Analysis and geometry on configuration spaces

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Abstract

In this paper foundations are presented to a new systematic approach to analysis and geometry for an important class of infinite dimensional manifolds, namely, configuration spaces. More precisely, a differential geometry is introduced on the configuration space Γ_X over a Riemannian manifold X. This geometry is "non-flat" even if $X = \mathbb{R}^d$. It is obtained as a natural lifting of the Riemannian structure on X. In particular, a corresponding gradient ∇^Γ, divergence div^Γ, and Laplace-Beltrami operator $H^\Gamma = -\mathrm{div}^\Gamma\nabla^\Gamma$ are constructed. The associated volume elements, i.e., all measures μ on Γ_X w.r.t. which ∇^Γ and div^Γ become dual operators on $L^2(\Gamma_X;\mu)$, are identified as exactly the mixed Poisson measures with mean measure equal to a multiple of the volume element dx on X. In particular, all these measures obey an integration by parts formula w.r.t. vector fields on Γ_X. The corresponding Dirichlet forms $\mathcal{E}^\Gamma_\mu$ on $L^2(\Gamma_X;\mu)$ are, therefore, defined. Each is shown to be associated with a diffusion process which is thus the Brownian motion on Γ_X and which is subsequently identified as the usual independent infinite particle process on X. The associated heat semigroup $(T^\Gamma_\mu(t))_{t>0}$ is calculated explicitly. It is also proved that the diffusion process, when started with μ, is time-ergodic (or equivalently $\mathcal{E}^\Gamma_\mu$ is irreducible or equivalently $(T^\Gamma_\mu(t))_{t>0}$ is ergodic) if and only if μ is Poisson measure $\pi_{z\,dx}$ with intensity $z\ dx$ for some $z \geq 0$. Furthermore, it is shown that the Laplace-Beltrami operator $H^\Gamma = -\mathrm{div}^\Gamma\nabla^\Gamma$ on $L^2(\Gamma_X;\pi_{z\,dx})$ is unitary equivalent to the second quantization of the Laplacian $-\triangle^X$ on X on the corresponding Fock space $\bigoplus_{n\geq 0} L^2(X; z\ dx)^{\hat{\otimes} n}$. As another direct consequence of our results we obtain a representation of the Lie-algebra of compactly supported vector fields on X on Poisson space. Finally, generalizations to the case where dx is replaced by an absolutely continuous measure and also to interacting particle systems on X are described, in particular, the case where the mixed Poisson measures μ are replaced by Gibbs measures of Ruelle-type on Γ_X.

Contents

1 Introduction

In this paper we introduce a natural (infinite dimensional) differential geometry on the configuration space Γ_X over a Riemannian manifold X, i.e., on

$$\Gamma_X := \{\gamma \subset X | \gamma \cap K \text{ is a finite set for every compact } K \subset X\}.$$

This geometry is constructed via a "lifting procedure" and is completely determined by the Riemannian structure on X (cf. Subsection 3.4). In particular, we obtain a corresponding gradient ∇^Γ, divergence div^Γ, and Laplace operator $\triangle^\Gamma = \mathrm{div}^\Gamma \nabla^\Gamma$. For details we refer to the main body of the paper (whose structure will be summarized below). We only mention here that the "tangent bundle" $T\Gamma_X$ of Γ_X is given as follows:

$$T_\gamma \Gamma_X := L^2(X \to TX; \gamma), \quad \gamma \in \Gamma_X,$$

i.e., the space of sections in the tangent bundle TX of X which are square-integrable with respect to γ. We note that here as usual $\gamma \in \Gamma_X$ is identified with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x$ on X (where $\varepsilon_x :=$ Dirac measure in x). Since each $T_\gamma\Gamma_X$ is thus a Hilbert space (endowed with the corresponding L^2-inner product $\langle\ ,\ \rangle_{T_\gamma\Gamma_X}$ coming from the measure γ), Γ_X obtains a Riemannian-type structure which is non-trivial (i.e., varies with γ) even when X is equal to Euclidean space $\mathbb{R}^d$.

There is an enormous literature on analysis and geometry on infinite dimensional spaces. In most cases the starting point was to replace X by a Hilbert (or Banach) space (resp. manifold). But at some point the analysis became extremely difficult because of lacking a good notion of "volume element", since as is well-known, there is no Lebesgue measure on infinite dimensional linear spaces. This is entirely different if one analyses the natural geometry of Γ_X. It turns out that one can identify the corresponding "volume elements" as classical objects (more precisely, well-known measures). This is the first main result of this paper, which also makes the above mentioned lifted differential geometry particularly interesting.

