Cofibration and Fibration Structures in Enriched Categories

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0 Introduction

This paper is motivated by the attempt to lay out the grounds for relative homological algebra for module spectra. Elmendorf, Kriz, Mandell, and May constructed an associative, commutative and unital smash product for spectra with some additional structure, called S-module spectra [6]. This good smash product allows the translation of concepts of classical homological algebra to the category ${}_S\mathcal{M}od$ of S-module spectra. Since ${}_S\mathcal{M}od$ is not abelian, the standard approach to derived functors does not apply. The most common frame work for doing homological algebra in the non-abelian case is Quillen's concept of a closed model category sytructure. Elmendorf, Kriz, Mandell and May construct a topological closed model structure on ${}_S\mathcal{M}od$ whose weak equivalences are maps inducing isomorphisms on homotopy groups and whose cofibrations are retracts of relative cell-objects. Relative cell-objects are constructed analogous to relative CW-complexes.

Relative homological algebra deals with weak equivalences which are genuine homotopy equivalences after forgetting part of the structure. Genuine homotopy equivalences often cannot be tested by a "set of small objects", and in the absence of small objects arguments one has trouble to define closed model structures.

Fortunately, one only needs parts of a closed model structure to define left or right derived functors: cofibration and fibration structures suffice [1], [13], and genuine homotopy equivalences fit nicely into such structures.

One of the most useful technical results for cofibrations of topological spaces is the "product theorem": if $i : A \to X$ and $j : B \to Y$ are cofibrations and one of them is closed then

$$(i \times Y, X \times j) : A \times Y \cup X \times B \longrightarrow X \times Y$$

is a cofibration. The condition that i or j is closed indicates that the proof of this result is not formal. In the absence of a product theorem alternative reasoning is necessary where-ever this result has been used in the corresponding topological case (e.g. see [6, VII. 6.8]).

Taking one of the properties of a closed cofibration as an axiom we introduce the notion of a strong cofibration. It turns out that the cofibrations in the closed model category structures of the categories of spectra defined by Elmendorf, Kriz, Mandell, and May are strong ones, and strong cofibrations are less rigid than relative cell objects: e.g. various forgetful functors map relative cell objects to strong cofibrations even in cases where relative cell objects are not preserved (see §6). Moreover, we derive a pairing theorem for strong cofibrations (Thm. 2.11), which specializes to the product theorem in the topological case.

Although this result holds more generally we prove it in the setting of enriched categories which are complete and cocomplete, tensored and cotensored.

The pairing theorem then is used to verify the axioms for fibrations and cofibration structures on our categories with genuine homotopy equivalences as weak equivalences.

We conclude with a list of categories which can be used for enriching, an analysis of cofibrations in categories of spectras and a few other examples to which the results can be applied.

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1 Enriched categories

To specify notations we recall the basic concepts of enriched category theory from [10].

Let $\mathcal{V} = (\mathcal{V}_0, \Box, \Phi)$ be a symmetric monoidal category with unit object Φ. We assume $\mathcal{V}$ to be *closed*, i.e. there is a bifunctor

$$[-,-] : \mathcal{V}_0^{op} \times \mathcal{V}_0 \to \mathcal{V}_0$$

satisfying the Hom-tensor adjunction, i.e. there is a natural isomorphism

1.1

$$\mathcal{V}_0(X \Box Y, Z) \cong \mathcal{V}_0(X, [Y, Z])$$

From these data we can deduce natural isomorphisms

1.2 (1) $\mathcal{V}_0(Y, Z) \cong \mathcal{V}_0(\Phi \Box Y, Z]) \cong \mathcal{V}_0(\Phi, [Y, Z])$

(2) $Z \cong [\Phi, Z]$

(3) $[X \Box Y, Z] \cong [X, [Y, Z]]$

As a sample we prove (3). For each A in $\mathcal{V}_0$ we have a sequence of natural bijections

$$\mathcal{V}_0(A, [X \Box Y, Z]) \cong \mathcal{V}_0(A \Box X \Box Y, Z]) \cong \mathcal{V}_0(A \Box X, [Y, Z]) \cong \mathcal{V}_0(A, [X, [Y, Z]])$$

which implies the result.

The composite isomorphism

$$\mathcal{V}_0(X\Box Y, Z) \cong \mathcal{V}_0(\Phi, [X\Box Y, Z]) \cong \mathcal{V}_0(\Phi, [X, [Y, Z]]) \cong \mathcal{V}_0(X, [Y, Z])$$

is the isomorphism (1.1). Following Kelly we call a map $f : \Phi \to Z$ in $\mathcal{V}_0$ an *element in* Z. Hence, in accordance with (1.2.1), an element of $[Y, Z]$ is a morphism in $\mathcal{V}_0$.

1.3 Definition: A $\mathcal{V}$-*category* is a category $\mathcal{C}$ whose morphism sets are replaced by objects in $\mathcal{V}$, composition is defined by maps

$$\mathcal{C}(B, C)\Box\mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

in $\mathcal{V}_0$ with identities $j_A : \Phi \to \mathcal{C}(A, A)$, subject to the obvious conditions. $\mathcal{V}$-*functors* and $\mathcal{V}$-*natural transformations* are defined in the obvious way.

Each $\mathcal{V}$-category $\mathcal{C}$ has an *underlying category* $\mathcal{C}_0$ with the same objects as $\mathcal{C}$ and $\mathcal{C}_0(A, B) = \mathcal{V}_0(\Phi, \mathcal{C}(A, B))$. A $\mathcal{V}$-functor $F : \mathcal{B} \to \mathcal{C}$ of $\mathcal{V}$-categories has an *underlying functor* $F_0 : \mathcal{B}_0 \to \mathcal{C}_0$ defined in the obvious way, and the same applies to $\mathcal{V}$-natural transformation. We also say that $\mathcal{C}$ is a $\mathcal{V}$-*enriched* version of $\mathcal{C}_0$.

We have a hom-functor

$$\mathcal{C}_0^{op} \times \mathcal{C}_0 \to \mathcal{V}_0, \quad (A, B) \mapsto \mathcal{C}(A, B).$$

If $f : \Phi \to \mathcal{C}(B, B')$ and $g : \Phi \to \mathcal{C}(A', A)$ are morphisms in $\mathcal{C}_0$ then f_* and g^* are defined by

$$f_* : \mathcal{C}(A, B) \cong \Phi\Box\, \mathcal{C}(A, B) \xrightarrow{f\Box id} \mathcal{C}(B, B')\Box\, \mathcal{C}(A, B) \to \mathcal{C}(A, B')$$

$$g^* : \mathcal{C}(A, B) \cong \mathcal{C}(A, B)\Box\Phi \xrightarrow{id\Box g} \mathcal{C}(A, B)\Box\, \mathcal{C}(A', A) \to \mathcal{C}(A', B)$$

Note that there is canonical $\mathcal{V}$-enriched version of $\mathcal{V}_0$, also denoted by $\mathcal{V}$: its objects are those of $\mathcal{V}_0$ and $\mathcal{V}(A, B) = [A, B]$.

1.4 Definition: A $\mathcal{V}$-category $\mathcal{C}$ is called $\mathcal{V}$-*complete* and $\mathcal{V}$-*cocomplete* if each functor $F : \mathcal{J} \to \mathcal{C}_0$, $\mathcal{J}$ a small indexing category, has a limit and a colimit in $\mathcal{C}_0$, and we have natural isomorphisms in $\mathcal{V}_0$

$$\begin{aligned} \mathcal{C}(X, \lim F) &\cong \lim \mathcal{C}(X, F) \\ \mathcal{C}(\operatorname{colim} F, X) &\cong \lim \mathcal{C}(F, X) \end{aligned}$$

1.5 Definition: A $\mathcal{V}$-category $\mathcal{C}$ is *tensored* and *cotensored* if there are functors

$$\begin{aligned} \mathcal{C}_0 \times \mathcal{V}_0 &\longrightarrow \mathcal{C}_0, \quad (X, K) \mapsto X \otimes K \\ \mathcal{C}_0 \times \mathcal{V}_0^{op} &\longrightarrow \mathcal{C}_0, \quad (X, K) \mapsto X^K \end{aligned}$$

and natural isomorphisms in $\mathcal{V}_0$

$$\mathcal{C}(X \otimes K, Y) \cong \mathcal{V}(K, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, Y^K)$$

We observe that $\mathcal{V}$ is tensored by $-\Box-$ and cotensored by $[-, -]$. From (1.5) and (1.4) we immediately deduce

1.6 There are natural isomorphisms in $\mathcal{C}_0$

$$X \otimes \Phi \cong X \cong X^{\Phi}$$
$$X \otimes (K \Box L) \cong (X \otimes K) \otimes L$$

1.7 If $\emptyset$ is initial in $\mathcal{C}_0$ and $*$ terminal, then

$$\emptyset \otimes K \cong \emptyset \text{ and } *^K \cong *$$

Sample proof:

$$\mathcal{C}_0(\emptyset \otimes K, Y) = \mathcal{V}_0(\Phi, \mathcal{C}(\emptyset \otimes K, Y)) \cong \mathcal{V}_0(\Phi, \mathcal{C}(\emptyset, Y^K)) = \mathcal{C}_0(\emptyset, Y^K) = \{*\}$$

.

