Non-bicritical Critical Snarks

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Abstract

Snarks are cubic graphs with chromatic index $\chi' = 4$. A snark G is called critical if $\chi'(G - \{v, w\}) = 3$ for any two adjacent vertices v and w, and it is called bicritical if $\chi'(G - \{v, w\}) = 3$ for any two vertices v and w. We construct infinite families of critical snarks which are not bicritical. This solves a problem stated by Nedela and Škoviera in [7].

1 Introduction

We are using standard graph theoretical terminology and notation in this paper. We define a *snark* to be a cubic graph with chromatic index $\chi' = 4$.
The study of reduction of snarks is as old as the study of snarks itself. The first reductions which are considered were edge-cut reduction and decomposition of snarks, cf. [2, 5, 6, 7, 8, 11, 12]. In [7] Nedela and Škoviera gave these two concepts a precise meaning, and we refer the reader to this paper for a more extensive introduction. In that paper the reduction of snarks is defined as follows.
Let $G = (V(G), E(G))$ be a snark and let $F \subset E(G)$ be an k-edge cut ($k \geq 0$) whose removal divides G into two components H_1 and H_2. If the chromatic index of one of the components is 4, say $\chi'(H_1) = 4$, then H_1 can be extended to a snark $H = (V(H), E(H))$ with $|V(H)| \leq |V(G)|$, and H is called a *k-reduction* of G. If $|V(H)| < |V(G)|$ then H is called a *proper* k-reduction of G.
A snark is called *k-irreducible* if it has no proper m-reduction for each $m < k$, and it is called *irreducible* if it k-irreducible for each $k \geq 1$.
A snark G is called *critical* if $\chi'(G - \{v, w\}) = 3$ for any two adjacent vertices $v, w \in V(G)$, and it is called *cocritical* if $\chi'(G - \{v, w\}) = 3$ for any two nonadjacent vertices v and w of G. Finally, a snark is called *bicritical* if it is critical and cocritical.
Nedela and Škoviera proved the following characterizations.

Theorem 1.1 ([7]) *Let G be a snark. Then the following statements hold true.*

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1. *If $5 \leq k \leq 6$, then G is k-irreducible if and only if it is critical.*

2. *If $k \geq 7$, then G is k-irreducible if and only if it is bicritical.*

Finally it turns out that

Theorem 1.2 ([7]) *A snark is irreducible if and only if it is bicritical.*

In [10] the author showed that there are bicritical of order n for each even $n \geq 92$. In that paper there is also given another characterization of bicritical snarks that yields a method for constructing all snarks starting from the bicritical ones.
In [1] it is shown that there are cocritical snarks which are not critical, and that there are snarks which are neither critical nor cocritical. Further there is no snark of order smaller than 30 that is critical and not cocritical, i.e. each critical snark on less than 30 vertices is bicritical and henceforth it is irreducible.
Nedela and Škoviera [7] state the following Problem.

Problem 1.3 ([7]) *Does there exist a snark that is critical but not bicritical? Equivalently, does there exist a 6-irreducible snark that is not irreducible?*

The main result of this paper is the following theorem, which answers the question above in the affirmative.

Theorem 1.4 *There are infinitely many critical snarks which are not bicritical.*

As a corollary we state

Corollary 1.5 *There are critical snarks which are not bicritical on n vertices for $n = 32$ and for each $n \in \{8k + 2 | k \geq 5\}$.*

To prove these two statements we construct infinite families of critical snarks which are not bicritical.

2 Construction of non-bicritical critical Snarks

The 7-block B_i is the graph obtained from the cycle C_6 on the vertices $v_{7i}, v_{7i+1}, \ldots, v_{7i+5}$ by adding a vertex v_{7i+6} and edges $v_{7i+6}v_{7i+1}$ and $v_{7i+6}v_{7i+4}$.
The Goldberg-graph F is obtained from three 7-blocks B_0, B_1 and B_2 by adding a vertex v_{21} and edges $v_{21}v_6, v_{21}v_{13}, v_{21}v_{20}$ and $v_0v_{12}, v_3v_9, v_7v_{19}, v_{10}v_{16}$. The graph is drawn in Figure 1.
Goldberg [5] showed that F is a 3-critical graph, where a graph is called *k-critical* ($k \geq 1$) if $\Delta(G) = k$, $\chi'(G) = k + 1$ and $\chi'(G - e) < \chi'(G)$ for each edge e of G. It is well known that each 3-critical graph contains a vertex of degree 2.