To describe this result more precisely, let us recall that (as is well-known, cf. e.g. [Ch]) the volume element m on X is the (up to constant multiples) unique positive Radon measure μ on X such that the gradient ∇^X and divergence div^X on X become dual operators on $L^2(X;\mu)$ (w.r.t. $\langle\ ,\ \rangle_{TX}$). In

Subsection 3.3 we prove that the probability measures μ on Γ_X, for which ∇^Γ and div^Γ (cf. above) become dual operators on $L^2(\Gamma_X;\mu)$ (w.r.t. $\langle\ ,\ \rangle_{T\Gamma_X}$), are exactly the mixed Poisson measures

$$\mu_{\lambda,m} \ := \ \int_{I\!R_+} \pi_{s\cdot m}\lambda(ds) \tag{1.1}$$

where λ is a probability measure on $I\!R_+$ (with finite first moment) and $\pi_{s\cdot m}$ is the Poisson measure on Γ_X with intensity $s\cdot m, s \geq 0$, i.e., the unique measure on Γ_X whose Laplace transform is given by

$$\int_{\Gamma_X} e^{\langle\varphi,\gamma\rangle}\pi_{s\cdot m}(d\gamma) \ = \ \exp(s\int_X (e^\varphi - 1)dm)$$

for all $\varphi \in C_0(X)$ (:= all continuous functions on X with compact support). Here we set

$$\langle\varphi,\gamma\rangle \ := \ \int_X \varphi\, d\gamma \ = \ \sum_{x\in\gamma}\varphi(x).$$

Thus the mixed Poisson measures are exactly the "volume elements" corresponding to our differential geometry on Γ_X.

We should mention that the "test" functions $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ (resp. "test" vector fields V) we consider as domains for our gradient ∇^Γ (resp div^Γ) above, are of cylinder type, i.e., $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ if and only if

$$\gamma \mapsto F(\gamma) \ = \ g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle)$$

for some $N \in I\!N, \varphi_1,\ldots,\varphi_N \in \mathcal{D} := C_0^\infty(X), g_F \in C_b^\infty(I\!R^N)$ (and V correspondingly, cf. (3.18)). Hence so far the analysis on Γ_X is basically finite dimensional. However, having identified the right volume elements $\mu_{\lambda,m}$ (cf. (1.1)) we can now do generic infinite dimensional analysis by introducing the first order Sobolev space $H_0^{1,2}(\Gamma_X;\mu_{\lambda,m})$ by closing the corresponding Dirichlet form

$$\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}(F,G) \ = \ \int_{\Gamma_X}\langle\nabla^\Gamma F,\nabla^\Gamma G\rangle_{T\Gamma_X}d\mu_{\lambda,m},$$
$$F,G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X),$$

on $L^2(\Gamma_X;\mu_{\lambda,m})$, i.e., a function $F \in H_0^{1,2}(\Gamma_X,\mu_{\lambda,m})$ is together with its gradient $\nabla^\Gamma F$ obtained as a limit in $L^2(\Gamma_X;\mu_{\lambda,m})$ of a sequence $F_n \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$

resp. $\nabla^\Gamma F_n, n \in \mathbb{N}$. Thus such F really depends on infinitely many points in X (cf. Subsections 4.1, 4.2). The corresponding Friedrichs' extensions of $-\triangle^\Gamma$ are denoted by $H^\Gamma_{\mu_{\lambda,m}}$. They are positive definite self-adjoint operators on $L^2(\Gamma_X;\mu_{\lambda,m})$. The heat semigroups $(\exp(-tH^\Gamma_{\mu_{\lambda,m}}))_{t\geq 0}$ generated by them can be calculated explicitly, as is done in Subsection 4.3, where we also study the ergodicity of these semigroups, i.e., the question whether

$$\exp(-tH^\Gamma_{\mu_{\lambda,m}})F \;\to\; \int F d\mu_{\lambda,m} \quad \text{as} \quad t\to\infty$$

in $L^2(\Gamma_X;\mu_{\lambda,m})$ for all $F\in L^2(\Gamma_X;\mu_{\lambda,m})$. We prove that we have ergodicity if and only if $\nabla^\Gamma F = 0$ for $F \in H^{1,2}_0(\Gamma_X;\mu_{\lambda,m})$ implies $F = \text{const}$ $\mu_{\lambda,m}$-a.e. which in turn is the case if and only if $\mu_{\lambda,m} = \pi_{s\cdot m}$ for some $s\geq 0$, i.e., $\mu_{\lambda,m}$ is a (pure) Poisson measure.

