The preprojective algebra of a quiver

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ABSTRACT. The preprojective algebra $\mathcal{P}_k(Q)$ of a quiver Q plays an important role in mathematics. We are going to present some descriptions of these algebras and their module categories which seem to be well-accepted by the experts, but for which we were unable to find complete proofs in the literature. In particular, we determine the fibre of the forgetful functor from the category of $\mathcal{P}_k(Q)$-modules to the category of kQ-modules in terms of the orbit algebra of a kQ-module with respect to the Auslander-Reiten translation. The aim of the present paper is to outline in which way the representation theory of kQ can be used in order to deal with varieties studied by Lusztig.

1. Introduction

Let k be a field. The algebras which we consider will be k-algebras, modules are usually left modules. If A is a k-algebra, we denote by $\operatorname{Mod} A$ the category of all A-modules, by $\operatorname{mod} A$ the full subcategory of all finite dimensional A-modules.

Quivers. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver; here, Q_0, Q_1 are finite sets, and $s, t\colon Q_1 \to Q_0$ are maps; the elements of Q_0 are called *vertices*, those of Q_1 are called *arrows*; given an arrow $\alpha \in Q_1$, one writes $\alpha\colon s\alpha \to t\alpha$ and calls $s\alpha$ its starting vertex and $t\alpha$ its terminal vertex. A path of length $l \geq 1$ is a sequence $(\alpha_1, \dots, \alpha_l)$ with $s\alpha_i = t\alpha_{i+1}$, for $1 \leq i < l$; the vertex $s\alpha_l$ is called its starting vertex and $t\alpha_1$ is called its terminal vertex; in addition, one also considers paths of length 0, they correspond bijectively to the vertices of Q. We denote by kQ the path algebra of the quiver Q with coefficient field k; here, the product of two paths is given by concatenation, whenever this is possible, and by zero otherwise. Note that the path algebra kQ has global dimension at most 1. Of course, kQ is finite dimensional if and only if there are no cyclic paths in Q (a cylic path is a path of length at least 1 with same starting and terminal vertex). Let kQ^+ be the subspace of kQ with basis the set of all paths of length at least 1; it is the ideal of kQ generated by the arrows.

The preprojective algebra of the quiver Q. Let $\overline{Q}$ be obtained from Q by adding for every arrow $\alpha\colon x \to y$ a formal inverse $\alpha^*\colon y \to x$, the set of new arrows will be denoted by Q_1^*. If Q has loops, then, by construction, the number of loops of $\overline{Q}$ is twice the number of loops of Q. (We may characterize $\overline{Q}$ as follows: it is a finite quiver with a fixpoint free involution ι on the set of arrows, such that for every arrow β of $\overline{Q}$ the starting point of $\iota(\beta)$ is the endpoint of β or, equivalently, the compositions $\beta\iota(\beta)$ and $\iota(\beta)\beta$ are defined; the quiver Q contains precisely one arrow in each ι-orbit of $\overline{Q}$.) We define on Q_1 an integer valued function $\varepsilon_Q\colon Q_1 \to \mathbb{Z}$

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by

$$\varepsilon_Q(\beta) = \begin{cases} 1 & \text{provided } \beta \in Q_1, \\ -1 & \text{provided } \beta \in Q_1^*. \end{cases}$$

We consider the following element

$$\rho = \sum_{\alpha \in \overline{Q}_1} \varepsilon_Q(\alpha)\alpha^*\alpha$$

in $k\overline{Q}$ and the ideal (ρ) generated by ρ. The algebra $\mathcal{P}_k(Q) = k\overline{Q}/(\rho)$ will be called the *preprojective algebra of the quiver* Q.

Tensor algebras of bimodules. If Λ is a ring and Ω a Λ-bimodule, let $\Lambda\langle\Omega\rangle$ denote the corresponding tensor algebra; it is the direct sum

$$\Lambda\langle\Omega\rangle = \bigoplus_{t\geq 0} \Omega^{\otimes t},$$

where $\Omega^{\otimes t}$ is the t-fold tensor power of Ω, with $\Omega^{\otimes 0} = \Lambda$, and $\Omega^{\otimes(t+1)} = \Omega^{\otimes t} \otimes \Omega$; the product of $a \in \Omega^{\otimes s}$ and $b \in \Omega^{\otimes t}$ in the tensor algebra is just $a \otimes b \in \Omega^{\otimes(s+t)}$, provided $s, t \geq 1$, and the scalar product ab otherwise. Note that Λ is a subring of $\Lambda\langle\Omega\rangle$, thus there is a forgetful functor from the category of all $\Lambda\langle\Omega\rangle$-modules to the category of Λ-modules. The ideal of $\Lambda\langle\Omega\rangle$ generated by Ω is called the *augmentation ideal.*

We denote by $D = \mathrm{Hom}_k(-, k)$ the usual k-duality: for example, starting with the right module kQ_{kQ}, we obtain the dual module $D(kQ_{kQ})$, and we may consider $\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ)$; this is a kQ-bimodule: we use as left module structure the canonical action of kQ on the right of $D(kQ_{kQ})$, and as right module structure the the canonical action of kQ on the right of ${}_{kQ}kQ$.

Theorem A. *Let Q be a quiver without cyclic paths. Let*

$$\Theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ).$$

The algebras $\mathcal{P}_k(Q)$ and $kQ\langle\Theta\rangle$ are isomorphic.

There exists an isomorphism whose restriction to kQ is the identity and which maps the ideal of $\mathcal{P}_k(Q)$ generated by the arrows of Q^* onto the augmentation ideal of $kQ\langle\Theta\rangle$.

Both algebras $\mathcal{P}_k(Q)$ and $kQ\langle\Theta\rangle$ have been studied by many mathematicians. The aim of Gelfand and Ponomarev [GP] was to construct an algebra A with the following property $(*)$: it contains kQ as a subalgebra and when considered as a left kQ-module, A decomposes as a direct sum of the indecomposable 'preprojective' kQ-modules, one from each isomorphism class (the definition of preprojective modules will be recalled in section 5 of the paper). Our joint paper [DR] with Dlab had the same aim in mind, but dealt with the more general case of a given species instead of a quiver. Since the algebra $\mathcal{P}_k(Q)$ has the property $(*)$, it became customary to call it the preprojective algebra of the quiver Q. But let us stress that for a fixed quiver Q, there may be several isomorphism classes of algebras A with property $(*)$, see the remark at the end of the paper. Algebras of the form $\mathcal{P}_k(Q)$ appear quite naturally in very diverse situations: Special cases of such algebras were considered by Kronheimer [K] when dealing with problems in differential geometry, and all of them play an important role in Lusztig's perverse sheaf approach to quantum groups [L1,L2,L3].

On the other hand, Baer, Geigle and Lenzing [BGL] have considered the algebras $kQ\langle\Theta\rangle$ under the name of preprojective algebras (see [BGL], Proposition 3.1); implicitly (and rightly) they assume that these are the algebras of interest. The assertion of Theorem A is well-accepted by the experts, but no complete proof seems to be available in the literature. Since this result will be needed in the sequel, our first aim is to provide a proof. We are going to derive it as a consequence of the Brenner-Butler-Gabriel theorem which describes the relationship between the Auslander-Reiten translation and the Coxeter functors of Bernstein-Gelfand-Ponomarev. This strategy of proof will not be surprizing, but we were astonished about the amount of additional calculations which seem to be necessary. Any reader is advised to look for a shorter proof. For example, one may consider the kQ-bimodule generated in $\mathcal{P}_k(Q)$ by the arrows of Q^* and one should try to show directly that this bimodule is isomorphic to Θ. We found it difficult to establish such an isomorphism.

If kQ is finite dimensional, then the Auslander-Reiten translations τ and τ^- on the category $\operatorname{mod} kQ$ of all finite dimensional kQ-modules are defined; since kQ has global dimension at most 1, these are endofunctors of $\operatorname{mod} kQ$:

$$\tau = D\operatorname{Ext}^1_{kQ}(-, {}_{kQ}kQ) \quad \text{and} \quad \tau^- = \operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), -).$$

Given a category $\mathcal{C}$ and two endofunctors F, G of $\mathcal{C}$, we denote by $\mathcal{C}(F,G)$ the following category: its objects are the pairs (C,c) where C is an object in $\mathcal{C}$ and $c\colon F(C) \to G(C)$ is a morphism in $\mathcal{C}$; given two objects (C,c) and (C',c'), a morphism $(C,c) \to (C',c')$ is a morphism $f\colon C \to C'$ such that $c' \circ F(f) = G(f) \circ c$.