1.8 Definition: We call a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{B}$ $\mathcal{V}$-left adjoint to the $\mathcal{V}$-functor $U : \mathcal{B} \to \mathcal{C}$ if there is a natural isomorphism

$$\mathcal{B}(F(C), B) \cong \mathcal{C}(C, U(B))$$

in $\mathcal{V}_0$ for all objects $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

Throughout this paper we make the following assumptions

1.9 Assumption on $\mathcal{V}$: It is a self-enriched closed symmetric monoidal category, $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete.

1.10 Assumption on $\mathcal{C}$: It is a $\mathcal{V}$-category with underlying category $\mathcal{C}_0$. We assume $\mathcal{C}_0$ to be $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete, tensored and cotensored.

1.11 Remark: If $\mathcal{V}_0$ has a 0-object then $\mathcal{C}_0$ has a 0-object: Since $\mathcal{V}_0(\Phi, K)$ is not empty for each object K in $\mathcal{V}_0$, the set $\mathcal{C}_0(X, Y) = \mathcal{V}_0(\Phi, \mathcal{C}(X, Y))$ is not empty for each pair of objects X, Y in $\mathcal{C}_0$. Hence the initial and terminal objects in $\mathcal{C}_0$, which exist by our assumptions, are isomorphic.

1.12 Examples: Each of the following categories satisfies the assumption on $\mathcal{V}$.

(1) $\mathcal{T}op$, the category of compactly generated spaces in the sense of [24, 5(ii)]. (The examples [24, 5(iii),(iv)] also satisfy the requirements on $\mathcal{V}$).
We define $X \Box Y = X \times Y$, and $[X, Y]$ to be the function space in the category.

(2) $\mathcal{SS}ets$, the category of simplicial sets with $X \Box Y = X \times Y$ and $[X, Y] = \mathrm{Hom}(X, Y)$, the internal Hom-functor (for details see [17]).

(3) The based versions $\mathcal{T}op_*$ and $\mathcal{SS}ets_*$ of (1) and (2). Here we have $X \Box Y = X \wedge Y$, but the same definitions for $[X, Y]$ with the obvious base points.

Additional examples will be given in §6.

2 Fibrations and cofibrations

Throughout this section let $\mathcal{C}$ be a $\mathcal{V}$-enriched category. We assume that $\mathcal{V}$ and $\mathcal{C}$ satisfy the assumptions (1.9) and (1.10). If there is no chance of confusion we denote its underlying category $\mathcal{C}_0$ by the same symbol $\mathcal{C}$.

Throughout this chapter we fix a class $\mathcal{T}$ of morphisms which we think of as *test morphisms* in $\mathcal{V}$.

2.1 Definition: (1) A map $p : E \to B$ in $\mathcal{C}$ is a $\mathcal{T}$*-fibration* if each solid arrow diagram with $j : K \to L$ in $\mathcal{T}$

$$\begin{tikzcd} X \otimes K \arrow[r] \arrow[d, "X\otimes j"'] & E \arrow[d, "p"] \\ X \otimes L \arrow[r] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

admits a dotted filler h making the triangles commute.
(2) An object X of $\mathcal{C}$ is $\mathcal{T}$*-fibrant* if $X \to *$ is a $\mathcal{T}$-fibration.
(3) A map $i : A \to X$ in $\mathcal{C}$ is a $\mathcal{T}$*-cofibration* if for any $j : K \to L$ in $\mathcal{T}$ and $g : A \otimes L \cup X \otimes K \to Z$ into a $\mathcal{T}$-fibrant object Z there is an extension h (we usually write $A \otimes L \cup X \otimes K$ for the pushout $A \otimes L \cup_{A\otimes K} X \otimes K$)

$$\begin{tikzcd} A \otimes L \cup X \otimes K \arrow[r, "g"] \arrow[d, "(i\otimes L, X\otimes j)"'] & Z \\ X \otimes L \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

(4) An object X of $\mathcal{C}$ is $\mathcal{T}$*-cofibrant* if $\emptyset \to X$ is a $\mathcal{T}$-cofibration.

2.2 Remark: The most important cases are $\mathcal{T} = \{0 \to [0,1]\}$ for $\mathcal{T}op$-enriched categories, and $\mathcal{T} = \{\Lambda^n_k \to \Delta^n |\ n \in \mathbb{N},\ 0 \le k \le n\}$ for $\mathcal{SS}ets$-enriched categories, where Λ^n_k is the simplicial subset of the standard simplicial n-simplex Δ^n generated by all faces except the k^{th} one. The first case will be discussed in greater detail in §4.

To stress that the two notions are almost dual we reformulate the definitions.

2.3 A morphism $i : A \to X$ has the *left lifting property* (LLP) for $p : E \to B$ if each commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

has a diagonal filler $h : X \to E$ making the diagram commute.
Dually, p has the *right lifting property* (RLP) for i if such an h exists.

Passing to adjoints using (1.5) we obtain

2.4 Lemma: The following are equivalent:
(1) $p : E \to B$ is a $\mathcal{T}$-fibration.
(2) p has the RLP for all $(X \otimes j) : X \otimes K \to X \otimes L$ with j in $\mathcal{T}$.
(3) $p^X : E^X \to B^X$ has the RLP for all $j : K \to L$ in $\mathcal{T}$ and all X.
(4) for all $j : K \to L$ in $\mathcal{T}$ a map $g : X \to E^K \times_{B^K} B^L$ lifts to E^L

$$\begin{tikzcd} & E^L \arrow[d, "{(E^j,p^L)}"] \\ X \arrow[ur, dotted, "h"] \arrow[r] & E^K \times_{B^K} B^L \end{tikzcd}$$

(5) for all $j : K \to L$ in $\mathcal{T}$ a map $(E^j, p^L) : E^L \to E^K \times_{B^K} B^L$ has a section.

2.5 Lemma: The following are equivalent:
(1) $i : A \to X$ is a $\mathcal{T}$-cofibration.
(2) i has the LLP for all $Z^j : Z^L \to Z^K$ with $j : K \to L$ in $\mathcal{T}$ and all $\mathcal{T}$-fibrant Z.
(3) $Z^i : Z^X \to Z^A$ has the RLP for all $j : K \to L$ in $\mathcal{T}$ and all $\mathcal{T}$-fibrant Z.

Renaming the corresponding notion of Gabriel and Zisman [8] we call a class $\mathcal{F}$ of morphisms *cosaturated* if the following conditions hold:
(1) All isomorphisms are in $\mathcal{F}$.
(2) Given a pushout with i in $\mathcal{F}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & X \arrow[d] \\ B \arrow[r, "j"] & Y \end{tikzcd}$$

then j belongs to $\mathcal{F}$ (i.e. $\mathcal{F}$ is closed under cobase change.)
(3) $\mathcal{F}$ is closed under retracts.
(4) $\mathcal{F}$ is closed under sequential colimits and arbitrary sums:
given a sequence

$$X_0 \to X_1 \to X_2 \to \dots$$

of morphisms in $\mathcal{F}$ the induced maps $X_n \to \operatorname{colim}_k X_k$ are in $\mathcal{F}$, and arbitrary sums of morphisms in $\mathcal{F}$ are in $\mathcal{F}$.

We call $\mathcal{F}$ *saturated* if (1) and (3) hold and the duals of (2) and (4):
(2*) Suppose (2) is a pullback diagram with j in $\mathcal{F}$ then i belongs to $\mathcal{F}$ (i.e. $\mathcal{F}$ is closed under base change).
(3*) $\mathcal{F}$ is closed under sequential limits and arbitrary products.

2.6 Notation: We denote by $\mathcal{T}_{cosat}$ and $\mathcal{T}_{sat}$ the cosaturation resp. saturation of $\mathcal{T}$, i.e. the smallest cosaturated (saturated) class of morphisms in $\mathcal{V}$ containing $\mathcal{T}$ and by **cof** and **fib** the classes of $\mathcal{T}$-cofibrations and $\mathcal{T}$-fibrations.

Since a class of morphisms defined by the LLP (RLP) with respect to some class of maps is cosaturated (saturated), we obtain from Lemmas 2.4 and 2.5

2.7 Lemma: (1) **cof** is cosaturated.
(2) $A \to X$ is a $\mathcal{T}$-cofibration $\Leftrightarrow$ $A \to X$ is a $\mathcal{T}_{cosat}$-cofibration.
(3) **fib** is saturated.
(4) $E \to B$ is a $\mathcal{T}$-fibration $\Leftrightarrow$ $E \to B$ s a $\mathcal{T}_{cosat}$-fibration.

For the proof of the pairing result mentioned in the introduction we need a stronger version of cofibrations.

2.8 Definition: Given morphisms $i : A \to X$ and $p : E \to B$ in $\mathcal{C}$, a morphism $j : K \to L$ in $\mathcal{T}$, and a commutative square

$$\begin{tikzcd} A \otimes L \cup X \otimes K \arrow[r] \arrow[d, "(i\otimes L, X\otimes j)"'] & E \arrow[d, "p"] \\ X \otimes L \arrow[r] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

We call $i : A \to X$ a *strong $\mathcal{T}$-cofibration* if the dotted filler h exists for all $j \in \mathcal{T}$ whenever p is a $\mathcal{T}$-fibration. We call p a *strong $\mathcal{T}$-fibrations* if the dotted filler h exists for all $j \in \mathcal{T}$ whenever i is a $\mathcal{T}$-cofibration.

