On the Cotangent Sheaf of Quot-Schemes

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It is the purpose of this short note to give a *global* description of the cotangent sheaf of Grothendieck's Quot-scheme in terms of a relative Ext-sheaf of the universal subsheaf and the universal quotient sheaf. As an application one gets a description of the cotangent sheaf of the moduli space of stable sheaves in terms of a relative Ext-sheaf involving only a universal family in case such a family exists.

1 Introduction

Let $\mathcal{H}$ be a coherent sheaf on a projective scheme X over an algebraically closed field k. In [1] Grothendieck constructed a projective k-scheme $\mathrm{Quot}(\mathcal{H}, P)$, whose closed points parametrise all possible quotient sheaves F of $\mathcal{H}$ with given Hilbert polynomial P with respect to some fixed ample line bundle $\mathcal{O}_X(1)$. Grothendieck also showed that the Zariski tangent space of $\mathrm{Quot}(\mathcal{H}, P)$ at a closed point $[\rho]$, corresponding to a quotient $\rho : \mathcal{H} \to F$ with kernel $K := \ker(\rho)$, is isomorphic to the space of homomorphisms from K to F:

$$T_{\mathrm{Quot}(\mathcal{H},P)}([\rho]) \cong \mathrm{Hom}(K, F). \tag{1}$$

The question arises how to describe globally the tangent sheaf, or rather the cotangent sheaf, of the Quot-scheme. The answer turns out as follows:

If X is smooth of dimension d and if ω_X denotes the dualising line bundle, we can use Serre duality to rewrite (1) as

$$\mathrm{Ext}^d(F, K \otimes \omega_X) \cong \Omega_{\mathrm{Quot}(\mathcal{H},P)/k}([\rho]). \tag{2}$$

In this form (2) can be globalised: Let $0 \to \mathcal{K} \to \mathcal{O}_{\mathrm{Quot}} \otimes \mathcal{H} \to \mathcal{F} \to 0$ be the universal exact sequence of sheaves on $\mathrm{Quot}(\mathcal{H}, P) \times X$. Let p and q denote the projections from $\mathrm{Quot}(\mathcal{H}, P) \times X$ to $\mathrm{Quot}(\mathcal{H}, P)$ and X, respectively. We claim that there is a natural isomorphism

$$\mathcal{E}xt^d_p(\mathcal{F}, \mathcal{K} \otimes q^*\omega_X) \cong \Omega_{\mathrm{Quot}(\mathcal{H},P)/k}. \tag{3}$$

Note that we do not make any regularity assumptions on $\mathrm{Quot}(\mathcal{H}, P)$. In fact, we will prove this result in the slightly more general framework of a smooth morphism $p : X \to S$ of Noetherian schemes. The main idea for the proof consists in identifying the Quot-scheme as the zero locus of a sheaf homomorphism on an appropriate product of relative Grassmann schemes.

As an application, we get a description of the cotangent sheaf of moduli spaces of stable sheaves: If M is the moduli space of stable sheaves with fixed Hilbert polynomial on the fibres of $X \to S$, and if $\tilde{F}$ is a universal family on $M \times_S X$, then there is an isomorphism $\mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \cong \Omega_{M/S}$.

In order to illustrate the method we begin with the discussion of the easier and also well-known case of the Grassmannian. Afterwards, we first briefly recall Grothendieck's construction of the Quot-scheme and then derive the description of the cotangent sheaf from a more precise formulation of Grothendieck's construction and relative duality.

2 The Grassmannian

Let S be a Noetherian scheme, and let $\alpha : N' \to N$ be a homomorphism of coherent $\mathcal{O}_S$-modules. If N is locally free, then there is a uniquely determined closed subscheme $S_0 \subset S$, the *vanishing locus* of α, such that an arbitrary morphism $g : S' \to S$ factors through S_0 if and only if $g^*(\alpha) = 0$. In fact, if $\alpha' : \mathcal{H}om(N, N') \to \mathcal{O}_S$ is the adjoint of α, then the ideal sheaf $\mathcal{I}_{S_0}$ of S_0 is the image of α'. In particular, if S is a T-scheme, we get an exact sequence $\mathcal{H}om(N|_{S_0}, N'|_{S_0}) \longrightarrow \Omega_{S/T}|_{S_0} \longrightarrow \Omega_{S_0/T} \longrightarrow 0$. This well-known principle can be used to describe the cotangent sheaf of the (relative) Grassmann scheme.

Recall the definition of a Grassmann scheme: If S is a Noetherian scheme, $\mathcal{H}$ a coherent $\mathcal{O}_S$-sheaf and r a positive integer, then there is a projective S-scheme $\pi : G := \mathrm{Grass}(\mathcal{H}, r) \to S$ representing the functor $\underline{\mathrm{Grass}}(\mathcal{H}, r) : (Sch/S) \to (Sets)$, which associates to a morphism $g : T \to S$ the set of all submodules K in $\mathcal{H}_T := g^*\mathcal{H}$ with rank r locally free quotient $\mathcal{H}_T/K$. Let $0 \to A \to \mathcal{H}_G \to B \to 0$ be the universal short exact sequence of $\mathcal{O}_G$-modules.