Corollary. *Let Q be a finite quiver without cyclic paths. Then, the categories $\operatorname{mod}\mathcal{P}_k(Q)$, $(\operatorname{mod} kQ)(\tau^-,1)$ and $(\operatorname{mod} kQ)(1,\tau)$ are isomorphic.*

This follows directly from Theorem A using Lemma 1 and Lemma 2 of section 3; all these considerations are quite obvious:

$$\operatorname{Mod} kQ\langle\Theta\rangle \simeq (\operatorname{Mod} kQ)(F,1) \simeq (\operatorname{Mod} kQ)(1,G),$$

where $F = \Theta \otimes -$ and $G = \operatorname{Hom}(\Theta, -)$. The restrictions of F, G to $\operatorname{mod} kQ$ are just the Auslander-Reiten translations τ^- and τ, respectively. This shows in which way Theorem A implies Corollary. But the reader should be aware that actually our method of proof is the reverse one: first, we are going to establish the assertion of Corollary, or better, the corresponding version for arbitrary, not necessarily finitely generated modules. We call this assertion Theorem B, see section 2. In section 3, we we derive Theorem A from Theorem B.

Representations of quivers. Let us consider for a moment an arbitrary finite quiver, possibly with cyclic paths. The category of kQ-modules may be described as the category of representations of Q. Recall that a representation (V,x) of Q is given by a Q_0-graded vector space V and a family $x = (x_\alpha)_\alpha$ of k-linear maps $x_\alpha\colon V_{s\alpha} \to V_{t\alpha}$. The family $(\dim V_i)_{i\in Q_0}$ is called the *dimension vector* of (V,x), the sum $\sum \dim V_i$ is called the dimension of (V,x). For every vertex $i \in Q_0$, we may consider the one-dimensional representation $E(i)$ with $E(i) = (V,x)$ where $V_i = k$, $V_j = 0$ for $j \neq i$ and $x_\alpha = 0$ for all arrows α. A finite dimensional representation (V,x) is said to be *nilpotent* provided it has a filtration whose factors are of the form $E(i_t)$ with $i_t \in Q_0$. The usual identification of the category of representations of Q with the category $\operatorname{Mod} kQ$ of kQ-modules attaches to the representation (V,x) of Q a corresponding kQ-module with underlying vector space $\bigoplus_{i\in Q_0} V_i$. The (finite dimensional) representation (V,x) is nilpotent if and only of some power of the ideal kQ^+ annihilates the corresponding kQ-module. It is easy to see that all the finite dimensional representations of Q are nilpotent if and only if there are no cyclic paths in Q

The affine space of representations with fixed dimension vector. If we fix a finite dimensional Q_0-graded vector space V, the set of all representations of Q of the form (V,x) forms the set

$$\mathcal{R}(Q;V) = \bigoplus_{\alpha\in Q_1} \operatorname{Hom}_k(V_{s\alpha}, V_{t\alpha});$$

of course, this is an affine space. The group

$$G(V) = \prod_{i\in Q_0} \operatorname{GL}(V_i)$$

operates on $\mathcal{R}(Q;V)$ via

$$(g * x)_\alpha = g_{t\alpha} x_\alpha g_{s\alpha}^{-1},$$

for $g = (g_i)_i \in G(V)$ and $x = (x_\alpha)_\alpha \in \mathcal{R}(Q;V)$, and two elements $x, y \in \mathcal{R}(Q;V)$ belong to the same $G(V)$-orbit if and only if the representations (V,x) and (V,y) are isomorphic. If I is an ideal of the path algebra kQ, we denote by $\mathcal{R}(Q,I;V)$ the subset of $\mathcal{R}(Q;V)$ of all elements x such that the kQ-module given by (V,x) is annihilated by I.

We denote by $\mathcal{R}_0(Q;V)$ the subset of $\mathcal{R}(Q;V)$ of all elements x such that (V,x) is a nilpotent representation. An element x belongs to $\mathcal{R}_0(Q;V)$ if and only if the zero element belongs to the closure of the orbit of x. Also, let $\mathcal{R}_0(Q,I;V)$ be the intersection of $\mathcal{R}(Q,I;V)$ and $\mathcal{R}_0(Q;V)$.

The projection π. Given a quiver Q with preprojective algebra $\mathcal{P}_k(Q)$, the path algebra kQ can be considered as a subalgebra of $\mathcal{P}_k(Q)$, thus the restriction yields a functor from the category of $\mathcal{P}_k(Q)$-modules to the category of kQ-modules. We write any representation of $\overline{Q}$ in the form $(V,x,\xi) = (V, x_\alpha, \xi_\alpha)_{\alpha\in Q_1}$; here, (V,x) is a representation of Q and (V,ξ) is a representation of Q^*. The resriction functor $\operatorname{Mod}\mathcal{P}_k(Q) \to \operatorname{Mod} kQ$ sends the representation (V,x,ξ) of $\overline{Q}$ to the representation (V,x) of Q. Our interest lies in the corresponding map

$$\pi\colon \mathcal{R}_0(\overline{Q},(\rho);V) \longrightarrow \mathcal{R}_0(Q;V),$$

we want to determine the fibers of this map. The varieties $\mathcal{R}_0(\overline{Q},(\rho);V)$ are those which have been referred to above: they play an important role in Lusztig's approach to quantum groups [L1,L2,L3].

The orbit ring of an object in an additive category with respect to an endofunctor. Let $\mathcal{A}$ be an additive category. Let F be an (additive) endofunctor of $\mathcal{A}$ and let X be an object of $\mathcal{A}$. The orbit ring $\mathcal{O}^F(X)$ is given by the graded abelian group

$$\mathcal{O}^F(X) = \bigoplus_{n\geq 0} \operatorname{Hom}(F^n(X), X)$$

with the multiplication

$$f * g = f \circ F^s(g)$$

for $f \in \operatorname{Hom}(F^s(X), X)$ and $g \in \operatorname{Hom}(F^t(X), X)$. It is a graded associative algebra with 1 (this construction is similar to that considered in [BGL]). The situation we are interested in, will be the following: $\mathcal{A}$ will be the category of representations of the quiver Q and $F = \tau^-$ will be an Auslander-Reiten translation. We will consider homogeneous elements of degree 1 which are nilpotent; thus it will be convenient to introduce the following notation:

$$\mathcal{N}^F(X) = \{f \in \operatorname{Hom}(F(X), X) \mid f \ \text{ is nilpotent in } \ \mathcal{O}^F(X)\}.$$

Theorem C. *Let Q be a quiver without cyclic paths. Let V be a finite dimensional Q_0-graded vector space and let x be an element of $\mathcal{R}(V;Q)$. Then $\pi^{-1}(x)$ may be identified with $\mathcal{N}^{\tau^-}(V,x)$.*

It will be shown in section 5 that Theorem C may be used very effectively in order to construct irreducible components of the varieties $\mathcal{R}_0(\overline{Q},(\rho);V)$, at least in the case of a tame quiver Q. Theorem C and its consequences have been presented first at ICRA VI, Ottawa 1992 and in lectures at Brandeis university in the same year. We are indebted to comments and suggestions by M. Auslander and G. Lusztig.

2. Proof of Theorem B

In the following, we fix a quiver Q without cyclic paths, and $\mathcal{C}$ will denote the category of all (not necessarily finitely generated) kQ-modules. We denote by Φ^-, Φ^+ the Coxeter functors (as introduced by Bernstein, Gelfand and Ponomarev); note that these functors are defined for all kQ-modules, not just the finitely generated ones (the definition of Φ^+ will be recalled in the next proof). We denote by T the endofunctor of $\mathcal{C}$ which sends (V,x) to $(V,-x)$.

In case categories $\mathcal{C}', \mathcal{C}''$ with 'canonical' functors $\Gamma'\colon \mathcal{C}' \to \mathcal{C}$ and $\Gamma''\colon \mathcal{C}'' \to \mathcal{C}$ are given, an isomorphism of categories $\Psi\colon \mathcal{C}' \to \mathcal{C}''$ will be said to be a $\mathcal{C}$-isomorphism provided we have $\Gamma' = \Gamma''\Psi$. For example, if F, G are endofunctors of $\mathcal{C}$, the forgetful functor which sends (C,c) to C will be considered as the canonical functor $\mathcal{C}(F,G) \to \mathcal{C}$. Similarly, for the category $\operatorname{Mod}\mathcal{P}_k(Q)$, the canonical functor $\operatorname{Mod}\mathcal{P}_k(Q) \to \mathcal{C}$ is the one induced by the canonical algebra homomorphism $kQ \to \mathcal{P}_k(Q)$.

