Reduction of Symmetric Configurations n_3

H.G. Carstens T. Dinski E. Steffen *

Abstract

Symmetric configurations n_3 are the bicubic graphs of girth ≥ 6. They are all reducible to the configuration 7_3, the Fano-Heawood-graph F i.e. the projective plane of order 2 by means of a Martinetti-like procedure.

1 Introduction

A *configuration* (v_r, b_k) is an incidence structure with the following properties:

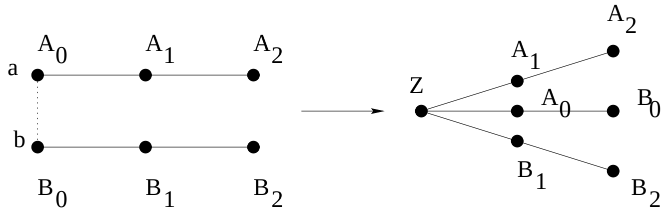
(a) There are v points and b lines.

(b) There are k points on each line and r lines through each point.

(c) Two different lines intersect at most once and two different points are connected by at most one line.

If $v = b$ (and hence $r = k$) then the configuration is called *symmetric* and denoted by v_k.

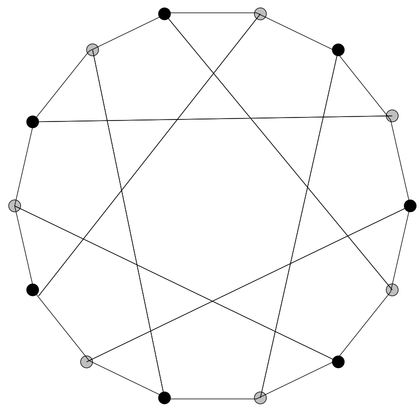
Configurations were defined by Reye [2] in 1876. In 1881 Kantor enumerated all symmetric configurations n_3 for $n \leq 10$. In 1887 Martinetti [1] gave a construction method for the symmetric configurations n_3.
Given a configuration n_3. If possible, take away two parallel (non intersecting) lines a and b and two points A_0 and B_0 from a and b resp. which are not connected. Remove the lines a and b and add a new point Z and three new lines $\{Z, A_0, A_2\}, \{Z, B_1, B_2\}, \{Z, A_0, B_0\}$ through Z. A configuration $(n+1)_3$ is obtained.



*Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld

This is not always possible. Martinetti called the configurations n_3 constructed in this way *reducible*. His method does not cover an infinite number of exeptions, called *irreducible*.

Some famous configurations n_3 are the Fano-plane 7_3, the Pappus Configuration 9_3 and the Desargues Configuration 10_3, which are important in projective and affine geometry. There is an interpretation of configurations as graphs. We could think of the lines as white vertices and the points as black ones. Two vertices are connected by an edge, if the corresponding line and point are incident. Since two lines intersect at most once and two points are connected by at most one line, the graph contains no cycle of length 4. The girth is greater or equal to 6. We will identify configurations and their graphs.
Symmetric configurations n_k are k-regular bipartite graphs. The configurations n_3 which we are interested in are 3-regular bipartite graphs, so called bicubic graphs, with girth ≥ 6. We give an example of this interpretation for the Fano-plane 7_3 which is the Heawood graph F.



Fano-Heawood Graph F

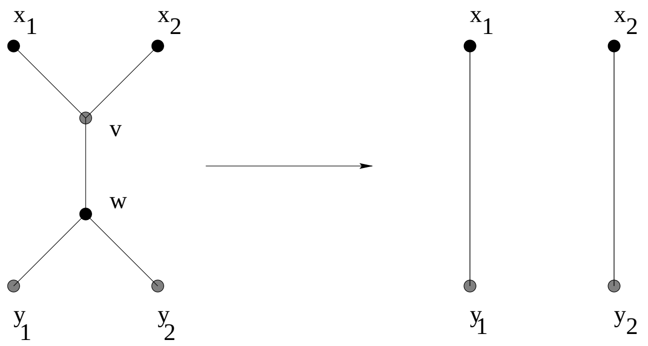
Martinetti's result with the infinity of exceptions motivated the development of a generalized reduction process by means of which every n_3 can be reduced to F. In this paper we shall prove the following theorem:

Theorem: There exists a finite number of generalized Martinetti-like reduction processes, such that every bicubic connected graph G with girth ≥ 6 different from the Fano-Heawood graph F can be reduced to a smaller one of the same kind, having two vertices less, and by iteration to F.