2.9 Lemma: (1) The class **sfib** of strong $\mathcal{T}$-fibrations is saturated.
(2) $i : A \to X$ is a strong $\mathcal{T}$-cofibration if it has the LLP for all morphisms $q = (E^j, p^L) : E^L \to E^K \times_{B^K} B^L$ where $p : E \to B$ is a $\mathcal{T}$-fibration and $j \in \mathcal{T}$. Hence the class **scof** of strong $\mathcal{T}$-cofibrations is cosaturated. $\square$

(1) follows from the definition and (2) by passing to adjoints. Taking $A = \emptyset$ or $B = *$ we see

2.10 Strong $\mathcal{T}$-cofibrations are $\mathcal{T}$-cofibrations. Each X in $\mathcal{C}$ is strongly $\mathcal{T}$-cofibrant, i.e. $\emptyset \to X$ is a strong $\mathcal{T}$-cofibration.
Strong $\mathcal{T}$-fibrations are $\mathcal{T}$-fibrations. Each $\mathcal{T}$-fibrant object X in $\mathcal{C}$ is strongly $\mathcal{T}$-fibrant, i.e. if $X \to *$ is a $\mathcal{T}$-fibration then it is a strong $\mathcal{T}$-fibration.

The main result of this section is

2.11 Pairing theorem for cofibrations: Suppose we are given
(i) $\mathcal{V}$-enriched categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, satisfying (1.10)
(ii) $\mathcal{V}$-functors

$$T : \mathcal{A} \times \mathcal{B} \to \mathcal{C}, \quad U : \mathcal{A}^{op} \times \mathcal{C} \to \mathcal{B}, \quad V : \mathcal{B}^{op} \times \mathcal{C} \to \mathcal{A}$$

(iii) Natural isomorphisms in $\mathcal{V}$

$$\mathcal{C}(T(A,B), C) \cong \mathcal{B}(B, U(A,C)) \cong \mathcal{A}(A, V(B,C))$$

Then:

(1) If $i : A \to X$ and $j : B \to Y$ are $\mathcal{T}$-cofibrations in $\mathcal{A}$ resp. $\mathcal{B}$ and at least one of them is strong, then the morphism

$$f : T(A,Y) \cup_{T(A,B)} T(X,B) \to T(X,Y)$$

induced by $T(i,Y)$ and $T(X,j)$ is a $\mathcal{T}$-cofibration in $\mathcal{C}$. If both i and j are strong, f is strong.

(2) If $j : B \to Y$ is a $\mathcal{T}$-cofibration in $\mathcal{B}$ and $p : E \to Z$ a $\mathcal{T}$-fibration in $\mathcal{C}$ and at least one of them is strong, then the morphism

$$g : V(Y, E) \to V(B, E) \times_{V(B,Z)} V(Y, Z)$$

induced by $V(j, E)$ and $V(Y, p)$ is a $\mathcal{T}$-fibration in $\mathcal{A}$. If both j and p are strong, g is strong.

The proof uses the following result

2.12 Lemma: Let $U : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ be a pair of $\mathcal{V}$-functors between $\mathcal{V}$-enriched categories satisfying (1.10). Suppose that F is $\mathcal{V}$-left adjoint to U. Then F preserves (strong) $\mathcal{T}$-cofibrations and U preserves (strong) $\mathcal{T}$-fibrations.

Proof: Note that F preserves colimits and tensors while U preserves limits and cotensors. Consider

$$\begin{tikzcd} X \otimes K \arrow[r, "f"] \arrow[d, "X\otimes k"'] & U(E) \arrow[d, "U(p)"] \\ X \otimes L \arrow[r, "g"] & U(B) \end{tikzcd}$$

with p a $\mathcal{T}$-fibration and $k : K \to L$ in $\mathcal{T}$. This square is adjoint to the square

$$\begin{tikzcd} F(X) \otimes K \arrow[r, "f"] \arrow[d, "F(X)\otimes k"'] & E \arrow[d, "p"] \\ F(X) \otimes L \arrow[r, "g"] & B \end{tikzcd}$$

Since p is a fibration this square has a diagonal filler, whose adjoint is a diagonal filler of the first square.
Hence U preserves $\mathcal{T}$-fibrations. In particular, it preserves $\mathcal{T}$-fibrant objects. Hence, again by passing to the adjoint situation, F preserves (strong) $\mathcal{T}$-cofibrations. The same argument implies that U preserves strong $\mathcal{T}$-fibrations. $\square$

Proof of the pairing theorem: Throughout this proof we assume that i is a strong $\mathcal{T}$-cofibration and p a $\mathcal{T}$-fibration.
Given a commutative diagram with $k : K \to L$ in $\mathcal{T}$ (we write $T(A, Y) \cup T(X, B)$ for $T(A, Y) \cup_{T(A,B)} T(X, B)$)

$$\begin{tikzcd} (T(A, Y) \cup T(X, B)) \otimes L \cup T(X, Y) \otimes K \arrow[r] \arrow[d, "(f\otimes L, T(X,Y)\otimes k)"'] & E \arrow[d] \\ T(X, Y) \otimes L \arrow[r] & Z \end{tikzcd} \tag{D.1}$$

we want to produce a filler $h : T(X, Y) \otimes L \to E$. Since $- \otimes L$ has a right adjoint it preserves colimits. Since $T(-, Y)$ has a $\mathcal{V}$-right adjoint it preserves tensors. Hence finding a filler for (D.1) is equivalent to finding a filler for

$$\begin{tikzcd} A \otimes L \cup X \otimes K \arrow[r] \arrow[d] & V(Y,E) \arrow[d, "g"] \\ X \otimes L \arrow[r] & V(B,E) \times_{V(B,Z)} V(Y,Z) \end{tikzcd} \tag{D.2}$$

Since i is strong if suffices to show that g is a $\mathcal{T}$-fibration in $\mathcal{A}$. This reduces the problem to the case that $A = \emptyset$. Since T preserves colimits in each variable $T(\emptyset, Y) = \emptyset$. Hence (D.1) reduces to

$$\begin{tikzcd} T(X,B) \otimes L \cup T(X,Y) \otimes K \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ T(X,Y) \otimes L \arrow[r] & Z \end{tikzcd} \tag{D.3}$$

which, by the same argument as above, is adjoint to

$$\begin{tikzcd} B \otimes L \cup Y \otimes K \arrow[r] \arrow[d] & U(X,E) \arrow[d, "U(X,p)"] \\ Y \otimes L \arrow[r] & U(X,Z) \end{tikzcd} \tag{D.4}$$

Since $U(X,-)$ has a $\mathcal{V}$-left adjoint it preserves $\mathcal{T}$-fibrations. Hence (D.4) admits a filler provided j is strong. So f is a strong $\mathcal{T}$-cofibration if i and j are strong.

If we only want to show that f is a $\mathcal{T}$-cofibration we go through the same argument with $Z = *$. Then $V(B,Z) = *$ and $U(X,Z) = *$, because the functors $V(B,-)$ and $U(X,-)$ preserve limits. But then (D.4) has the required filler if j is just a $\mathcal{T}$-cofibration.

The proof of part (2) is dual. □

We apply the pairing theorem to the functors

$$\begin{array}{ll} T : \mathcal{C} \times \mathcal{V} \to \mathcal{C}, & (X,K) \mapsto X \otimes K \\ V : \mathcal{C} \times \mathcal{V}^{op} \to \mathcal{C}, & (K,X) \mapsto X^K \\ U : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}, & (X,Y) \mapsto C(X,Y) \end{array}$$

2.13 Corollary: Let $i : K \to L$ be a strong $\mathcal{T}$-cofibration in $\mathcal{V}$.

(1) If $j : A \to X$ is a (strong) $\mathcal{T}$-cofibration in $\mathcal{C}$, so is

$$f = (j \otimes L, X \otimes i) : A \otimes L \cup X \otimes K \to X \otimes L$$

(2) If $p : E \to B$ is a (strong) $\mathcal{T}$-fibration in $\mathcal{C}$, so is

$$g = (E^i, p^L) : E^L \to E^K \times_{B^K} B^L$$

2.14 Corollary: Given a $\mathcal{T}$-cofibration $j : A \to X$ and a $\mathcal{T}$-fibration $p : E \to B$ in $\mathcal{C}$ and at least one of them is strong, then

$$(j^*, p_*) : \mathcal{C}(X,E) \to \mathcal{C}(A,E) \times_{\mathcal{C}(A,B)} \mathcal{C}(X,B)$$

is a fibration in $\mathcal{V}$. If j and p are strong then (j^*, p_*) is strong. □

2.15 Remark: These corollaries allow the construction of strong $\mathcal{T}$-cofibrations and -fibrations from strong $\mathcal{T}$-cofibration and -fibrations in $\mathcal{V}$.

3 Model type structures

For the definition and investigation of derived functors on non-abelian categories Quillen's model category structures are the most familiar tools [17]. In the absence of a small object argument such structures are difficult to find. But one can do with less: fibration or cofibration structures suffice [1]. In this section we analyse whether $\mathcal{T}$-cofibrations and $\mathcal{T}$-fibrations give rise to such structures.

Our weak equivalences will be genuine homotopy equivalences defined by a cylinder object in $\mathcal{V}$.