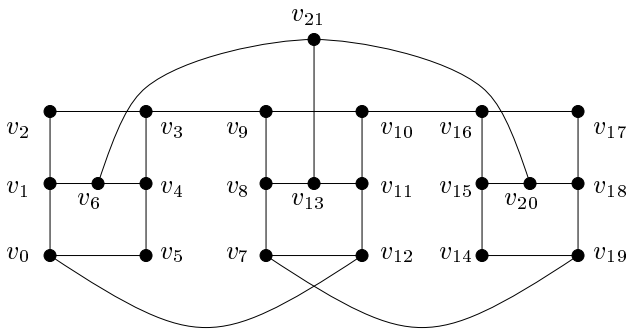

Figure 1. Goldberg-graph F

Next we define the Hajós-union of two 3-critical graphs, a method for constructing new 3-critical graphs from smaller ones, cf. [4].
Let H_1 and H_2 be 3-critical graphs. Then H_1 contains a vertex v of degree 2, say with neighbors u and w. Let $e = xy$ be an edge of H_2. The graph H obtained from $H_1 - v$ and $H_2 - e$ by adding edges ux and wy is called the *Hajós-union* of H_1 and H_2, and it is a 3-critical graph.
The flower snark J_{2k+1}, constructed by Isaacs in [6], is the graph with vertex set $VJ_{2k+1} = \{a_i, b_i, c_i, d_i | i = 0, 1, \ldots, 2k\}$ and edge set $EJ_{2k+1} = \{b_ia_i, b_ic_i, b_id_i; a_ia_{i+1}; c_id_{i+1}; d_ic_{i+1} | i = 0, 1, \ldots, 2k\}$.

Theorem 2.1 ([9]) *For each $k \geq 2$ the graph $J_{2k+1} - v$ is 3-critical for every vertex v.*

Let G be a snark and let $e = vw$ be an edge. Then e is called *good* in G if $\chi'(G - \{v, w\}) = 3$. The following theorems are weaker formulations of theorems proved in [9].

Theorem 2.2 ([9]) *Let G be a snark and let $e = vw$ be an edge of G. Then e is good in G if and only if there is a 2-factor of G with exactly two odd cycles connected by e.*

Theorem 2.3 ([9]) *A snark G is critical if and only if for each edge e there is a 2-factor of G with exactly two odd cycles connected by e.*

We define two classes of graphs: For $k \geq 2$ let G_k be the graph obtained from $F - \{v_7v_{12}\}$ and $J_{2k+1} - \{a_0, b_0\}$ by adding two vertices v and w and edges v_7c_0, $v_{12}d_0$, vv_2, vv_{14}, va_{2k}, wv_{17}, wv_5 and wa_1. Graph G_2 is drawn in Figure 2.
Let G'_k be the graph with $V(G'_k) = V(G_k)$ and $E(G'_k) = (E(G_k) - \{v_7c_0, v_{12}d_0\}) \cup \{v_7d_0, v_{12}c_0\}$.
Let $\psi : V(G_k) - \{v, w\} \to V(G'_k) - \{v, w\}$ be the function which is the identity on vertices v_i, $i = 0, \ldots, 21$, and which maps a_i, b_i, c_j and d_j to a_{2k+1-i}, b_{2k+1-i}, d_{2k+1-j} and c_{2k+1-j}, respectively, $i = 1, \ldots, k$; $j = 0, \ldots, k$. It is easy to see that ψ is an isomorphism.