Theorem B$'$. *There is a $\mathcal{C}$-isomorphism Ψ from the category of all $\mathcal{P}_k(Q)$-modules to the category $\mathcal{C}(1, T\Phi^+)$.*

Proof: We will have to consider various summations where the index sets are sets of arrows. Usually, we will distinguish the arrows from Q and from Q^*. It will be convenient to consider as index sets only sets of arrows from Q, for example, the element ρ will be used in the form

$$\rho = \sum_{\alpha\in Q_1} \alpha^*\alpha - \sum_{\beta\in Q_1} \beta\beta^*$$

This convention allows us to delete the reference to Q_1.

We assume that the quiver Q has n vertices. Then ρ can be written as the sum of n elements, namely of the elements

$$\rho_i = \sum_{s\alpha=i} \alpha^*\alpha - \sum_{t\beta=i} \beta\beta^*$$

with $i \in Q_0$.

A $\mathcal{P}_k(Q)$-module may be considered as a representation of the quiver $\overline{Q}$ satisfying the relation ρ. We write any representation of $\overline{Q}$ in the form $(V, x, \xi) = (V, x_\alpha, \xi_\alpha)_{\alpha\in Q_1}$; here, (V, x) is a representation of Q and (V, ξ) is a representation of Q^*.

Recall that Φ^+ denotes one of the Coxeter functors for the quiver Q as introduced by Bernstein, Gelfand and Ponomarev. Given a representation (V, x) of Q, the representation $\Phi^+(V, x)$ is constructed as follows: Since we assume that Q has no oriented cycles, we may assume that we use as vertex set the set $Q_0 = \{1, 2, \dots, n\}$ such that for any arrow β we have $s\beta > t\beta$. Inductively, we define vector spaces W_i, for any vertex i, and linear maps $y_\beta \colon W_{t\beta} \to V_{s\beta}$ and $z_\beta \colon W_{s\beta} \to W_{t\beta}$ such that the sequences

$$(*) \qquad 0 \longrightarrow W_i \xrightarrow{(y_\alpha, z_\beta)_{\alpha\beta}} \bigoplus_{t\alpha=i} V_{s\alpha} \oplus \bigoplus_{s\beta=i} W_{t\beta} \xrightarrow{(x_\alpha, y_\beta)_{\alpha\beta}} V_i$$

are exact. Let us fix some vertex i. By induction, we may assume that the vector spaces W_j with $j < i$, the maps $y_\beta \colon W_{t\beta} \to V_{s\beta}$ for all arrows β with $s\beta \le i$ and the maps $z_\alpha \colon W_{s\alpha} \to W_{t\alpha}$ for all arrows α with $t\alpha < i$ are already defined. In particular, the right map of $(*)$ is defined; we denote this map by ε_i. We define W_i as the kernel of ε_i, in this way, we obtain corresponding maps $y_\alpha \colon W_i \to V_{s\alpha}$, for every arrow α with $t\alpha = i$ and $z_\beta \colon W_i \to W_{t\beta}$ for every arrow β with $s\beta = i$. Then, by definition, $\Phi^+(V, x) = (W, z)$, and therefore $T\Phi^+(V, x) = (W, -z)$.

Consider now an object of $\mathcal{C}(1, T\Phi^+)$. It is of the form $((V, x), \psi)$ where (V, x) belongs to $\mathcal{C}$ and ψ is a map $\psi \colon (V, x) \to (W, -z)$. We define $\xi_\beta = y_\beta\psi_{t\beta}$, this is a linear map $V_{t\beta} \to V_{s\beta}$. In order to show that the representation (V, x, ξ) of $\overline{Q}$ satisfies the relation ρ, we calculate:

$$\begin{aligned}
\sum_{t\alpha=i} x_\alpha\xi_\alpha - \sum_{s\beta=i} \xi_\beta x_\beta &= \sum_{t\alpha=i} x_\alpha y_\alpha \psi_{t\alpha} - \sum_{s\beta=i} y_\beta\psi_{t\beta}x_\beta \\
&= \sum_{t\alpha=i} x_\alpha y_\alpha \psi_{t\alpha} - \sum_{s\beta=i} y_\beta(-z_\beta)\psi_{s\beta} \\
&= -\left(\sum_{t\alpha=i} x_\alpha y_\alpha + \sum_{s\beta=i} y_\beta(-z_\beta) \right) \psi_i = 0.
\end{aligned}$$

Here, we have used that $\psi_{t\beta}x_\beta = -z_\beta\psi_{s\beta}$, since $\psi = (\psi_i)_i$ is a homomorphism between representations of Q, and the exactness of $(*)$.

Conversely, assume that the representation (V,x,ξ) of $\overline{Q}$ satisfies the relation ρ. Inductively, we are going to define linear maps $\psi_i\colon V_i \to W_i$ such that the conditions $\psi_{t\beta}x_\beta = -z_\beta\psi_{s\beta}$ and $\xi_\beta = y_\beta\psi_{t\beta}$ are satisfied for all arrows β. Assume that the maps ψ_j with $j<i$ have been constructed satisfying $\psi_{t\beta}x_\beta = -z_\beta\psi_{s\beta}$ for all arrows β with $s\beta < i$ and $\xi_\alpha = y_\alpha\psi_{t\alpha}$ for all arrows α with $t\alpha < i$. Consider the map

$$(\xi_\alpha, -\psi_{t\beta}x_\beta)_{\alpha\beta}\colon V_i \longrightarrow \bigoplus_{t\alpha=i} V_{s\alpha} \oplus \bigoplus_{s\beta=i} W_{t\beta}.$$

Its composition with ε_i is zero, as the following calculation shows:

$$\sum_{t\alpha=i} x_\alpha\xi_\alpha - \sum_{s\beta=i} y_\beta\psi_{t\beta}x_\beta = \sum_{t\alpha=i} x_\alpha\xi_\alpha - \sum_{s\beta=i}\xi_\beta x_\beta = 0.$$

As a consequence, we can factor it through the kernel W_i of ε_i. We obtain a map $\psi_i\colon V_i \to W_i$ such that, on the one hand, we have $\xi_\alpha = y_\alpha\psi_{t\alpha}$ for all arrows α with $t\alpha = i$, and, on the other hand, we have $\psi_{t\beta}x_\beta = -z_\beta\psi_{s\beta}$ for all arrows β with $s\beta = i$. This shows that $\psi = (\psi_i)_i$ is a kQ-module homomorphism $(V,x) \to (W,-z)$. If we want to stress that ψ has been derived from ξ, we write $\psi = \psi_\xi$. Starting with $\mathcal{P}_k(Q)$-module (V,x,ξ), let $\Psi(V,x,\xi) = ((V,x),\psi_\xi)$.

Altogether, we see that Ψ furnishes, for every representation (V,x) of Q, a bijection between the $\mathcal{P}_k(Q)$-modules (V,x,ξ) and the maps $\psi\colon (V,x)\to(W,-z)$.

Consider now two $\mathcal{P}_k(Q)$-modules (V,x,ξ) and (V',x',ξ') with corresponding maps $\psi = \psi_\xi\colon (V,x)\to(W,-z)$ and $\psi' = \psi_{\xi'}\colon (V',x')\to(W',-z')$. Since these are kQ-module homomorphisms, the equations

$$\psi_{t\alpha}x_\alpha = -z_\alpha\psi_{s\alpha} \tag{1}$$

$$\psi'_{t\alpha}x'_\alpha = -z'_\alpha\psi'_{s\alpha} \tag{2}$$

are satisfied for all arrows α of Q. According to the definition of Ψ, we have

$$\xi_\beta = y_\beta\psi_{t\beta} \tag{3}$$

$$\xi'_\beta = y'_\beta\psi'_{t\beta} \tag{4}$$

for any arrow β.