2 Symmetric Configurations n_3

We give a reduction procedure for the class of connected finite bicubic graphs with girth ≥ 6. Using this procedure every graph n_3 can be reduced to a graph $(n-1)_3$, as long as it is not in the base, the set of minimal elements. It is well known that the base consists of the Fano-Heawood graph $7_3 = F$ only. All graphs n_3 could be produced by reversing the process.

Definition 2.1 Let G be a symmetric configuration n_3. G is called *edge-reducible* iff there is an edge $vw \in EG$ such that $G \setminus vw$ has a supergraph H, $V(G \setminus vw) = VH$, and H is a configuration $(n-1)_3$.



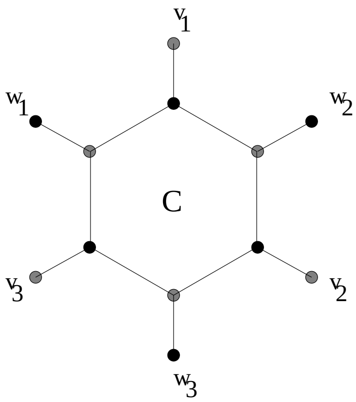
Lemma 2.1 Every symmetric configuration n_3, having an edge not in a cycle of length 6 is edge-reducible.

Proof: Let G be a symmtric configuration n_3. $vw \in EG$, vw not in a cycle of length 6.

Remove vw from G, and join x_1, y_1 and x_2, y_2 by new edges as above. The graph constructed in such a way is bicubic and has girth ≥ 6.•

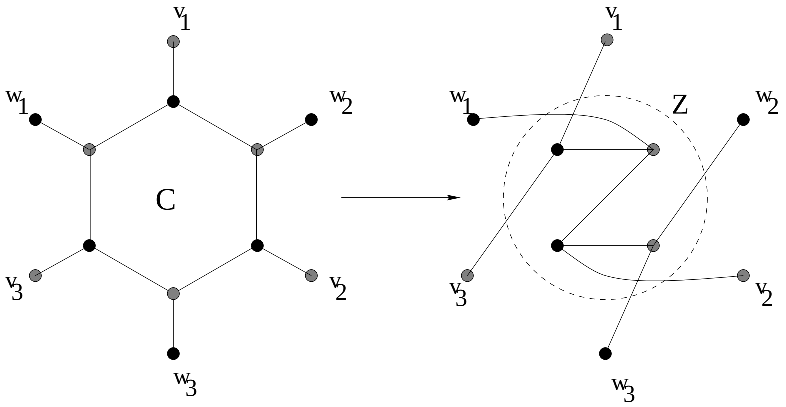
Next we shall try to remove 6-cycles from configurations which are not edge-reducible.

Let G be such a configuration n_3 and C a 6-cycle in G.



There is only one bipartite graph Z on 4 vertices,namely the path Z of length 3, such that $(G \setminus C) \cup Z$ can be extended to a $(n-1)_3$-configuration H with $VH = V(G \setminus C) \cup V$.

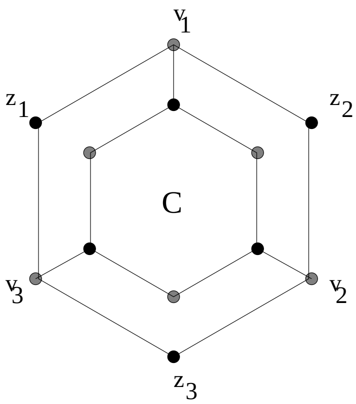
Definition 2.2 Let G be a configuration n_3 and C a 6-cycle in G. G is *Z-reducible*, iff the graph we obtain by removing C from G and attaching the graph Z as in the following figure is a configuration $(n-1)_3$.