Since on X there is a natural diffusion process intrinsically determined by the geometry, namely Brownian motion on X, it is at hand to ask whether the same is true for our geometry on Γ_X. This question is easily answered using the standard theory of Dirichlet forms (cf. [MR1]) and a recent result in [Yo 96]. More precisely, in Section 6 we construct from $\mathcal{E}^\Gamma_{\mu_{\lambda,m}}$ (in complete analogy to constructing Brownian motion on X) a corresponding diffusion process on Γ_X which is generated by (an extension of) $-\triangle^\Gamma$, i.e., we obtain the Brownian motion on Γ_X. The ergodicity results mentioned above translate directly into ergodicity properties of this diffusion w.r.t. time-translations (cf. Subsection 6.3). But most importantly this Brownian motion is identified as a classical object. It is just the independent infinite particle process obtained by taking countably many independent Brownian motions on X. The latter process has been known for more than 40 years (cf. [D, Do 1]) and has been studied quite intensively (see e.g. [M-L, ST]). But so far it was not realized that it is just Brownian motion on Γ_X w.r.t. the natural geometry inherited from X. We emphasize however, that we obtain this result easily as a by-product of our considerations. It was not the main aim of this work.

What we have described so far can be summarized as the identification of the intrinsic geometry of Γ_X. There is also another kind of geometry of Γ_X which is based on fixing a pure Poisson measure $\pi_{s\cdot m}$ (as reference measure) and considering the isometry between $L^2(\Gamma_X;\pi_{s\cdot m})$ and the corresponding Fock space

$$\bigoplus_{n\geq 0} L^2(X; s\cdot m)^{\hat{\otimes} n},$$

where $L^2(X; s \cdot m)^{\hat{\otimes} n}$ denotes the n-fold symmetric tensor product of $L^2(X; s \cdot m)$. This can be called the extrinsic geometry of Γ_X (equipped with $\pi_{s \cdot m}$). In Section 5 we clarify the exact relation of these two pictures. Our main result is to prove that $H^{\Gamma}_{\pi_{s \cdot m}}$ is unitary equivalent (under the above isometry) to the second quantization of $-\triangle^X$.

We would also like to draw the reader's attention to Subsection 3.5. It is well-known (see [VGG]) that there is a canonical unitary representation on Poisson space, i.e., $L^2(\Gamma_X; \pi_{s \cdot m})$, of the group of diffeomorphisms $\mathrm{Diff}_0(X)$ on X which are equal to the identity outside a compact. On the basis of our results described above, we provide a corresponding representation of the associated Lie algebra of compactly supported vector fields. We also exhibit explicit formulas for the corresponding generators. This Lie-algebra ("algebra of currents") plays a significant role both in quantum field theory and infinite dimensional representation theory.

We would like to mention at this point that though we assume the manifold X to be non-compact, many of the results of this paper, in particular, the ones in Subsection 3.5 also hold in the compact case. However, e.g. for proving the ergodicity results in Subsection 4.3 it is crucial that X is non-compact.

The presentation of our results so far might give the impression that finding the right geometry on Γ_X by the lifting procedure happened just by coincidence. This was, however, not the case. In fact, we started with the Poisson measure π_m and tried to understand "its geometry", i.e., quasi-invariance properties under flows, integration by parts, corresponding divergence etc. and only afterwards we realized that this so found geometry is in fact given independently of the Poisson measure π_m by a lifting procedure. Therefore, we proceed in this paper as follows. We implement three steps by which one usually can identify the "differential geometry of a measure":

Step 1: Identification of flows on Γ_X under which π_m is quasi-invariant.

Step 2: Find the associated "vector fields" on Γ_X and the corresponding directional derivatives (which simultaneously gives an integration by parts formula for π_m). Here we thus obtain the pre-tangent bundle $(T\Gamma_X)^\circ$ given by
$(T_\gamma \Gamma_X)^\circ :=$ all compactly supported vector fields on X, $\gamma \in \Gamma$.

