \times A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) \to A^{\mathbb{Z}}(x^{[n]}, z^{[l]})$$

is given as follows:

(i) If $m = n, l = m$, then this is the composition of A:

$$A(y, z) \times A(x, y) \to A(x, z).$$

(ii) If $m = n, l = m + 1$, then this is given by the right A-module structure of $DA(-, ?)$:

$$DA(z, y) \times A(x, y) \to DA(z, x).$$

(iii) If $m = n+1, l = m$, then this is given by the left A-module structure of $DA(-,?)$:

$$A(y,z) \times DA(y,x) \longrightarrow DA(z,x).$$

(iv) Otherwise the composition is zero.

(2) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $A^{\mathbb{Z}}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow A^{\mathbb{Z}}, x \mapsto x^{[n]}$, the embedding functor.

(3) The Nakayama automorphism ν_A of $A^{\mathbb{Z}}$ is defined by
- $\nu_A(x^{[n]}) := x^{[n+1]}$;
- $\nu_A(f^{[n]}) := f^{[n+1]}$; and
- $\nu_A(\varphi^{[n]}) := \varphi^{[n+1]}$

for all $x \in A$ and for all $f \in A(x,y)$, $\phi \in DA(y,x)$ with $x, y \in A$.

Note that if a spectroid A is locally bounded, then so is $A^{\mathbb{Z}}$.

1.3. **Display-functors.** The following is immediate from the definition above.

Lemma. *Let A be a locally bounded spectroid, Q its quiver and $\Phi\colon kQ \longrightarrow A$ a display-functor* ([7, 8.1, 8.3a]). *Then a display-functor $\hat{\Phi}\colon k\hat{Q} \longrightarrow A^{\mathbb{Z}}$ for $A^{\mathbb{Z}}$ is given as follows.*

(1) *$\hat{Q}$ is obtained from the disjoint union $Q \times \mathbb{Z}$ of copies $Q^{[i]}$ of Q by adding $d_{y,x}$ arrows $\alpha^{(t)\ [i]}_{y,x}\colon x^{[i]} \longrightarrow y^{[i+1]}$ $(t = 1, \dots, d_{y,x})$ for all $i \in \mathbb{Z}$, where $d_{y,x} := \dim(\operatorname{soc}_A A \cap \operatorname{soc} A_A)(y,x)$ and $(\operatorname{soc}_A A \cap \operatorname{soc} A_A)(y,x)$ is the subspace of $A(y,x)$ consisting of elements μ with $\mu\alpha = 0 = \beta\mu$ for all $\alpha \in (\operatorname{rad} A)(-,y)$ and all $\beta \in (\operatorname{rad} A)(x,-)$.*

(2) *Put $\{\beta^{(t)}_{y,x}\}_{t=1}^{e_{y,x}}$ $(e_{y,x} := \dim A(y,x))$ to be a basis of $A(y,x)$ with $\{\beta^{(t)}_{y,x}\}_{t=1}^{d_{y,x}}$ a basis of $(\operatorname{soc}_A A \cap \operatorname{soc} A_A)(y,x)$ and $\{\beta^{(t)\ *}_{y,x}\}_{t=1}^{e_{y,x}}$ to be the basis of $DA(y,x)$ dual to $\{\beta^{(t)}_{y,x}\}_{t=1}^{e_{y,x}}$. Then $\hat{\Phi}$ is defined by*
- *$\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$;*
- *$\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$; and*
- *$\hat{\Phi}(\alpha^{(t)\ [i]}_{y,x}) := (\beta^{(t)\ *}_{y,x})^{[i]}$*

for all $i \in \mathbb{Z}$, all vertices x, y of Q, all arrows α of Q and all $t = 1, \dots, d_{y,x}$.

Remark. In the above $\hat{Q}$ is uniquely determined by Q and A. But $\hat{\Phi}$ depends on the choice of a basis of $A(x,y)$ for each $x, y \in A$.

1.4. **The Nakayama automorphism of $k\hat{Q}$.** Let $\Phi\colon kQ \longrightarrow A$ be a display-functor for a locally bounded spectroid A and $\hat{\Phi}\colon k\hat{Q} \longrightarrow A^{\mathbb{Z}}$ a display-functor for $A^{\mathbb{Z}}$ induced from Φ as in 1.3. The *Nakayama automorphism* ν_Q of $\hat{Q}$ is defined by

- $\nu_Q(x^{[i]}) := x^{[i+1]}$;
- $\nu_Q(\alpha^{[i]}) := \alpha^{[i+1]}$; and
- $\nu_Q(\alpha^{(t)\ [i]}_{y,x}) := \alpha^{(t)\ [i+1]}_{y,x}$

for all $i \in \mathbb{Z}$, all vertices x, y of Q, all arrows α of Q and all $t = 1, \dots, d_{y,x}$.

The following is immediate from definitions.

Lemma. *For $\hat{\Phi}\colon k\hat{Q} \to A^{\mathbb{Z}}$ above the diagram*

$$\begin{array}{ccc} k\hat{Q} & \xrightarrow{\hat{\Phi}} & A^{\mathbb{Z}} \\ {\scriptstyle k\nu_Q}\downarrow & & \downarrow{\scriptstyle \nu_A} \\ k\hat{Q} & \xrightarrow[\hat{\Phi}]{} & A^{\mathbb{Z}} \end{array}$$

is commutative.

1.5. **Commutative directed quivers.** A quiver with relations (Q, I) is called a *commutative directed quiver* if

(i) Q is a (locally finite) quiver without oriented cycles with the property that for each arrow $\alpha\colon x \to y$ in Q, α is the only path from x to y in Q; and
(ii) The admissible ideal ([7, 8.3 Example 2]) I of kQ contains all the differences of parallel paths (= paths with the same tail and the same head).

In this case for any two vertices x, y of Q, either $I(x, y)$ is spanned by all the differences of paths from x to y or else $I(x, y) = kQ(x, y)$, i.e., I is defined by the full commutativity relations and some zero relations. If Q is an oriented tree, then clearly any quiver with relations (Q, I) is a commutative directed quiver. Note that all quivers with relations in 2.4 are commutative directed quivers, for which we will need to compute repetitions.

A display-functor $\Phi\colon kQ \to A$ for a locally bounded spectroid A is called *commutative directed* if $(Q, \operatorname{Ker}\Phi)$ is a commutative directed quiver.

Assume that A has such a display-functor. Then we have $0 \le d_{y,x} \le e_{y,x} \le 1$ for all vertices x, y of Q, thus $d_{y,x} = e_{y,x} = 1$ if $d_{y,x} \ne 0$ (1.3). Hence if we decide to choose $\beta^{(1)}_{y,x}$ as the image of a path under Φ, then $\hat{\Phi}$ is uniquely determined by Φ. Namely the description of $\hat{\Phi}$ becomes as in the next Lemma, which gives a simple way to compute a quiver presentation of $A^{\mathbb{Z}}$ (Cf. [4, 5.2]). As $\hat{\Phi}$ we always use this in the sequel.

Recall that a path μ from y to x in a quiver with relations (Q, I) is called *complete* if $\mu \notin I$ but $\alpha\mu, \mu\beta \in I$ for each arrow α of Q with tail x and each β with head y.

Lemma. *Let $\Phi\colon kQ \to A$ be a commutative directed display-functor for a locally bounded spectroid A with $I := \operatorname{Ker}\Phi$. Then*

(1) *Φ uniquely induces the following commutative directed display-functor $\hat{\Phi}\colon k\hat{Q} \to A^{\mathbb{Z}}$ for $A^{\mathbb{Z}}$:*
   (i) *$\hat{Q}$ is obtained from the disjoint union $Q \times \mathbb{Z}$ of copies $Q^{[i]}$ of Q by adding arrows $\alpha^{[i]}_{y,x}\colon x^{[i]} \to y^{[i+1]}$ for all $i \in \mathbb{Z}$, whenever there is a complete paths μ from y to x in (Q, I).*
   (ii) *$\hat{\Phi}$ is defined as follows:*
   - $\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$;
   - $\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$; *and*
   - $\hat{\Phi}(\alpha^{[i]}_{y,x}) := (\Phi(\mu)^*)^{[i]}$

   for all $i \in \mathbb{Z}$, all vertices x of Q, all arrows α of Q and all pairs (x, y) of vertices of Q such that there is a complete path μ from y to x in (Q, I).

(2) $\operatorname{Ker}\hat{\Phi}$ *is equal to the ideal* $\hat{I}$ *defined by the full commutativity relations on* $\hat{Q}$ *and the zero relations* $\mu = 0$ *for those paths* μ *of* $\hat{Q}$ *for which there is no path from the head of* μ *to the* $x^{[i+1]}$ *with* $x^{[i]}$ *the tail of* μ.

Proof. (1) is clear. We give an outline of the proof of (2); the details are left to the reader. Write $\bar{\mu} := \Phi(\mu)$ for short for all μ in kQ. First the following three statements are easily verified:

(a) Let μ be a complete path from a to b and γ a path from x to b in Q. Then $\bar{\mu}^*\bar{\gamma} \neq 0$ implies that there is a path λ from a to x with $\bar{\mu} = \bar{\gamma}\bar{\lambda}$ and $\bar{\mu}^*\bar{\gamma} = \bar{\lambda}^*$;

(b) Dually, let μ be a complete path from a to b and ε a path from a to y in Q. Then $\bar{\varepsilon}\bar{\mu}^* \neq 0$ implies that there is a path κ from y to b with $\bar{\mu} = \bar{\kappa}\bar{\varepsilon}$ and $\bar{\varepsilon}\bar{\mu}^* = \bar{\kappa}^*$; and

(c) For any vertices x, y of Q, $A(x, y) \neq 0$ iff there exist both a path from x to y in Q and a path from $y^{[i]}$ to $x^{[i+1]}$ in $Q^{\mathbb{Z}}$ for some $i \in \mathbb{Z}$.

Now using the statements (a) and (b) we see that $\operatorname{Ker}\hat{\Phi} \supseteq \hat{I}$. Then the statement (c) shows that $\dim(kQ^{\mathbb{Z}}/I^{\mathbb{Z}})(x^{[i]}, y^{[j]}) = \dim A^{\mathbb{Z}}(x^{[i]}, y^{[j]})$ for all vertices $x^{[i]}, y^{[j]}$ of $Q^{\mathbb{Z}}$. As a consequence we have $\operatorname{Ker}\hat{\Phi} = \hat{I}$. □

2. Main results

2.1. **Types.** Let Δ be a Dynkin graph. Then following Bretscher, Läser and Riedtmann [4] we denote by m_Δ the "Loewy length" of the mesh category $k(\mathbb{Z}\Delta)$, i.e., the smallest integer m such that all paths in $\mathbb{Z}\Delta$ of length at least m is zero modulo the mesh ideal. The value of m_Δ is known as $m_{A_n} = n, m_{D_n} = 2n - 3, m_{E_6} = 11, m_{E_7} = 17, m_{E_8} = 29$.

Let Λ be any representation-finite selfinjective algebra ($\neq k$). Then by Riedtmann [17], the stable Auslander-Reiten quiver ${}_s\Gamma_\Lambda$ of Λ has the form $\mathbb{Z}\Delta/\langle\phi\tau^{-r}\rangle$ with $r \geq 1$, where Δ is a Dynkin graph A_n ($n \geq 1$), D_n ($n \geq 4$) or E_n ($n = 6, 7, 8$), τ is the translation of $\mathbb{Z}\Delta$ and ϕ is an automorphism of $\mathbb{Z}\Delta$ with a fixed vertex (Cf. 5.2). We set $\Delta(\Lambda) := \Delta, f(\Lambda) := r/m_\Delta$ and $t(\Lambda) :=$ the order of ϕ, and call them the *tree class*, the *frequency* and the *torsion order* of Λ, respectively. The triple $\operatorname{typ}(\Lambda) := (\Delta(\Lambda), f(\Lambda), t(\Lambda))$ is called the *type* of Λ. Note that the type of Λ is uniquely determined by ${}_s\Gamma_\Lambda$. We set $\operatorname{typ}(\mathrm{RFSS}) := \{\operatorname{typ}(\Lambda) | \Lambda \in \mathrm{RFSS}\}$.

Remark. (1) Denote by $n(\Lambda)$ the number of vertices of $\Delta(\Lambda)$ for all $\Lambda \in \mathrm{RFSS}$. Then by [4, 2.3] we have $n(\Lambda)f(\Lambda) = \operatorname{rank} K_0(\Lambda)$.

(2) If Λ is a Brauer tree algebra given by a Brauer tree with e edges and with multiplicity m, then $\operatorname{typ}(\Lambda) = (A_{me}, 1/m, 1)$.

(3) By Riedtmann [19], if the algebra Λ is non-standard, then $\operatorname{char} k = 2$ and $\operatorname{typ}(\Lambda) = (D_{3m}, 1/3, 1)$ for some $m \geq 2$. Conversely if $\operatorname{char} k = 2$ and $m \geq 2$, then there exists a non-standard algebra Λ with $\operatorname{typ}(\Lambda) = (D_{3m}, 1/3, 1)$.

2.2. **Main results.** We now state our main results. The proof of the following theorem is given in section 5.

Theorem. (1) *Let* $\Lambda, \Lambda' \in \mathrm{RFSS}$. *Then the following are equivalent.*

(a) Λ *and* Λ' *are derived equivalent;*

(b) Λ *and* Λ' *are stably equivalent;*

(c) ${}_{\mathrm{s}}\Gamma_\Lambda \cong {}_{\mathrm{s}}\Gamma_{\Lambda'}$; *and*
(d) $\mathrm{typ}(\Lambda) = \mathrm{typ}(\Lambda')$.

(2) *The set* typ(RFSS) *is equal to the disjoint union* $\mathcal{L}$ *of the following sets:*
$\{(A_n, s/n, 1) | n, s \in \mathbb{N}\}$;
$\{(A_{2p+1}, s, 2) | p, s \in \mathbb{N}\}$;
$\{(D_n, s, 1) | n, s \in \mathbb{N}, n \geq 4\}$;
$\{(D_{3m}, s/3, 1) | m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$;
$\{(D_n, s, 2) | n, s \in \mathbb{N}, n \geq 4\}$;
$\{(D_4, s, 3) | s \in \mathbb{N}\}$;
$\{(E_n, s, 1) | n = 6, 7, 8, s \in \mathbb{N}\}$; *and*
$\{(E_6, s, 2) | s \in \mathbb{N}\}$.