3.1 Definition: A *cylinder object* in $\mathcal{V}$ is an object I equipped with morphisms $i_\varepsilon : \Phi \to I$, $\varepsilon = 0,1$, and $\pi : I \to \Phi$ such that $\pi \circ i_\varepsilon = id$, $\varepsilon = 0,1$. We call i_0 and i_1 the elements 0 and 1 of I.

3.2 Definition: A cylinder object (I, i_0, i_1, π) is called *good* if it satisfies the following axioms.

(Z1) $i_0 : \Phi \to I$ is in $\mathcal{T}$.

(Z2) $(i_0, i_1) : \Phi \sqcup \Phi \to I$ is a strong $\mathcal{T}$-cofibration in $\mathcal{V}$.

Throughout the rest of the paper we assume that $\mathcal{V}$ has a good cylinder object. Cylinder objects induce a homotopy relation in the obvious way.

Since I is a good cylinder object the usual argument implies (e.g. see [1, I. 3.5]).

3.3 Lemma: If Y is $\mathcal{T}$-fibrant, homotopy is an equivalence relation on $\mathcal{C}(X,Y)$. □

Our first result is

3.4 Proposition: Let $\mathcal{C}$ satisfy (1.10), let I be a good cylinder object and $P : \mathcal{C} \to \mathcal{C}$ the path space functor $P(X) = X^I$. Then $(\mathcal{C}_f, \mathbf{fib}, P, *)$ and $(\mathcal{C}_f, \mathbf{sfib}, P, *)$ are P-categories in the sense of [1, §I. 3a], where $\mathcal{C}_f$ is the full subcategory of $\mathcal{C}$ of $\mathcal{T}$-fibrant object.

For the reader's convenience we recall the definition.

3.5 Definition: A *P-category* is a category $\mathcal{C}$ together with a subcategory **fib** of fibrations, a path space functor $P : \mathcal{C} \to \mathcal{C}$ with natural transformations $\pi_0, \pi_1 : P \to Id$ and $i : Id \to P$, and a terminal object $*$. The following axioms hold:

(P1) Path space axiom: $\pi_0 \circ i = \pi_1 \circ i = id$.

(P2) Pullback axiom: pullbacks along fibrations exist and are preserved by P. The class **fib** is closed under base change. $P(*) = *$.

(P3) Fibration axiom: each isomorphism is a fibration, each map $X \to *$ is a fibration. Each fibration has the homotopy lifting property.

(P4) Relative path space axiom: for a fibration $f : X \to B$ the map

$$(Pf, (\pi_0, \pi_1)) : P(X) \longrightarrow P(B) \times_{(B \times B)} (X \times X)$$

is a fibration.

(P5) Interchange axiom: there is a natural transformation $\tau : P \circ P \to P \circ P$. such that for all X and $\varepsilon = 0, 1$

$$\pi_\varepsilon(P(X)) \circ \tau(X) = P(\pi_\varepsilon(X)) \text{ and } P(\pi_\varepsilon(X)) \circ \tau(X) = \pi_\varepsilon(P(X)).$$

I-structures are defined dually with P replaced by a cylinder functor I and **fib** replaced by a subcategory **cof** of cofibrations. (For an explicit list of axioms see [1, §I. 3].)

Proof of 3.4: Note that $\pi_\varepsilon : P \to Id$ and $i : Id \to P$ are induced by $i_\varepsilon : \Phi \to I$ and $\pi : I \to \Phi$. Since **fib** and **sfib** are saturated and each $\mathcal{T}$-fibrant object is strongly $\mathcal{T}$-fibrant the axioms hold. For (P4) observe that $(Pf, (\pi_0, \pi_1))$ is isomorphic to

$$(f^I, X^{(j_0, j_1)}) : X^I \longrightarrow B^I \times_{B^{\Phi \sqcup \Phi}} X^{\Phi \sqcup \Phi}$$

Since $(j_0, j_1) : \Phi \sqcup \Phi \to I$ is a strong $\mathcal{T}$-cofibration this map is a (strong) $\mathcal{T}$-fibration for (strong) $\mathcal{T}$-fibrations $f : X \to B$ by the pairing theorem. $\square$

P-categories admit fibration structures [1, I. 3a.4], which allow the definitions of right derived functors. We again recall the definition (we change the terminology slightly).

3.6 Definition: A *fibration structure* on a category $\mathcal{C}$ consists of subcategories **we** of *weak equivalences* and **fib** of *fibrations* and a class $\mathcal{M}$ of *m-cofibrant* objects (m for "model"). We assume that $\mathcal{C}$ has a terminal object $*$. The following axioms hold

(F1) If $g \circ f$ is defined and any two of $f, g, g \circ f$ are in **we** so is the third. Isomorphism are in **fib** $\cap$ **we**.

(F2) Pullbacks of maps $X \to B$ along fibrations $E \to B$ exist, and **fib** and **fib** $\cap$ **we** are closed under base change.

(F3) Every $f : X \to Y$ has a factorization $f = p \circ i$ into a fibration p and a weak equivalence i.

(F4) (i) For every X there is an m-cofibrant Y and a map $f : Y \to X$ in **fib** $\cap$ **we**.

(ii) Any map $p : X \to Y$ in **fib** $\cap$ **we** with Y m-cofibrant has a section.

Cofibration structures are defined dually: we assume that $\mathcal{C}$ has an initial object, **fib** is replaced by a category **cof** of *cofibrations* and $\mathcal{M}$ is a class of *m-fibrant* objects.

3.7 Cogluing Axiom: A fibration category satisfies the *cogluing axiom* if in any pullback diagram

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{p}"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

with p a fibration and f a weak equivalence, the map $\overline{f}$ is a weak equivalence.

The *Gluing Axiom* for cofibration categories is defined dually.

For the rest of this section we assume that $\mathcal{C}$ satisfies (1.10) and has a good cylinder object.

3.8 Corollary: ([1, I. 3a,4]) $(\mathcal{C}_f, \mathbf{eq}, \mathbf{fib}, \mathcal{M})$ and $(\mathcal{C}_f, \mathbf{eq}, \mathbf{sfib}, \mathcal{M})$ are fibration structures on $\mathcal{C}_f$ satisfying the cogluing axiom with **eq** the category of homotopy equivalences and $\mathcal{M} = ob\mathcal{C}_f$.

The result has a number of consequences:

3.9 Cogluing Lemma: ([1, II. 1.2]) Given a commutative diagram of $\mathcal{T}$-fibrant objects

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] & X_2 \arrow[l] \arrow[d] \\ B_1 \arrow[r] & B_0 & B_2 \arrow[l] \end{tikzcd}$$

If there is a $\mathcal{T}$-fibration in each row and all vertical maps are homotopy equivalences, then the induced map

$$X_1 \times_{X_0} X_2 \longrightarrow B_1 \times_{B_0} B_2$$

is a homotopy equivalence. $\square$

3.10 Proposition: [1, I.3.12] Let $f : Y \to B$ be a map of $\mathcal{T}$-fibrant objects. Let Pf be the mapping path space, i.e.

$$\begin{tikzcd} Pf \arrow[r, "\overline{f}"] \arrow[d, "\overline{\pi_0}"'] \arrow[dr, phantom, "\text{pullback}"] & B^I \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & B \\ Y \arrow[r, "f"] & B & \end{tikzcd}$$

Then $\pi_1 \circ \overline{f}$ is a strong $\mathcal{T}$-fibration and the map $Y \to Pf$ induced by $(id_Y, \iota_B \circ f)$ is a homotopy equivalence ($\iota_B : B \to B^I$ is induced by the projection $\pi : I \to \Phi$).

3.11 Proposition: [1, II.1.12] Let $p : E \to B$ be a $\mathcal{T}$-fibration and a homotopy equivalence between $\mathcal{T}$-fibrant objects. Then p is a fiberwise deformation retraction. $\square$

The dual ($\mathcal{T}$-cofibration) situation is less pleasant because $\mathcal{T}$-cofibrations do not satisfy the homotopy extension property in general. In particular, the homotopy relation is not an equivalence relation. We need additional axioms.

3.12 Additional cylinder axioms:

(Z3) $i_1 : \Phi \to I$ is in $\mathcal{T}$.

(Z4) $i_1 : \Phi \to I$ is a strong deformation retract (see Def. 3.14 below).

To obtain an equivalence relation for arbitrary $\mathcal{C}(X, Y)$ we define new cylinder objects.

3.13 General cylinder objects: There are several operations which create new cylinder objects from given ones: firstly, we may interchange the structure data i_0 and i_1. We denote the result by $-I$. Secondly, given a cylinder object (J, j_0, j_1, ρ) with "inclusions" $j_\varepsilon : \Phi \to J$ and "retraction" $\rho : J \to \Phi$ the pushout of

$$J \xleftarrow{j_1} \Phi \xrightarrow{j_2} I$$

is a new cylinder object $(\overline{J}, \overline{j}_0, \overline{j}_1, \overline{\rho})$ with the obvious $\overline{\rho} : \overline{J} \to \Phi$, with $\overline{j}_0 : \Phi \to \overline{J}$ induced by $j_0 : \Phi \to J$, and $\overline{j}_1 : \Phi \to \overline{J}$ induced by $i_\eta : \Phi \to I$ where $\{\varepsilon, \eta\} = \{0, 1\}$.

3.14 Definition: $f, g : X \to Y$ are called *homotopic* if they are in the same equivalence class with respect to the equivalence relation generated by the homotopy relation. *Relative homotopy* is defined similarly and *strong deformation retractions* are understood in this sense.