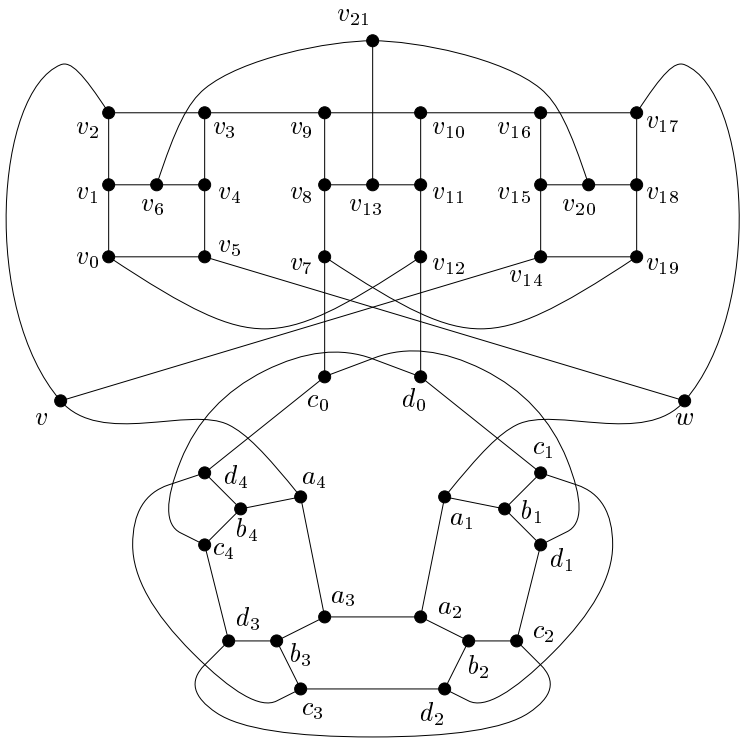

Figure 2. Graph G_2

Theorem 2.4 *For each $k \geq 2$ the graphs G_k and G'_k are critical snarks and they are not bicritical.*

Proof. Graph $G_k - \{v, w\}$ is the Hajós-union of F and $J_{2k+1} - a_0$, where b_0 is removed from $J_{2k+1} - a_0$ in the construction. Since $G_k - \{v, w\}$ is isomorphic to $G'_k - \{v, w\}$, both graphs are 3-critical. Henceforth G_k and G'_k are snarks, and they are are not bicritical.
We have to show that G_k and G'_k are critical, i.e. we show that each edge of these graphs is good. The idea is to define spanning 2-factors which contain precisely two odd cycles. Then follows with Theorem 2.2 that each edge connecting these odd cycles is good.
First we define a function $\phi : V(G_k) \to V(G_k)$ by permutation cycles (v_0, v_{19}), (v_1, v_{18}), (v_2, v_{17}), (v_3, v_{16}), (v_4, v_{15}), (v_5, v_{19}), (v_6, v_{20}), (v_7, v_{12}), (v_8, v_{11}), (v_9, v_{10}), and (a_i, a_{2k+1-i}), (b_i, b_{2k+1-i}), for $i = 1, \ldots k$, and (c_i, d_{2k+1-i}), (d_i, c_{2k+1-i}) for $i = 0, 1, \ldots, k$, where the indices are added modulo $2k + 1$.
It is easy to see that ϕ is an automorphism on G_k as well as on G'_k, and we will make use of this fact in the following. We begin with proving the statement for G_k.

Claim 1: The edges of $E_1 = \{v_0v_1, v_2v_3, v_4v_6, v_{13}v_{21}, v_{15}v_{20}, v_{16}v_{17}, v_{18}v_{19}, vv_{14}, wv_5\}$, and the edges a_ib_i, b_ic_i, b_id_i and c_id_{i+1} for $i = 0, 1, \ldots, 2k$ are good in G_k.
Proof. For $i = 1, \ldots, 2k-1$ let A_i be the path $a_1, a_2, \ldots, a_i, b_i, d_i, c_{i+1}, b_{i+1}, a_{i+1}, a_{i+2}, \ldots$, a_{2k-1}, a_{2k}, and let B_i be the path $d_0, c_1, b_1, d_1, c_2, b_2, d_2, c_3, \ldots$, d_{i-1}, c_i, d_{i+1}, c_{i+2}, b_{i+2}, d_{i+2}, $c_{i+3}, \ldots$, d_{2k-1}, c_{2k}, b_{2k}, d_{2k}, c_0.

For $i = 1, \dots 2k-1$ let $C_i = v, v_2, v_1, v_6, v_{21}, v_{20}, v_{18}, v_{17}, w, A_i, v$, and let $K_i = v_7, v_{19}, v_{14}, v_{15}, v_{16}, v_{10}, v_{11}, v_{13}, v_8, v_9, v_3, v_4, v_5, v_0, v_{12}, B_i, v_7$.
These two cycles form a spanning 2-factor of G_k, and they are connected by the edges of E_1, and b_ic_i, $b_{i+1}d_{i+1}$, $c_{i-1}d_i$, $c_{i+1}d_{i+2}$ and a_jb_j for all $1 \leq j \leq 2k$ and $j \neq i, i+1$. Choosing an appropriate i and applying ϕ on C_{2k-1} and K_{2k-1} shows that for each edge e of the assertion there is a 2-factor with precisely two odd cycles connected by e.