By (1) above we may say that $\mathrm{typ}(\Lambda)$ is a "complete invariant" for derived equivalence in RFSS.

We give the proof of the next statement in section 6.

Proposition. *Let Λ and Λ' be representation-finite selfinjective algebras. Assume that Λ is standard but Λ' is non-standard. Then Λ is not stably equivalent to Λ'.* (*Hence they are not derived equivalent.*)

This immediately implies the following.

Corollary 1. *Let Λ and Λ' be selfinjective algebras. If $\Lambda \in$ RFSS and Λ is stably equivalent to Λ', then $\Lambda' \in$ RFSS.*

Combining the theorem and the corollary above we get the following statement (1); and the statement (2) follows from (1) by Rickard [16, Corollary 5.5].

Corollary 2. *Let Λ and Λ' be selfinjective algebras and assume that $\Lambda \in$ RFSS. Then*
(1) *Λ and Λ' are derived equivalent iff they are stably equivalent; and*
(2) *If Λ and Λ' are stably equivalent, then there exists a stable equivalence of Morita type between Λ and Λ'.*

Next we fix a representative algebra for each type in $\mathcal{L}$. To do this we need the following two subsections.

2.3. The repetition of an automorphism.

Lemma. *Let $\psi\colon A \to B$ be an isomorphism of locally bounded spectroids. Denote by $\psi_x^y\colon A(y,x) \to B(y,x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\psi^{\mathbb{Z}}\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ as follows.*

For each $x^{[i]} \in A^{\mathbb{Z}}, \psi^{\mathbb{Z}}(x^{[i]}) := (\psi x)^{[i]}$;
For each $f^{[i]} \in A^{\mathbb{Z}}(x^{[i]}, y^{[i]}), \psi^{\mathbb{Z}}(f^{[i]}) := (\psi f)^{[i]}$; *and*
For each $\varphi^{[i]} \in A^{\mathbb{Z}}(x^{[i]}, y^{[i+1]}), \psi^{\mathbb{Z}}(\varphi^{[i]}) := (D((\psi_x^y)^{-1})(\varphi))^{[i]} = (\varphi \circ (\psi_x^y)^{-1})^{[i]}$.

Then
(1) *$\psi^{\mathbb{Z}}$ is an isomorphism.*
(2) *Given an isomorphism $\rho\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$, the following are equivalent.*
(a) $\rho = \psi^{\mathbb{Z}}$;

(b) *ρ satisfies the following.*
(i) $\rho\nu_A = \nu_B\rho$;
(ii) $\rho(A^{[0]}) = A^{[0]}$;
(iii) *The diagram*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "1^{[0]}"'] & B \arrow[d, "1^{[0]}"] \\ A^{[0]} \arrow[r, "\rho"'] & B^{[0]} \end{tikzcd}$$

is commutative; and
(iv) $\rho(\varphi^{[0]}) = (\varphi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\varphi \in DA(y,x)$.

Proof. (1) This is shown by the same way as in the proof of [1, Theorem 1.5].
(2) Straightforward. □

2.4. **Some quivers with relations and automorphisms.**

2.4.1. *Class A_n.* Let $\vec{A}_n$ $(n \geq 1)$ be the quiver:

$$1 \to 2 \to \cdots \to n.$$

Define an automorphism γ_n of $B^{\mathbb{Z}}$ for $B := k\vec{A}_n$ by the quiver automorphism determined by $\gamma_n(x^{[i]}) := (x+1)^{[i]}$ for all $1 \leq x \leq n-1$ and $\gamma_n(n^{[i]}) := 1^{[i+1]}$.

Let $\vec{A}_p^p$ $(p \geq 1)$ be the quiver:

$$(-p) \leftarrow \cdots \leftarrow (-1) \leftarrow 0 \to 1 \to \cdots \to p.$$

Define an automorphism $\phi_{A_{2p+1}}$ of $\vec{A}_p^p$ by $x \mapsto -x$; also define an automorphism ϕ_B of $B := k\vec{A}_p^p$ by $\phi_{A_{2p+1}}$.

2.4.2. *Class D_n.* Let $\vec{D}_n$ $(n \geq 4)$ be the quiver:

$$\begin{tikzcd} & & & n \\ 1 & 2 \arrow[l] & \cdots \arrow[l] & n-2 \arrow[l] \arrow[u] \arrow[r] & n-1. \end{tikzcd}$$

Define an automorphism ϕ_{D_n} of $\vec{D}_n$ by the permutation $(n-1, n)$, and an automorphism ϕ_B of $B := k\vec{D}_n$ by ϕ_{D_n}. For the case $n = 4$, define further an automorphism ω_{D_4} of $\vec{D}_4$ by the permutation $(1,3,4)$, and an automorphism ω of $B := k\vec{D}_4$ by ω_{D_4}.

Let (K_m, J_m) be the following quiver

$$\begin{tikzcd} 2(0) \arrow[d] & 1(0) \arrow[l] \arrow[d] \\ \vdots \arrow[d] & 1(1) \arrow[r] \arrow[d] & 2(1) \arrow[r] & \cdots \arrow[r] & m(1) \\ m(0) \arrow[r] & 1(2) \arrow[r] & 2(2) \arrow[r] & \cdots \arrow[r] & m(2) \end{tikzcd}$$

with the commutativity relation and the zero relation $m(0) \to 1(2) \to 2(2) = 0$. Define an automorphism μ_m of $B^{\mathbb{Z}}$ for $B := k(K_m, J_m)$ by the quiver automorphism determined by $\mu_m(x(a)^{[i]}) := x(a+1)^{[i]}$ for $a = 0$ and 1, and $\mu_m(x(2)^{[i]}) := x(0)^{[i+1]}$.

2.4.3. *Class E_n.* Let $\vec{E}_n$ $(n = 6, 7, 8)$ be the quiver

$$\begin{array}{ccccc} & & n & & \\ & & \uparrow & & \\ 1 \leftarrow 2 \leftarrow & & 3 & \to \cdots \to n-1. \end{array}$$

Define an automorphism ϕ_{E_6} of $\vec{E}_6$ by the permutation (1,5)(2,4), and an automorphism ϕ_B of $B := k\vec{E}_6$ by ϕ_{E_6}.

2.5. **Representatives.** For each $(\Delta, f, t) \in \mathcal{L}$, we fix an algebra $\Lambda(\Delta, f, t)$ as follows.
$\Lambda(A_n, s/n, 1) := B^{\mathbb{Z}}/\langle\gamma_n^s\rangle$, where $B = k\vec{A}_n$;
$\Lambda(A_{2p+1}, s, 2) := B^{\mathbb{Z}}/\langle\phi_B^{\mathbb{Z}}\nu_B^s\rangle$, where $B = k\vec{A}_p^p$;
$\Lambda(D_n, s, 1) := B^{\mathbb{Z}}/\langle\nu_B^s\rangle$, where $B = k\vec{D}_n$;
$\Lambda(D_{3m}, s/3, 1) := B^{\mathbb{Z}}/\langle\mu_m^s\rangle$, where $B = k(K_m, J_m)$ and $3 \nmid s$;
$\Lambda(D_n, s, 2) := B^{\mathbb{Z}}/\langle\phi_B^{\mathbb{Z}}\nu_B^s\rangle$, where $B = k\vec{D}_n$;
$\Lambda(D_4, s, 3) := B^{\mathbb{Z}}/\langle\omega^{\mathbb{Z}}\nu_B^s\rangle$, where $B = k\vec{D}_4$;
$\Lambda(E_n, s, 1) := B^{\mathbb{Z}}/\langle\nu_B^s\rangle$, where $B = k\vec{E}_n$; and
$\Lambda(E_6, s, 2) := B^{\mathbb{Z}}/\langle\phi_B^{\mathbb{Z}}\nu_B^s\rangle$, where $B = k\vec{E}_6$.

Remark. The algebra $\Lambda(A_n, s/n, 1)$ is the selfinjective Nakayama algebra $N_{s,n}$ with exactly s isoclasses of simple modules and of Loewy length $n+1$; and the algebra $\Lambda(A_{2p+1}, s, 2)$ is the "canonical Möbius" algebra $M_{p,s}$ defined in [1, Definition 5.5] (or [18, Fig. 36], [21, Figure 2]).

We will show in section 5 that $\Lambda(\mathrm{typ}(\Lambda))$ and Λ are derived equivalent, which is an essential part of the proof of the main theorem. It follows from this that the algebra $\Lambda(\Delta, f, t)$ actually has type (Δ, f, t) for each $(\Delta, f, t) \in \mathcal{L}$.

3. Computations in a formal additive hull

Let A be a spectroid and G a group of automorphisms of A. Then $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, A)$ has a natural G-action as defined in [1]. Under this action, however, it is not easy to give a G-stable tilting spectroid for A explicitly because the set $\mathcal{X}_A$ of representable functors of A is not closed under this G-action in a strict sense. To make the computation easy we introduce a category $\oplus A$ with a G-action, which is equivalent to $\mathrm{pro}\, A$ with the G-action. In this paper we will use $\oplus A$ instead of $\mathrm{pro}\, A$, and all computations will be done in this setting.

3.1. **A formal additive hull of a spectroid.** Let A be a spectroid. Then by $\oplus A$ we denote the following category ([7, 2.1 Example 8]). Objects are finite sequences $(x_1, \dots, x_n)$ of objects of A; $(\oplus A)((x_1, \dots, x_n), (y_1, \dots, y_m)) := \{(\mu_{ji})_{i,j} | \mu_{ji} \in A(x_i, y_j), \text{for all } 1 \le i \le n, 1 \le j \le m\}$ for all objects $(x_1, \dots, x_n), (y_1, \dots, y_m)$; and the composition is given by the matrix multiplication. We regard that A is contained in $\oplus A$ by the embedding $(f\colon x \to y) \mapsto ((f)\colon (x) \to (y))$ for all f in A.

Remark. Define a functor $\eta_A\colon \oplus A \to \operatorname{pro} A$ by $(x_1,\dots,x_n) \mapsto A(-,x_1)\oplus\cdots\oplus A(-,x_n)$ and $(\mu_{ji})_{i,j} \mapsto (A(-,\mu_{ji}))_{i,j}$. Then η_A is an equivalence.

Lemma. *Let $F\colon A \to B$ be a functor of spectroids. Then F naturally induces functors $\oplus F\colon \oplus A \to \oplus B$ and $\tilde{F} := \mathcal{H}^{\mathrm{b}}(\oplus F)\colon \mathcal{H}^{\mathrm{b}}(\oplus A) \to \mathcal{H}^{\mathrm{b}}(\oplus B)$, which are equivalences if F is an isomorphism.*

Proof. $\oplus F$ is defined by $(x_1,\dots,x_n) \mapsto (Fx_1,\dots,Fx_n)$ and $(\mu_{ji}) \mapsto (F\mu_{ji})$ for all objects $(x_1,\dots,x_n)$ and for all morphisms (μ_{ji}) in $\oplus A$. $\square$

3.2. **The space $\mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$.** From the proof of [1, Theorem 1.5] we see that for any $T, U \in \mathcal{H}^{\mathrm{b}}(\oplus A)$ and any $i \in \mathbb{Z}$, $\mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$ is isomorphic to $D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T)$. In the next section we need an explicit form of this isomorphism.

Lemma. *Let A be a spectroid and let $T, U \in \mathcal{H}^{\mathrm{b}}(\oplus A), i \in \mathbb{Z}$. Then we have an isomorphism*

$$\Theta^{[i]}\colon \mathcal{H}^{\mathrm{b}}(\oplus A)(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) \to D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T)$$

of vector spaces defined by $x \mapsto \overline{\theta}^{[i]}(x,-)$, where

$$\overline{\theta}^{[i]}\colon \mathcal{H}^{\mathrm{b}}(\oplus A)(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) \times \mathcal{H}^{\mathrm{b}}(\oplus A)(U,T) \to k$$

is a non-degenerate bilinear form defined by $\overline{\theta}^{[i]}(\overline{\eta^{[i]}}, \overline{f}) := \sum_{p\in\mathbb{Z}}(-1)^p\eta^p(f^p)$ for all $\eta = (\eta^p)_{p\in\mathbb{Z}}, f = (f^p)_{p\in\mathbb{Z}}$ with $\eta^p \in D(\oplus A)(U^p, T^p), f^p \in (\oplus A)(U^p, T^p)$.