Clearly, f and g are homotopic iff there is a cylinder object J and and a *homotopy* $H : X \otimes J \to Y$ with $H \circ (X \otimes j_0) = f$ and $H \circ (X \otimes j_1) = g$.

3.15 If I is a good cylinder object satisfying Axiom 3.13 (Z3) and J is an arbitrary cylinder object then, by construction, j_0 and j_1 are in $\mathcal{T}_{cosat}$. Moreover, by Reedy's patching lemma [3], $(j_0, j_1) : \Phi \sqcup \Phi \to J$ is a strong $\mathcal{T}$-cofibration.

3.16 Proposition: Let I be a good cylinder object satisfying Axioms 3.13. Then $(\mathcal{C}, \mathbf{eq}, \mathbf{cof}, \mathcal{M})$ and $(\mathcal{C}, \mathbf{eq}, \mathbf{scof}, \mathcal{M})$ with $\mathbf{eq}$ the class of homotopy equivalences and $\mathcal{M}$ the class of $\mathcal{T}$-fibrant objects satisfies all axioms of a cofibration structure apart from the dual of (F4(i)).

The duals of Axioms (F1) and (F2) clearly hold. The duals of (F3) and (F4(ii)) are consequences of the following two results.

3.17 Proposition: Let I be a good cylinder object satisfying Axioms 3.13 and Mf the mapping cylinder of $f : A \to X$, i.e.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "i_1"] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d, "\overline{i}"] \\
A \arrow[r, "i_0"] & A \otimes I \arrow[r, "\overline{f}"] & Mf
\end{tikzcd}$$

Then $\overline{f} \circ i_0$ is a strong $\mathcal{T}$-cofibration and the retraction $q : Mf \to X$ of $\overline{i}$ induced by $f \circ \pi$ and id_X is a strong deformation retraction. Observe that $q \circ \overline{f} \circ i_0 = f$.

Proof: Consider the pushout

$$\begin{tikzcd}
 & A \sqcup A \arrow[r] \arrow[d] & A \otimes I \arrow[d] \\
A \arrow[r] & X \sqcup A \arrow[r] & Mf
\end{tikzcd}$$

By the pairing theorem $A \sqcup A \to A \otimes I$ is a strong $\mathcal{T}$-cofibration. Since X is strongly $\mathcal{T}$-cofibrant $A \to Mf$ is a strong $\mathcal{T}$-cofibration. The strong deformation retraction $I \to \Phi$ of i_1 induces a deformation $relX$

$$Mf \otimes J \to Mf$$

from id_{Mf} into $\overline{i} \circ q$. □

3.18 Lemma: Let I be a good cylinder object satisfying Axioms 3.13 and let $i : A \to X$ be a $\mathcal{T}$-cofibration and a homotopy equivalence.

(1) If A is $\mathcal{T}$-fibrant, then i admits a retraction $r : X \to A$.

(2) If A and X are $\mathcal{T}$-fibrant, then r is a strong deformation retraction.

Proof: (1) Let $p : X \to A$ be a homotopy inverse of i and $H : A \otimes J \to A$ a homotopy from $p \circ i$ to id_A. Since A is $\mathcal{T}$-fibrant and $0 \to J$ is in $\mathcal{T}_{cosat}$ the map $H \cup p : A \otimes J \cup X \otimes 0 \to A$ has an extension $K : X \otimes J \to A$. The map $r = K \circ (X \otimes j_1) : X \to A$ is the required retraction.

(2) Since A and X are $\mathcal{T}$-fibrant we have a retraction r of i and a homotopy $G : X \otimes I \to X$ from $i \circ r$ to id. For $f, g : X \to Y$ let $[X \otimes I, Y]_f^g$ denote the set of homotopy classes $rel(X \otimes \partial I)$ of homotopies from f to g (we assume Y to be $\mathcal{T}$-fibrant so that (3.3) applies). Since i is a homotopy equivalence we have a bijection

$$(i \otimes I)^* : [X \otimes I, X]_{ior}^{id} \to [A \otimes I, X]_i^i.$$

Assume G to be the homotopy, which maps to the constant one. Hence there is a homotopy of homotopies $rel(A \otimes \partial I)$

$$L : A \otimes I \otimes I \to X$$

from $G \circ (i \otimes I)$ to the constant homotopy on i.

Since X is $\mathcal{T}$-fibrant and $A \otimes I \cup X \otimes \partial I \to X \otimes I$ is a $\mathcal{T}$-cofibration by the pairing theorem, the map

$$L \cup i \circ r \circ \pi \cup \pi \cup G : A \otimes I \otimes I \cup X \otimes 0 \otimes I \cup X \otimes 1 \otimes I \cup X \otimes I \otimes 0 \to X$$

has an extension $\hat{L} : X \otimes I \otimes I \to X$. The restriction of $\hat{L}$ to $X \otimes I \otimes 1$ defines a homotopy $relA$ from $i \circ r$ to id. □

Since we have both fibrations and cofibrations we are also interested in IP-categories which are defined as follows.

3.19 Definition: $\mathcal{C}$ is an *IP-category* if it has the structure $(\mathcal{C}, \mathbf{cof}, \mathbf{fib}, I, P, \emptyset, \star)$ satisfying the axioms

(IP1) $(\mathcal{C}, \mathbf{cof}, I, \emptyset)$ is an I-category and $(\mathcal{C}, \mathbf{fib}, P, \star)$ a P-category.

(IP2) I is left adjoint to P.

(IP3) Relative homotopy lifting property: let $i : A \to X$ be a cofibration and $p : E \to Z$ be a fibration then the diagram

$$\begin{tikzcd} I(A) \cup X \arrow[r] \arrow[d] & E \arrow[d] \\ I(X) \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

has a (dotted) filler. ($I(A) \cup X$ is the pushout of $X \xleftarrow{i} A \xrightarrow{i_0} I(A)$).

4 The minimal case

We still make the general assumption that $\mathcal{V}$ and $\mathcal{C}$ satisfy (1.9) and (1.10) and that we are given a good cylinder object in $\mathcal{V}$. Hence the minimal case is

$$\mathcal{T} = \{i_0 : \Phi \to I\}$$

In this case $\mathcal{T}$-fibrations and -cofibrations are formally those one is used to from $\mathcal{T}op$. So we drop $\mathcal{T}$ from the notation. Since i_0 has π as retraction every object X in $\mathcal{C}$ is fibrant, and the set-up of §2 is symmetric in fibrations and cofibrations. Hence every result dualizes. In particular, we get

4.1 Proposition: If $\mathcal{T}$ is minimal and I is a good cylinder object then $(\mathcal{C}, \mathbf{cof}, \mathbf{sfib}, - \otimes I, (-)^I, \emptyset, *)$ and $(\mathcal{C}, \mathbf{scof}, \mathbf{fib}, - \otimes I, (-)^I, \emptyset, *)$ are IP-categories (cf. Def. 3.19).

4.2 Corollary: If $\mathcal{T}$ is minimal and I is a good cylinder object, then $(\mathcal{C}, \mathbf{eq}, \mathbf{cof})$ and $(\mathcal{C}, \mathbf{eq}, \mathbf{scof})$ are cofibration categories satisfying the gluing axiom with all objects m-fibrant, and $(\mathcal{C}, \mathbf{eq}, \mathbf{fib})$ and $(\mathcal{C}, \mathbf{eq}, \mathbf{sfib})$ are fibration categories satisfying the cogluing axiom with all objects m-cofibrant (cf. Def. 3.6).

4.3 Proposition: If $\mathcal{T}$ is minimal and I is a good cylinder objects the results (3.9), (3.10), (3.11) and (3.18) and their duals hold without any restrictions on the objects involved.

We mention in passing that the minimal case applies to categories enriched in $\mathcal{T}op, \mathcal{HT}op, \mathcal{T}op_*$ and $\mathcal{HT}op_*$ (see §5), the case we are most interested in.

In the minimal case we have additional characterizations of (strong) cofibrations and (strong) fibration, which are more in the spirit of a closed model category setting.

4.4 Proposition: 1. $j \in \mathbf{cof} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{sfib} \cap \mathbf{eq}$

2. $p \in \mathbf{sfib} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{cof} \cap \mathbf{eq}$
3. $j \in \mathbf{cof} \cap \mathbf{eq} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{sfib}$
4. $j \in \mathbf{scof} \cap \mathbf{eq} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{fib}$
5. $p \in \mathbf{fib} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{scof} \cap \mathbf{eq}$
6. $j \in \mathbf{scof} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{fib} \cap \mathbf{eq}$
7. $p \in \mathbf{fib} \cap \mathbf{eq} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{scof}$
8. $p \in \mathbf{sfib} \cap \mathbf{eq} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{cof}$

The proof makes use of

4.5 Lemma: If $j : A \to X$ is a (strong) cofibration then $(j \otimes I, i_0) : A \otimes I \cup X \otimes 0 \to X \otimes I$ is a (strong) cofibration and homotopy equivalence.
If $p : E \to B$ is a (strong) fibration then $(p^I, \pi_0) : E^I \to B^I \times_B E$ is a (strong) fibration and homotopy equivalence.