Let $pr_1, pr_2 : G \times_S G \to G$ denote the projections to the two factors. Any S-morphism $f : T \to G \times_S G$ factors through the diagonal $\Delta \subset G \times_S G$ if and only if the induced quotients $\mathcal{H}_T \to f^*pr_i^*B$ are equivalent, i.e. have the same kernel. Hence the diagonal is the vanishing locus of the composite homomorphism

$$pr_1^*A \longrightarrow \mathcal{H}_{G \times_S G} \longrightarrow pr_2^*B.$$

As B is locally free, we get a natural surjection $\mathcal{H}om(pr_2^*B, pr_1^*A) \longrightarrow \mathcal{I}_\Delta$, which restricted to the diagonal yields a surjection $\varphi_\mathcal{H} : \mathcal{H}om(B, A) \longrightarrow \Omega_{G/S}$.

Theorem 2.1 — *$\varphi_\mathcal{H} : \mathcal{H}om(B, A) \longrightarrow \Omega_{\mathrm{Grass}(\mathcal{H},r)/S}$ is an isomorphism.*

Proof. Assume first that $\mathcal{H}$ is free of rank n, i.e. $\mathcal{H} \cong \mathcal{O}_S^n$. Then $G \cong S \times_\mathbb{Z} \mathrm{Grass}(\mathbb{Z}^n, r)$ is smooth over S of relative dimension $r(n-r)$. Moreover, A is locally free of rank $n-r$. Hence both $\mathcal{H}om(B, A)$ and $\Omega_{G/S}$ are locally free sheaves of the same rank $r(n-r)$. As a surjective homomorphism of locally free sheaves of the same rank $\varphi_\mathcal{H}$ must be an isomorphism.

If $\mathcal{H}$ is arbitrary, we may always assume that $\mathcal{H}$ is the quotient of a free sheaf $\mathcal{H}'$, since the question whether $\varphi_\mathcal{H}$ is an isomorphism is certainly local in S. The surjection $\rho : \mathcal{H}' \to \mathcal{H}$ induces a closed immersion $G := \mathrm{Grass}(\mathcal{H}, r) \longrightarrow G' := \mathrm{Grass}(\mathcal{H}', r)$. Let $0 \to A' \to \mathcal{H}'_{G'} \to B' \to 0$ be the tautological exact sequence on G'. Then G

is the vanishing locus of the composite homomorphism $\ker(\rho)_{G'} \to \mathcal{H}'_{G'} \to B'$. As above we get a surjection $\tilde{\varphi} : \mathcal{H}om_G(B, \ker(\rho)_G) \longrightarrow \mathcal{I}_G/\mathcal{I}_G^2$, where $\mathcal{I}_G$ is the ideal sheaf of G in G'. (Note that $B'|_G = B$.) This leads to a commutative diagram with exact rows and surjective vertical arrows:

$$\begin{array}{ccccccc}
\mathcal{I}_G/\mathcal{I}_G^2 & \longrightarrow & \Omega_{G'/S}|_G & \longrightarrow & \Omega_{G/S} & \longrightarrow & 0 \\
\Big\uparrow \tilde{\varphi} & & \Big\uparrow \varphi_{\mathcal{H}'}|_G & & \Big\uparrow \varphi_{\mathcal{H}} & & \\
\mathcal{H}om(B, \ker(\rho)_G) & \longrightarrow & \mathcal{H}om(B, \mathcal{H}'_G) & \longrightarrow & \mathcal{H}om(B, \mathcal{H}_G) & \longrightarrow & 0.
\end{array}$$

Diagram chase shows that if $\varphi_{\mathcal{H}'}$ is an isomorphism, then so is $\varphi_{\mathcal{H}}$. $\square$

3 The Quot-Scheme

Let $p : X \to S$ be a smooth projective morphism of Noetherian schemes of relative dimension d. Let $\mathcal{O}_X(1)$ be an invertible $\mathcal{O}_X$-sheaf on X that is very ample relative to S. Replacing it by some tensor power if necessary, we may assume that the graded algebra $\mathcal{S} = \oplus \mathcal{S}_i$, $\mathcal{S}_i := p_*\mathcal{O}_X(i)$, is generated by $\mathcal{S}_1$. Moreover, let $\omega_{X/S}$ be the relative dualising sheaf.

Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-sheaf, and let P be a numerical polynomial, i.e. $P(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$. Grothendieck [1] showed that there is a projective S-scheme $Q := \mathrm{Quot}(\mathcal{H}, P)$ that represents the set-valued functor on S-schemes, which associates to a morphism $g : S' \to S$ the set of all coherent subsheaves $K' \subset \mathcal{H}_{S'} := \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{H}$ such that $\mathcal{H}_{S'}/K'$ is S'-flat with Hilbert polynomial P with respect to $\mathcal{O}_X(1)$. Let $0 \to \mathcal{K} \to \mathcal{H}_Q \to \mathcal{F} \to 0$ be the universal short exact sequence of sheaves on $Q \times_S X$. Let $p : Q \times_S X \to Q$ and $q : Q \times_S X \to X$ denote the projetions.