Proof. Put $T = (T^p, d_T^p)$ and $U = (U^p, d_U^p)$. Denote by $\mathcal{H}om_A(U,T)$ the Hom-complex, i.e., $\mathcal{H}om_A(U,T) := (\mathcal{H}om_A^p(U,T), d^p)$, where $\mathcal{H}om_A^p(U,T) := \prod_{-i+j=p}(\oplus A)(U^i, T^j)$ (note that this is a finite product) and $d^p : \mathcal{H}om_A^p(U,T) \to \mathcal{H}om_A^{p+1}(U,T)$ is defined by $(f^i : U^i \to T^j) \mapsto d_T^j \circ f^i - (-1)^p f^{i+1} \circ d_U^i$. Denote also by $\mathcal{H}om_{A^{\mathbb{Z}}}(X,Y) := (\mathcal{H}om_{A^{\mathbb{Z}}}^p(X,Y), \delta^p)$ the Hom-complex of each $X, Y \in \mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})$. Then $H^0\mathcal{H}om_A(U,T) = \mathcal{H}^{\mathrm{b}}(\oplus A)(U,T)$ and $H^0\mathcal{H}om_{A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) = \mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$. For each $s \in \mathbb{Z}$ define a map

$$\theta^s\colon \mathcal{H}om_{A^{\mathbb{Z}}}^s(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) \times \mathcal{H}om_A^{-s}(U,T) \to k$$

by $\theta^s((\eta^{p\ [i]})_p, (f^p)_p) := \sum_{p\in\mathbb{Z}}(-1)^{p+[s/2]}\eta^p(f^{p+s})$ for all $(\eta^{p\ [i]})_p \in \mathcal{H}om_{A^{\mathbb{Z}}}^s(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$ and all $(f^p)_p \in \mathcal{H}om_A^{-s}(U,T)$, where $\eta^p \in D(\oplus A)(U^{p+s}, T^p)$ and $f^p \in (\oplus A)(U^p, T^{p-s})$ (so $f^{p+s} \in (\oplus A)(U^{p+s}, T^p)$), and the notation $[q]$ for each real number q is defined as usual by the conditions that $[q] \in \mathbb{Z}$ and that $[q] \le q < [q]+1$. By looking at the shape of θ^s, it is clear that θ^s are non-degenerate bilinear forms for all $s \in \mathbb{Z}$. Hence for each $s \in \mathbb{Z}$ the θ^s induces an isomorphism

$$\Theta^s\colon \mathcal{H}om_{A^{\mathbb{Z}}}^s(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) \to D\mathcal{H}om_A^{-s}(U,T)$$

by defining $\Theta^s(\eta) := \theta^s(\eta, -)$. Noting that $[s/2] \equiv [(s+1)/2] + s \pmod 2$ for all $s \in \mathbb{Z}$, it is not hard to see that the diagram

$$\begin{array}{ccc}
\mathcal{H}om^s_{A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) & \xrightarrow{\delta^s} & \mathcal{H}om^{s+1}_{A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) \\
{\scriptstyle\Theta^s}\downarrow & & \downarrow{\scriptstyle\Theta^{s+1}} \\
D\mathcal{H}om^{-s}_A(U,T) & \xrightarrow[D(d^{-s-1})]{} & D\mathcal{H}om^{-s-1}_A(U,T)
\end{array}$$

is commutative. Therefore Θ^0 induces an isomorphism $\bar{\Theta}^0$ in the commutative diagram

$$\begin{array}{ccc}
H^0\mathcal{H}om^s_{A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) & \xrightarrow{\bar{\Theta}^0} & H_0 D\mathcal{H}om_A(U,T) \\
\Vert & & \downarrow{\scriptstyle\text{can.}} \\
\mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) & \xrightarrow[\Theta^{[i]}]{} & DH^0\mathcal{H}om_A(U,T).
\end{array}$$

As a consequence $\Theta^{[i]}$ is an isomorphism for all $i \in \mathbb{Z}$. $\square$

For each $T, U \in \mathcal{H}^{\mathrm{b}}(\oplus A)$ and each $i \in \mathbb{Z}$ we define a linear map

$$\Xi^{[i]} \colon D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T) \to \mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$$

as the inverse of $\Theta^{[i]}$. Then every element of $\mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U))$ has the form $\Xi^{[i]}(\varphi)$ for some unique $\varphi \in D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T)$.

Corollary. *Let ϕ_A be an automorphism of a spectroid A. Then we have $\tilde{\phi_A^{\mathbb{Z}}}(\Xi^{[i]}(\varphi)) = \Xi^{[i]}((D(\tilde{\phi_A}^{-1}))(\varphi))$ for all $U, T \in \mathcal{H}^{\mathrm{b}}(\oplus A)$, all $\varphi \in D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T)$ and all $i \in \mathbb{Z}$.*

Proof. It is not hard to check that the diagram

$$\begin{array}{ccc}
\mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\mathbb{1}}^{[i]}(T), \tilde{\mathbb{1}}^{[i+1]}(U)) & \xrightarrow{\tilde{\phi_A^{\mathbb{Z}}}} & \mathcal{H}^{\mathrm{b}}(\oplus A^{\mathbb{Z}})(\tilde{\phi_A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i]}(T)), \tilde{\phi_A^{\mathbb{Z}}}(\tilde{\mathbb{1}}^{[i+1]}(U))) \\
{\scriptstyle\Theta^{[i]}}\downarrow & & \downarrow{\scriptstyle\Theta^{[i]}} \\
D\mathcal{H}^{\mathrm{b}}(\oplus A)(U,T) & \xrightarrow[D(\tilde{\phi_A}^{-1})]{} & D\mathcal{H}^{\mathrm{b}}(\oplus A)(\tilde{\phi_A}U, \tilde{\phi_A}T)
\end{array}$$

is commutative, from which the assertion follows. $\square$

4. A covering technique

4.1. **Tilting spectroids.** Our setting needs a small change of the definition of tilting spectroid. The following statement follows from [1, Proposition 1.1] (Cf. Keller [11], Rickard [14]).

Proposition. *Let A and B be spectroids. Then the following are equivalent:*

(1) *There is a triangle equivalence $\mathcal{D}(\operatorname{Mod} B) \to \mathcal{D}(\operatorname{Mod} A)$; and*
(2) *There is a full subcategory E of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ such that*
    (a) *For any $T, U \in E$ and any $n \neq 0$, $\mathcal{H}^{\mathrm{b}}(\oplus A)(T, U[n]) = 0$;*
    (b) *$\operatorname{tri}(E)$ contains A; and*

(c) *E is isomorphic to B.*

Definition. We say that spectroids A and B are *derived equivalent* if one of the equivalent conditions above holds. In (2) the triple (A, E, B) is called a *tilting triple* and $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus A)$ is called a *tilting spectroid* for A.

4.2. **Derived equivalence of repetitions.** [1, Theorem 1.5] is interpreted as follows.

Theorem. *If (A, E, B) is a tilting triple of spectroids, then $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is also a tilting triple, where $\hat{E}$ is the full subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ consisting of the $\tilde{\mathbb{1}}^{[n]}(T)$ with $T \in E$, $n \in \mathbb{Z}$.*

Remark. In the theorem above we can form $E^{\mathbb{Z}}$ because E is a spectroid. Then we have an isomorphism $\xi_E \colon \hat{E} \to E^{\mathbb{Z}}$ defined by

- $\xi_E(\tilde{\mathbb{1}}^{[n]}(T)) := T^{[n]}$;
- $\xi_E(\tilde{\mathbb{1}}^{[n]}(f)) := f^{[n]}$; and
- $\xi_E(\Xi^{[n]}(\varphi)) := \varphi^{[n]}$

for all $n \in \mathbb{Z}$, all $T \in E$, all $f \in E(T, U)$ and all $\varphi \in DE(U, T)$ with $T, U \in E$. If we identify $\hat{E}$ and $E^{\mathbb{Z}}$ through ξ_E, then $\hat{\psi} = \psi^{\mathbb{Z}}$ (Cf. (c) in the proof of [1, Theorem 1.5]).

Now let $\psi \colon E \to B$ be an isomorphism. Then we have an isomorphism $\hat{\psi} := \psi^{\mathbb{Z}} \circ \xi_E \colon \hat{E} \to B^{\mathbb{Z}}$. By the definition of the repetition of an isomorphism in 2.3 and the definition of the Nakayama automorphism it is easy to see that $\tilde{\nu}_A T \in \hat{E}$ for all $T \in E$ and that the diagram

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\hat{\psi}} & B^{\mathbb{Z}} \\ {\scriptstyle \tilde{\nu}_A}\big\downarrow & & \big\downarrow{\scriptstyle \nu_B} \\ \hat{E} & \xrightarrow[\hat{\psi}]{} & B^{\mathbb{Z}} \end{array}$$

is commutative.

4.3. **Pushdown of a tilting spectroid.** Throughout this subsection R is a locally bounded spectroid, G a group of automorphisms of R and $F \colon R \to R/G$ the canonical Galois covering functor. In this case $\mathcal{H}^{\mathrm{b}}(\oplus R)$ is an aggregate.

The functor F induces natural isomorphisms

$$F_{\bullet} \circ \eta_R \to \eta_{R/G} \circ \oplus F \quad \text{and} \quad F_{\bullet} \circ \mathcal{H}^{\mathrm{b}}(\eta_R) \to \mathcal{H}^{\mathrm{b}}(\eta_{R/G}) \circ \tilde{F}.$$

Up to these the following diagrams are commutative:

$$\begin{array}{ccc} \oplus R & \xrightarrow{\eta_R} & \operatorname{pro} R \\ {\scriptstyle \oplus F}\big\downarrow & & \big\downarrow{\scriptstyle F_{\bullet}} \\ \oplus R/G & \xrightarrow[\eta_{R/G}]{} & \operatorname{pro} R/G, \end{array} \qquad \begin{array}{ccc} \mathcal{H}^{\mathrm{b}}(\oplus R) & \xrightarrow{\mathcal{H}^{\mathrm{b}}(\eta_R)} & \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R) \\ {\scriptstyle \tilde{F}}\big\downarrow & & \big\downarrow{\scriptstyle F_{\bullet}} \\ \mathcal{H}^{\mathrm{b}}(\oplus R/G) & \xrightarrow[\mathcal{H}^{\mathrm{b}}(\eta_{R/G})]{} & \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R/G). \end{array}$$

The set $\oplus G := \{\oplus g | g \in G\}$ is a group of automorphisms of $\oplus R$, and $\mathcal{H}^{\mathrm{b}}(\oplus G) := \{\tilde{g} | g \in G\}$ is a group of automorphisms of $\mathcal{H}^{\mathrm{b}}(\oplus R)$. Further we have group isomorphisms $G \to \oplus G, g \mapsto \oplus g$ and $G \to \mathcal{H}^{\mathrm{b}}(\oplus G), g \mapsto \tilde{g}$. Using these isomorphisms we consider that G acts on $\oplus R$ and on $\mathcal{H}^{\mathrm{b}}(\oplus R)$. Each $g \in G$ induces natural isomorphisms

$$ {}^{g}(-) \circ \eta_R \to \eta_R \circ (\oplus g) \quad \text{and} \quad \hat{g} : {}^{g}(-) \circ \mathcal{H}^{\mathrm{b}}(\eta_R) \to \mathcal{H}^{\mathrm{b}}(\eta_R) \circ \tilde{g}. $$

Up to these the following diagrams are commutative:

$$
\begin{array}{ccccccc}
\oplus R & \xrightarrow{\eta_R} & \operatorname{pro} R & & \mathcal{H}^{\mathrm{b}}(\oplus R) & \xrightarrow{\mathcal{H}^{\mathrm{b}}(\eta_R)} & \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R) \\
{\scriptstyle \oplus g}\downarrow & & \downarrow{\scriptstyle {}^{g}(-)} & & {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle {}^{g}(-)} \\
\oplus R & \xrightarrow[\eta_R]{} & \operatorname{pro} R, & & \mathcal{H}^{\mathrm{b}}(\oplus R) & \xrightarrow[\mathcal{H}^{\mathrm{b}}(\eta_R)]{} & \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R).
\end{array}
$$

We call a full subcategory E of $\mathcal{H}^{\mathrm{b}}(\oplus R)$ *G-stable* if $\tilde{g}T \in E$ for all $g \in G$ and all $T \in E$. Recall that a full subcategory E' of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$ is called *G-stable up to isomorphisms* if ${}^{g}T$ is isomorphic to some object of E' for all $g \in G$ and all $T \in E'$. Here we give the interpretation of [1, Theorem 3.5]:

Theorem. *Let R be a locally bounded spectroid, G a group of automorphisms of R and $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus R)$ a tilting spectroid for R. If G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\oplus R)$ and E is G-stable, then R/G is derived equivalent to E/G.*

To prove this we use the following three lemmas.

Lemma 1. *The group G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\oplus R)$ iff G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$.*

Proof. This is clear from the diagram above. □

Corollary. *If G acts freely on R and G is torsion-free, then G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\oplus R)$.*

Proof. This follows from the above lemma and [1, Lemma 2.11]. □

We write $\eta' := \mathcal{H}^{\mathrm{b}}(\eta_R) \colon \mathcal{H}^{\mathrm{b}}(\oplus R) \to \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$ for short. Clearly η' is a triangle equivalence.

Lemma 2. *Let $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus R)$ be a G-stable tilting spectroid for R. Then the full subcategory E' of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$ consisting of $\eta'(X)$ with $X \in E$ is a tilting spectroid for R which is G-stable up to isomorphisms.*

Proof. Since η' is a triangle equivalence and $\eta'(x) = R(-, x)$ for all $x \in R$, we have that E' is a tilting spectroid for R. Since ${}^{g}\eta'(X) \cong \eta'(\tilde{g}X)$ for all $g \in G$ and all $X \in E$, E' is G-stable up to isomorphisms. □

Lemma 3. *Let $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus R)$ be a G-stable tilting spectroid for R. Then E/G is isomorphic to E'/G.*

Proof. η' restricts to an isomorphism $\eta' \colon E \to E'$. Put N to be a complete set of representatives of G-orbits of objects of E. Then $\operatorname{obj} E' = \{\eta'(\tilde{g}X) | X \in N, g \in G\}$. Put $\bar{E}$ to be the full subcategory of $\mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$ defined by the object set $\{{}^{g}\eta'(X) | X \in N, g \in G\}$. Then $\bar{E}$ is G-stable by construction and is a tilting spectroid for R because E' is. Recall that a G-action on E', denoted by $g * (-)$ for all $g \in G$, was induced from the G-action on $\bar{E}$ by identifying E' and $\bar{E}$ through some sequence $(v_T)_{T \in E'}$

of isomorphisms $v_T : T \to v(T)$ in $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, R)$ with $v(T) \in \bar{E}$, where v_T is chosen as identity of T if $T \in \bar{E}$. Namely the sequence $(v_T)_{T \in E'}$ gives an isomorphism $v : E' \to E$ of spectroids defined by $T \mapsto v(T)$ and $v(f) := v_U f v_T^{-1}$ for all $f : T \to U$ in E'. Then for each $g \in G$, the new action is defined by $v(g * T) = {}^g v(T)$ and $v(g * f) = {}^g v(f)$ for all $f : T \to U$ in E. Here we choose $(v_T)_{T \in E'}$ as follows. Let $T \in E'$. Then $T = \eta'(\tilde{g}X)$ for a unique $g \in G$ and a unique $X \in N$. Set $v(T) := {}^g\eta'(X)$ and $v_T := \hat{g}_X^{-1} : \eta'(\tilde{g}X) \xrightarrow{\sim} {}^g\eta'(X)$. Note that $\eta'(\tilde{g}X) = {}^g\eta'(X)$ iff g is the identity element of G. This choice makes the diagram

$$(*) \qquad \begin{array}{ccc} E & \xrightarrow{\eta'} & E' \\ {\scriptstyle \tilde{t}}\downarrow & & \downarrow{\scriptstyle t*(-)} \\ E & \xrightarrow[\eta']{} & E' \end{array}$$

commutative for all $t \in G$, from which we have $E/G \cong E'/G$. In fact, it is enough to show that

(a) $t * \eta'(\tilde{g}X) = \eta'(\tilde{t}\tilde{g}X)$; and
(b) $t * \eta'(f) = \eta'(\tilde{t}f)$

for all $X, Y \in N$, all $g, h, t \in G$ and all $f : \tilde{g}X \to \tilde{h}Y$ in E.