Let us start with an arbitrary kQ-homomorphism $f\colon (V,x)\to(V',x')$, thus $f=(f_i)_i$, where $f_i\colon V_i \to V'_i$ are k-linear maps such that

$$f_{t\alpha}x_\alpha = x'_\alpha f_{s\alpha} \tag{5}$$

for any arrow $\alpha\in Q_1$. Inductively, we construct maps $g_i\colon W_i \to W'_i$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & W_i \arrow[r, "(y_\alpha{,}z_\beta)_{\alpha\beta}"] \arrow[d, "g_i"] & \bigoplus_{t\alpha=i} V_{s\alpha} \oplus \bigoplus_{s\beta=i} W_{t\beta} \arrow[r, "(x_\alpha{,}y_\beta)_{\alpha\beta}"] \arrow[d, "\bigoplus f_{s\alpha}\oplus\bigoplus g_{t\beta}"] & V_i \arrow[d, "f_i"] \\
0 \arrow[r] & W'_i \arrow[r, "(y'_\alpha{,}z'_\beta)_{\alpha\beta}"] & \bigoplus_{t\alpha=i} V'_{s\alpha} \oplus \bigoplus_{s\beta=i} W'_{t\beta} \arrow[r, "(x'_\alpha{,}y'_\beta)_{\alpha\beta}"] & V'_i
\end{tikzcd}$$

The maps g_i which we obtain in this way satisfy the following conditions:

$$f_{s\alpha}y_\alpha = y'_\alpha g_{t\alpha}, \tag{6}$$
$$g_{t\alpha}z_\alpha = z'_\alpha g_{s\alpha}. \tag{7}$$

Of course, the last equality just expresses the fact that $g = (g_i)$ is a kQ-module homomorphism $(W, z) \to (W', z')$. But then g is also a kQ-module homomorphism $(W, -z) \to (W', -z')$.

First, let us assume that f is a $\mathcal{P}_k(Q)$-module homomorphism, thus

$$f_{s\alpha}\xi_\alpha = \xi'_\alpha f_{t\alpha} \tag{a}$$

for all arrrows α. By induction on i, we want to show that $g_i\psi_i = \psi'_i f_i$. In order to show this equality, it is sufficient to see that both

$$y'_\alpha \psi'_i f_i = y'_\alpha g_i\psi_i,$$
$$z'_\beta \psi'_i f_i = z'_\beta g_i\psi_i$$

for all arrows α with $t\alpha = i$ and all arrows β with $s\beta = i$ are satisfied (since $(y_\alpha, z_\beta)_{\alpha\beta}$ is a monomorphism). Fix some i, and let us assume that

$$g_j\psi_j = \psi'_j f_j \text{ for } j < i \tag{b}$$

holds. Now, using (4), (a), (3), (6) we have

$$y'_\alpha \psi'_{t\alpha} f_{t\alpha} = \xi'_\alpha f_{t\alpha} = f_{s\alpha}\xi_\alpha = f_{s\alpha}y_\alpha\psi_{t\alpha} = y'_\alpha g_{t\alpha}\psi_{t\alpha}.$$

Next, consider an arrow β with $s\beta = i$. Since $t\beta < i$, we can use (b) for $j = t\beta$. Altogether, we use (2), (5), (b), (1) and (7):

$$\begin{aligned} z'_\beta\psi'_{s\beta}f_{s\beta} &= -\psi'_{t\beta}x'_\beta f_{s\beta} = -\psi'_{t\beta}f_{t\beta}x_\beta \\ &= -g_{t\beta}\psi_{t\beta}x_\beta = g_{t\beta}z_\beta\psi_{s\beta} = z'_\beta g_{s\beta}\psi_{s\beta}. \end{aligned}$$

This shows that $g_i\psi_i = \psi'_i f_i$ holds for all i, thus $f\colon ((V, x), \psi) \longrightarrow ((V', x'), \psi')$ is a morphism in the category $\mathcal{C}(1, T\Phi^+)$.

Conversely, assume that f is a morphism $((V, x), \psi) \longrightarrow ((V', x'), \psi')$. in the category $\mathcal{C}(1, T\Phi^+)$. This means that

$$g_i\psi_i = \psi'_i f_i \tag{c}$$

for all vertices i of Q. Using (3), (6), (c), (4), we see that

$$f_{s\beta}\xi_\beta = f_{s\beta}y_\beta\psi_{t\beta} = y'_\beta g_{t\beta}\psi_{t\beta} = y'_\beta\psi'_{t\beta}f_{t\beta} = \xi'_\beta f_{t\beta},$$

thus $f\colon (V, x, \xi) \longrightarrow (V', x', \xi')$ is a homomorphism of $k\overline{Q}$-modules.

Theorem B follows immediately from Theorem B$'$, using the Brenner-Butler-Gabriel theorem: the restriction of the functor $T\Phi^+$ to the finitely generated kQ-modules ist just the Auslander-Reiten translation τ, see [G], Proposition 5.3.

3. Proof of Theorem A

We are going to use some additional $\mathcal{C}$-isomorphisms of categories. Let us formulate the corresponding results as Lemma 1 and Lemma 2. These assertions are obvious.

Lemma 1. *Let F, G be a pair of adjoint endofunctors of the category $\mathcal{C}$. Then there is a $\mathcal{C}$-isomorphism from $\mathcal{C}(F,1)$ to $\mathcal{C}(1,G)$.*

Lemma 2. *Let Λ be a ring and Ω a Λ-bimodule. Let $\mathcal{C}$ be the category of all Λ-modules and F the tensor functor $\Omega \otimes -$. Then there is a $\mathcal{C}$-isomorphism from $\mathcal{C}(F,1)$ to the category of all $\Lambda\langle\Omega\rangle$-modules.*

For later reference, let us write down such a $\mathcal{C}$-isomorphism in detail. An object of $\mathcal{C}(F,1)$ is of the form (C,c), where C is a Λ-module and $c\colon \Omega\otimes_\Lambda C \to C$ is a Λ-module homomorphism. In order to consider C as a $\Lambda\langle\Omega\rangle$-module, we have to define a bilinear map $\mu\colon \Lambda\langle\Omega\rangle \otimes C \to C$. Since $\Lambda\langle\Omega\rangle$ is the direct sum of the Λ-bimodules $\Omega^{\otimes t}$, it is sufficient to define maps $\mu_t\colon \Omega^{\otimes t}\otimes C \to C$. We do this inductively. The map $\mu_0\colon \Lambda\otimes C \to C$ is by definition the scalar multiplication of the Λ-module C; if μ_t is already defined, let $\mu_{t+1} = \mu_t \circ (1_{\Omega^t}\otimes f)$. In this way, we define a functor from $\mathcal{C}(F,1)$ to the category $\operatorname{Mod}\Lambda\langle\Omega\rangle$. Conversely, given a $\Lambda\langle\Omega\rangle$-module M, then we take the restriction of the scalar multiplication to $\Omega\otimes M$.

Let us consider now again the case where Q is a quiver without cyclic paths and $\mathcal{C} = \operatorname{Mod} kQ$. We consider the image $\Theta' = T\Phi^-({}_{kQ}kQ)$ of the 'regular representation' $M = {}_{kQ}kQ$ of kQ under the functor $T\Phi^-$. Of course, Θ', as an object in $\mathcal{C}$, is a left kQ-module; the endomorphisms of M are mapped under the functor $T\Phi^-$ to endomorphisms of Θ'. The endomorphism ring of M is just the opposite algebra of kQ, via the right multiplication; in this way, Θ' becomes also a right kQ-module. Thus, Θ' is a kQ-bimodule. We have noted already in the introduction that also $\operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ)$ is a kQ-bimodule.

Lemma 3. *The kQ-bimodules*

$$T\Phi^-({}_{kQ}kQ) \quad \textit{and} \quad \operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ)$$

are isomorphic.

Proof: According to the Brenner-Butler-Gabriel theorem, the restriction of the functor $T\Phi^-$ to the finitely generated kQ-modules ist just the Auslander-Reiten translation τ^-. But the endofunctor τ^- may be identified with $\operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), -)$.

Recall that the functor Φ^- is left adjoint to the functor Φ^+, thus $T\Phi^-$ is left adjoint to $T\Phi^+$. Also, since the functor $T\Phi^-$ has an adjoint functor, it is a tensor product functor, namely $\Theta\otimes -$, where Θ is the kQ-bimodule $\Theta = T\Phi^-({}_{kQ}kQ)$. But according to lemma 3, this bimodule is just $\operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ)$.