Theorem 3.1 — *There is a natural isomorphism of $\mathcal{O}_{\mathrm{Quot}(\mathcal{H},P)}$-sheaves*

$$\Phi : \mathcal{E}xt_p^d(\mathcal{F}, \mathcal{K} \otimes q^*\omega_{X/S}) \longrightarrow \Omega_{\mathrm{Quot}(\mathcal{H},P)/S}.$$

We begin with an inspection of Grothendieck's construction of Q: For each integer m let $\mathcal{K}_m := p_*\mathcal{K}(m)$ etc., and consider the sequences

$$0 \to \mathcal{K}_m \to (\mathcal{H}_Q)_m \to \mathcal{F}_m \to 0. \tag{4}$$

Then there is an integer m_0 such that for all $m \geq m_0$ we have:
(i) The sequence (4) is exact;
(ii) $\mathcal{K}_m$ generates the graded $\mathcal{S}$-module $\oplus_{m' \geq m} \mathcal{K}_{m'}$;
(iii) $\mathcal{F}_m$ is locally free of rank $P(m)$; and
(iv) $(\mathcal{H}_Q)_m = (\mathcal{H}_m)_Q$.
By (i), the graded $\mathcal{S}$-module $\oplus_{m' \geq m} \mathcal{F}_{m'}$, and hence $\mathcal{F}$, is determined by $\oplus_{m' \geq m} \mathcal{K}_{m'}$. By (ii), the submodule $\oplus_{m' \geq m} \mathcal{K}_{m'} \subset \oplus_{m' \geq m} (\mathcal{H}_Q)_{m'}$ is determined by $\mathcal{K}_m \subset (\mathcal{H}_Q)_m$. And by (iii) and (iv) the submodule $\mathcal{K}_m \subset (\mathcal{H}_m)_Q$ is determined by a classifying morphism $\iota_m : Q \to G_m := \mathrm{Grass}(\mathcal{H}_m, P(m))$. Altogether this implies that ι_m is in fact a closed immersion. The next step consists in showing that, for a suitable sequence of

integers $n_0 < \ldots < n_\ell$, the image of the immersion $(\iota_0, \ldots, \iota_\ell) : Q \to \prod G_{n_i}$ is the zero locus of an appropriate sheaf homomorphism.

For each integer m let $0 \longrightarrow A_m \xrightarrow{\alpha_m} (\mathcal{H}_m)_{G_m} \xrightarrow{\beta_m} B_m \longrightarrow 0$ be the universal sequence on G_m. For any pair of integers $m' > m$ consider the homomorphism

$$\psi_{m,m'} : \mathcal{S}_{m'-m} \otimes pr_1^* A_m \longrightarrow pr_2^* B_{m'}$$

of sheaves on the product $G_m \times G_{m'}$, which is defined as the composition $\psi_{m,m'} := pr_2^*(\beta_{m'}) \circ \sigma_{\mathcal{H}} \circ (\mathrm{id}_{\mathcal{S}_{m'-m}} \otimes pr_1^*(\alpha_m))$, where $\sigma_{\mathcal{H}} : \mathcal{S}_{m'-m} \otimes \mathcal{H}_m \to \mathcal{H}_{m'}$ is the multiplication map.

Lemma 3.2 — *There are integers $m_0 \leq n_0 < n_1 < \ldots < n_\ell$ such that the image of the closed immersion*

$$\iota = (\iota_{n_i})_{i=0}^{\ell} : \mathrm{Quot}(\mathcal{H}, P) \longrightarrow \prod_{i=0}^{\ell} \mathrm{Grass}(p_* \mathcal{H}(n_i), P(n_i))$$

is an open and closed subscheme of the vanishing locus of the homomorphism $\psi := \bigoplus_{i=1}^{\ell} \psi_{n_0,n_i} : \bigoplus_{i=1}^{\ell} \mathcal{S}_{n_i - n_0} \otimes pr_0^* A_{n_0} \longrightarrow \bigoplus_{i=1}^{\ell} pr_i^* B_{n_i}$.

Proof. Let $n_0 \geq m_0$ (so that the conditions (i) to (iv) above are satisfied), and let $G := G_{n_0}$. The inclusion $\alpha_{n_0} : A_{n_0} \to (\mathcal{H}_{n_0})_G$ induces a homomorphism of graded $\mathcal{S}$-sheaves

$$\chi : \mathcal{S}(-n_0) \otimes A_{n_0} \to \oplus_{n \geq 0} (\mathcal{H}_n)_G,$$

which in turn corresponds to a homomorphisms of coherent sheaves

$$\chi' : \mathcal{O}_X(-n_0) \otimes_{\mathcal{O}_S} A_{n_0} \longrightarrow \mathcal{H}_G$$

on $G \times_S X$. Let $C := \mathrm{coker}(\chi)$, and let $\mathcal{C} := \mathrm{coker}(\chi')$. In general, $\mathcal{C}$ is *not* flat over G. But there is a *flattening stratification* of G for $\mathcal{C}$ [6]: The set $\mathcal{P}$ of Hilbert polynomials $\{P(\mathcal{C}_g) | \, g \in G\}$ is finite, and there is a decomposition of G into finitely many locally closed subschemes $G_{P'}$, $P' \in \mathcal{P}$, such that a morphism $f : T \to G$ factors through

$$j : \coprod_{P' \in \mathcal{P}} G_{P'} \longrightarrow G$$

if and only if $\mathcal{C}_T$ is flat. Moreover, there is an integer $m_1 \geq n_0$ such that each $G_{P'}$ is characterized as the maximal locally closed subscheme in G such that $C_n|_{G_{P'}}$ is locally free of rank $P'(n)$ for all $n \geq m_1$.