(a). This is clear because both hand sides are in the spectroid E' and both are isomorphic to ${}^{tg}\eta'(X)$.

(b). By definition we have a commutative diagram

$$\begin{array}{ccc} t * \eta'(\tilde{g}X) & \xrightarrow{t*\eta'(f)} & t * \eta'(\tilde{h}Y) \\ \| & & \| \\ \eta'(\tilde{t}\tilde{g}X) & \xrightarrow{t*\eta'(f)} & \eta'(\tilde{t}\tilde{h}Y) \\ {\scriptstyle (\hat{tg})_X}\uparrow & & \uparrow{\scriptstyle (\hat{th})_Y} \\ {}^{tg}\eta'(X) & \xrightarrow{{}^t v(\eta'(f))} & {}^{th}\eta'(Y) \\ {\scriptstyle {}^t\hat{g}_X}\downarrow & & \downarrow{\scriptstyle {}^t\hat{g}_Y} \\ {}^t\eta'(\tilde{g}X) & \xrightarrow[{}^t\eta'(f)]{} & {}^t\eta'(\tilde{h}Y) \\ {\scriptstyle \hat{t}_{\tilde{g}X}}\downarrow & & \downarrow{\scriptstyle \hat{t}_{\tilde{h}Y}} \\ \eta'(\tilde{t}\tilde{g}X) & \xrightarrow[\eta'(\tilde{t}f)]{} & \eta'(\tilde{t}\tilde{h}Y). \end{array}$$

By the naturality of $\hat{g}$ we have $\hat{t}_{\tilde{g}X} \circ {}^t\hat{g}_X = (\widehat{tg})_X$ for all $X \in \mathcal{H}^{\mathrm{b}}(\oplus R)$, from which (b) follows. $\square$

Proof of Theorem 4.3. By Lemma 1, G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, R)$. By Lemma 2, E' is a tilting spectroid for R which is G-stable up to isomorphisms. By [1, Theorem 3.5], R/G is derived equivalent to E'/G. By Lemma 3, $E'/G \cong E/G$. Hence R/G is derived equivalent to E/G. $\square$

Remark. (1) This theorem can be proved without using the equivalence $\eta'\colon \mathcal{H}^{\mathrm{b}}(\oplus R) \to \mathcal{H}^{\mathrm{b}}(\operatorname{pro} R)$. First we can show that $\tilde{F}\colon \mathcal{H}^{\mathrm{b}}(\oplus R) \to \mathcal{H}^{\mathrm{b}}(\oplus R/G)$ restricts to a Galois covering $\tilde{F}\colon \operatorname{ind}\mathcal{H}^{\mathrm{b}}(\oplus R) \to \operatorname{ind}^1\mathcal{H}^{\mathrm{b}}(\oplus R/G)$ as in [1, Theorem 2.9], where $\operatorname{ind}\mathcal{H}^{\mathrm{b}}(\oplus R)$ is a G-invariant spectroid of $\mathcal{H}^{\mathrm{b}}(\oplus R)$ and $\operatorname{ind}^1\mathcal{H}^{\mathrm{b}}(\oplus R/G)$ is the full subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus R/G)$ consisting of $\tilde{F}(T)$ with $T \in \operatorname{ind}\mathcal{H}^{\mathrm{b}}(\oplus R)$. Second if $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus R)$ is a G-stable tilting spectroid for R, then we can show that its "pushdown" $\tilde{E}$ (= the full subcategory of $\operatorname{ind}^1\mathcal{H}^{\mathrm{b}}(\oplus R/G)$ formed by $\tilde{F}(T)$ with $T \in E$) is a tilting spectroid for R/G and that $\tilde{E} \cong E/G$.

(2) The commutativity of the diagram $(*)$ in Lemma 3 above guarantees that writing ${}^gX$ for $g * X$ (as done in [1, sections 5, 6]) makes no confusion if we regard the isomorphism ${}^gR(-,x) := R(g^{-1}(-),x) \to R(-,gx)$ induced by g as the identity for all $g \in G$ and all $x \in R$.

4.4. **Derived equivalence of quotients of repetitions.**

4.4.1. *Cyclic groups of infinite order.* From Theorem 4.2, Corollary 4.3 and Theorem 4.3 we obtain the following:

Proposition. *Let (A,E,B) be a tilting triple of locally bounded spectroids with an isomorphism $\psi\colon E \to B$, g an automorphism of $A^{\mathbb{Z}}$ and h an automorphism of $B^{\mathbb{Z}}$. Then $A^{\mathbb{Z}}/\langle g\rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle h\rangle$ if*

1. *g is of infinite order and $\langle g\rangle$ acts freely on $A^{\mathbb{Z}}$;*
2. *$\hat{E}$ is $\langle\tilde{g}\rangle$-stable; and*
3. *The diagram*

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\hat{\psi}} & B^{\mathbb{Z}} \\ {\scriptstyle\tilde{g}}\downarrow & & \downarrow{\scriptstyle h} \\ \hat{E} & \xrightarrow[\hat{\psi}]{} & B^{\mathbb{Z}} \end{array}$$

is commutative.

4.4.2. *A power of the Nakayama automorphism.* [1, Theorem 4.1] follows easily from 4.2 and Proposition 4.4.1:

Proposition ([1, Theorem 4.1])**.** *Let A and B be locally bounded spectroids. If A is derived equivalent to B, then $A^{\mathbb{Z}}/\langle\nu_A^n\rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle\nu_B^n\rangle$ for all $n \in \mathbb{Z}$.*

4.4.3. *A product of a twist and a power of the Nakayama automorphism.*

Proposition. *Let (A,E,B) be a tilting triple of locally bounded spectroids, $\psi\colon E \xrightarrow{\sim} B$ an isomorphism. Further let ϕ_A and ϕ_B be automorphisms of A and B, respectively.*

Assume that E is $\langle\tilde{\phi}_A\rangle$-stable, and that the diagram

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\tilde{\phi}_A"'] & B \arrow[d, "\phi_B"] \\ E \arrow[r, "\psi"'] & B \end{tikzcd}$$

is commutative. Then for any $n \neq 0$, $A^{\mathbb{Z}}/\langle\phi_A^{\mathbb{Z}}\nu_A^n\rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle\phi_B^{\mathbb{Z}}\nu_B^n\rangle$.

Proof. By Proposition 4.4.1, it is enough to verify the following three facts:

(1) $\phi_A^{\mathbb{Z}}\nu_A^n$ is of infinite order and $\langle\phi_A^{\mathbb{Z}}\nu_A^n\rangle$ acts freely on $A^{\mathbb{Z}}$;
(2) $\hat{E}$ is $\langle\widetilde{\phi_A^{\mathbb{Z}}}\tilde{\nu}_A^n\rangle$-stable; and
(3) The diagram

$$\begin{tikzcd} \hat{E} \arrow[r, "\hat{\psi}"] \arrow[d, "\widetilde{\phi_A^{\mathbb{Z}}}\tilde{\nu}_A^n"'] & B^{\mathbb{Z}} \arrow[d, "\phi_B^{\mathbb{Z}}\nu_B^n"] \\ \hat{E} \arrow[r, "\hat{\psi}"'] & B^{\mathbb{Z}} \end{tikzcd}$$

is commutative.

(1). This is clear from $n \neq 0$.

(2). Let $T \in E$ and $i \in \mathbb{Z}$. Then by definitions we have $(\widetilde{\phi_A^{\mathbb{Z}}}\tilde{\nu}_A^n)\tilde{\mathbb{1}}^{[i]}(T) = \widetilde{\phi_A^{\mathbb{Z}}}\tilde{\mathbb{1}}^{[i+n]}(T) = \tilde{\mathbb{1}}^{[i+n]}(\phi_A T) \in \hat{E}$.

(3). Consider the diagram

$$\begin{tikzcd} \hat{E} \arrow[r, "\xi_E"] \arrow[d, "\widetilde{\phi_A^{\mathbb{Z}}}"'] & E^{\mathbb{Z}} \arrow[r, "\psi^{\mathbb{Z}}"] \arrow[d, "(\tilde{\phi}_A|E)^{\mathbb{Z}}"] & B^{\mathbb{Z}} \arrow[d, "\phi_B^{\mathbb{Z}}"] \\ \hat{E} \arrow[r, "\xi_E"'] & E^{\mathbb{Z}} \arrow[r, "\psi^{\mathbb{Z}}"'] & B^{\mathbb{Z}}. \end{tikzcd}$$

Then by the definition in 2.3 and Corollary 3.2 the left square is commutative. Again by 2.3 the commutativity of the right square follows from the commutativity assumption. Hence we have a commutative diagram

$$\begin{tikzcd} \hat{E} \arrow[r, "\hat{\psi}"] \arrow[d, "\widetilde{\phi_A^{\mathbb{Z}}}"'] & B^{\mathbb{Z}} \arrow[d, "\phi_B^{\mathbb{Z}}"] \\ \hat{E} \arrow[r, "\hat{\psi}"'] & B^{\mathbb{Z}}, \end{tikzcd}$$

which together with the diagram in 4.2 proves the assertion. □

5. Proof of Theorem 2.2

5.1. **Proof of Theorem 2.2(2).** (1) typ(RFSS) $\subseteq \mathcal{L}$.

Let $\Lambda \in$ RFSS. Put $\Gamma_\Lambda \cong (\mathbb{Z}\Delta)_{\mathcal{C}}/\langle \phi\tau^{-r}\rangle$ with $r \geq 1$, where $\mathcal{C}$ is a configuration of $\mathbb{Z}\Delta$ of period e and $\phi\tau^{-r}$ is an automorphism of $(\mathbb{Z}\Delta)_{\mathcal{C}}$ such that ${}_s\Gamma_\Lambda \cong \mathbb{Z}\Delta/\langle\phi\tau^{-r}\rangle$ as in 2.1. Note that $e \mid r$.

Case (i): $\Delta = A_n$ (for some $n \geq 1$). In this case Λ is wreath-like ($\Leftrightarrow t(\Lambda) = 1$) or Möbius ($\Leftrightarrow t(\Lambda) = 2$). If Λ is wreath-like, then since $m_\Delta = n$, we have $\mathrm{typ}(\Lambda) = (A_n, r/n, 1) \in \mathcal{L}$. Next if Λ is Möbius, then $n = 2p+1$ for some $p \in \mathbb{N}$, and $e = n$, thus $r = sn$ for some $s \in \mathbb{N}$. Hence $\mathrm{typ}(\Lambda) = (A_{2p+1}, s, 2) \in \mathcal{L}$.

Case (ii):$\Delta = D_n$ (for some $n \geq 4$). In this case $m_\Delta = 2n-3$ and by [4, 1.6] $(2n-3)/e \in \{1,3\}$. If $2n-3$ divides r, say $r = s(2n-3)$, then $\mathrm{typ}(\Lambda) = (D_n, s, t(\Lambda))$ with $t(\Lambda) \in \{1,2,3\}$, and $t(\Lambda) = 3$ implies $n = 4$. Hence $\mathrm{typ}(\Lambda) \in \mathcal{L}$. If $2n-3$ does not divide r, then $(2n-3)/e = 3$ and so $n = 3m,\ e = 2m-1$ for some $m \in \mathbb{N}$ and $r = s(2m-1)$ with $3 \nmid s$. In this case $t(\Lambda) = 1$ again by [4, 1.6], whence $\mathrm{typ}(\Lambda) = (D_{3m}, s/3, 1) \in \mathcal{L}$.

Case (iii):$\Delta = E_n$ (for some $n \in \{6,7,8\}$). In this case m_Δ is a prime number and $e \neq 1$. Thus $r = sm_\Delta$ with $s \in \mathbb{N}$, and $\mathrm{typ}(\Lambda) = (E_n, s, t(\Lambda))$ with $t(\Lambda) \in \{1,2\}$. Here $t(\Lambda) = 2$ implies $n = 6$. Hence $\mathrm{typ}(\Lambda) \in \mathcal{L}$.

(2) $\mathrm{typ}(\mathrm{RFSS}) \supseteq \mathcal{L}$.

Let $(\Delta, f, t) \in \mathcal{L}$. Then we can construct a translation quiver of the form $\mathbb{Z}\Delta/\langle\phi\tau^{-fm_\Delta}\rangle$ with t the order of ϕ, where ϕ is defined as in 5.2 below. If there is a configuration $\mathcal{C}$ of $\mathbb{Z}\Delta$ such that $\mathcal{C}$ is stable both under ϕ and under τ^{-fm_Δ}, then this $\mathcal{C}$ gives an algebra Λ with ${}_s\Gamma_\Lambda \cong \mathbb{Z}\Delta/\langle\phi\tau^{-fm_\Delta}\rangle$, and we have $(\Delta, f, t) = \mathrm{typ}(\Lambda) \in \mathrm{typ}(\mathrm{RFSS})$. It is not hard to see that our representative algebras with $s = 1$ in (2.5) supply such configurations. $\square$

5.2. **Proof of Theorem 2.2(1).** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear. For the implication (d) $\Rightarrow$ (a) it is enough to show that Λ is derived equivalent to $\Lambda(\mathrm{typ}(\Lambda))$ for all $\Lambda \in$ RFSS.