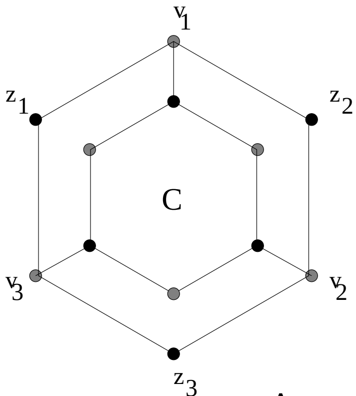
It is easy to see, that Z-reducibility requires, that at least one pair of $v_i w_j$ does not have a common neighbor or that there are not exactly 3 edges $v_i w_j$, $1 \leq i, j \leq 3$.

We deal with the non Z-reducible case.

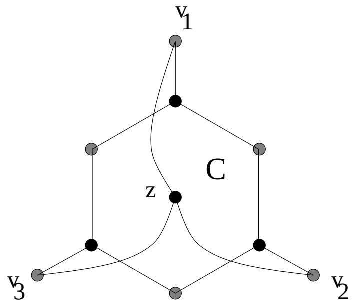
If all pairs (v_i, v_j) of vertices have a common neighbor, then G has a double-6-cycle as a subgraph,



where z_i, z_j may be the same vertex. There are two possibilities, $| \{z_1, z_2, z_3\} |= 3$ or $| \{z_1, z_2, z_3\} |= 1$, hence we have two subgraphs A_1, A_2 which are the obstacles for Z-reduction:

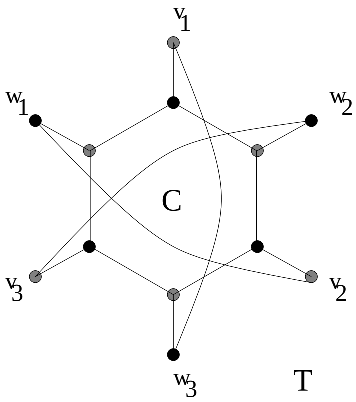


A1



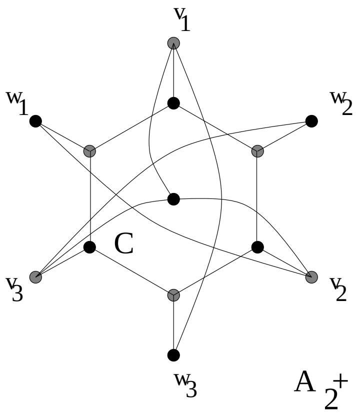
A 2

If in G there are exactly three edges $v_i w_j$, then there is a subgraph of G isomorphic to the following T.



T

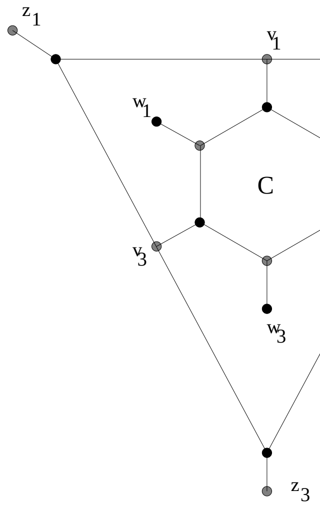
A_1 and T can not occur simultaneously on any 6-cycle C in G, but a combination "$A_2 + T$" of A_2 and T is possible.