Then $\iota_{n_0}(Q) = G_P$ as subschemes in G. Note that *a priori* the subscheme G_P is only locally closed and that an additional argument involving the valuative criterion is needed to conclude that Q is proper over S and hence that G_P is closed in G. In particular, there is an open neighbourhood U of $\iota_{n_0}(Q)$ in G and an integer $m_2 \geq m_1$ such that for all $n \geq m_2$ and all $u \in U$ we have $P(\mathcal{C}_u, n) = \dim_{k(u)}(C_n(u)) \leq P(n)$; and if we denote by $Z_n \subset U$ the maximal closed subset in U such that $C_n|_{Z_n}$ is locally free of rank $P(n)$, then $\iota(Q) = \bigcap_{n \geq m_2} Z_n$. As G is Noetherian, it suffices to take finitely many integers $m_2 \leq n_1 < \ldots < n_\ell$ such that $\iota(Q) = \bigcap_{i=1}^{\ell} Z_{n_i}$. For each n_i we conclude as follows: The coherent sheaf C_{n_i} has rank $\leq P(n_i)$ everywhere on U. Hence, for any morphism $f : T \to U$ a *sur*jection $f^*C_{n_i} \to N$ onto a locally free sheaf N of rank $P(n_i)$ is necessarily an *iso*morphism, so that there is at most one such surjection for each f. This implies that $\mathrm{Grass}(C_{n_i}|_U, P(n_i)) \to U$ is a closed immersion with scheme theoretic image Z_{n_i}. In fact, the inclusion $Z_{n_i} \to U$ factors through a closed immersion into $U \times G_{n_i}$, whose image is precisely the vanishing locus of $\psi_{n_0,n_i}|_{U\times G_{n_i}}$. It follows that $\iota(Q) \subset U \times \prod_{i=1}^{\ell} G_{n_i}$ is the vanishing locus of $\psi|_{U\times\prod_{i=1}^{\ell} G_{n_i}}$. Replacing U by $G = G_{n_0}$ we get the assertion of the lemma. $\square$

Remark 3.3 — It is tempting to eliminate the phrase 'an open and closed subscheme of' from the lemma. But I do not know of a good reason to justify this.

Corollary 3.4 — *There is an exact sequence of $\mathcal{O}_Q$-sheaves*

$$\bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{S}_{n_i-n_0} \otimes \mathcal{K}_{n_0}) \xrightarrow{\Lambda} \bigoplus_{i=0}^{\ell} \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{K}_{n_i}) \longrightarrow \Omega_{Q/S} \longrightarrow 0.$$

Proof. This follows from Lemma 3.2, Theorem 2.1, and the fact that $\iota^*(A_{n_i}) = \mathcal{K}_{n_i}$ and $\iota^*(B_{n_i}) = \mathcal{F}_{n_i}$. $\square$

Next, we need to identify the entries of the $(\ell+1)\times\ell$-matrix Λ. Let $m' > m$ be integers. The vanishing locus $Z \subset G_m \times G_{m'}$ of the homomorphism $\psi_{m,m'}$ is the maximal closed subscheme such that $\sigma_{\mathcal{H}}$ maps $\mathcal{S}_{m'-m} \otimes pr_1^* A_m|_Z$ to $pr_2^* A_{m'}|_Z$ and hence induces multiplication maps $\sigma_A : \mathcal{S}_{m'-m} \otimes pr_1^* A_m|_Z \to pr_2^* A_{m'}|_Z$ and $\sigma_B : \mathcal{S}_{m'-m} \otimes pr_2^* B_{m'}^\vee|_Z \to pr_1^* B_m^\vee|_Z$. Then one easily checks:

Lemma 3.5 — *There is a commutative diagram of $\mathcal{O}_Z$-sheaves*

$$\begin{array}{ccc}
\mathcal{I}_Z/\mathcal{I}_Z^2 & \longrightarrow & \mathrm{pr}_1^*\Omega_{G_m/S}|_Z \oplus pr_2^*\Omega_{G_{m'}/S}|_Z \\
\uparrow & & \uparrow \cong \\
(pr_2^* B_{m'}^\vee \otimes \mathcal{S}_{m'-m} \otimes pr_1^* A_m)|_Z & \xrightarrow{\lambda} & pr_1^*(B_m^\vee \otimes A_m)|_Z \oplus pr_2^*(B_{m'}^\vee \otimes A_{m'})|_Z,
\end{array}$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in $G_m \times_s G_{m'}$, and the homomorphism λ is given by $\lambda(f) = (-f \circ \sigma_B, \sigma_A \circ f)$. $\square$