Let $\Lambda \in$ RFSS with $\mathrm{typ}(\Lambda) = (\Delta, f, t) \in \mathcal{L}$. Then $\Gamma_\Lambda \cong (\mathbb{Z}\Delta)_{\mathcal{C}}/\Pi$, where ϕ is an automorphism of $\mathbb{Z}\Delta$ with order $t \in \{1,2,3\}$, $\Pi = \langle\phi\tau^{-fm_\Delta}\rangle$ and $\mathcal{C}$ is a Π-stable configuration of $\mathbb{Z}\Delta$. Define $\vec{\Delta}$ as follows. If $(\Delta, t) = (A_{2p+1}, 2)$ let $\vec{\Delta} = \vec{A}^p_p$; otherwise, let $\vec{\Delta} = \vec{A}_n, \vec{D}_n$ or $\vec{E}_n$ according as $\Delta = A_n, D_n$ or E_n, respectively. Introduce the coordinate of $\mathbb{Z}\Delta$ by identifying $\mathbb{Z}\Delta = \mathbb{Z}\vec{\Delta}$ so that an automorphism η of $\vec{\Delta}$ induces an automorphism $\mathbb{Z}\eta$ of $\mathbb{Z}\Delta$ by defining $(\mathbb{Z}\eta)(i,p) := (i, \eta(p))$ for all $i \in \mathbb{Z}$ and all vertices p of $\vec{\Delta}$. Then without loss of generality we may take ϕ as follows. When $t = 1$, $\phi = \mathbb{1}$; when $t = 2$, $\phi = \mathbb{Z}\phi_\Delta$; and when $t = 3$, $\phi = \mathbb{Z}\omega_{D_4}$ (2.4).

Now $\mathcal{C}$ is stable under ϕ. In fact, this is trivial in the case $\phi = \mathbb{1}$ and otherwise $f \in \mathbb{N}$ so the stability of $\mathcal{C}$ under ϕ follows from that under both Π and τ^{m_Δ}. Thus ϕ can be uniquely extended to an automorphism of $\mathbb{Z}\Delta_{\mathcal{C}}$. We regard ϕ as this extension. Set R to be the universal cover of Λ, i.e., R is the full subcategory of the mesh category $k(\mathbb{Z}\Delta_{\mathcal{C}})$ consisting of the projective vertices of $\mathbb{Z}\Delta_{\mathcal{C}}$. Since $\mathcal{C}$ is Π-stable, Π acts on R and we have $\Lambda \cong R/\Pi$. As in [4, section 2] construct an algebra A as the full subcategory of R consisting of those projective vertices p of $\mathbb{Z}\Delta_{\mathcal{C}}$ for which there exist both a path from a vertex of $\tau^{m_\Delta}\vec{\Delta}$ to p and a path from p to a vertex of $\vec{\Delta}$. Since A is stable under ϕ by construction, ϕ restricts to an automorphism ϕ_A of A. Let Q be the quiver of A. We may take the object set of A as the vertex set of Q. Then ϕ_A defines an automorphism ϕ_Q of Q. For display-functor $\Phi\colon kQ \to A$ we always assume that Φ is the identity on vertices.

5.2.1. *A description of Λ in terms of A when $f \in \mathbb{N}$.* Here we show that $\Lambda \cong R/\Pi \cong A^{\mathbb{Z}}/\langle \phi_A^{\mathbb{Z}} \nu_A^f \rangle$ if $f \in \mathbb{N}$. This follows immediately from the following statement.

Lemma. *In the setting as above, there exists an isomorphism $\lambda\colon R \to A^{\mathbb{Z}}$ such that the diagram*

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\phi\tau^{-f}m_\Delta"'] & A^{\mathbb{Z}} \arrow[d, "\phi_A^{\mathbb{Z}}\nu_A^f"] \\ R \arrow[r, "\lambda"'] & A^{\mathbb{Z}} \end{tikzcd}$$

commutes.

Proof. First we show the following.

Claim. *There is a commutative directed display-functor $\Phi\colon kQ \to A$ such that the diagram*

$$\begin{tikzcd} kQ \arrow[r, "\Phi"] \arrow[d, "k\phi_Q"'] & A \arrow[d, "\phi_A"] \\ kQ \arrow[r, "\Phi"'] & A \end{tikzcd}$$

is commutative.

When the torsion order t of Λ is 1, we can take any commutative directed display-functor $\Phi\colon kQ \to A$, whose existence is guaranteed by Bongartz [2]. When $t \neq 1$, we may assume that Q is an oriented tree. In fact, if Δ is of class A_n, then this is clear (e.g. Happel-Ringel [9] or [4, section 6]; Q was called *pedigree* in [4]). If Δ is of class D_n ($n \geq 4$), then since $t \neq 1$ we may assume that the configuration $\mathcal{C}$ contains $(-1, n-1)$ and $(-1, n)$ by [4, Lemma 1.6]. This implies that Q is obtained by adding two arrows $a \to n-1$ and $a \to n$ and two vertices $n-1$ and n to an oriented tree (a pedigree) with exactly $n-2$ vertices including the vertex a, as explained in [4, 7.4]. Therefore Q itself is an oriented tree also in this case. If Δ is of class E_n, then since $t \neq 1$, we have $n = 6$. Looking at the classification of the configurations of $\mathbb{Z}E_6$ listed in [4, section 8], we see that there are only five of them with $t \neq 1$ (Note that if Q becomes an oriented tree for $\mathcal{C}$, then the same holds for $\mathcal{C}^{\mathrm{op}}$). It is easy to see that these five configurations give Q as an oriented tree if we shift the configurations suitably. Now we give a desired display-functor $\Phi\colon kQ \to A$ as follows. Let $\{\alpha_i\colon x_i \to y_i | i = 1, \dots, p\}$ be a complete set of representatives of ϕ_Q orbits of the arrows of Q. For each $i = 1, \dots, p$ define $\Phi(\alpha_i) := \bar{\mu}_i$ for some path μ_i from x_i to y_i in $\mathbb{Z}\Delta_{\mathcal{C}}$ such that $\operatorname{rad} A(x_i, y_i) = k\bar{\mu}_i \oplus \operatorname{rad}^2 A(x_i, y_i)$, where $\mu_i \in \bar{\mu}_i \in A$. For each $0 \leq j < t$ define $\Phi(\phi_Q^j \alpha_i) := \phi_A^j(\bar{\mu}_i)$. Then we clearly have $\Phi(\phi_Q \alpha) = \phi_A(\Phi\alpha)$ for all arrows α in Q. This shows that the Φ gives the desired commutative diagram, and also this Φ is commutative directed because Q is an oriented tree. This finishes the proof of the claim.

We set $I := \operatorname{Ker}\Phi$. Now by [3, 3.2] R admits a Π-invariant multiplicative basis, i.e., we can choose a basis-vector ${}_y\varepsilon_x$ in each nonzero morphism space $R(x,y)$ in such a way that

(i) ${}_z\varepsilon_y \cdot {}_y\varepsilon_x = {}_z\varepsilon_x$ if $R(y,z) \circ R(x,y) \neq 0$; and
(ii) $\rho({}_y\varepsilon_x) = {}_{\rho y}\varepsilon_{\rho x}$ for all $x, y \in R$ and all $\rho \in \Pi$.

Then by [4, Proposition 5.2] the quiver of R is $\hat{Q}$ which is determined by Q and A. Let $\Psi\colon k\hat{Q} \to R$ be the display-functor for R defined by $\Psi(x^{[i]}) = (\tau^{-m_\Delta})^i x$ for all vertices $x^{[i]}$ of $\hat{Q}$ and $\Psi(\alpha) := {}_y\varepsilon_x$ for all arrows $\alpha\colon x \to y$ in $\hat{Q}$. Since $({}_y\varepsilon_x)$ is a multiplicative basis of R, the display-functor Ψ is commutative directed. Hence again by [4, Proposition 5.2] we have $\operatorname{Ker}\Psi = \hat{I}$. On the other hand by Lemma 1.5 we have a display-functor $\hat{\Phi}\colon k\hat{Q} \to A^{\mathbb{Z}}$ with $\operatorname{Ker}\hat{\Phi} = \hat{I}$. Define an isomorphism $\lambda\colon R \to A^{\mathbb{Z}}$ by the commutativity of the diagram:

$$\begin{tikzcd} k\hat{Q} \arrow[r, equal] \arrow[d, "\Psi"'] & k\hat{Q} \arrow[d, "\hat{\Phi}"] \\ R \arrow[r, "\lambda"'] & A^{\mathbb{Z}}. \end{tikzcd}$$

Let $\phi_{\hat{Q}}$ be an automorphism of $\hat{Q}$ defined by ϕ. To show the required commutativity of the diagram in the assertion, it is enough to show the following:

Claim. *The diagram*

$$\begin{tikzcd} R \arrow[d, "\phi\tau^{-fm_\Delta}"'] & k\hat{Q} \arrow[l, "\Psi"'] \arrow[r, "\hat{\Phi}"] \arrow[d, "k\phi_{\hat{Q}}\nu_{\hat{Q}}^f"] & A^{\mathbb{Z}} \arrow[d, "\phi_A^{\mathbb{Z}}\nu_A^f"] \\ R & k\hat{Q} \arrow[l, "\Psi"] \arrow[r, "\hat{\Phi}"'] & A^{\mathbb{Z}} \end{tikzcd}$$

is commutative.

Since $({}_y\varepsilon_x)$ is Π-invariant, the commutativity of the left square holds. Further by Lemma 1.4 we have a commutative diagram

$$\begin{tikzcd} k\hat{Q} \arrow[r, "\hat{\Phi}"] \arrow[d, "k\nu_{\hat{Q}}"'] & A^{\mathbb{Z}} \arrow[d, "\nu_A"] \\ k\hat{Q} \arrow[r, "\hat{\Phi}"'] & A^{\mathbb{Z}}. \end{tikzcd}$$

Hence it remains to show that the commutativity of the diagram

$$\begin{tikzcd} k\hat{Q} \arrow[r, "\hat{\Phi}"] \arrow[d, "k\phi_{\hat{Q}}"'] & A^{\mathbb{Z}} \arrow[d, "\phi_A^{\mathbb{Z}}"] \\ k\hat{Q} \arrow[r, "\hat{\Phi}"'] & A^{\mathbb{Z}}. \end{tikzcd}$$

The commutativity on vertices is obvious. The commutativity on arrows of the form $\alpha^{[i]}\colon x^{[i]} \to y^{[i]}$ directly follows from the previous Claim. Now let $\alpha^{[i]} := \alpha_{y,x}^{[i]}\colon x^{[i]} \to y^{[i+1]}$ be an arrow in $\hat{Q}$. By definition of $\hat{Q}$ there exists a complete path μ from y to x in (Q, I). Then

$$\phi_A^{\mathbb{Z}}(\hat{\Phi}(\alpha^{[i]})) = \phi_A^{\mathbb{Z}}((\Phi(\mu)^*)^{[i]}) = (\Phi(\mu)^* \circ (\phi_{A}{}_x^y)^{-1})^{[i]}.$$

And

$$\hat{\Phi}(k\phi_{\hat{Q}}(\alpha^{[i]})) = \hat{\Phi}(\phi_{\hat{Q}}(\alpha^{[i]})) = (\Phi(\phi_Q\mu)^*)^{[i]}$$

because $\phi_{\hat{Q}}(\alpha^{[i]})\colon (\phi_Q x)^{[i]} \to (\phi_Q y)^{[i]}$ and $\phi_Q\mu$ is a complete path from $\phi_Q y$ to $\phi_Q x$. Hence it is enough to show that $\Phi(\mu)^* \circ (\phi_{A}{}_x^y)^{-1} = \Phi(\phi_Q\mu)^*$. But again by the previous Claim, we have $\phi_{A}{}_x^y(\Phi(\mu)) = \Phi(\phi_Q\mu)$, i.e., $(\phi_{A}{}_x^y)^{-1}(\Phi(\phi_Q\mu)) = \Phi(\mu)$. This gives the desired equality because the both hand sides are elements of $DA(\phi_A y, \phi_A x)$, and they map the basis $\Phi(\phi_Q\mu)$ of $A(\phi_A y, \phi_A x)$ to 1. This finishes the proof of the lemma. □

5.2.2. *The case where the frequency f is a natural number.* Set $M(x) := k(\mathbb{Z}\Delta_{\mathcal{C}})(-, x)|_R$ for all $x \in \mathbb{Z}\Delta_{\mathcal{C}}$. Then $\{M(x) | x \in \mathbb{Z}\Delta_{\mathcal{C}}\}$ is a complete set of representatives of indecomposable R-modules ([4, 1.2]). Let $\operatorname{ind} R$ be the full subcategory of $\operatorname{mod} R$ formed by the set $\{M(x) | x \in \mathbb{Z}\Delta_{\mathcal{C}}\}$. Then we have an isomorphism $M\colon k(\mathbb{Z}\Delta_{\mathcal{C}}) \to \operatorname{ind} R$. Let Σ be the full subquiver of $\mathbb{Z}\Delta_{\mathcal{C}}$ consisting of those vertices x for which $k(\mathbb{Z}\Delta_{\mathcal{C}})(p, x) = 0$ for all projective vertices p of $\mathbb{Z}\Delta_{\mathcal{C}}$ not belonging to A. Using extension by 0, we can identify the indecomposable A-modules with the indecomposable R-modules with support in A. Then by [4, 2.1] $\{M(x) | x \in \Sigma\}$ is a complete set of representatives of indecomposable A-modules, and for any arrow $\alpha\colon x \to y$ in Σ, $M(\bar{\alpha})\colon M(x) \to M(y)$ is irreducible in $\operatorname{mod} A$, where $\alpha \in \bar{\alpha} \in k(\mathbb{Z}\Delta_{\mathcal{C}})(x, y)$. Thus the Auslander-Reiten quiver Γ_A of A is obtained from $M(\Sigma)$ by adding some arrows with a suitable translation. Then this Γ_A contains $M(\vec{\Delta})$ as a section by [4, Proposition 2.1]. Hence A is tilted from $B := k\vec{\Delta}$ in a classical sense, namely the direct sum S of all $M(x)$ with $x \in \vec{\Delta}$ is a tilting module for A with $\operatorname{End}_A(S) = B$. For each $x \in \vec{\Delta}$, construct a minimal projective resolution of $M(x)$ of the form

$$0 \to \eta_A(P_2(x)) \xrightarrow{\eta_A(p_2(x))} \eta_A(P_1(x)) \xrightarrow{\eta_A(p_1(x))} \eta_A(P_0(x)) \xrightarrow{\ \varepsilon\ } M(x) \to 0,$$

where

$$H(x) := (\cdots \to 0 \to P_2(x) \xrightarrow{p_2(x)} P_1(x) \xrightarrow{p_1(x)} P_0(x) \to 0 \to \cdots) \in \mathcal{H}^{\mathrm{b}}(\oplus A).$$

Then $\vec{\Delta}$ yields a tilting triple (A, E, B) by setting E to be the full subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ formed by $H(x)$ with $x \in \vec{\Delta}$. If $t = 1$, then by 5.2.1 $\Lambda \cong A^{\mathbb{Z}}/\langle \nu_A^f \rangle$, and $A^{\mathbb{Z}}/\langle \nu_A^f \rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle \nu_B^f \rangle = \Lambda(\operatorname{typ}(\Lambda))$ by 4.4.2. Thus the assertion holds.