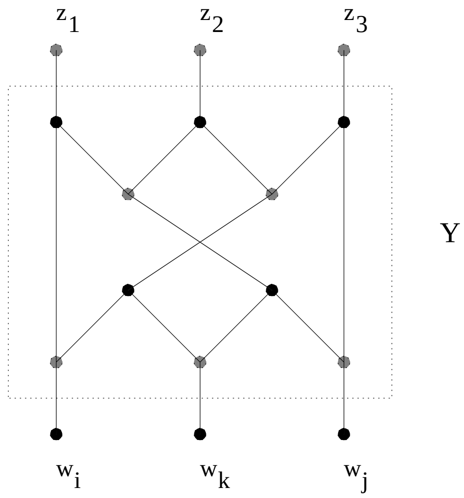
A 2+ T

We now find reductions for these special situations.

Case A_1: Assume that G contains A_1

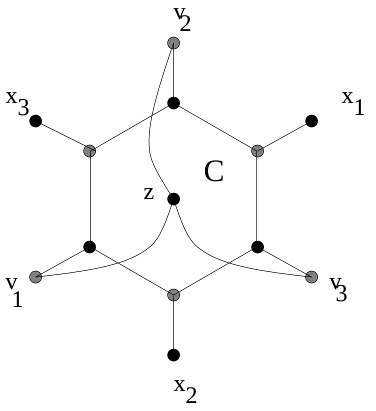


Obviously $\mid \{z_1, z_2, z_3\} \mid = 3$ and there are $i, j \in \{1, 2, 3\}$, $i \neq j$,such that $z_1 w_i, z_3 w_j \notin EG$.
Remove C and the cycle $C' = v_1 z_1 v_2 z_2 v_3 z_3 v_1$ from G and join $G \setminus (C \cup C')$ with a new graph Y on 10 vertices as follows:



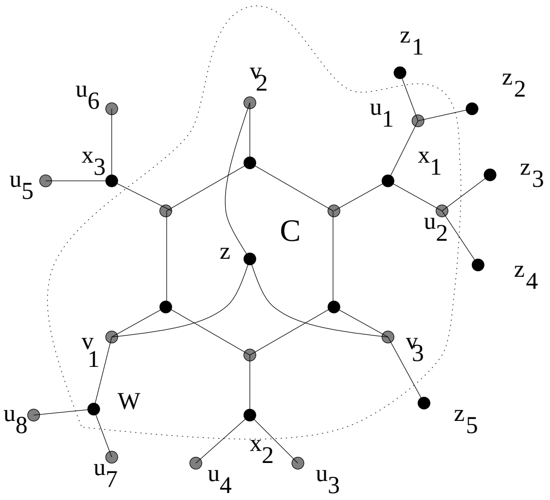
where $1 \leq k \leq 3$ and $k \notin \{i, j\}$. This graph is bipartite and has no cycle of length 4 and has two vertices less then G.

Case A_2: Assume that G contains A_2.



We distinguish two subcases:

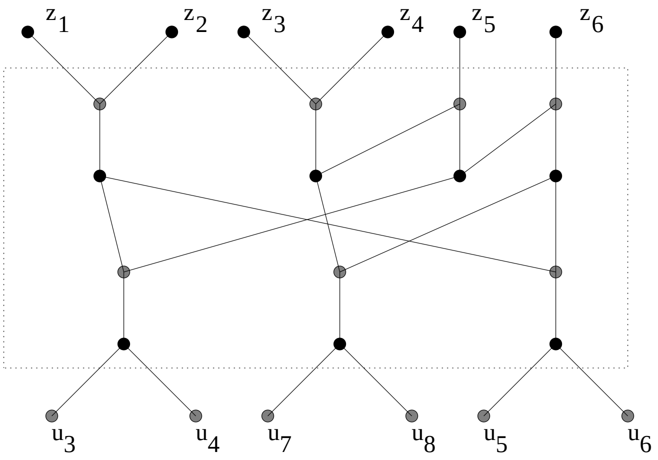
Subcase A_2 **and not** T: There is a vertex w in G and $i \in \{1, 2, 3\}$ such that $v_i w \in EG$ and $w \neq x_i$. The situation is as in the following figure with $i = 1$:



It is obvious that $u_i \neq v_1$ for $i = 1, 2$, otherwise a cycle C_4 would occur. The following inequalities hold:

$$\begin{aligned} z_1 &\neq z_2, z_3, z_4 \\ z_2 &\neq z_3, z_4 \\ z_3 &\neq z_4 \\ u_i &\neq u_{i+1} \qquad for \quad i = 3, 5, 7 \end{aligned}$$

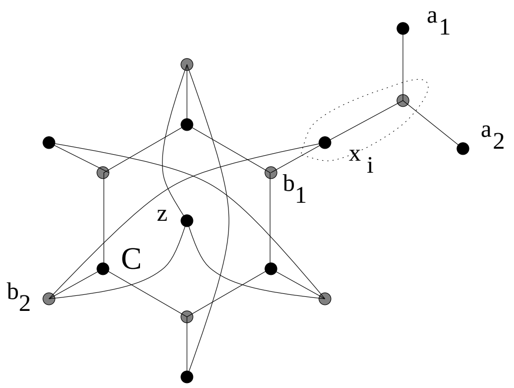
We may assume that $z_5 \neq z_3$ and $z_5 \neq z_4$. We remove the indicated subgraph U_a from G and join $G \setminus U_a$ with a new graph on 14 vertices as in the picture:



This is a 3-regular bipartite graph with no cycle of length 4. Since 16 vertices are removed from G and 14 new ones are added, there are only $\mid VG \mid -2$ vertices in the new graph.

Subcase A_2 and T: Assume the graph $A_2 + T$ is contained in G.

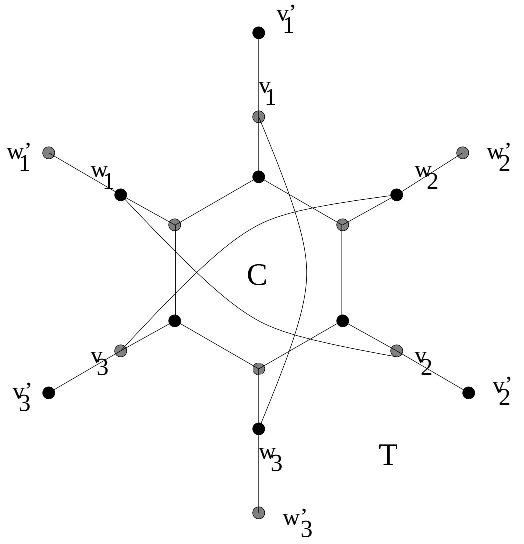
For all v_i, $i \in \{1,2,3\}$, there is an edge $v_i x_j$, $j \in \{1,2,3\}$. So we are in the following situation:



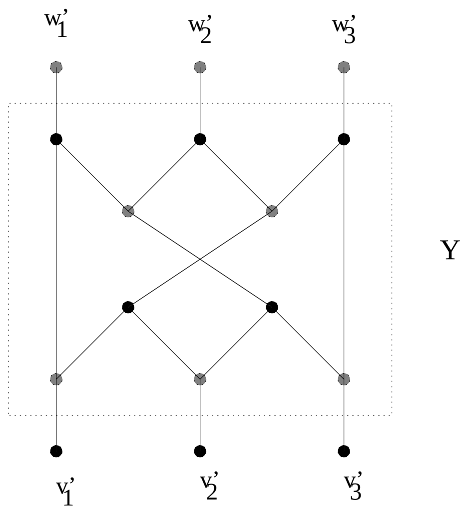
Assuming, that G is not the Fano-Heawood-graph F, there is a vertex $u \in VG \setminus VA_2$ and $i \in \{1,2,3\}$, such that $a_1, a_2 \in VG \setminus VA_2$ with $ua_1, ua_2 \in EG$. Remove the edge $x_i u$ and connect $a_i b_i$, $i = 1,2$, by a new edge. No cycle of length

4 will appear, since the neighbors of b_1 and b_2 in A_2 all have degree 3, there can not be a common neighbor with a_1 or a_2.
We again obtain a 3-regular bipartite graph on $\mid VG \mid -2$ vertices with no cycle of length 4.