Therefore, let

$$\lambda_i' : \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{S}_{n_i-n_0} \otimes \mathcal{K}_{n_0}) \to \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{K}_{n_i})$$

and

$$\lambda_i'' : \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{S}_{n_i-n_0} \otimes \mathcal{K}_{n_0}) \to \mathcal{H}om(\mathcal{F}_{n_0}, \mathcal{K}_{n_0})$$

be the homomorphisms induced by the multiplication maps

$$\mathcal{S}_{n_i-n_0} \otimes \mathcal{K}_{n_0} \to \mathcal{K}_{n_i} \qquad \text{and} \qquad \mathcal{S}_{n_i-n_0} \otimes \mathcal{F}_{n_i}^\vee \to \mathcal{F}_{n_0}^\vee,$$

respectively. As a consequence of Lemma 3.5 we see that the entries of Λ in Corollary 3.4 are given as follows: $\Lambda_{0i} = -\lambda_i''$ and $\Lambda_{ii} = \lambda_i'$, for $i = 1, \ldots, \ell$, and $\Lambda_{jk} = 0$ else.

Let $\mathcal{K}'$ be the kernel of the surjection $p^*\mathcal{K}_{n_0}(-n_0) \to \mathcal{K}$. There is an integer $m_3 \geq m_2$ such that for all $n \geq m_3$ the higher direct images $R^i p_* \mathcal{K}'(n)$, $i > 0$, vanish and $p^*\mathcal{K}'_n \to \mathcal{K}'(n)$ is surjective. Thus, p_* applied to

$$0 \to \mathcal{K}'(n) \to p^*\mathcal{K}_{n_0} \otimes \mathcal{O}_X(n - n_0) \to \mathcal{K}(n) \to 0$$

yields exact sequences

$$0 \to \mathcal{K}'_n \to \mathcal{K}_{n_0} \otimes \mathcal{S}_{n-n_0} \to \mathcal{K}_n \to 0.$$

In particular, assuming that $n_1 > m_3$, we see that for all $i > 0$ the homomorphism λ_i' is surjective and has kernel $\mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{K}'_{n_i})$. This yields:

Corollary 3.6 — *The cotangent sheaf $\Omega_{Q/S}$ is isomorphic to the cokernel of the homomorphism*

$$\overline{\Lambda} : \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{F}_{n_i}, \mathcal{K}'_{n_i}) \longrightarrow \mathcal{H}om(\mathcal{F}_{n_0}, \mathcal{K}_{n_0}),$$

where the i-th component of $\overline{\Lambda}$ is induced by the inclusion $\mathcal{K}'_{n_i} \subset \mathcal{S}_{n_i - n_0} \otimes \mathcal{K}_{n_0}$ and the natural map $\mathcal{F}^\vee_{n_i} \otimes \mathcal{S}_{n_i - n_0} \to \mathcal{F}^\vee_{n_0}$. □

Proof of Theorem 3.1. The assumption that X/S be smooth and the fact that $\mathcal{F}$ is Q-flat imply that $\mathcal{E}xt^i_p(\mathcal{F}, N) = 0$ for $i > d$ and any coherent sheaf N. Thus, $\mathcal{E}xt^d_p(\mathcal{F}, \,.\,)$ is right exact, and, moreover, it commutes with base change. Let $\omega = \omega_{X_Q/Q} = q^*\omega_{X/S}$ be the relative dualising sheaf. Applying $\mathcal{E}xt^d_p(\mathcal{F}, \,.\, \otimes \omega)$ to the exact sequence

$$\bigoplus_{i=1}^{\ell} p^*\mathcal{K}'_{n_i}(-n_i) \longrightarrow p^*\mathcal{K}_{n_0}(-n_0) \longrightarrow \mathcal{K} \longrightarrow 0$$

we get a new exact sequence

$$\begin{aligned} \textstyle\bigoplus_{i=1}^{\ell} \mathcal{E}xt^d_p(\mathcal{F}, p^*\mathcal{K}'_{n_i}(-n_i) \otimes \omega) \xrightarrow{\widetilde{\Lambda}} \mathcal{E}xt^d_p(\mathcal{F}, p^*\mathcal{K}_{n_0}(-n_0) \otimes \omega) \longrightarrow \\ \longrightarrow \mathcal{E}xt^d_p(\mathcal{F}, \mathcal{K} \otimes \omega) \longrightarrow 0. \end{aligned}$$

By relative duality [2] we have isomorphisms

$$\mathcal{E}xt^d_p(\mathcal{F}, p^*\mathcal{K}_{n_0}(-n_0) \otimes \omega) \cong \mathcal{E}xt^d_p(\mathcal{F}(n_0), p^*\mathcal{K}_{n_0} \otimes \omega) \cong \mathcal{H}om(p_*\mathcal{F}(n_0), \mathcal{K}_{n_0}),$$

and similarly,

$$\mathcal{E}xt^d_p(\mathcal{F}, p^*\mathcal{K}'_{n_i}(-n_i) \otimes \omega) \cong \mathcal{E}xt^d_p(\mathcal{F}(n_i), p^*\mathcal{K}'_{n_i} \otimes \omega) \cong \mathcal{H}om(p_*\mathcal{F}(n_i), \mathcal{K}'_{n_i}).$$