Proof: By the pairing theorem $(j \otimes I, i_0) : A \otimes I \cup X \otimes 0 \to X \otimes I$ is a (strong) cofibration because $0 \to I$ is a strong cofibration. Now $A \otimes I \cup X \otimes 0$ is the mapping cylinder of $j : A \to X$. Hence $(j \otimes \pi, id_X)$ in the diagram

$$\begin{tikzcd} A \otimes I \cup X \otimes 0 \arrow[rr] \arrow[dr, "(j\otimes\pi, id_X)"'] & & X \otimes I \arrow[dl, "X\otimes\pi"] \\ & X & \end{tikzcd}$$

is a homotopy equivalence by the dual of (3.10). Since $X \otimes \pi$ is a homotopy equivalence (apply the dual of (3.10) to id_X) the map $A \otimes I \cup X \otimes 0 \to X \otimes I$ is a homotopy equivalence.
The second part is dual. □

Proof of 4.4: We call fibrations or cofibrations which are also homotopy equivalences *trivial* fibrations or cofibrations. We prove (1), (2), (3) and (4); the other results are dual.

(1) "$\Leftarrow$": $\pi_0 : Z^I \to Z$ is a strong fibration and a homotopy equivalence by (4.5) with $B = *$.

"$\Rightarrow$": Let $j : A \to X$ be a cofibration and $p : E \to B$ a trivial strong fibration. By (3.11) there is a section $s : B \to E$ of p and a fiber homotopy equivalence

$$s \circ p \simeq_B id_E$$

Take a map $(f, g) : j \to p$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

The morphism $k = s \circ g : X \to E$ makes the lower triangle commutative and the upper commute by a homotopy

$$H : A \otimes I \to E, \quad H : k \circ j \simeq f$$

with $p \circ H = p \circ f \circ (A \otimes \pi)$. Since p is a strong fibration, the diagram

$$\begin{tikzcd} X \otimes 0 \cup A \otimes I \arrow[r, "{(k,H)}"] \arrow[d] & E \arrow[d, "p"] \\ X \otimes I \arrow[r, "g \circ \pi_X"'] & B \end{tikzcd}$$

admits a lift $K : X \otimes I \to E$. The morphisms $K \circ i_1 : X \to E$ is the required filler h.

(2) "$\Leftarrow$": Let $j : A \to X$ be a cofibration. We have to show that

$$\begin{tikzcd} A \otimes I \cup X \otimes 0 \arrow[r] \arrow[d, "{(j \otimes I, i_0)}"'] & E \arrow[d, "p"] \\ X \otimes I \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

admits a filler. By (4.5) the map $(j \otimes I, i_0)$ is a trivial cofibration. Hence the filler exists.

"$\Rightarrow$": Consider the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

with j a trivial cofibration and p a strong fibration. By the dual of (3.18) there is a retraction $r : X \to A$ of j and a deformation *rel* A

$$H : X \otimes I \to X, \quad H : j \circ r \overset{A}{\simeq} id_X$$

with $H \circ (j \otimes I) = j \circ (A \otimes \pi)$. Since $p \in \mathbf{sfib}$ the diagram

$$\begin{tikzcd} A \otimes I \cup X \otimes 0 \arrow[r, "(f\circ\pi_A, f\circ r)"] \arrow[d] & E \arrow[d, "p"] \\ X \otimes I \arrow[r, "g\circ H"'] \arrow[ur, dotted] & B \end{tikzcd}$$

admits a filler $K : X \otimes I \to E$. The morphisms $K \circ i_1 : X \to E$ is the required lift.

(3),(4) "$\Rightarrow$" follows from (2) resp. (5), which is dual to (1).

"$\Leftarrow$": Suppose that $j : A \to X$ has the LLP for all (strong) fibrations p. Then j is a cofibration because $Z^{i_0} : Z^I \to Z$ is a strong fibration for all Z by the pairing theorem. Since all objects in $\mathcal{C}$ are strongly fibrant there is a retraction $r : X \to A$, and the diagram

$$\begin{tikzcd} A \otimes I \cup X \otimes 0 \arrow[r, "(j\circ(A\otimes\pi), j\circ r)"] \arrow[d, "(j\otimes I, i_0)"'] & X \arrow[d] \\ X \otimes I \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

admits a filler, which is a deformation retraction. This proves case (3). For (4) it remains to show that j is a strong cofibration. If $p : E \to B$ is a fibration then $(p^I, E^{i_0}) : E^I \to B^I \times_B E$ is a fibration by the pairing theorem. Hence j is a strong cofibration. □

4.6 Addendum to the Pairing Theorem: Suppose that $\mathcal{T}$ is minimal, that I is a good cylinder object, and that the assumptions of the pairing theorem hold. Then, in the notation of the pairing theorem:
(1) If i or j is a homotopy equivalence, then f is a homotopy equivalence.
(2) If j or p is a homotopy equivalence, then g is a homotopy equivalence.

Proof: We consider a commutative diagram

$$\begin{tikzcd} T(A,Y) \cup T(X,B) \arrow[r] \arrow[d, "f"'] & E \arrow[d, "p"] \\ T(X,Y) \arrow[r] & Z \end{tikzcd}$$

with p a strong fibration. Finding a filler is equivalent to finding a filler for

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & V(Y,E) \arrow[d, "g"] \\ X \arrow[r] & V(B,E) \times_{V(B,Z)} V(Y,Z) \end{tikzcd}$$

By the pairing theorem g is a fibration. Hence this diagram has a filler provided i is a homotopy equivalence. Now apply the previous proposition.

If j is a homotopy equivalence we interchange the roles of i and j: we suppose that j is a strong cofibration and that i is a cofibration and a homotopy equivalence. Then g is a strong fibration by the pairing theorem and the required filler again exists.

The proof of the second part of (2) is dual. □

4.7 Proposition: The IP-categories of Proposition 4.1 satisfy the *Continuity Axiom* [1, p.182], i.e. given a commutative ladder

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & \cdots \\ B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & \cdots \end{tikzcd}$$

such that each f_i and g_i is a cofibration and each h_i a homotopy equivalence then $\operatorname{colim} h_i : \operatorname{colim} A_i \to \operatorname{colim} B_i$ is a homotopy equivalence. Suppose h_0, each f_i and g_i and each induced map

$$q_i : B_i \cup_{A_i} A_{i+1} \longrightarrow B_{i+1}$$

is a cofibration, then $\operatorname{colim} h_i$ is a cofibration. If each f_i, g_i, and q_i is a strong cofibration, then $\operatorname{colim} h_i$ is a strong cofibration.
The dual statements hold, too.

We leave the formal proof as an exercise.

5 Basic examples

In this section we list examples of possible categories $\mathcal{V}$. Our main interest are topologically enriched categories.

5.1 $\mathcal{T}op$

Recall from (1.12) that $\mathcal{T}op$ is a self-enriched category satisfying Assumption (1.9). We are in the minimal case with I the unit interval. Cofibrations and fibrations are the usual ones. Strong cofibrations are exactly the closed cofibrations [23, Prop. 1]. Since $\{0,1\} \subset [0,1]$ is a closed cofibration, I is a good cylinder object.
We do not know of a similar characterization of strong fibrations but we can show

5.1.1 Proposition: Any fibration $p : E \to B$ with B Hausdorff is strong.

Proof: Let $A \to X$ be a cofibration and

$$\begin{tikzcd} X \times 0 \cup A \times I \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"] & B \end{tikzcd}$$

a commutative diagram. Since $X \times 0 \cup A \times I$ is a retract of $X \times I$ the map f in particular extends to a map $\overline{f} : X \times 0 \cup \overline{A} \times I \to E$. Since B is Hausdorff and $X \times 0 \cup A \times I$ dense in $X \times 0 \cup \overline{A} \times I$ we have $p \circ \overline{f} = g \mid X \times 0 \cup \overline{A} \times I$. Since $\overline{A} \subset X$ is a strong cofibration there is a filler $G : X \times I \to E$ extending $\overline{f}$ and lifting g. $\square$

Our pairing theorem provides the following result, which is a minor extension of [22, Thm. 10].

5.1.2 Proposition: Let $i : A \subset X$ be a cofibration and $p : E \to B$ a fibration. If A is closed or B Hausdorff the natural map

$$E^X \longrightarrow E^A \times_{B^A} B^X$$

is a fibration.

5.2 $\mathcal{T}op_*$

$\mathcal{T}op_*$ is $\mathcal{T}op_*$-enriched and satisfies Assumption 1.9.

We do not know of any characterization of strong cofibrations like in (5.1) but we can distinguish a large class of them: If $j : A \to X$ is a closed cofibration in $\mathcal{T}op_*$ of well-pointed spaces (i.e. the inclusion of the base point is a closed cofibration) then j is a strong cofibration in $\mathcal{T}op_*$: by [23, Prop. 9] the map j is also a closed cofibration in $\mathcal{T}op$. Since any fibration in $\mathcal{T}op_*$ is also a fibration in $\mathcal{T}op$ the required filler for (2.8) with $(L, K) = (I, 0)$ exists in $\mathcal{T}op$. Since we started with a diagram in $\mathcal{T}op_*$ this filler factors through the reduced cylinder.

Again our pairing theorem gives a minor extension of a well-known result [23, Prop. 12].