Case T: Assume G contains T

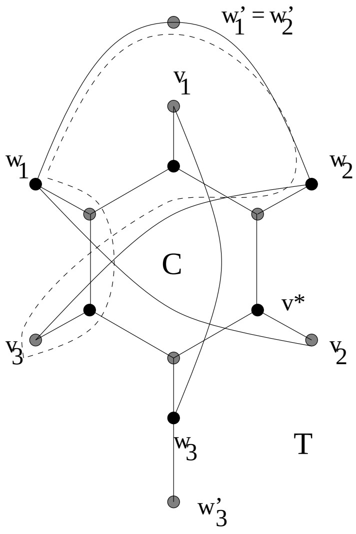


There are no edges $v_1'w_3', v_2'w_1'$ or $v_3'w_2'$ in G. Let C^+ the subgraph of G induced by C and all neighbors of C.
If $\mid \{v_1', v_2', v_3', w_1', w_2', w_3'\} \mid = 6$ and there are no edges $v_i'w_j'$ then one can remove C^+ and join the graph Y to $G \setminus C^+$.



The new graph is 3-regular and bipartite and has girth ≥ 6. It has $\mid VG \mid -2$ vertices.

1. Subcase: If two w'_i are identical and different from the third, say $w'_1 = w'_2$ and $w'_3 \neq w'_1$, then consider the following graph:

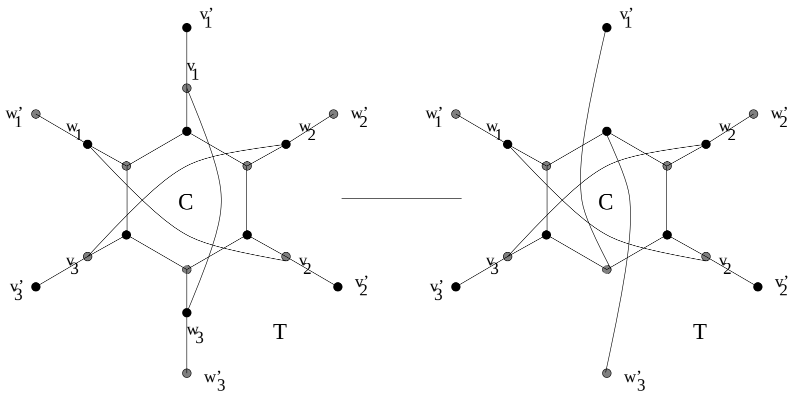


v^* is a neighbor of all white neighbors of the dotted cycle. Fortunately this occurs in case A_2.

2. Subcase: If $w'_1 = w'_2 = w'_3$, the situation is the same as in case $A_2 + T$.

3. Subcase: There are edges $v'_i w'_j$.

a: If there are exactly 6 neighbors of C, we remove an edge $v'_i w'_j$ and add two new edges as shown in the picture.



The new graph is bicubic and contains no cycle of length 4. The number of vertices is reduced by 2.

b: If $w'_1 = w'_2$ and $w'_1 \neq w'_3$, then the situation is like the one in the first subcase.

c: If $w'_1 = w'_2 = w'_3$, then we are in the second subcase.

Theorem 2.1 There exists a finite number of generalized Martinetti-like reduction processes, such that every bicubic connected graph G with girth ≥ 6 different from the Fano-Heawood graph F can be reduced to a smaller one of the same kind, having two vertices less and by iteration to F.

Proof: If there is an edge, which is not contained in a cycle of length 6, then G is edge-reducible. If all edges belong to some 6-cycle, then G is Z-reducible or there are subgraphs isomorphic to A_1, A_2, T or $A_2 + T$ in G and can be reduced with one of the procedures described above. •

References

[**1**] V. Martinetti, Sulle configurazioni piane μ_3, Annali di mat. pura ed applicata 15 (1887),1-26

[**2**] H. Gropp, On the history of configurations, Conference San Sebastian, September 1990