With respect to these isomorphisms the homomorphism $\widetilde{\Lambda}$ equals $\overline{\Lambda}$. We conclude:

$$\mathcal{E}xt^d_p(\mathcal{F}, \mathcal{K} \otimes \omega) \cong \operatorname{coker}(\widetilde{\Lambda}) \cong \operatorname{coker}(\overline{\Lambda}) \cong \Omega_{Q/S}.$$

This finishes the proof. □

Remark 3.7 — Let $\rho' : \mathcal{H}_{S'} \to \mathcal{F}'$ be an S'-flat quotient with Hilbert polynomial P and let $g : S' \to Q$ be the classifying morphism. Then $g_X^*\mathcal{K} \cong \ker(\rho') =: \mathcal{K}'$ and $g_X^*\mathcal{F} \cong \mathcal{F}'$, and moreover, $g^*\mathcal{E}xt_p^d(\mathcal{F}, \mathcal{K}\otimes\omega) \cong \mathcal{E}xt_p^d(\mathcal{F}', \mathcal{K}'\otimes\omega)$. An argument similar to the one above yields a homomorphism $\mathcal{E}xt_p^d(\mathcal{F}', \mathcal{K}'\otimes\omega) \to \Omega_{S'/S}$ which, using the isomorphism of the main theorem, is the differential dg of g. This is the analogue of the Kodaira-Spencer map.

As a consequence of Theorem 3.1 we also recover Grothendieck's description of the Zariski tangent space and the smoothness criterion as a corollary to the base change theorem (cf. [1]):

Corollary 3.8 — *Assume that S is of finite type over a field of characteristic zero, and that $\mathcal{H}$ is S-flat. Let $s \in S$ be a closed point, and let $q \in Q$ be a closed point in the fibre over s corresponding to a short exact sequence $0 \to K \to \mathcal{H}_s \to F \to 0$. Then the Zariski tangent space of the fibre Q_s at q is given by $T_qQ_s \cong \mathrm{Hom}(K, F)$. If $\mathrm{Ext}^1(K, F) = 0$, then Q is smooth over S at q.*

Proof. Since $\mathcal{E}xt_p^d(\mathcal{F}, \,.\,)$ commutes with base change, one gets

$$\begin{aligned} \Omega_{Q_s/k(s)}(q) &\cong \Omega_{Q/S}(q) \cong \mathcal{E}xt_p^d(\mathcal{F}, \mathcal{K}\otimes\omega)(q) \\ &\cong \mathrm{Ext}^d(\mathcal{F}_q, \mathcal{K}_q\otimes\omega) \cong \mathrm{Hom}(\mathcal{K}_q, \mathcal{F}_q)^\vee. \end{aligned}$$

Dualising one finds $T_qQ_s \cong \mathrm{Hom}(K, F)$. If $\mathrm{Ext}^{d-1}(F, K\otimes\omega) = \mathrm{Ext}^1(K, F)^\vee$ vanishes, then the homomorphism $\mathcal{E}xt_p^{d-1}(\mathcal{F}, \mathcal{K}\otimes\omega)\otimes k(q) \to \mathrm{Ext}^{d-1}(F, K\otimes\omega)$ is surjective for trivial reasons. The flatness assumption for $\mathcal{H}$ implies that $\mathcal{K}$ is Q-flat. Therefore, the base change theorem for the relative Ext sheaves applies and says that $\mathcal{E}xt_p^d(\mathcal{F}, \mathcal{K}\otimes\omega) \cong \Omega_{Q/S}$ is locally free at q. As the characteristic of the base field is zero, the local freeness of $\Omega_{Q/S}$ at q implies that Q is locally smooth over S at q. □

4 Moduli Spaces of Stable Sheaves

We keep the notations and assumptions on $p : X \to S$ of the previous section, and assume in addition that p has geometrically integral fibres and that S is of finite type over a field of characteristic zero. Let δ be the degree of the polynomial P. Let F be a sheaf on the fibre $X_s = X \times_S \mathrm{Spec}(k(s))$ for some point $s \in S$. Recall that F is geometrically stable, if for any field extension $k(s) \subset \tilde{k}$ and any nontrivial proper coherent subsheaf $F' \subset \tilde{k}\otimes F$ one has

$$r(F)\cdot P(F', n) < r(F')\cdot P(F, n) \text{ for all } n \gg 0,$$

where $r(F')$ is the multiplicity of F', i.e. the coefficient of n^δ in the Hilbert polynomial $P(F', n)$. Simpson [7] showed that there is quasi-projective S-scheme $M = M_{X/S}(P)$, the *moduli space of stable sheaves*, that corepresents the functor which associates to an S-scheme S' the set of isomorphism classes of S'-flat families of geometrically stable sheaves with Hilbert polynomial P.