Claim 2: Edges d_ic_{i+1}, $i = 0, \dots, 2k$, and the elements of $E_2 = \{v_3v_9, v_6v_{21}, v_0v_{12}, wv_{17}\}$ and of $E_2' = \{v_{10}v_{16}, v_{20}v_{21}, v_7v_{19}, vv_2\}$ are good in G_k.
Proof. For each even $2 \leq i \leq k-1$ define $A_i = a_1, b_1, c_1, d_2, b_2, a_2, a_3, \dots, c_{i-1}, d_i, b_i, a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, d_{i+3}, b_{i+3}, a_{i+3}, \dots, d_{2k-1}, b_{2k-1}, a_{2k-1}, a_{2k}$ and $B_i = c_0, d_1, c_2, \dots, c_i, d_{i+1}, b_{i+1}, c_{i+1}, d_{i+2}, \dots, c_{2k-1}, d_{2k}, b_{2k}, c_{2k}, d_0$.
Then for each even $2 \leq i \leq k-1$ the cycles $C_i = v, v_2, v_3, v_4, v_6, v_1, v_0, v_5, w, A_i, v$ and $K_i = v_{12}, v_{11}, v_{13}, v_{21}, v_{20}, v_{15}, v_{14}, v_{19}, v_{18}, v_{17}, v_{16}, v_{10}, v_9, v_8, v_7, B_i, v_{12}$ form a spanning 2-factor of G_k.
These two cycles are connected by the elements of E_2, d_ic_{i+1}, $d_{i+1}c_{i+2}$, d_0c_1 and $d_{2k-1}c_{2k}$. Since $\phi[E_2] = E_2'$, $\phi(d_0c_1) = c_0d_{2k}$, $\phi(d_{2k-1}c_{2k}) = c_2d_1$ and $\phi(d_ic_{i+1}) = c_{2k+1-i}d_{2k-i}$ the assertion follows for the edges of the assertion as in claim 1.

Claim 3: The elements of $E_3 = \{v_0v_5, v_1v_6, v_3v_4, v_{14}v_{19}, v_{15}v_{16}, v_{18}v_{20}\}$ and edges a_ia_{i+1}, $i = 1, \dots 2k-1$, are good in G_k.

Proof. We start with defining three paths:
$P = v, v_2, v_3, v_4, v_5, w, v_{17}, v_{18}, v_{20}, v_{21}, v_6, v_1, v_0, v_{12}, v_{11}, v_{13}, v_8, v_9, v_{10}, v_{16}, v_{15}, v_{14}, v_{19}, v_7$,
$P_1 = v, v_{14}, v_{15}, v_{20}, v_{21}, v_6, v_4, v_5, w$, and
$P_2 = v_7, v_{19}, v_{18}, v_{17}, v_{16}, v_{10}, v_{11}, v_{13}, v_8, v_9, v_3, v_2, v_1, v_0, v_{12}$.

First we prove the assertion for G_2. One 2-factor consists of $C = P_1, a_1, b_1, c_1, d_2, c_3, d_4, b_4, a_4, v$ and $K = P_2, d_0, c_4, d_3, b_3, a_3, a_2, b_2, c_2, d_1, c_0, v_7$. Another one consists of $C' = P, c_0, d_1, c_2, d_3, b_3, a_3, a_4, v$ and of $K' = d_0, c_1, b_1, a_1, a_2, b_2, d_2, c_3, d_4, b_4, c_4, d_0$. Cycles C and K are connected by a_1a_2, a_3a_4 and the edges of E_3, and a_2a_3 connects C' and K'.
Thus the statement is true for G_2, and we may assume that $k \geq 3$ in the following.
If $1 \leq i \leq k$ is odd then we define paths A_i and B_i as follows: $A_i = a_1, b_1, d_1, c_2, b_2, a_2, a_3, b_3, \dots, b_i, d_i, c_{i+1}, d_{i+2}, c_{i+3}, b_{i+3}, a_{i+3}, \dots, b_{2k-1}, d_{2k-1}, c_{2k}, b_{2k}, a_{2k}$, and $B_i = d_0, c_1, d_2, c_3, \dots, c_i, d_{i+1}, b_{i+1}, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, d_{i+3}, c_{i+4}, \dots, c_{2k-1}, d_{2k}, c_0$.
If $1 \leq i \leq k$ is even then define these path as follows: $A_i = a_1, a_2, b_2, d_2, c_3, b_3, a_3, a_4 \dots b_i, d_i, c_{i+1}, d_{i+2}, c_{i+3}, b_{i+3}, a_{i+3}, \dots, c_{2k-1}, b_{2k-1}, a_{2k-1}, a_{2k}$, and $B_i = d_0, c_1, b_1, d_1, c_2, \dots, c_i, d_{i+1}, b_{i+1}, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, d_{i+3}, \dots, d_{2k-1}, c_{2k}, b_{2k}, d_{2k}, c_0$.
For $1 \leq i \leq k$ we define $C_i = P_1, A_i$ and $K_i = P_2, B_i$. Then C_i and K_i are a 2-factor of G_k and the edges of E_3, a_ia_{i+1}, $a_{i+2}a_{i+3}$ connect K_i with C_i. As in claim 1 it follows by choosing appropriate i and applying the automorphism ϕ that each edge of this assertion is good in G_k.