So assume that $t \neq 1$, i.e., $t \in \{2,3\}$. Then for each $x \in \vec{\Delta}$ and for each $p \in A$, we have

$$\begin{aligned}({}^{\phi_A}M(x))(p) &= M(x)(\phi_A^{-1}p) = M(x)(\phi^{-1}p)\\ &= k(\mathbb{Z}\Delta_{\mathcal{C}})(\phi^{-1}p, x)\\ &\cong k(\mathbb{Z}\Delta_{\mathcal{C}})(p, \phi x) = M(\phi x)(p).\end{aligned}$$

Thus for each $x \in \vec{\Delta}$, we have ${}^{\phi_A}M(x) \cong M(\phi x)$ as A-modules. Hence $\tilde{\phi}_A H(x) \cong H(\phi x)$ for all $x \in \vec{\Delta}$. This shows that E is $\langle\tilde{\phi}_A\rangle$-stable up to isomorphisms. Choose a complete set Δ_0' of representatives of ϕ-orbits in the vertex set of $\vec{\Delta}$, and reset $H(\phi^i x) := \tilde{\phi}_A^i H(x)$ for all $x \in \Delta_0'$ and all i with $0 \leq i < t$. Then (A, E, B) is still a tilting triple and E turns out to be $\langle\tilde{\phi}_A\rangle$-stable. Since $\tilde{\phi}_A H(x) = H(\phi x) = H(\phi_B x)$ for all $x \in \vec{\Delta}$, we can construct an isomorphism $\psi\colon E \to B$ such that $\psi\tilde{\phi}_A = \phi_B\psi$. Hence by 4.4.3, we have $\Lambda \cong A^{\mathbb{Z}}/\langle\phi_A^{\mathbb{Z}}\nu_A^f\rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle\phi_B^{\mathbb{Z}}\nu_B^f\rangle = \Lambda(\mathrm{typ}(\Lambda))$.

5.2.3. *The case where the frequency f is not a natural number.* In this case we have either $\mathrm{typ}(\Lambda) = (A_n, s/n, 1)$ with $n \nmid s$ or $\mathrm{typ}(\Lambda) = (D_{3m}, s/3, 1)$ with $3 \nmid s$. In the first case the assertion has been shown in [1, Proposition 5.3].

So let $\mathrm{typ}(\Lambda) = (D_{3m}, s/3, 1)$ with $3 \nmid s$. Then ${}_s\Gamma_\Lambda \cong \mathbb{Z}D_{3m}/\langle\tau^{-s(2m-1)}\rangle$. By [4], $A^{\mathbb{Z}}/\langle\nu_A\rangle \cong D(\bar{Q}, \bar{Q}, \bar{Q})$ for some looped Brauer quiver $\bar{Q}$. Define an algebra A' by the quiver Q' in Figure 1 with the commutativity relation and the zero relations $\alpha\beta = 0 = \beta\alpha$ and $m(0) \to 1(2) \to v = 0$, where $Q(a)$ are pairwise isomorphic and for each $a = 0, 1, 2$, $Q(a)$ is obtained from $\bar{Q}$ by deleting one arrow from each cycle in $\bar{Q}$ and is a full subquiver of the quiver in Figure 2 containing the α-path L_a from $1(a)$ to $m(a)$. Then $A'^{\mathbb{Z}}/\langle\nu_{A'}\rangle \cong D(\bar{Q}, \bar{Q}, \bar{Q}) \cong A^{\mathbb{Z}}/\langle\nu_A\rangle$. Hence also $A'^{\mathbb{Z}} \cong A^{\mathbb{Z}} \cong R$ by Hughes and Waschbüsch [10, Proposition 2.7]. Here we reset $A := A'$ (so $Q := Q'$). We construct a nice tilting spectroid E for A in a similar way as in [1, section 5]. Let x be a vertex of A. Then x is in $Q(a)$ for a unique $a \in \{0, 1, 2\}$. Further there is a unique shortest path from a vertex $x_0 \in L_a$ to x in Q:

$$x_0 \overbrace{\xrightarrow{\beta}\xrightarrow{\beta}\cdots\xrightarrow{\beta}}^{a_0} x_1 \overbrace{\xrightarrow{\alpha}\xrightarrow{\alpha}\cdots\xrightarrow{\alpha}}^{a_1} x_2 \xrightarrow{\beta}\cdots\to\cdots\to x_t = x,$$

where $x_0, x_1, \ldots, x_{t-1}$ are junctions if $t > 0$. We then put

$$\mathcal{H}^{\mathrm{b}}(\oplus A) \ni T_x := (\cdots \to 0 \to x_0 \xrightarrow{\beta^{a_0}} x_1 \xrightarrow{\alpha^{a_1}} x_2 \to \cdots \to x_t \to 0 \to \cdots),$$

where x_0 is in degree zero. Introduce a linear order $\leq$ on the vertex set of $Q(a)$ for all $a = 0, 1, 2$ lexicographically as follows: Let $x, y \in Q(a)$ and let

$$(*) \quad \begin{cases} T_x = (\cdots \to 0 \to x_0 \xrightarrow{\beta^{a_0}} x_1 \xrightarrow{\alpha^{a_1}} x_2 \to \cdots \to x_t = x \to 0 \to \cdots) \\ T_y = (\cdots \to 0 \to y_0 \xrightarrow{\beta^{b_0}} y_1 \xrightarrow{\alpha^{b_1}} y_2 \to \cdots \to y_u = y \to 0 \to \cdots). \end{cases}$$

(i) If x_0 is a predecessor of y_0, then we define $x \leq y$.
(ii) Assume that $x_0 = y_0, \ldots, x_i = y_i$ for $i \geq 0$. If $y_i = y$, then define $x \leq y$. In case $x_i \neq x$ and $y_i \neq y$, if x_{i+1} is a predecessor of y_{i+1}, then define $x \leq y$.

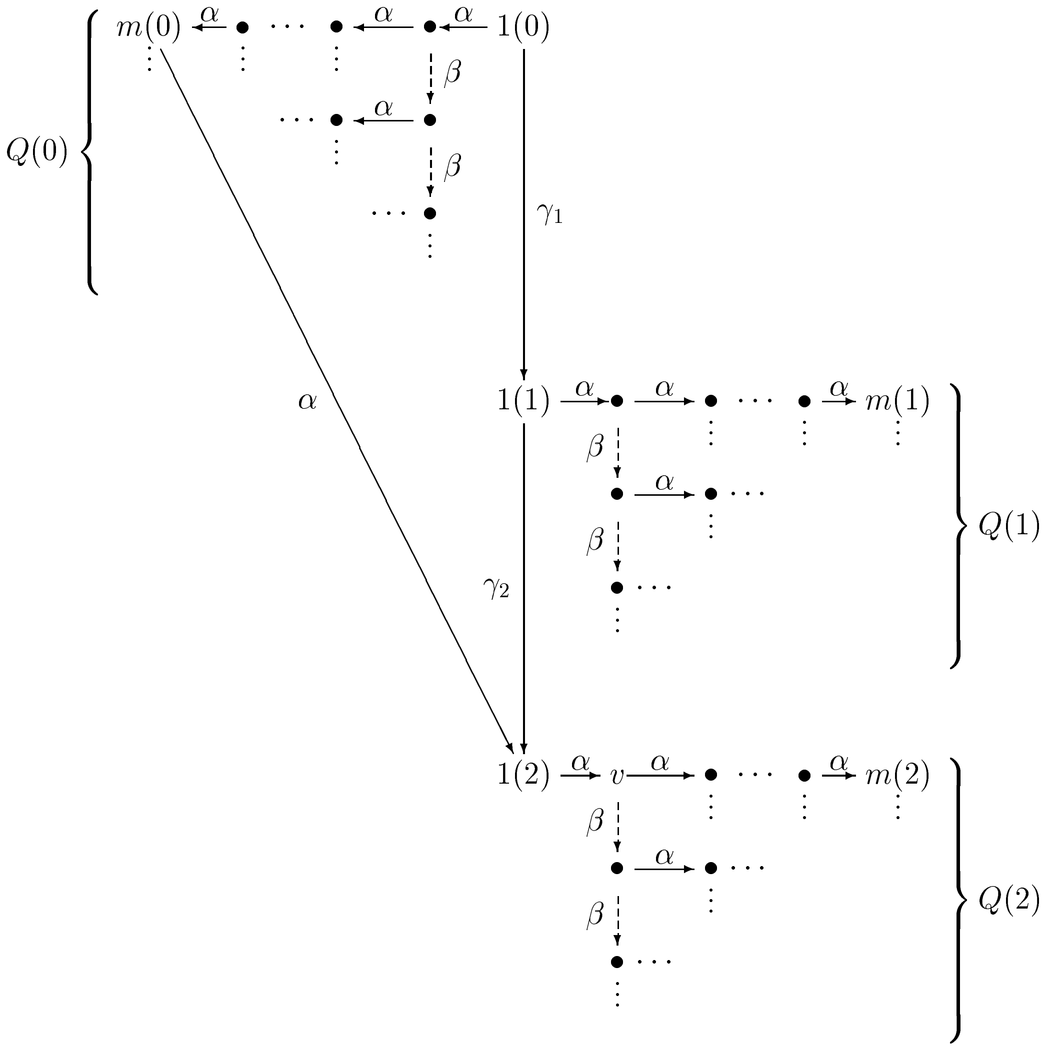

FIGURE 1

Give a name to each vertex of $Q(a)$ as $i(a)$ with $1 \leq i \leq m$ so that $i(a) < j(a)$ iff $i < j$. Let E be the full subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ defined by $\{T_x | x \in A\}$.

We show that (A, E, B) is a tilting triple, where $B = k(K_m, J_m)$ (2.5).

(a) For each $x \in A$, $(\underline{x}) \in \operatorname{tri}(E)$.

Here the underline indicates the place of degree zero. By construction we have a triangle of the form

$$(\underline{x}) = (\underline{x}_t) \longrightarrow T_{x_t} \longrightarrow T_{x_{t-1}} \longrightarrow$$

in $\mathcal{H}^{\mathrm{b}}(\oplus A)$ if $(\underline{x}) \notin E$. Hence in any case we have $(\underline{x}) \in \operatorname{tri}(E)$.

(b) For each $x, y \in A$ and for each $i \neq 0$, $\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y[i]) = 0$.

First note that $\dim A(x, y) \leq 1$ for all $x, y \in A$. If $x, y \in Q(a)$ for some $a = 0, 1, 2$, then the argument in [1, section 5] applies. Let $x \in Q(a)$, $y \in Q(b)$ with $a, b \in \{0, 1, 2\}$ and $a \neq b$, and let $i \in \mathbb{Z}$. Assume that we have $0 \neq \overline{f} \in \mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y[i])$. Then we show that $i = 0$. Put $f = (f^j)$ with $f^j : x_j \longrightarrow y_{j+i}$. Then $f^j \neq 0$ for some $j \geq 0$. Since

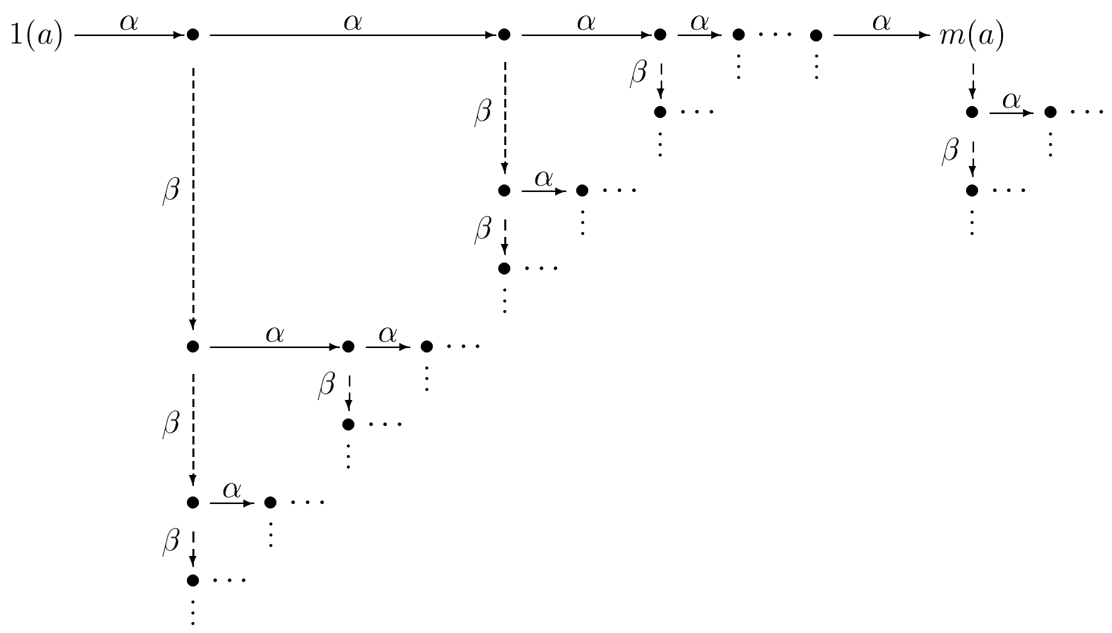

FIGURE 2

$0 \neq f^j \in A(x_j, y_{j+i})$, we must have $(a,b) \in \{(0,1),(1,2),(0,2)\}$. But then $x_j \in L_a$ and $y_{j+i} \in L_b$ by the structure of A. Thus $j = 0$ and $j+i = 0$, whence $i = 0$.