Theorem 4.1 — *Suppose there exists a universal family $\tilde{F}$ on $M \times_S X$. Then there is an isomorphism $\Phi_M : \mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \longrightarrow \Omega_{M/S}$.*

This theorem is a consequence of the more general Theorem 4.2 below. In order to formulate it, we need to introduce some more notations (for a detailed presentation see also [3]): Because of the boundedness of the family of stable sheaves there is an integer m such that any geometrically stable sheaf F on the fibres of p with Hilbert polynomial P is m-regular. Let $\mathcal{H} := \mathcal{O}_X(-m)^{\oplus P(m)}$, let $Q := \mathrm{Quot}(\mathcal{H}, P)$ and let $0 \to \mathcal{K} \to \mathcal{H}_Q \to \mathcal{F} \to 0$ be the universal sequence on $Q \times_S X$. Let $R \subset Q$ be the maximal open subscheme such that for all points $s \in R$ the sheaf $\mathcal{F}_s$ is geometrically stable and such that the surjection $\mathcal{H}_R \to \mathcal{F}_R$ induces an isomorphism $\mathcal{O}_R^{\oplus P(m)} \to \mathcal{F}_m|_R$. Note that this and the regularity assumption for stable sheaves imply that

$$\mathcal{E}xt_p^i(\mathcal{F}_R, \mathcal{H}_R \otimes \omega) = 0 \text{ for all } i \neq d \tag{5}$$

and, by relative duality, that there is an isomorphism

$$t : \mathcal{E}xt_p^d(\mathcal{F}_R, \mathcal{H}_R \otimes \omega) \xrightarrow{\cong} \mathcal{H}om(\mathcal{F}_m|_R, \mathcal{O}_R^{\oplus P(m)}) \xrightarrow{\cong} \mathcal{E}nd(\mathcal{O}_R^{\oplus P(m)}) \tag{6}$$

The group $\mathrm{GL}(P(m))$ acts on Q (and on R) on the right via composition. Moreover, the centre acts trivially, so that the $\mathrm{GL}(P(m))$-action induces a $\mathrm{PGL}(P(m))$-action on R. Using Geometric Invariant Theory, Simpson [7] showed that there is a $\mathrm{PGL}(P(m))$-invariant morphism $\pi : R \to M$ which is a principal fibre bundle, i.e. there is an étale surjective morphism $M' \to M$ so that $\pi' : R' := M' \times_M R \to M'$ is equivariantly isomorphic to the projection $M' \times \mathrm{PGL}(P(m)) \to M'$. We can now state:

Theorem 4.2 — *There is a natural* $\mathrm{GL}(P(m))$*-equivariant isomorphism*

$$\Phi' : \mathcal{E}xt_p^{d-1}(\mathcal{F}_R, \mathcal{F}_R \otimes \omega) \longrightarrow \pi^*\Omega_{M/S}.$$

Proof. Let $\sigma : Y \times_S G \to Y$ be an action of an algebraic group G/S on an S-scheme Y. The pull-back of the composite homomorphism

$$\sigma^*\Omega_{Y/S} \xrightarrow{d\sigma} pr_1^*\Omega_{Y/S} \oplus pr_2^*\Omega_{G/S} \xrightarrow{pr} pr_2^*\Omega_{G/S}$$

under the closed immersion $(\mathrm{id}, e) : Y \to Y \times_S G$ is a homomorphism $\mu : \Omega_{Y/S} \to \mathcal{O}_Y \otimes \mathfrak{g}^\vee$ to the dual of the Lie algebra $\mathfrak{g}$ of G which describes the infinitesimal action of G. Returning to the situation of the theorem we see that the following sequence is exact:

$$0 \longrightarrow \pi^*\Omega_{M/S} \xrightarrow{d\pi} \Omega_{R/S} \xrightarrow{\mu} \mathcal{O}_R \otimes \mathfrak{pgl}(P(m))^\vee \longrightarrow 0. \tag{7}$$

Indeed, after some étale base change if necessary, we may assume that π is a projection $M \times \mathrm{PGL}(P(m)) \to M$, in which case the assertion is clear. Now consider the following commutative diagram with equivariant maps, exact rows and isomorphic vertical arrows:

$$\begin{array}{ccccccc}
\Omega_{R/S} & \xrightarrow{\mu} & \mathcal{O}_R \otimes \mathfrak{gl}(P(m))^\vee & \to & \mathcal{O}_R & \to & 0 \\
\uparrow \Phi & & \uparrow t & & \uparrow tr & & \\
\mathcal{E}xt_p^d(\mathcal{F}_R, \mathcal{K}_R \otimes \omega) & \to & \mathcal{E}xt_p^d(\mathcal{F}_R, \mathcal{H}_R \otimes \omega) & \to & \mathcal{E}xt_p^d(\mathcal{F}_R, \mathcal{F}_R \otimes \omega) & \to & 0,
\end{array}$$

where $tr : \mathcal{E}xt_p^d(\mathcal{F}_R, \mathcal{F}_R \otimes \omega) \to R^d p_* \omega \cong \mathcal{O}_R$ is the trace map. Since by (5) $\mathcal{E}xt_p^{d-1}(\mathcal{F}_R, \mathcal{H}_R \otimes \omega) = 0$, and since the kernel of μ is isomorphic to $\pi^*\Omega_{M/S}$ by (7), we see that Φ induces an equivariant isomorphism $\Phi' : \mathcal{E}xt_p^{d-1}(\mathcal{F}_R, \mathcal{F}_R \otimes \omega) \to \pi^*\Omega_{M/S}$. This proves Theorem 4.2. $\square$