Altogether, we see that there are $\mathcal{C}$-isomorphisms

$$\operatorname{Mod}\mathcal{P}_k(Q) \to \mathcal{C}(1, T\Phi^+) \to \mathcal{C}(T\Phi^-, 1) \to \operatorname{Mod} kQ\langle\Theta\rangle;$$

let us denote the composition by Ψ'. We want to see that Ψ' is induced by some algebra isomorphism $\eta\colon kQ\langle\Theta\rangle \to \mathcal{P}_k(Q)$. We may compose the canonical functors

$\operatorname{Mod}\mathcal{P}_k(Q) \to \mathcal{C}$ and $\operatorname{Mod} kQ\langle\Theta\rangle \to \mathcal{C}$ with the forgetful functor $\mathcal{C} \to \operatorname{Mod} k$ and apply the following Lemma.

Lemma 4. *Let R, R' be k-algebras and let*

$$\Gamma\colon \operatorname{Mod} R \to \operatorname{Mod} k \quad \text{and} \quad \Gamma'\colon \operatorname{Mod} R' \to \operatorname{Mod} k$$

be the forgetful functors. Assume that there exists an equivalence $\Psi\colon \operatorname{Mod} R \to \operatorname{Mod} R'$ such that $\Gamma = \Gamma'\Psi$. Then there is an algebra isomorphism $R' \to R$ which induces Ψ.

Proof: The image $\Psi({}_RR)$ is a progenerator of the category $\operatorname{Mod} R'$, in particular a faithful and balanced module (this means that the canonical map from R' into the 'double centralizer' is bijective). On the other hand, the underlying vector space $\Gamma'\Psi({}_RR)$ of the module $\Psi({}_RR)$ is the same as the underlying vector space $\Gamma({}_RR)$ of R, since we assume that $\Gamma = \Gamma'\Psi$. Given $r \in R$, let $\rho_r\colon \Gamma({}_RR) \to \Gamma({}_RR)$ be the right multiplication by r. The endomorphism ring of ${}_RR$ is just the set of elements ρ_r with $r \in R$. The set of elements of R has to be considered here in various ways; in order to avoid confusion, we denote by E the set of right multiplications ρ_r with $r \in R$ (of course, $E = \operatorname{End}({}_RR)$, and this endomorphism ring ring is isomorphic to the opposite ring R^o of R). Since $\Gamma'\Psi(\rho_r) = \Gamma(\rho_r)$, we see that the image of the endomorphism ring of ${}_RR$ under the functor Ψ is again the set E. Since Ψ is a full functor, E is the set of all endomorphisms of $\Psi({}_RR)$. The double centralizer of $\Psi({}_RR)$ is the endomorphism ring of the E-module $\Gamma({}_RR)$. But the endomorphism ring of the E-module $\Gamma({}_RR)$ is just the given ring R operating on $\Gamma({}_RR)$ via left multiplication. This shows that the double centralizer of $\Psi({}_RR)$ is the ring R. On the other hand, since $\Psi({}_RR)$ is a balanced R'-module, its double centralizer is also isomorphic to R'. This shows that the rings R and R' are isomorphic. (Actually, our proof gives an identification of R with the double centralizer of $\Psi({}_RR)$ and thus a fixed isomorphism $R' \to R$.)

Let $\eta\colon kQ\langle\Theta\rangle \to \mathcal{P}_k(Q)$ be the algebra isomorphism which induces the $\mathcal{C}$-isomorphism

$$\Psi'\colon \operatorname{Mod}\mathcal{P}_k(Q) \longrightarrow \operatorname{Mod} kQ\langle\Theta\rangle$$

Let us consider first the isomorphism $\Psi\colon \operatorname{Mod}\mathcal{P}_k(Q) \to \mathcal{C}(1, T\Phi^+)$ with $\Psi(V,x,\xi) = ((V,x),\psi_\xi)$. We have $\xi = 0$ if and only if $\psi_\xi = 0$. There are corresponding assertions for the isomorphisms $\mathcal{C}(1,T\Phi^+) \to \mathcal{C}(T\Phi^-,1)$ and $\mathcal{C}(T\Phi^-,1) \to \operatorname{Mod} kQ\langle\Theta\rangle$. Altogether, we see: if (V,x,ξ) is a $\mathcal{P}_k(Q)$-module, then $\xi = 0$ if and only if the $kQ\langle\Theta\rangle$-module $\Psi'(V,x,\xi)$ is annihilated by the augmentation ideal. As a consequence, the augmentation ideal of $kQ\langle\Theta\rangle$ is mapped under η onto the ideal of $\mathcal{P}_k(Q)$ generated by the arrows of Q^*. Also, since Ψ' is a $\mathcal{C}$-isomorphism, it follows that the restriction of η to kQ is the identity.

This completes the proof of Theorem A.

4. Proof of Theorem C

Let I be the ideal of $\mathcal{P}_k(Q)$ generated by the arrows of Q^*. A finite dimensional $\mathcal{P}_k(Q)$-module M is nilpotent if and only if M is annihilated by some power I^s of I. Namely, the simple modules $E(i)$ with $i \in Q_0$ are annihilated by I, thus any module having a filtration of length l with factors of the form $E(i_t)$ is annihilated by I^l. On the other hand, if M is annihilated by I^s, then any composition factor of M is annihilated by I and therefore a simple kQ-module. But since kQ is finite dimensional, the only simple kQ-modules are those of the form $E(i_t)$.

Consider now the general case of a ring Λ and a Λ-bimodule Ω. The $\Lambda\langle\Omega\rangle$-modules are just of the form (M, f) where M is a Λ-module and $f\colon \Omega \otimes_\Lambda M \to M$ is a Λ-homomorphism. Of course, such a map f may also be considered as an element of the orbit ring $\mathcal{O}^F(M)$ for the functor $F = \Omega \otimes -$. There is the following relationship:

Proposition 1. *Let M be a Λ-module and let $f\colon \Omega \otimes_\Lambda M \to M$ be a Λ-homomorphism. The map f is nilpotent as an element of the orbit ring $\mathcal{O}^F(M)$ if and only if the $\Lambda\langle\Omega\rangle$-module (M, f) is annihilated by some power of the augmentation ideal.*

Proof. The s-fold power of f in the orbit ring $\mathcal{O}^F(M)$ is the composition of the following maps

$$\Omega^{\otimes s} \otimes M \xrightarrow{1_{\Omega^{s-1}} \otimes f} \Omega^{\otimes s-1} \otimes M \to \cdots \to \Omega \otimes M \xrightarrow{f} M.$$

But this is also the restriction μ_s of the scalar multiplication of the $\Lambda\langle\Omega\rangle$-module (M, f) to $\Omega^{\otimes s} \otimes M$. Thus, we see that the s-fold power f^{*s} of f in the orbit ring $\mathcal{O}^F(M)$ is zero if and only if the map μ_s is the zero map. The inductive definition of the maps μ_s shows that $\mu_s = 0$ implies $\mu_t = 0$ for all $t \geq s$. This shows that $f^{*s} = 0$ in the orbit ring $\mathcal{O}^F(M)$ if and only if the s-fold power of the augmentation ideal annihilates the $\Lambda\langle\Omega\rangle$-module (M, f).

Theorem C is an immediate consequence of Proposition 1 and the previous considerations.

5. Application

We want to give some indications in which way Theorem C can be used. Let Q be a connected quiver without cyclic paths and let us consider only finite dimensional kQ-modules. Given such a module M, let $\mathcal{O}(M) = \mathcal{O}^{\tau^-}(M)$ denote the orbit ring with respect to the Auslander-Reiten translation τ^-. Recall that $\mathcal{O}(M)$ is a graded ring and we are mainly interested in the set $\mathcal{O}(M)_1 = \operatorname{Hom}(\tau^- M, M)$ of elements of degree 1. Also, we denote the subset $\mathcal{N}^{\tau^-}(M)$ just by $\mathcal{N}(M)$. We are going to derive some recipes for calculating $\mathcal{N}(M)$.