(c) $E \cong B$.

By the argument of [1, section 5], the full subcategory E_a of E formed by $\{T_x | x \in Q(a)\}$ is presented by the quiver $1(a) \to 2(a) \to \cdots \to m(a)$ with no relations for all $a = 0,1,2$. Namely, for each $x, y \in Q(a)$, we have

$$\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) \cong \begin{cases} k & \text{if } x \leq y\text{; and} \\ 0 & \text{otherwise.} \end{cases}$$

Since we have arrows $1(0) \xrightarrow{\gamma_1} 1(1)$, $1(1) \xrightarrow{\gamma_2} 1(2)$ and $m(0) \xrightarrow{\alpha} 1(2)$ in Q, we have nonzero maps $T_{1(0)} \xrightarrow{\overline{(\gamma_1)}} T_{1(1)}$, $T_{1(1)} \xrightarrow{\overline{(\gamma_2)}} T_{1(2)}$ and $T_{m(0)} \xrightarrow{\overline{(\alpha)}} T_{1(2)}$ in $\mathcal{H}^{\mathrm{b}}(\oplus A)$ where $\overline{(-)}$ denotes the homotopy class of $(-)$. Let $x \in Q(a), y \in Q(b)$ and $a < b$ in $\{0,1,2\}$. Put T_x, T_y as in $(*)$. Then we have

Claim. (i) *$\mathcal{H}^{\mathrm{b}}(\oplus A)(T_y, T_x) = 0$;*

(ii) *If $(a,b) = (0,1)$, then*

$$\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) \cong \begin{cases} k & \textit{if } x = 1(0) \\ 0 & \textit{otherwise} \end{cases};$$

(iii) *If $(a,b) = (1,2)$, then*

$$\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) \cong \begin{cases} k & \textit{if } x = 1(1) \\ 0 & \textit{otherwise.} \end{cases}$$

(iv) *If* $(a, b) = (0, 2)$, *then*

$$\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) \cong \begin{cases} k & \textit{if } y = 1(2) \\ 0 & \textit{otherwise.} \end{cases}$$

In fact, (i): if this is not true, there is $0 \neq f = (f^i) \in \mathcal{C}^{\mathrm{b}}(\oplus A)(T_y, T_x)$, which yields $Q(b) \ni y_i \xrightarrow{f^i \neq 0} x_i \in Q(a)$ for some i, a contradiction.

(ii). If $x \neq 1(0)$, then $x_0, x_1, \ldots, x_t \in Q(0)\backslash\{1(0)\}$. Therefore $\mathcal{C}^{\mathrm{b}}(\oplus A)(T_x, T_y) = 0$. Thus $\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) = 0$. Next let $x = 1(0)$. Then $\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) = \mathcal{C}^{\mathrm{b}}(\oplus A)(T_x, T_y)$ $= A(1(0), y_0) = k\alpha^p\gamma_1 \neq 0$ for some p.

(iii). The same proof as for (ii) works.

(iv). If $y \neq 1(2)$, then $y_0, y_1, \ldots, y_t \in Q(2)\backslash\{1(2)\}$. Thus $\mathcal{C}^{\mathrm{b}}(\oplus A)(T_x, T_y) = 0$, whence $\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) = 0$. Next let $y = 1(2)$. Then $\mathcal{H}^{\mathrm{b}}(\oplus A)(T_x, T_y) = \mathcal{C}^{\mathrm{b}}(\oplus A)(T_x, T_y) = A(x_0, 1(2)) = k\alpha^p \neq 0$ for some $p \geq 1$.

Hence the quiver of E coincides with K_m. By Claim above we have a display-functor $\Upsilon\colon kK_m \to E$ defined by the following figure:

$$\begin{tikzcd}
T_{2(0)} \arrow[d, "f_2"'] & T_{1(0)} \arrow[l, "f_1"'] \arrow[d, "\overline{(\underline{\gamma_1})}"] & & & \\
\vdots \arrow[d, "f_{m-1}"'] & T_{1(1)} \arrow[r, "g_1"] \arrow[d, "\overline{(\underline{\gamma_2})}"] & T_{2(1)} \arrow[r, "g_2"] & \cdots \arrow[r, "g_{m-1}"] & T_{m(1)} \\
T_{m(0)} \arrow[r, "\overline{(\underline{\alpha})}"'] & T_{1(2)} \arrow[r, "h_1"'] & T_{2(2)} \arrow[r, "h_2"'] & \cdots \arrow[r, "h_{m-1}"'] & T_{m(2)},
\end{tikzcd}$$

where f_i, g_j, h_l are defined as in [1, section 5]. Note that h_1 has the form $\overline{(\underline{\alpha})}$ for $\alpha\colon 1(2) \to v$. By the relations of A, we have $h_1\overline{(\underline{\alpha})} = 0$ and $\overline{(\underline{\gamma_2})}\,\overline{(\underline{\gamma_1})} = \overline{(\underline{\alpha})} f_{m-1} \cdots f_2 f_1$. Hence the display-functor Υ induces a full functor $\bar{\Upsilon}\colon B := k(K_m, J_m) \to E$ that is bijective on objects. Comparing the dimension of each Hom-space, we see that $\bar{\Upsilon}$ is faithful, i.e., $\bar{\Upsilon}$ is an isomorphism. Thus $E \cong B$. As a consequence (A, E, B) is a tilting triple.

Define a display-functor $\Psi\colon k\hat{Q} \to R$ by a $\langle\tau^{-(2m-1)}\rangle$-invariant multiplicative basis of R as in 5.2.1. Let $\Phi\colon kQ \to A$ be a commutative directed display-functor with $I := \operatorname{Ker}\Phi$. The automorphism $\tau^{-(2m-1)}$ on R defines an automorphism f of $\hat{Q}$. As easily seen (e.g. use arguments in Roggon [20, 2.2, 2.4]) the action of f on vertices is as follows.

$$f(x(a)^{[i]}) := \begin{cases} x(a+1)^{[i]} & a = 0, 1 \\ x(0)^{[i+1]} & a = 2. \end{cases}$$

This f defines an automorphism g of $A^{\mathbb{Z}}$ through $\hat{\Phi}$. Thus we have a commutative diagram

$$\begin{tikzcd} R \arrow[d, "\tau^{-(2m-1)}"'] & k\hat{Q} \arrow[l, "\Psi"'] \arrow[r, "\hat{\Phi}"] \arrow[d, "kf"'] & A^{\mathbb{Z}} \arrow[d, "g"] \\ R & k\hat{Q} \arrow[l, "\Psi"] \arrow[r, "\hat{\Phi}"'] & A^{\mathbb{Z}}. \end{tikzcd}$$

with $\operatorname{Ker}\Psi = \hat{I} = \operatorname{Ker}\hat{\Phi}$. Hence $\Lambda \cong A^{\mathbb{Z}}/\langle g^s\rangle$. Put $\psi := \bar{\Upsilon}^{-1}\colon E \to B$. Then it is easy to check that $\hat{E}$ is $\langle \tilde{g}\rangle$-stable and that the diagram

$$\begin{tikzcd} \hat{E} \arrow[r, "\hat{\psi}"] \arrow[d, "\tilde{g}"'] & B^{\mathbb{Z}} \arrow[d, "\mu_m"] \\ \hat{E} \arrow[r, "\hat{\psi}"'] & B^{\mathbb{Z}} \end{tikzcd}$$

is commutative. Hence by Proposition 4.4.1, $\Lambda \cong A^{\mathbb{Z}}/\langle g^s\rangle$ is derived equivalent to $B^{\mathbb{Z}}/\langle \mu_m^s\rangle = \Lambda(\mathrm{typ}(\Lambda))$.

As a result, in any case Λ is derived equivalent to $\Lambda(\mathrm{typ}(\Lambda))$. □

6. Proof of Proposition 2.2

Assume that Λ is stably equivalent to Λ'. Then $\operatorname{char} k = 2$ and their type have to be $(D_{3m}, 1/3, 1)$ for some $m \geq 2$ by Remark 2.1(3). In this proof we use the coordinates introduced in Riedtmann [18, 1.3] for the vertices of $\mathbb{Z}D_n$. We put $\tilde{\Gamma} := \mathbb{Z}D_{3m}$, $\Gamma := \tilde{\Gamma}/\langle \tau^{2m-1}\rangle$, and $\pi\colon \tilde{\Gamma} \to \Gamma$ to be the canonical morphism and $k(\pi)\colon k(\tilde{\Gamma}) \to k(\Gamma)$ to be the canonical covering functor. The Auslander-Reiten quivers of Λ and Λ' have the forms $\Gamma_{\mathcal{C}} := \tilde{\Gamma}_{\mathcal{C}}/\langle \tau^{2m-1}\rangle$ and $\Gamma_{\mathcal{C}'} := \tilde{\Gamma}_{\mathcal{C}'}/\langle \tau^{2m-1}\rangle$, respectively, for some $\langle \tau^{2m-1}\rangle$-stable configurations $\mathcal{C}$ and $\mathcal{C}'$ of $\tilde{\Gamma}$ containing $(0, n-1)$, where $n := 3m$. Then the spectroids of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$ have the forms $k(\Gamma_{\mathcal{C}})$ and $k\Gamma_{\mathcal{C}'}/J_{\mathcal{C}'}$, respectively, where $k(\Gamma_{\mathcal{C}})$ is the mesh category of $\Gamma_{\mathcal{C}}$ and $J_{\mathcal{C}'}$ is the ideal of $k\Gamma_{\mathcal{C}}$ defined in [19, 3.3]. By π we also denote the canonical morphism $\tilde{\Gamma}_{\mathcal{C}'} \to \Gamma_{\mathcal{C}'}$. To make clear the definition of $J_{\mathcal{C}'}$ and for later use, we introduce the following notations. We put x_{ij} to be the vertex of $\tilde{\Gamma}$ with the coordinate (i,j). We denote arrows of $\tilde{\Gamma}$ as follows:

$$x_{ij} \xrightarrow{\alpha_{ij}} x_{i,j+1} \xrightarrow{\alpha'_{ij}} x_{i+1,j}$$

for $1 \leq j \leq n-2$ and

$$x_{i,n-2} \xrightarrow{\alpha_{i,n-1}} x_{in} \xrightarrow{\alpha'_{i,n-1}} x_{i+1,n-2}.$$

Further we denote by θ_z the mesh-relation $\sum \alpha\cdot\sigma\alpha$, where z is a stable vertex of $\tilde{\Gamma}$, the sum is taken over all arrows $\alpha\colon z' \to z$, and $\sigma\alpha$ is the arrow $\tau z \to z'$. Then noting that any modified mesh-relation ([19, 2.3]) coincides with the corresponding mesh-relation because of $\operatorname{char} k = 2$, the ideal $J_{\mathcal{C}'}$ is generated by $\pi\theta_z$ with $z \notin \langle \tau^{2m-1}\rangle x_{1,n-1}$ and by the element

$$\pi(\alpha_{1,n-2}\alpha'_{0,n-2}) + \pi(\kappa\iota) - \pi(\xi),$$

where we put $x_{0,n-1} \xrightarrow{\iota} x^*_{0,n-1} \xrightarrow{\kappa} x_{1,n-1}$ with $x^*_{0,n-1}$ a projective vertex in $\tilde{\Gamma}_{C'}$ and

$$\xi := \alpha_{2m,n-2}\alpha_{2m,n-3}\alpha'_{2m-1,n-3}\cdots\alpha_{3,n-3}\alpha'_{2,n-3}\alpha_{2,n-3}\alpha'_{1,n-3}\alpha'_{0,n-2}.$$

Therefore the spectroids of $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda'$ have the forms $k(\Gamma)$ and $k\Gamma/J$, respectively, where J is obtained from $J_{C'}$ by substituting 0 for paths through projective vertices. Since Λ and Λ' are stably equivalent, we have an isomorphism $\rho\colon k\Gamma/J \to k(\Gamma)$. The both quivers of $k\Gamma/J$ and $k(\Gamma)$ are isomorphic to Γ. Thus ρ induces an automorphism of Γ, thus an automorphism λ of $k(\Gamma)$. Hence by substituting $\lambda^{-1}\rho$ for ρ if necessary, we may assume that ρ is the identity on objects. Let $F\colon k\Gamma \to k(\Gamma)$ be the composite of the canonical morphism $k\Gamma \to k\Gamma/J$ and ρ. Then F is a display-functor with $\mathrm{Ker}\, F = J$. Set $g := \tau^{-(2m-1)}$. For a pair of vertices x, y of $\tilde{\Gamma}$ with an arrow $\alpha\colon x \to y$ in $\tilde{\Gamma}$, we have $k(\tilde{\Gamma})(x, g^i y) = 0$ for all $i \geq 2$ because $2 \cdot 2(2m-1) > 2(3m) - 3 = m_{D_{3m}}$. Hence we have an isomorphism $k(\pi)\colon k(\tilde{\Gamma})(x,y) \oplus k(\tilde{\Gamma})(x, gy) \to k(\Gamma)(\pi x, \pi y)$. Thus $\mathrm{rad}\, k(\Gamma)(\pi x, \pi y) = k(\pi)(\bar{\alpha}) \oplus k(\pi)(k(\tilde{\Gamma})(x, gy))$. Here $\bar{\alpha}$ denotes the morphism of $k(\tilde{\Gamma})$ containing α. We give a basis of $k(\tilde{\Gamma})(x, gy)$ and of $k(\tilde{\Gamma})(x, g\tau^{-1}x)$. Set $A_i := \alpha'_{i,n-2}\alpha_{i,n-2}$ and $B_i := \alpha'_{i,n-1}\alpha_{i,n-1}$. Then paths of the form $A_{i+s}\cdots B_{i+3}A_{i+2}B_{i+1}A_i$ or $B_{i+s}\cdots B_{i+3}A_{i+2}B_{i+1}A_i$ is called a *high path of type 1*, and paths of the form $A_{i+s}\cdots A_{i+3}B_{i+2}A_{i+1}B_i$ or $B_{i+s}\cdots A_{i+3}B_{i+2}A_{i+1}B_i$ is called a *high path of type 2*. For a vertex $x = x_{ij}$ of $\tilde{\Gamma}$, we call $\begin{cases} x_{i,n-2} & \text{if } 1 \leq j \leq n-2 \\ x_{i+1,n-2} & \text{if } j \in \{n-1, n\} \end{cases}$ the *right station* of x; and call $\begin{cases} x_{i+j-n+2,n-2} & \text{if } 1 \leq j \leq n-2 \\ x_{i,n-2} & \text{if } j \in \{n-1, n\} \end{cases}$ the *left station* of x. We call a pair (x, y) of vertices of $\tilde{\Gamma}$ *far* if the first coordinate of the right station of x is smaller than that of the left station of y. For any pair (x, y) of vertices of $\tilde{\Gamma}$ we define $\eta(x, y)$ and $\xi(x, y)$ as follows. If (x, y) is not far, then we set $\eta(x, y) = 0$; if it is far, then we define $\eta(x, y)$ to be the path obtained by composing the following paths:

(i) the unique path from x to the right station of x;
(ii) the high path of type 1 from the right station of x to the left station of y; and
(iii) the unique path from the left station of y to y.