5.2.1 Proposition: Let $A \subset X$ be a closed cofibration in $\mathcal{T}op_*$ with A and X well-pointed and $B \subset Y$ an arbitrary cofibration in $\mathcal{T}op_*$ then

$$A \wedge Y \cup X \wedge B \longrightarrow X \wedge Y$$

is a cofibration in $\mathcal{T}op_*$

5.3 $\mathcal{HT}op$ and $\mathcal{HT}op_*$

The results of Examples (5.1) and (5.2) also hold if we replace $\mathcal{T}op$ by the category $\mathcal{HT}op$ of compactly generated weak Hausdorff spaces. For the properties needed we refer the reader to [11].

5.4 $\mathcal{SS}ets$ and $\mathcal{SS}ets_*$

$\mathcal{S}Sets$ is self-enriched, tensored and cotensored (1.12). Let Δ^n denote the standard simplicial n-simplex. Then Δ^1 is the canonical candidate for a cylinder object. Unfortunately $\{0,1\} \subset \Delta^1$ does not have the homotopy extension property for all objects. Hence the "minimal case" does not apply.

The minimal choice for $\mathcal{T}$ which makes Δ^1 a good cylinder object is the set of inclusions $\Lambda^n_k \to \Delta^n$, $n \in \mathbb{N}$, $0 \leq k \leq n$ (see (2.2)). Then $\mathcal{T}$-fibrations are exactly the Kan fibrations and all inclusions $A \to X$ of simplicial subsets are strong $\mathcal{T}$-cofibrations. In particular, Δ^1 is good and satisfies Axioms 3.12.

The situation for $\mathcal{S}Sets_*$ is similar.

5.5 $\mathcal{SA}b$

The category $\mathcal{SA}b$ of simplicial abelian groups is a self-enriched category satisfying (1.9): $\mathcal{SA}b(X,Y)_n$ is the abelian group of all simplicial homomorphisms $X \otimes \mathbb{Z}(\Delta^n) \to Y$. The simplicial exponential law (e.g. see [17]) specializes to an exponential law enriched in $\mathcal{SA}b$

$$\mathcal{SA}b(X \otimes Y, Z) \cong \mathcal{SA}b(Y, \mathcal{SA}b(X,Z))$$

(the tensor product is taken dimensionwise). This tensor product and the internal Hom-functor make $\mathcal{SA}b$ tensored and cotensored. $\mathbb{Z}(\Delta^1)$ is the canonical candidate for a cylinder object. Since each simplicial abelian group is a Kan complex, $\mathbb{Z}(\Delta^1)$ is a good cylinder object in the minimal case. Inclusions of simplicial abelian subgroups are strong cofibrations.

5.6 $\mathcal{C}plx_R$

Let R be a commutative ring and $\mathcal{C}plx_R$ the category of chain complexes of R-modules. The usual tensor product of chain complexes and the Hom-complex make $\mathcal{C}plx_R$ a self-enriched category satisfying (1.9). The cellular chain complex of the unit interval is our candidate for a cylinder object I. If follows from [5] that it is a good cylinder object in the "minimal case". For further information see [1, §I. 6], [5], [17].

5.7 $\mathcal{C}at$

Let $\mathcal{C}at$ be the category of small categories. The usual product of categories and the functor categories make $\mathcal{C}at$ a self-enriched category satisfying (1.9). The most common candidate for a cylinder object is the category

$$\mathcal{L}_1 : 0 \longrightarrow 1$$

$\mathcal{L}_1$ is not a good cylinder object for the "minimal case"; we are in a situation similar to the one for $\mathcal{S}Sets$.
Another reasonable, though less interesting cylinder object is the category $\mathcal{I}s$ consisting of exactly one non-trivial isomorphism

$$0 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 1$$

Since a functor $\mathcal{I}s \times \mathcal{I}s \to \mathcal{D}$ into any category $\mathcal{D}$ is completely determined by its restriction to $0 \times \mathcal{I}s \cup 1 \times \mathcal{I}s \cup \mathcal{I}s \times 0$, we get a good cylinder object in the "minimal case".

6 Spectra

In 1996 Elmendorff, Kriz, Mandell, and May constructed a category of spectra admitting an associative, commutative, and unital smash product [6]. This simplifies a number of constructions on spectra and enables one to develop a reasonable homological algebra.

Let $\mathcal{S}p$ denote the category of spectra in the sense of Lewis, May, and Steinberger [12] and $\mathcal{L}$ the linear isometry operad. An $\mathcal{L}$-spectrum E is a spectrum together with an action

$$\mathcal{L}(1) \ltimes E \to E$$

of the monoid $\mathcal{L}(1)$ on E. The category $\mathcal{LS}p$ of $\mathcal{L}$-spectra has an associative and commutative smash product $- \wedge_{\mathcal{L}} -$ which, unfortunately, is not unital.

Let S be the sphere spectrum. An $\mathcal{L}$-spectrum E, for which the natural map $S \wedge_{\mathcal{L}} E \to E$ is an isomorphism, is called an S-module. The smash-product $-\wedge_{\mathcal{L}}-$ restricts to an associative, commutative and unital smash product $-\wedge_S-$ on the subcategory ${}_S\mathcal{M}$ of S-module spectra. Using $- \wedge_S -$ it is now clear how to define ring spectra R as algebras over S, algebra spectra over R, R-modules or bimodules.

Let A and B be S-algebra spectra and ${}_A\mathcal{M}_B$ the category of left A-right B-module spectra. This category as well as $\mathcal{S}p$ and $\mathcal{LS}p$ is $\mathcal{HT}op_*$-enriched satisfying (1.10). If X is in $\mathcal{HT}op_*$ and $\Sigma^\infty X$ is its suspension spectrum, then

$$\begin{aligned} M \otimes X &:= M \wedge X \\ M^X &= F(X, M) \end{aligned}$$

for $\mathcal{S}p$ and $\mathcal{LS}p$ (see [12, I.3]), and

$$M \otimes X := M \wedge X \cong M \wedge_S \Sigma^\infty X \qquad M^X = S \wedge_{\mathcal{L}} F(X, M)$$

for ${}_S\mathcal{M}$ and ${}_A\mathcal{M}_B$ make these categories tensored and cotensored.

Let R be a commutative S-algebra. Let ${}_R\mathcal{CA}lg$ and ${}_R\mathcal{A}lg$ denote the categories of (commutative) R-algebras. These categories are $\mathcal{HT}op$-enriched and satisfy (1.10). The verification of the properties is not difficult. For details see [6] and [15].

We note in passing that for a commutative S-algebra R the category ${}_R\mathcal{M}$ of left R-module spectra also is self-enriched with the smash product $- \wedge_R -$ as tensor and the function spectrum $F_R(-,-)$ as internal Hom-functor.

Elmendorf, Kriz, Mandell, and May pay particular interest to cell objects in these categories for their study of derived functors. The forgetful functor $U_1 : \mathcal{LS}p \to \mathcal{S}p$ has a left adjoint

$$\mathbb{L} : \mathcal{S}p \to \mathcal{LS}p, \qquad E \mapsto \mathcal{L}(1) \ltimes E.$$

Let ${}_A(S^n)_B = A \wedge_S \mathbb{L}(\Sigma^\infty S^n) \wedge_S B$. A pair (M, K) is called a *relative cell (A, B)-bimodule* if M is the colimit of a sequence

$$K = M_0 \to M_1 \to \dots$$

in ${}_A\mathcal{M}_B$ and M_{n+1} is obtained from M_n by attaching cells, i.e. there is a pushout

6.1

$$\begin{tikzcd}
\bigvee_{i\in J} {}_A(S^{q_i})_B \arrow[rr, "j"] \arrow[d] & \text{pushout} & \bigvee_{i\in J} C({}_A(S^{q_i})_B) \arrow[d] \\
M_n \arrow[rr] & & M_{n+1}
\end{tikzcd}$$

where $C(-)$ is the cone functor. J may be empty. Call M a *cell bimodule* if $(M,*)$ is a relative cell bimodule.

The derived category $\mathcal{D}({}_A\mathcal{M}_B)$ is obtained from ${}_A\mathcal{M}_B$ by formally investing weak equivalences, i.e. maps in ${}_A\mathcal{M}_B$ which are weak equivalences of underlying spectra. The derived category exists and is equivalent to the homotopy category of cell (A,B)-bimodules [6]. Moreover, $\mathcal{D}({}_S\mathcal{M})$ is equivalent to Boardman's stable homotopy category [2].

Let R be a commutative S-algebra. Then there are functors

$$\mathbb{T}: {}_R\mathcal{M} \to {}_R\mathcal{A}lg \text{ and } \mathbb{P}: {}_R\mathcal{M} \to {}_R\mathcal{CA}lg$$

which are left adjoint to the forgetful functors. A pair (X,A) in ${}_R\mathcal{CA}lg$ resp. ${}_R\mathcal{A}lg$ is called a *(commutative) relative cell algebra* if X is the colimit of a sequence

$$A = X_0 \to X_1 \to \dots$$

in ${}_R\mathcal{CA}lg$ resp. ${}_R\mathcal{A}lg$ and X_{n+1} is obtained from X_n by attaching cells:

6.2

$$\begin{tikzcd}
\bigvee_{i\in J} \mathbb{P}({}_RS^{q_i}) \arrow[rr, "j"] \arrow[d] & \text{pushout} & \bigvee_{i\in J} \mathbb{P}(C_RS^{q_i}) \arrow[d] \\
X_n \arrow[rr] & & X_{n+1}
\end{tikzcd}$$

(use $\mathbb{T}$ instead of $\mathbb{P}$ for ${}_R\mathcal{A}lg$).