First of all, observe the following: if we consider two kQ-modules M, N, then we can write $\mathcal{O}(M \oplus N)_1$ in matrix form as follows:

$$\mathcal{O}(M \oplus N)_1 = \begin{bmatrix} \operatorname{Hom}(\tau^- M, M) & \operatorname{Hom}(\tau^- N, M) \\ \operatorname{Hom}(\tau^- M, N) & \operatorname{Hom}(\tau^- N, N) \end{bmatrix}.$$

Let M be a kQ-module. Recall that M is said to be *preprojective* provided there exists some $n \geq 0$ such that $\tau^n(M) = 0$. Similarly, M is said to be *preinjective* provided there exists some $n \geq 0$ such that $\tau^{-n}(M) = 0$. Finally, M is said to be *regular* provided no indecomposable direct summand of M is preprojective or preinjective. In case kQ is representation finite, all the modules are both preprojective and preinjective, otherwise the only module which is both preprojective and preinjective is the zero module. Any module M is isomorphic to a module of the form

$P \oplus R \oplus I$, where P is preprojective, R is regular and I is preinjective, and in case kQ is representation infinite, such a decomposition is unique up to isomorphism.

Proposition 2. *Let P be preprojective, R regular and I preinjective. If kQ is representation finite, we assume in addition that $I = 0$. Then*

$$\mathcal{O}(P \oplus R \oplus I)_1 = \begin{bmatrix} \mathrm{Hom}(\tau^- P, P) & 0 & 0 \\ \mathrm{Hom}(\tau^- P, R) & \mathrm{Hom}(\tau^- R, R) & 0 \\ \mathrm{Hom}(\tau^- P, I) & \mathrm{Hom}(\tau^- R, I) & \mathrm{Hom}(\tau^- I, I) \end{bmatrix}$$

and

$$\mathcal{N}(P \oplus R \oplus I) = \begin{bmatrix} \mathrm{Hom}(\tau^- P, P) & 0 & 0 \\ \mathrm{Hom}(\tau^- P, R) & \mathcal{N}(R) & 0 \\ \mathrm{Hom}(\tau^- P, I) & \mathrm{Hom}(\tau^- R, I) & \mathrm{Hom}(\tau^- I, I) \end{bmatrix}.$$

Proof: The first assertion follows directly from the well-known structure of the module category of a finite dimensional hereditary algebra. The triangular form of these matrices implies that the nilpotency of elements has to be checked only for P, R and I separately. Now assume that $\tau^{-n} I = 0$ for some $n \geq 0$. Then, for any $f \in \mathrm{Hom}(\tau^- I, I)$, the n-fold power f^{*n} in the orbit ring $\mathcal{O}(I)$ is zero. Similarly, let us assume that $\tau^n P = 0$ for some $n \geq 0$. Since τ^{-n} is left adjoint to τ^n, it follows that $\mathrm{Hom}(\tau^{-n} P, P) \simeq \mathrm{Hom}(P, \tau^n P) = 0$. Thus for any element in $f \in \mathrm{Hom}(\tau^- P, P)$, we also have $f^{*n} = 0$ in $\mathcal{O}(P)$.

This shows that it is sufficient to consider the case of a regular module R. In general, the problem of determining $\mathcal{N}(R)$ inside $\mathcal{O}(R)_1$ seems to be difficult. *Let us assume from now on that kQ is tame,* thus the underlying graph of Q is of the form $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7$ or $\widetilde{E}_8$.

Recall that in the tame case, all regular kQ-modules are τ-periodic. A module M is said to be *aperiodic* provided there does not exist any indecomposable module N of τ-period p such that the direct sum $\bigoplus_{i=0}^{p-1} \tau^i N$ is isomorphic to a direct summand of M. This terminology is due to Lusztig [L2] (in [R2] we used the word 'separated'). A module $M = P \oplus R \oplus I$ with P preprojective, R regular and I preinjective is aperiodic, if and only if R is aperiodic. This means that the problem of aperiodicity only concerns the regular modules.

Proposition 3. *Let Q be a tame quiver without cyclic paths. Let M be a kQ-module. Then $\mathcal{N}(M) = \mathcal{O}(M)_1$ if and only if M is aperiodic.*

Proof. First, let us assume that M is not aperiodic. Let N be an indecomposable module of τ-period p such that the direct sum $M' = \bigoplus_{i=0}^{p-1} \tau^i N$ is a direct summand of M. It is sufficient to construct an element f of $\mathcal{O}(M')_1$ which does not belong to $\mathcal{N}(M')$. Since N is τ-periodic with τ-period p, it follows that $\tau^- M'$ is isomorphic to M' (the direct summands of $\tau^- M'$ are the same as those of M', but cyclically shifted). Let $f \colon \tau^- M' \to M'$ be an isomorphism. Then $\tau^{-t} f$ is an isomorphism, for all natural numbers t. The t-fold power f^{*t} of f is just the composition of these isomorphisms and therefore non-zero, for all t.

Conversely, assume that $\mathcal{N}(M)$ is a proper subset of $\mathcal{O}(M)_1$. Write $M = P \oplus R \oplus I$ with P preprojective, R regular and I preinjective. According to Proposition 2, there exists an element $f \colon \tau^- R \to R$ which does not belong to $\mathcal{N}(R)$. Write

$R = \bigoplus_{i=1}^{m} R_i$ with indecomposable modules R_i. Any regular module is τ-periodic, thus there is some integer $q \geq m$ such that $\tau^q R$ is isomorphic to R. Consider the q-fold power f^{*q} of f in $\mathcal{O}(R)$. We may consider f^{*q} as a homomorphism $R \to R$. If this endomorphism f^{*q} of R belongs to the radical of $\operatorname{End}(R)$, then it is nilpotent, and then also the corresponding power of f^{*q} vanishes in $\mathcal{O}(R)$. Since we assume that f does not belong to $\mathcal{N}(R)$, it follows that f^{*q} does not belong to the radical of $\operatorname{End}(R)$. Recall that f^{*q} has a factorization of the following kind:

$$R = \tau^{-q} R \to \tau^{-q+1} R \to \cdots \to \tau^{-} R \to R,$$

It follows that there is a sequence of numbers $i(j)$ with $1 \leq i(j) \leq m$ and $0 \leq j \leq q$, such that there are isomorphisms

$$\tau^{-q} R_{i(q)} \to \tau^{-q+1} R_{i(q-1)} \to \cdots \to \tau^{-} R_{i(1)} \to R_{i(0)}.$$

Since $q \geq m$, we deal with at least $m+1$ modules $R_{i(j)}$, thus at least two of these modules have to be equal, say $R_{i(j)}$ and $R_{i(j+p)}$, where $0 \leq j < j+p \leq q$. Let $N = R_{i(j)}$. Then, for $1 \leq t \leq p$, we apply τ^{j+t} to the isomorphism

$$\tau^{-j-t} R_{i(j+t)} \to \tau^{-j} R_{i(j)} = \tau^{-j} N$$

and obtain an isomorphism

$$R_{i(j+t)} \to \tau^{t} N.$$

Since $R_{i(j+p)} = R_{i(j)} = N$, we see that N is of τ-period p. Of course, we may assume that p is chosen minimal, thus the modules $R_{i(j)}, R_{i(j+1)}, \ldots R_{i(j+p-1)}$ are pairwise different, and their direct sum is a direct summand of R. This shows that $\bigoplus_{t=0}^{p-1} \tau^t N$ is isomorphic to a direct summand of R, and thus of M. Altogether, we have shown that M is not aperiodic.

The full subcategory of all regular modules is abelian, the regular modules without a proper regular submodule are called *simple regular*, they are the simple objects in this subcategory (but they are not necessarily simple kQ-modules). For any simple regular module S and any $n \in \mathbb{N}_1$, there exists (up to isomorphism) a unique indecomposable module $S[n]$ which has a filtration of length n such that all the factors are simple regular and such that $S = S[1]$ occurs as a submodule of $S[n]$; in this way, one obtains all the indecomposable regular modules. A kQ-module M is called *homogeneous* provided $\tau M' = M'$ for any indecomposable direct summand of M. The homogeneous modules are regular, and such a module is simple regular if and only if its endomorphism ring is a division ring. If S is simple regular and $l \in \mathbb{N}_1$, then $S[l]$ is homogeneous if and only if S is homogeneous. There are always infinitely many isomorphism classes of homogeneous modules which are simple regular; in case k is an infinite field, then there are even infinitely many such modules with endomorphism ring equal to k and all have the dimension vector $\mathbf{h}$.