Similarly if (x, y) is not far, then we set $\xi(x, y) = 0$; if it is far, then we define $\xi(x, y)$ to be the path obtained by composing the following paths:

(i) the unique path from x to the right station of x;
(ii) the high path of type 2 from the right station of x to the left station of y; and
(iii) the unique path from the left station of y to y.

Note that the pairs (x, gy) and $(x, g\tau^{-1}x)$ are far if there is an arrow $\alpha\colon x \to y$ in $\tilde{\Gamma}$ and if $k(\tilde{\Gamma})(x, gy) \neq 0$. It is easy to see that if a pair (x, y) is far, then for any path from x to y which is nonzero in $k(\tilde{\Gamma})$ is equal to $\eta(x, y)$, $\xi(x, y)$ or $\eta(x, y) + \xi(x, y)$ modulo the mesh ideal. This implies that for a far pair (x, y) if $\dim k(\tilde{\Gamma})(x, y) = 1$, then $\{\bar{\eta}(x, y)\}$ or $\{\bar{\xi}(x, y)\}$ forms a basis of $k(\tilde{\Gamma})(x, y)$; and if $\dim k(\tilde{\Gamma})(x, y) = 2$, then

$\{\bar{\eta}(x,y), \bar{\xi}(x,y)\}$ forms a basis of $k(\tilde{\Gamma})(x,y)$. Hence we can write

$$\begin{cases} F(\pi\alpha_{ij}) = a_{ij}k(\pi)\bar{\alpha}_{ij} + b_{ij}k(\pi)\bar{\eta}(x_{ij}, gx_{i,j+1}) + c_{ij}k(\pi)\bar{\xi}(x_{ij}, gx_{i,j+1}) \\ F(\pi\alpha'_{ij}) = a'_{ij}k(\pi)\bar{\alpha}'_{ij} + b'_{ij}k(\pi)\bar{\eta}(x_{i,j+1}, gx_{i+1,j}) + c'_{ij}k(\pi)\bar{\xi}(x_{i,j+1}, gx_{i+1,j}) \end{cases}$$

for all i, j. Since F is a display-functor, we have

(a) $$a_{ij} \neq 0, a'_{ij} \neq 0$$

for all i, j. We agree to put $b_{ij} = 0$ if $\bar{\eta}(x_{ij}, gx_{i,j+1}) = 0$; and to put $c_{ij} = 0$ if $\bar{\xi}(x_{ij}, gx_{i,j+1}) = 0$ or $\bar{\eta}(x_{ij}, gx_{i,j+1}) = \bar{\xi}(x_{ij}, gx_{i,j+1})$. We use the same conventions also for b'_{ij}, c'_{ij}. Now since $F(J) = 0$, we have

(b) $$\begin{cases} F(\pi(\theta_z)) = 0 & \text{for all meshes } \theta_z \text{ with } z \notin \langle g\rangle x_{1,n-1}\text{; and} \\ F(\pi(\alpha_{1,n-2})\pi(\alpha'_{0,n-2})) = F(\pi(\xi)). \end{cases}$$

By looking at starting and stopping functions it is not hard to verify the following.

Claim. *Let $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ be in $\tilde{\Gamma}$ with $x = \tau z$ and $\alpha = \sigma\beta$. Then*

- *if $\bar{\eta}(x, gy) \neq 0$, then $0 \neq \overline{g\beta}\bar{\eta}(x, gy) \in \{\bar{\eta}(x, gz), \bar{\xi}(x, gz)\}$;*
- *if $\bar{\xi}(x, gy) \neq 0$, then $0 \neq \overline{g\beta}\bar{\xi}(x, gy) \in \{\bar{\eta}(x, gz), \bar{\xi}(x, gz)\}$;*
- *if $\bar{\eta}(y, gz) \neq 0$, then $0 \neq \bar{\eta}(y, gz)\bar{\alpha} \in \{\bar{\eta}(x, gz), \bar{\xi}(x, gz)\}$; and*
- *if $\bar{\xi}(y, gz) \neq 0$, then $0 \neq \bar{\xi}(y, gz)\bar{\alpha} \in \{\bar{\eta}(x, gz), \bar{\xi}(x, gz)\}$.*

Put $d(\pi(\alpha_{ij})) := \frac{b_{ij}}{a_{ij}} + \frac{c_{ij}}{a_{ij}}$ and $d(\pi(\alpha'_{ij})) := \frac{b'_{ij}}{a'_{ij}} + \frac{c'_{ij}}{a'_{ij}}$. Further for a mesh θ_z put $d(\pi(\theta_z)) := \sum d(\pi(\alpha))$, where the sum is taken over all arrows $\pi(\alpha)$ involved in $\pi(\theta_z)$). Then using the claim above it follows from (a) and (b) that

(c) $$d(\pi(\theta_z)) = 0 \quad \text{for all meshes } \theta_z \text{ with } z \notin \langle g\rangle x_{1,n-1}\text{; and}$$

(d) $$d(\pi(\theta_{x_{1,n-1}})) + c = 0 \quad \text{for some } c \neq 0.$$

In fact, put for an arrow $\alpha\colon x \longrightarrow y$ in $\tilde{\Gamma}$,

$$a(\alpha) := a_{ij}, b(\beta) := b_{ij}, c(\alpha) := c_{ij},$$
$$\breve{\alpha} := k(\pi)(\bar{\alpha}_{ij}), p(x,y) := k(\pi)\bar{\eta}(x_{ij}, gx_{i,j+1}), q(x,y) := k(\pi)\bar{\xi}(x_{ij}, gx_{i,j+1})$$

if $\alpha = \pi\alpha_{ij}$ (so $x = x_{ij}, y = x_{i,j+1}$); and put

$$a(\alpha) := a'_{ij}, b(\beta) := b'_{ij}, c(\alpha) := c'_{ij},$$
$$\breve{\alpha} := k(\pi)(\bar{\alpha}'_{ij}), p(x,y) := k(\pi)\bar{\eta}(x_{i,j+1}, gx_{i+1,j}), q(x,y) := k(\pi)\bar{\xi}(x_{i,j+1}, gx_{i+1,j})$$

if $\alpha = \pi\alpha'_{ij}$ (so $x = x_{i,j+1}, y = x_{i+1,j}$) for short. Then for any arrow $\alpha\colon x \longrightarrow y$ in $\tilde{\Gamma}$ we have

$$F(\pi\alpha) = a(\alpha)\breve{\alpha} + b(\alpha)p(x,y) + c(\alpha)q(x,y), \quad a(\alpha) \neq 0.$$

Further by the claim above we have the following for arrows $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\tilde{\Gamma}$ with $x = \tau z$ and $\alpha = \sigma\beta$,

- If $p(x,y) \neq 0$, then $0 \neq \breve{\beta}p(x,y) \in \{p(x,z), q(x,z)\}$;
- If $q(x,y) \neq 0$, then $0 \neq \breve{\beta}q(x,y) \in \{p(x,z), q(x,z)\}$;

- If $p(y,z) \neq 0$, then $0 \neq p(x,y)\breve{\alpha} \in \{p(x,z), q(x,z)\}$; and
- If $q(y,z) \neq 0$, then $0 \neq q(x,y)\breve{\alpha} \in \{p(x,z), q(x,z)\}$.

We show only the statement (c) because noting that $\bar{\xi} = \bar{\xi}(x_{0,n-1}, x_{2m,n-1}) \neq 0$, the proof of (d) is similar and simpler. Now let $\theta_z := \sum_{i=1}^{r} \beta_i\alpha_i$ be a mesh-relation in $\tilde{\Gamma}$ with $x \xrightarrow{\alpha_i} y_i \xrightarrow{\beta_i} z$ and $z \notin \langle g\rangle x_{1,n-1}$. Then $1 \leq r \leq 3$. It follows from $F(\pi(\theta_z)) = 0$ that

$$\sum_{i=1}^{r}[a(\beta_i)\breve{\beta}_i + b(\beta_i)p(y_i,z) + c(\beta_i)q(y_i,z)][a(\alpha_i)\breve{\alpha}_i + b(\alpha_i)p(x,y_i) + c(\alpha_i)q(x,y_i)] = 0.$$

Note that the length of $\eta(x, gy)$ is $4m-3$ for each arrow $\alpha\colon x \to y$ in $\tilde{\Gamma}$. Then $p(y_i,z)p(x,y_i) = p(y_i,z)q(x,y_i) = q(y_i,z)p(x,y_i) = q(y_i,z)q(x,y_i) = 0$ because $2(4m-3) > 6m-3 = m_{D_{3m}}$. Hence we have

$$\sum_{i=1}^{r} a(\beta_i)a(\alpha_i)\breve{\beta}_i\breve{\alpha}_i + \sum_{i=1}^{r} a(\beta_i)b(\alpha_i)\breve{\beta}_i p(x,y_i) + a(\beta_i)c(\alpha_i)\breve{\beta}_i q(x,y_i) \\ + b(\beta_i)a(\alpha_i)p(y_i,z)\breve{\alpha}_i + c(\beta_i)a(\alpha_i)q(y_i,z)\breve{\alpha}_i = 0.$$

This implies

$$\sum_{i=1}^{r} a(\beta_i)a(\alpha_i)\breve{\beta}_i\breve{\alpha}_i = 0; \tag{1}$$

and $A = 0, Bp(x,z) + Cq(x,z) = 0$, where

$$A := \sum_{p(x,y_i)=0} a(\beta_i)b(\alpha_i) + \sum_{q(x,y_i)=0} a(\beta_i)c(\alpha_i) + \sum_{p(y_i,z)=0} b(\beta_i)a(\alpha_i) + \sum_{q(y_i,z)=0} c(\beta_i)a(\alpha_i);$$

$$B := \sum_{\substack{p(x,y_i)\neq 0 \\ \breve{\beta}_i p(x,y_i)=p(x,z)}} a(\beta_i)b(\alpha_i) + \sum_{\substack{q(x,y_i)\neq 0 \\ \breve{\beta}_i q(x,y_i)=p(x,z)}} a(\beta_i)c(\alpha_i) \\ + \sum_{\substack{p(y_i,z)\neq 0 \\ p(x,y_i)\breve{\alpha}_i=p(x,z)}} b(\beta_i)a(\alpha_i) + \sum_{\substack{q(y_i,z)\neq 0 \\ q(y_i,z)\breve{\alpha}_i=p(x,z)}} c(\beta_i)a(\alpha_i); \text{ and}$$

$$C := \sum_{\substack{p(x,y_i)\neq 0 \\ \breve{\beta}_i p(x,y_i)=q(x,z)}} a(\beta_i)b(\alpha_i) + \sum_{\substack{q(x,y_i)\neq 0 \\ \breve{\beta}_i q(x,y_i)=q(x,z)}} a(\beta_i)c(\alpha_i) \\ + \sum_{\substack{p(y_i,z)\neq 0 \\ p(x,y_i)\breve{\alpha}_i=q(x,z)}} b(\beta_i)a(\alpha_i) + \sum_{\substack{q(y_i,z)\neq 0 \\ q(y_i,z)\breve{\alpha}_i=q(x,z)}} c(\beta_i)a(\alpha_i).$$

Note that (i) if B appears, then $p(x,z) \neq 0$; (ii) if C appears, then $q(x,z) \neq 0$ and $p(x,z) \neq q(x,z)$; and (iii) if both B and C appears, then $p(x,z), q(x,z)$ are linearly independent. In any case we have $B + C = 0$. As a consequence $A + B + C = 0$, thus

$$\sum_{i=1}^{r} a(\beta_i)b(\alpha_i) + a(\beta_i)c(\alpha_i) + b(\beta_i)a(\alpha_i) + c(\beta_i)a(\alpha_i) = 0. \tag{2}$$

The equality (1) yields $a(\beta_1)a(\alpha_1) = a(\beta_2)a(\alpha_2)$ when $r = 2$, and $a(\beta_1)a(\alpha_1) = a(\beta_2)a(\alpha_2) = a(\beta_3)a(\alpha_3)$ when $r = 3$ by mesh-relations. Hence it follows from (2) that

$$\sum_{i=1}^{r} \frac{b(\alpha_i)}{a(\alpha_i)} + \frac{c(\alpha_i)}{a(\alpha_i)} + \frac{b(\beta_i)}{a(\beta_i)} + \frac{c(\beta_i)}{a(\beta_i)} = 0,$$

Thus $d(\pi(\theta_z)) = 0$ as desired.

Then (c) and (d) implies that

$$\sum d(\pi(\theta_z)) + c = 0,$$

where the sum is taken over all meshes $\pi(\theta_z)$ in Γ. But we have

$$\sum d(\pi(\theta_z)) = 2 \sum d(\pi(\alpha)) = 0,$$

where the second sum is taken over all arrows $\pi(\alpha)$ involved in Γ. Hence we have $c = 0$, a contradiction. □

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