All these categories have closed model structures in Quillen's sense [17]: the weak equivalences are maps which are weak equivalences of underlying spectra, the cofibrations, called *q-cofibrations* in [6] to distinguish them from the usual cofibrations, are precisely the retracts of relative cell objects. The *q-fibrations* are the Serre fibrations for $\mathcal{S}p$ and $\mathcal{LS}p$, and the morphisms $f: X \to Y$ for which $F(id,f): F_{\mathcal{L}}(S,X) \to F_{\mathcal{L}}(S,Y)$ is a Serre fibration of spectra in all other cases.

Since q-cofibrations are not preserved by some of the forgetful functors change of rings arguments and the development of relative homological algebra is awkward in this setting. The situation is much better for (strong) cofibrations.

In the following diagram the functors U_i are the forgetful functors and R is a commutative S-algebra

6.3

$$
\begin{tikzcd}[row sep=small]
{}_R\mathcal{CA}lg \arrow[dr, "U_5"] & & & & \\
& {}_R\mathcal{M} \arrow[r, "U_3"] & {}_S\mathcal{M} \arrow[r, "U_2"] & \mathcal{LS}p \arrow[r, "U_1"] & \mathcal{S}p \\
{}_R\mathcal{A}lg \arrow[ur, "U_4"'] & & & &
\end{tikzcd}
$$

6.4 Lemma: [6]

(1) U_1 has the left adjoint $\mathbb{L}$ and $U_1 \circ \mathbb{L}$ has a right adjoint.

(2) U_2 has a right adjoint $E \mapsto S \wedge_{\mathcal{L}} E$, which in turn is left adjoint to $M \mapsto F_{\mathcal{L}}(S, M)$.

(3) U_3 has a left adjoint $M \mapsto R \wedge_S M$ and a right adjoint $M \mapsto F_S(R, M)$.

(4) U_4 has the left adjoint $\mathbb{T}$, U_5 has the left adjoint $\mathbb{P}$.

As a consequence we obtain :

6.5 Proposition: (1) U_1 preserves cofibrations and (strong) fibrations. The functors $\mathbb{L}$ and $U_1 \circ \mathbb{L}$ preserve (strong) cofibrations (in the sense of (2.8)).

(2) The functor U_2 preserves (strong) cofibrations. The functor $S \wedge_{\mathcal{L}} -$ preserves (strong) cofibrations and (strong) fibrations.

(3) The functor U_3 preserves (strong) cofibrations and (strong) fibrations.

Proof: Since $U_1 \circ \mathbb{L}$ has a right adjoint the functor U_1 creates all limits and colimits. Hence it preserves pushouts and tensors and therefore cofibrations. The other statements follow from (2.12). □

The situations for U_4 and U_5 are more complicated. We postpone their treatment and consider first the module situation.

Given S-algebras A, B, and C, possibly commutative, and a map of S-algebras $f : B \to A$ we have an obvious forgetful functor

$$U : {}_A\mathcal{M}_C \to {}_B\mathcal{M}_C$$

U has a left adjoint $M \mapsto A \wedge_B M$ and a right adjoint $M \mapsto F_B(A, M)$. Hence we have

6.6 Proposition: The forgetful functor $U : {}_A\mathcal{M}_C \to {}_B\mathcal{M}_C$ preserves (strong) cofibrations and (strong) fibrations. □

The smash-product

$${}_A\mathcal{M}_B \times {}_B\mathcal{M}_C \to {}_A\mathcal{M}_C, \quad (M, N) \mapsto M \wedge_B N$$

satisfies the requirements of (2.11). We obtain the following pairing result, which simplifies arguments in [6].

6.7 Proposition: Suppose we are given cofibrations $i : K \to M$ in ${}_A\mathcal{M}_B$ and $j : L \to N$ in ${}_B\mathcal{M}_C$, and a fibration $p : Y \to Z$ in ${}_A\mathcal{M}_C$, then

(1) if i or j is strong, then

$$f : K \wedge_B N \cup M \wedge_B L \to M \wedge_B N$$

is a cofibration in ${}_A\mathcal{M}_C$. If, in addition, one is a homotopy equivalence, then f is a homotopy equivalence. If both i and j are strong then f is strong.

(2) if j or p is strong, then

$$g : F_C(N, Y) \to F_C(L, Y) \times_{F_C(L,Z)} F_C(N, Z)$$

is a fibration in ${}_A\mathcal{M}_B$. If, in addition, one is a homotopy equivalence, then g is a homotopy equivalence. If both j and p are strong then g is strong.

□

Corollary (2.13) gives the following useful result.

6.8 Proposition: Let $i : K \to L$ be a closed cofibration in $\mathcal{HT}op_*$ of well-pointed spaces. Then

(1) If $j : M \to N$ is a (strong) cofibration in ${}_A\mathcal{M}_B$, so is

$$f : M \wedge N \cup N \wedge K \to N \wedge L.$$

If, in addition, j or i is a homotopy equivalence, so is f.

(2) If $p : M \to N$ is a (strong) cofibration in ${}_A\mathcal{M}_B$, so is

$$g : M^L \to M^K \times_{N^K} N^L$$

If, in addition, p or i is a homotopy equivalence, so is g.

We return to algebras. Since every R-module is strongly cofibrant (2.10), the inclusions of wedge summands of R-modules are strong cofibrations in ${}_R\mathcal{M}$. Hence the argument of [6, VII. 3.9] in fact shows

6.9 Lemma: For any pushout diagram

$$\begin{tikzcd} \mathbb{T}M \arrow[r] \arrow[d] & X \arrow[d, "i"] \\ \mathbb{T}CM \arrow[r] & Y \end{tikzcd}$$

in ${}_R\mathcal{A}lg$ the map i is a strong cofibration of R-modules. The same result holds in ${}_R\mathcal{CA}lg$ with $\mathbb{T}$ replaced by $\mathbb{P}$. □

6.10 Corollary: The functors U_4 and U_5 map q-cofibrations to strong cofibrations. $\square$

We finally note

6.11 Proposition: In all categories of Diagram (6.3) the q-cofibrations are strong cofibrations.

Proof: Since **scof** is closed under retracts and each q-cofibration is a retract of a relative cell object it suffices to consider a relative cell object (M, A). This case reduces to proving that j in (6.1) and (6.2) is a strong cofibration. Since the functors $\mathbb{T}, \mathbb{P}, \mathbb{L}, R \wedge_S -$, and $S \wedge_{\mathcal{L}} -$ are left adjoints they preserve strong cofibrations and we are reduced to showing that $\Sigma^\infty S^n \to C\Sigma^\infty S^n$ is a strong cofibration. This follows from (6.7) because this map is isomorphic to $S \wedge S^n \to S \wedge CS^n$. $\square$

7 Other examples

In this final section we list a few categories to which we intend to apply our results to.

7.1 Algebras: Let Θ be a topological algebraic theory and $\Theta - \mathcal{T}op$ the category of Θ-algebras in $\mathcal{T}op$ (also called Θ-spaces). By [18] $\Theta - \mathcal{T}op$ is $\mathcal{T}op$-enriched, $\mathcal{T}op$-complete, $\mathcal{T}op$-cocomplete, tensored and cotensored: $X^K = \mathcal{T}op(K, X)$ with the obvious Θ-structure while $X \otimes K$ is the pushout

$$\begin{array}{ccc} \coprod_{k \in K} F(X) & \longrightarrow & F(X \times K) \\ \downarrow & & \downarrow \\ \coprod_{k \in K} X & \longrightarrow & X \otimes K \end{array}$$

in $\Theta - \mathcal{T}op$ where $F : \mathcal{T}op \to \Theta - \mathcal{T}op$ is the free functor. If Θ has exactly one 0-ary operation a Θ-algebra is naturally based so that its underlying space can be considered an object of $\mathcal{T}op_*$. In this case $\Theta - \mathcal{T}op$ is $\mathcal{T}op_*$-enriched, complete and cocomplete, tensored and cotensored with the obvious modifications in the definitions of $X \otimes K$ and X^K.

7.2 Topological modules: This is a special case of (7.1). Of course, our results on module spectra equally well hold for topological modules over topological rings. If we replace the sphere spectrum S by the ring $\mathbb{Z}$ (or, more generally, any commutative ground ring k), the smash product $- \wedge_R -$ by $\otimes_k$, the function spectrum $F_R(M, N)$ by $\mathrm{Hom}_k(M, N)$ and if we delete $\mathcal{LS}p$ and $\mathcal{S}p$ from our diagram (6.3) the analysis of the rest of (6.3) holds.

7.3 Diagrams in $\mathcal{T}$op and $\mathcal{T}$op$_*$: Let $\mathcal{A}$ and $\mathcal{B}$ be topologically enriched small indexing categories. The associated categories ${}^{\mathcal{A}}\mathcal{T}op^{\mathcal{B}}$ of continuous $(\mathcal{A} \times \mathcal{B}^{op})$-diagrams is canonically $\mathcal{T}op$-enriched, $\mathcal{T}op$-complete, $\mathcal{T}op$-cocomplete, tensored and cotensored. Tensors and cotensors are formed "pointwise".

The based case is analogous.

7.4 G-spaces: Let G be a topological monoid or group. Then the categories $G - \mathcal{T}op$ of G-spaces and $G - \mathcal{T}op_*$ of based G-spaces are special cases of (7.3).

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