Proof of Theorem 4.1. If $\tilde{F}$ is a universal family on $M \times_S X$, then there is a line bundle A on R, together with a $\mathrm{GL}(P(m))$-linearisation, such that $\pi^*\tilde{F}$ is equivariantly isomorphic to $\mathcal{F}_R \otimes p^*A$. It follows that there are equivariant isomorphisms

$$\pi^*\mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \cong \mathcal{E}xt_p^{d-1}(\mathcal{F}_R \otimes p^*A, \mathcal{F}_R \otimes p^*A \otimes \omega) \cong \mathcal{E}xt_p^{d-1}(\mathcal{F}_R, \mathcal{F}_R \otimes \omega).$$

The resulting isomorphism $\pi^*\mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \longrightarrow \pi^*\Omega_{M/S}$ is equivariant and therefore descends to an isomorphism $\Phi_M : \mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \longrightarrow \Omega_{M/S}$. $\square$

Remark 4.3 — The action of $\mathrm{GL}(P(m))$ on $\mathcal{E}xt_p^i(\mathcal{F}_R, \mathcal{F}_R \otimes \omega)$ is induced by conjugation. This implies that the centre of $\mathrm{GL}(P(m))$ acts trivially. As a consequence, the sheaves $\mathcal{E}xt_p^i(\mathcal{F}_R, \mathcal{F}_R \otimes \omega)$ descend to coherent sheaves $\mathcal{E}^i$ on M, even if a universal family $\tilde{F}$ on $M \times_S X$ does not exist. (If it does, there are, of course, isomorphisms $\mathcal{E}^i \cong \mathcal{E}xt_p^i(\tilde{F}, \tilde{F} \otimes \omega)$.) Using these notations, the trace map always gives an isomorphism $\mathcal{E}^d \cong \mathcal{O}_M$, and Theorem 4.2 implies $\mathcal{E}^{d-1} \cong \Omega_{M/S}$.

Analogously to the case of the Quot-scheme one recovers the well-known description of the Zariski tangent space of the moduli space and a smoothness criterion [4, 5, 7]:

Corollary 4.4 — *Let $s \in S$ be a closed point, and let $[F] \in M$ be a closed point in the fibre over s corresponding to a stable sheaf F on the fibre X_s. Then the Zariski tangent space of M_s at $[F]$ is given by $T_{[F]}M \cong \mathrm{Ext}^1(F, F)$. Moreover, if the trace homomorphism $tr : \mathrm{Ext}^2(F, F) \to H^2(X_s, \mathcal{O}_{X_s})$ is injective, M is locally smooth over S at $[F]$.*

Proof. Assume that there is a universal family $\tilde{F}$ on $M \times_S X$. Since the trace homomorphism $tr : \mathcal{E}xt_p^d(\tilde{F}, \tilde{F} \otimes \omega) \to \mathcal{O}_M$ is an isomorphism, the base change theorem holds for $\mathcal{E}xt^{d-1}$. It follows that

$$\begin{aligned} \Omega_{M_s/k(s)}([F]) &\cong \Omega_{M/S}([F]) \cong \mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega)([F]) \\ &\cong \mathrm{Ext}^{d-1}(F, F \otimes \omega) \cong \mathrm{Ext}^1(F, F)^\vee \end{aligned}$$

and hence $T_{[F]}M_s \cong \mathrm{Ext}^1(F, F)$. Let $i : R^\nu p_*(\omega) \to \mathcal{E}xt_p^\nu(\tilde{F}, \tilde{F} \otimes \omega)$ be the natural morphism induced by the $\mathcal{O}_M$-module structure of $\tilde{F}$ [3, Ch.10]. It is dual to the trace homomorphism. Thus, to say that the trace $tr : \mathrm{Ext}^2(F, F) \to H^2(X_s, \mathcal{O}_{X_s})$ is injective is equivalent to the assertion that the map labelled $i_{[F]}$ in the following commutative diagram is surjective:

$$\begin{array}{ccc} R^{d-2}p_*(\omega)([F]) & \longrightarrow & H^{d-2}(X_s, \omega) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_{[F]}} \\ \mathcal{E}xt_p^{d-2}(\tilde{F}, \tilde{F} \otimes \omega)([F]) & \xrightarrow{\varphi} & \mathrm{Ext}^{d-2}(F, F \otimes \omega) \end{array}$$

But then φ must be surjective as well, so that the base change theorem implies that $\mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \cong \Omega_{M/S}$ is locally free at $[F]$. Since the characteristic of the base field is zero, this shows that M is locally smooth over S at $[F]$.

If there is no universal family on $M \times_S X$ one can use essentially the same argument on R, working with $\mathcal{F}_R$ instead. $\square$

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