The dimension vectors of kQ-modules may be considered as elements of the Grothendieck group $K_0(kQ)$ of all finite length modules modulo exact sequences. The dimension vectors of the indecomposable kQ-modules are the positive vectors of an affine root system, and $\mathbf{h}$ is just the smallest positive imaginary root. The corresponding quadratic form on $K_0(kQ) = \mathbb{Z}^{Q_0}$ will be denoted by q.

Proposition 4. *Let Q be a tame quiver without cyclic paths. Let H be a homogeneous kQ-module. Then $\mathcal{O}(H)_1 = \mathrm{Hom}(\tau^- H, H)$ can be identified with the set $\mathrm{End}(H)$ of all endomorphisms of H. Under this identification, the set $\mathcal{N}(H)$ corresponds to the set of nilpotent endomorphisms.*

In particular, if H is the direct sum of pairwise non-isomorphic indecomposable modules, then $\mathcal{N}(H)$ is identified with the radical of the endomorphism ring $\mathrm{End}(H)$ of H.

Remark: If one writes $\mathcal{O}(H)_1 = \mathrm{End}(H)$, one has to be aware that this is an equality only of additive groups (or vector spaces), but that the multiplicative structures are different: $\mathrm{End}(H)$ is a ring, whereas the multiplication of elements from $\mathcal{O}(H)_1$ yields elements in $\mathcal{O}(H)_2$.

Proof of Proposition 4: Consider the full subcategory of all homogeneous kQ-modules. On this subcategory, the Auslander-Reiten translation is the identity functor. It follows that for any homogeneous module H, the orbit ring $\mathcal{O}(H)$ is the polynomial ring $\mathrm{End}(H)[t]$ in one commuting indeterminate t with coefficients in $\mathrm{End}(H)$, and $\mathcal{O}(H)_1 = \mathrm{End}(H)\cdot t$. Of course, given any ring Λ and its polynomial ring $\Lambda[t]$, an element $c \in \Lambda$ is nilpotent if and only if $c\cdot t$ is a nilpotent element of $\Lambda[t]$. This shows the first assertion.

For the second assertion, consider any finite length module M which is a direct sum of pairwise non-isomorphic indecomposable modules. Then the factor ring of $\mathrm{End}(M)$ modulo its radical is a finite product of division rings, whereas all elements of the radical are nilpotent. It follows that the radical of $\mathrm{End}(M)$ is the set of all nilpotent endomorphisms.

Proposition 5. *Let Q be a tame quiver without cyclic paths. Let M be an aperiodic and H a homogeneous kQ-module. If we assume that H is the direct sum of r pairwise non-isomorphic indecomposable modules, then $\mathcal{N}(M\oplus H)$ is a subspace of $\mathcal{O}(M\oplus H)_1$ of codimension r.*

Proof: This is a direct consequence of Propositions 2 and 4.

The previous considerations, in particular Proposition 3, show the importance of the aperiodic representations of a tame quiver. Let us outline in which way we may use these considerations for the construction of irreducible components.

Let M be an aperiodic kQ-module and $n \in \mathbb{N}_0$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ a partition of n. We are going to consider the kQ-modules X which are isomorphic to a module of the form

$$M \oplus S_1[\lambda_1] \oplus \cdots \oplus S_s[\lambda_s],$$

with homogeneous modules $S_1, \dots, S_s$ with endomorphism ring k. All these modules have the same dimension vector, namely $\mathbf{dim}\, M + n\mathbf{h}$. In particular, we are interested in those modules, where $S_1, \dots S_s$ are pairwise non-isomorphic. Fix a Q_0-graded vector space V with dimension vector $\mathbf{dim}\, M + n\mathbf{h}$. We denote by $\Xi(M,\lambda)$ the subset of $\mathcal{R}(Q;V)$ of all representations of Q which are of the form

$$M \oplus S_1[\lambda_1] \oplus \cdots \oplus S_s[\lambda_s]$$

where $S_1, \dots S_s$ are pairwise non-isomorphic homogeneous modules with endomorphism ring k.

Proposition 6. *Let Q be a tame quiver without cyclic paths. Let M be an aperiodic kQ-module and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ a partition. If X belongs to $\Xi(M, \lambda)$, then $e = \dim \operatorname{End}(X)$ and $d = \dim \mathcal{N}(X)$ only depend on M and λ, and we have*

$$e - d - s = q(\mathbf{dim}\, X).$$

Proof: In order to calculate e, we use the bilinear form $\langle -, - \rangle$ on $K_0(kQ)$ defined by

$$\langle \mathbf{dim}\, N_1, \mathbf{dim}\, N_2, \rangle = \dim \operatorname{Hom}(N_1, N_2) - \dim \operatorname{Ext}^1(N_1, N_2)$$

for kQ-modules N_1, N_2; the corresponding quadratic form is just q.

Let $X = M \oplus H$. The dimension e of $\operatorname{End}(M \oplus H)$ is the sum of the dimensions of $\operatorname{End}(M)$, $\operatorname{End}(H)$, $\operatorname{Hom}(M, H)$ and $\operatorname{Hom}(H, M)$. The endomorphism ring of H is the product of the endomorphism rings of the modules $S_i[\lambda_i]$, since $\operatorname{Hom}(S_i[\lambda_i], S_j[\lambda_j]) = 0$ for $i \neq j$. Now, $\operatorname{End}(S_i[\lambda_i])$ is a uniserial algebra of dimension λ_i, thus $\dim \operatorname{End}(H) = \sum_{i=1}^{s} \lambda_i = n$.

Let $M = P \oplus R \oplus I$ with P preprojective, R regular and I preinjective. Note that $\operatorname{Hom}(R \oplus I, H) = 0 = \operatorname{Ext}^1(P, H)$, thus

$$\begin{aligned} \dim \operatorname{Hom}(M, H) &= \dim \operatorname{Hom}(P, H) = \dim \operatorname{Hom}(P, H) - \dim \operatorname{Ext}^1(P, H) \\ &= \langle \mathbf{dim}\, P, \mathbf{dim}\, H \rangle = \langle \mathbf{dim}\, P, n\mathbf{h} \rangle. \end{aligned}$$

Similarly, $\operatorname{Hom}(H, P \oplus R) = 0 = \operatorname{Ext}^1(H, I)$ yield

$$\dim \operatorname{Hom}(H, M) = \langle \mathbf{dim}\, H, \mathbf{dim}\, I \rangle = \langle n\mathbf{h}, \mathbf{dim}\, I \rangle.$$

Altogether, we see that

$$e = \dim \operatorname{End}(M) + n + \langle \mathbf{dim}\, P, n\mathbf{h} \rangle + \langle n\mathbf{h}, \mathbf{dim}\, I \rangle$$

only depends on M and n.

On the other hand, let us consider $\dim \operatorname{Ext}^1(X, X)$. The Auslander-Reiten formula identifies $\operatorname{Ext}^1(X, X)$ with the k-dual of $\operatorname{Hom}(\tau^- X, X) = \mathcal{O}(X)_1$. According to Proposition 5, $\mathcal{N}(X)$ is a subspace of $\mathcal{O}(X)_1$ of codimension s. This shows that

$$\dim \operatorname{Ext}^1(X, X) = \dim \mathcal{N}(X) + s = d + s.$$

It follows that

$$q(\mathbf{dim}\, X) = \dim \operatorname{End}(X) - \dim \operatorname{Ext}^1(X, X) = e - d - s.$$

In particular, since $\mathbf{dim}\, X$ as well as e and s only depend on M and λ, the same is true for d. This completes the proof.

Corollary (Lusztig). *Assume that k is an algebraically closed field. Let Q be a tame quiver without cyclic paths. Let M be an aperiodic kQ-module and fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$. Then the closure of $\pi^{-1}(\Xi(M, \lambda))$ is an irreducible component of $\mathcal{R}_0(\overline{Q}, (\rho); V)$.*

This result has been shown (with some minor restrictions on Q) by Lusztig in [L2], Proposition 4.14. It is an essential prerequisite for his description of the affine quantum groups. The proof presented here is streamlined in order to allow the parallel discussion of all the affine quantum groups (related to arbitrary symmetrizable and not necessarily symmetric generalized Cartan matrices). We want to use these considerations in order to provide a Hall algebra treatment of these quantum groups, as outlined in [R3].