Claim 4: The elements of $E_4 = \{v_1v_2, v_8v_{13}, v_{10}v_{11}\}$ and of $E_4' = \{v_{18}v_{17}, v_{11}v_{13}, v_8v_9\}$ are

good in G_k.
Proof. Define cycles $C_1 = v, v_2, v_3, v_4, v_5, w, v_{17}, v_{16}, v_{10}, v_9, v_8, v_7, v_{19}, v_{18}, v_{20}, v_{15}, v_{14}, v$;
$C_2 = v_0, v_1, v_6, v_{21}, v_{13}, v_{11}, v_{12}, v_0$, and
$C_3 = c_0, d_1, b_1, a_1, a_2, \ldots, a_{2k}, b_{2k}, c_{2k}, d_0, c_1, d_2, b_2, c_2, d_3, b_3, \ldots, c_{2k-1}, d_{2k}, c_0$.
These three cycles form a 2-factor of G_k, where C_3 is of even length. Hence C_1 and C_2 are of odd length and they are connected by the edges of E_4. It holds $\phi[E_4] = E'_4$, and hence the edges of E'_4 are also good in G_k.

Claim 5: The elements of $E_5 = \{v_9v_{10}, v_{15}v_{14}, v_{11}v_{12}\}$ and of $E'_5 = \{v_4v_5, v_8v_7\}$ are good in G_k.
Proof. Since $J_{2k+1} - a_0$ is hamiltonian (cf. [3]) there is a hamiltonian path P in $J_{2k+1} - \{a_0, b_0\}$ from c_0 to d_0. Let $C = v_{12}, v_0, v_1, v_6, v_4, v_5, w, v_{17}, v_{18}, v_{19}, v_{14}, v, v_2, v_3, v_9, v_8, v_7$, P, v_{12}, and $K = v_{13}, v_{21}, v_{20}, v_{15}, v_{16}, v_{10}, v_{11}, v_{13}$.
These two cycles form a spanning 2-factor of G_k, and the edges of E_5 connect C and K. Since $\phi[E_5] = E'_5$ the edges of E'_5 are good in G_k.

Claim 6: Edges c_0v_7, d_0v_{12}, wa_1 and va_{2k} are good in G_k.
Proof. Let P be defined as in the proof of claim 3. Define cycles $C = P, c_0, d_1, c_2, b_2, d_2, c_3, \ldots, d_{2k-2}, c_{2k-1}, d_{2k}, b_{2k}, a_{2k}, v$, and $K = d_0, c_1, b_1, a_1, a_2, \ldots, a_{2k-2}, a_{2k-1}, b_{2k-1}, d_{2k-1}, c_{2k}, d_0$. Edges d_0v_{12} and a_1w connect these two cycles. Further $\phi(wa_1) = va_{2k}$ and $\phi(d_0v_{12}) = c_0v_7$ and hence the assertion holds true.

Claims 1 - 6 show that G_k satisfies the condition of Theorem 2.3, and hence it is a critical snark.

In case of G'_k we argue analogously. Let R be a path with start vertex x and end vertex y, then $\overline{R}$ denotes the reversed path from y to x.
Claims 1, 2 and 3 ($k \geq 3$) can be proved similarly when $\overline{B_i}$ is substituted for B_i in the definition of K_i.
To prove Claim 3 for G'_2 it is obvious how to define C and K, having their definition for G_2. To show that a_2a_3 is good in G'_2 we define $C' = P, d_0, c_1, d_2, c_3, b_3, a_3, a_4, v$ and $K' = c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, c_4, b_4, d_4, c_0$, which is a 2-factor of G'_k.
Claim 4 holds true for G'_k, since v_7c_0 and $v_{12}d_0$ are elements of the complementary 1-factor of C_1, C_2 and C_3, which also form a 2-factor of G'_k.
Substituting $\overline{P}$ for P in the definition of C yields an analogous proof of claim 5 for G'_k.
To show that c_0v_{12}, d_0v_7, wa_1 and va_{2k} are good in G'_k, we argue as in the proof of claim 6 for G_k by using cycles C and K, defined as follows: $C = P, d_0, c_1, d_2, b_2, c_2, d_3, \ldots, c_{2k-2}, d_{2k-1}, c_{2k}, b_{2k}, a_{2k}, v$, and $K = c_0, d_1, b_1, a_1, a_2 \ldots, a_{2k-2}, a_{2k-1}, b_{2k-1}, c_{2k-1}, d_{2k}, c_0$.
This finishes the proof that G'_k is a critical snark.