Proof of Corollary: Recall that $G(V) = \prod_{i \in Q_0} \mathrm{GL}(V_i)$ operates on $\mathcal{R}(Q;V)$ so that the $G(V)$-orbits are just isomorphism classes. By definition, $\Xi(M,\lambda)$ is a union of isomorphism classes, thus of $G(V)$-orbits. All these orbits have the same dimension, namely

$$v = \dim G(V) - \dim \mathrm{End}(X),$$

where $X = (V,x)$ is an element of $\Xi(M,\lambda)$. The isomorphism classes of the homogeneous, simple regular modules can be parametrized by a cofinite subset of the projective line $\mathbb{P}_1(k)$. It follows that $\Xi(M,\lambda)$ is irreducible and of dimension $v+s$. For every X in $\Xi(M,\lambda)$, the fiber $\pi^{-1}(X) = \mathcal{N}(X)$ is an affine space and its dimension d is independent of X. There are two consequences: since the fibres are irreducible and have equal dimension, $\pi^{-1}(\Xi(M,\lambda))$ is irreducible as well. Second, the dimension of $\pi^{-1}(\Xi(M,\lambda))$ is

$$v + s + d = \dim G(V) - \dim \mathrm{End}(X) + s + d = \dim G(V) - q(\mathbf{dim}\, X).$$

However, we also know that

$$q(\mathbf{dim}\, X) = \dim G(V) - \dim \mathcal{R}(Q;V),$$

thus the dimension of $\pi^{-1}(\Xi(M,\lambda))$ is equal to $r = \dim \mathcal{R}(Q;V)$. On the other hand, the variety $\mathcal{R}_0(Q,(\rho);V)$ has dimension r (see [L1]), thus any closed irreducible subset of dimension r has to be an irreducible component.

6. Final remark

As we have mentioned in the introduction, the early investigations of Gelfand and Ponomarev, and of Dlab and myself, were aiming at algebras A which contain a quiver algebra kQ as a subalgebra and such that A when considered as a left kQ-module, decomposes as a direct sum of the indecomposable preprojective kQ-modules, each occurring with multiplicity one. The problem whether there are several possible choices was not discussed explicitly. The construction presented in [DR] starts with what is called a 'modulated graph', but any non-trivial quiver Q gives rise to a wealth of modulated graphs (the bilinear forms needed may be chosen quite arbitrarily). A suitable choice will always produce the preprojective algebra $\mathcal{P}_k(Q)$ as considered in the present paper, but other choices may yield algebras which are not isomorphic to $\mathcal{P}_k(Q)$. We are going to exhibit a corresponding example.

The problem considered here may be phrased differently, as follows: in the definition of $\mathcal{P}_k(Q)$, we have used the sign function ε_Q. Of course, one may also consider the algebra $\mathcal{P}'_k(Q) = k\overline{Q}/(\rho')$, where

$$\rho' = \sum_{\alpha \in \overline{Q}_1} \alpha^* \alpha.$$

The same calculations as above show that the category of all $\mathcal{P}'_k(Q)$-modules is equivalent to the category $\mathcal{C}(1, \Phi^+)$ and therefore to the category $\mathcal{C}(\Phi^-, 1)$, where, as before, $\mathcal{C} = \operatorname{Mod} kQ$. As a consequence, $\mathcal{P}'_k(Q)$ is isomorphic to the tensor algebra of the kQ-bimodule $\Phi^-({}_{kQ}kQ)$ (where the right kQ-module structure comes from the canonical action of kQ on the right of ${}_{kQ}kQ$). The algebra $\mathcal{P}'_k(Q)$ is one of those which have kQ embedded as a subalgebra and which decompose as left kQ-module into a direct sum of all the indecomposable preprojective kQ-modules. It is easy to see that for Q a tree, the algebras $\mathcal{P}_k(Q)$ and $\mathcal{P}'_k(Q)$ will be isomorphic. However, in general this is no longer true.

The example to be considered is the case of the affine quiver Q of type $\widetilde{A}_{12}$.

$$\begin{tikzcd} & \circ \arrow[dr, "\beta"] & \\ \circ \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & \circ \end{tikzcd}$$

As Gabriel [G] has pointed out, this is the typical (and smallest) example of a quiver where the endofunctors Φ^- and τ^- are not equivalent, provided the characteristic of k is different from 2: then there exists a 3-dimensional kQ-module S such that the images $\Phi^-(S)$ and $\tau^-(S)$ are not isomorphic. We will use such a module S in order to prove that the algebras $\mathcal{P} = \mathcal{P}_k(Q)$ and $\mathcal{P}' = \mathcal{P}'_k(Q)$ are not isomorphic.

For $\lambda \in k$, let S_λ be the following representation of kQ

$$\begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] \arrow[rr, "\lambda"'] & & k \end{tikzcd}$$

On the one hand, a direct (and easy) calculation shows that $\Phi^-(S_\lambda) = S_{-\lambda}$. On the other hand, S_λ is clearly simple regular. Since its dimension vector is $\mathbf{h}$, we conclude that S_λ is homogeneous. As a consequence, $\tau^-(S_\lambda)$ is isomorphic to S_λ. This shows that for $\lambda \neq 0$ (and characteristic different from 2), the kQ-modules $\Phi^-(S_\lambda)$ and $\tau^-(S_\lambda)$ are not isomorphic. We fix some $\lambda \neq 0$ and let $S = S_\lambda$.

Recall that the $\mathcal{P}'_k(Q)$-modules can be considered as pairs (C, c), where C is a kQ-module and $c \colon \Phi^-(M) \to M$ is a kQ-module homomorphism. We consider $S = (S, 0)$ as a $\mathcal{P}'_k(Q)$-module and claim that $\operatorname{Ext}^1_{\mathcal{P}'}(S, S) = k$. In order to see this, consider an exact sequence

$$0 \to (S, 0) \to (C, c) \to (S, 0) \to 0.$$

Since $0 \to S \to C \to S \to 0$ is an exact sequence of kQ-modules, we see that C is either isomorphic to $S[2]$ or to $S \oplus S$. But then $f \colon \Phi^-(C) \to C$ has to be the zero map. Namely, It follows from $\Phi^-(S_\lambda) = S_{-\lambda}$ that $\Phi^-(S_\lambda[2]) = S_{-\lambda}[2]$, and therefore we have both $\operatorname{Hom}(\Phi^-(S), S) = 0$ and $\operatorname{Hom}(\Phi^-(S[2]), S[2]) = 0$. Consequently, we can identify $\operatorname{Ext}^1_{\mathcal{P}'}(S, S)$ with $\operatorname{Ext}^1_{kQ}(S, S) = k$. Note that the module S is indecomposable, has length 3 and dimension 3.

On the other hand, we are going to show that *for every indecomposable $\mathcal{P}$-module M of dimension* 3 *and length* 3, *the vector space* $\operatorname{Ext}^1_{\mathcal{P}}(M, M)$ *is at least* 2-*dimensional.* First, we have to analyse the possibilities for M. Its dimension vector has to be $\mathbf{h} = (1\ 1\ 1)$, since otherwise we would deal with a module for the preprojective algebra of a quiver of type A_2, however such an algebra has no indecomposable modules of dimension 3. Also, for any arrow α of Q, at most one of the elements α, α^* acts non-trivially on M, since otherwise the length of M is at most 2. For the same reason, it is impossible that all the elements α, β, γ^* act non-trivially; and similarly, not all the elements $\alpha^*, \beta^*, \gamma$ act non-trivially. It follows that there is a subquiver Q' of $\overline{Q}$ which again is of the form $\widetilde{A}_{12}$ such that all the arrows outside of Q' operate trivially on M. This shows that without loss of generality, we may assume that M is a kQ-module with dimension vector (1 1 1).

In particular, we see that M as a kQ-module is an indecomposable regular module with dimension vector $\mathbf{h}$. We want to show that $\mathrm{Ext}^1_{\mathcal{P}}(M, M)$ has dimension at least 2. First of all, since $\mathrm{Ext}^1_{kQ}(M, M) \neq 0$, there is a non-split exact sequence

$$0 \to M \to M' \to M \to 0$$

of kQ-modules. Second, there is a non-zero homomorphism $\phi\colon \tau^-(M) \to M$, thus we may construct the following object in the category $\mathcal{C}(\tau^-, 1)$

$$M'' = \left(M \oplus M, \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} \right)$$

and there is an obvious exact sequence

$$0 \to M \to M'' \to M \to 0.$$

Since M'' is indecomposable, this sequence also does not split. Clearly, the objects M' and M'' are not isomorphic. This completes the proof.

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