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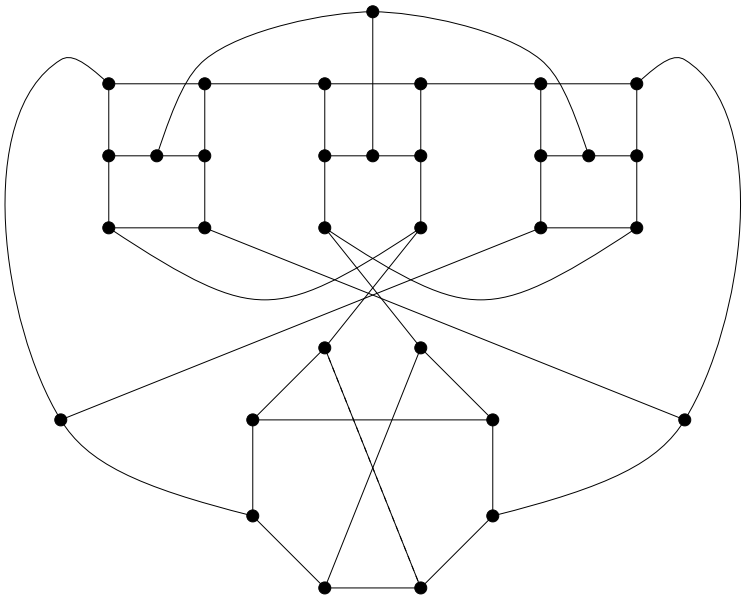

Figure 3. Graph H'

Graphs H and H', see Figure 3, are obtained as G_k and G'_k by using the Petersen graph instead of J_{2k+1} in the construction. The order of these graphs is 32, and it is easy to see that they are critical but not bicritical snarks. The order of G_k (G'_k) is $22 + 4(2k + 1) = 8(k + 3) + 2$, $k \geq 2$, and hence there are non-bicritical critical snarks of order n for each $n \in \{8k + 2 | k \geq 5\}$. These results and Theorem 2.4 imply Corollary 1.5.

References

[1] G. Brinkmann, E. Steffen, Snarks and Reducibility, to appear in *Ars Combinatoria*

[2] P. J. Cameron, A. G. Chetwynd, J. J. Watkins, Decomposition of Snarks, *J. of Graph Theory* **11** (1987) 13-19

[3] S. Fiorini, Hypohamiltonian Snarks, *in* "Graphs and other Combinatorial Topics" (M. Fiedler ed.) *Texte zur Mathem.* **59**, Teubner, Leipzig (1983), 70-75

[4] S. Fiorini, R. J. Wilson, "Edge-colourings of graphs" Research notes in math. no. **17**, Pitman, London, San Francisco, Melbourne (1977)

[5] M. K. Goldberg, Construction of class 2 graphs with maximum vertex degree 3, *J. of Comb. Theory, Ser. B* **31** (1981), 282-291

[6] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colorable, *Am. Math. Monthly* **82** (1975) 221-239

[7] R. Nedela, M. Škoviera, Decompositions and Reductions of Snarks, *J. of Graph Theory* **22** (1996) 253-279

[8] M. Preissmann, C-minimal Snarks, *Annals of Discrete Mathematics* **17** (1983) 559-565

[9] E. Steffen, Classifications and Characterizations of Snarks, submitted

[10] E. Steffen, On bicritical Snarks, Preprint 97-027, SFB 343, Universität Bielefeld (1997)

[11] J. J. Watkins, Snarks, *Ann. New York Acad. Sci.* **576** (1989) 606-622

[12] J. J. Watkins, R. J. Wilson, A Survey of Snarks *in* "Graph theory, Combinatorics and Applications" (Y. Alavi et al. eds.) Wiley, New York (1991) 1129-1144