Step 3: Find an inner product $\langle\ ,\ \rangle_{T_\gamma\Gamma_X}$ on $(T_\gamma\Gamma_X)^\circ$ so that for all functions in $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ the directional derivatives come from a corresponding gradient ∇^Γ (cf. Definition 3.3; here Riesz's representation theorem must be used, so we have to complete $(T_\gamma\Gamma_X)^\circ$ w.r.t. $\langle\ ,\ \rangle_{T_\gamma\Gamma_X}$, i.e., get the corresponding Hilbert space $L^2(X \to TX;\gamma)$).

From then on the definition of div^Γ is straightforward. We refer to Subsections 3.1, 3.2 for details. The lifting procedure is then explained in Subsection 3.4.

Thus the contents of Sections 3–6 has been described. It remains to add that Section 2 consists of necessary preliminaries and that all results above have been proved in the more general case where m is replaced by a positive Radon measure σ, absolute continuous w.r.t. m under some weak assumptions on the Radon-Nikodym derivative $\rho = \frac{d\sigma}{dm}$. We emphasize at this point that the transition semigroup of the corresponding diffusion process, which is called distorted Brownian motion, is not necessarily Feller. Furthermore, as in the case $m = \sigma$, it is not necessarily conservative, in particular, σ is in general not an invariant measure of the process.

Finally, we would like to mention that this paper is the basis for many future directions of research. First of all we would like to stress that most of the results, in particular those in Subsection 3.3 on integration by parts characterization, as well as the ergodicity results in Theorem 4.3 extend in a natural way to the case where the mixed Poisson measures are replaced by Gibbs measures of Ruelle type. Of course, the divergence changes drastically. The corresponding results are contained in [AKR 4,5] which also motivated the present work. Furthermore, even for the Poisson case, the "usual" perturbation theory of Dirichlet forms which yields new non-symmetric, even non-sectorial or time-dependent Dirichlet forms, is applicable to obtain entirely new interacting infinite particle processes on X (cf. [R, Section 6] and [St]). Alternatively, in many cases the corresponding processes can be obtained via stochastic differential equations on Γ_X because the theory of such equations can now be carried out on the basis of the geometric structure of Γ_X analyzed in this paper. Furthermore, we can replace X by a manifold with boundary. In particular, the integration by parts formula in [CP, Theorem 1.4] (see also the corresponding result in [Pri]) can be interpreted as a special case of Theorem 3.1 below with our manifold X being just [0,1] (resp.

$[0,\infty)$ in [Pri]). The tangent bundle and hence the gradient in [CP], [Pri] are, however, totally different from ours, since the authors consider other test functions than $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$. We can also replace X by a manifold which is infinite dimensional itself. For example one can take X = loop space over a Riemannian manifold to obtain diffusions which are interacting loop-valued processes (cf. [AKMR, KLRRS] and also the concluding remarks in [AKR 1]).

All results of this paper have been announced in [AKR 1,2] (see also [R, Section 6] for the definition of ∇^Γ). Apart from a series of lectures at Bielefeld University in Summer 1996 they have also been presented at international conferences in Barcelona in June'96, Ascona in September'96, Trento in January'97, and Rome in February'97.

2 Measures on configuration spaces

In this section we describe some facts about probability measures on configuration spaces which are necessary later on. Let us mention that such measures are called states in the statistical physics of continuous systems. In probability theory they are known as random point processes.

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by $T_x(X)$; and the tangent bundle endowed with its natural differentiable structure will be denoted $T(X) = \cup_{x\in X} T_x(X)$. The Riemannian metric on X associates to each $x \in X$ an inner product on $T_x(X)$, which we denote by $\langle\cdot,\cdot\rangle_{T_x}$. The associated norm will be denoted by $|\cdot|_{T_x}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ-algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X), \mathcal{B}(X)$ respectively, which have compact closures.

2.1 The configuration space over a manifold

The configuration space Γ_X over the manifold X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X = \{\gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}.$$

Here $|A|$ denotes the cardinality of a set A. For $\Lambda \subset X$ we sometimes use the shorthand γ_Λ for $\gamma \cap \Lambda$ and define

$$\Gamma_\Lambda \ := \ \{\gamma \in \Gamma_X \mid \gamma \cap (X \setminus \Lambda) = \emptyset\}.$$

The reader should watch, however, that if Λ is a sub-manifold not equal to X there is an ambiguity in the notation (consider e.g. the case where Λ is relatively compact). As in the standard literature we shall, however, never consider sub-manifolds of X and always understand Γ_Λ in the above sense.

We can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure

$$\sum_{x \in \gamma} \varepsilon_x \ \in \ \mathcal{M}_p(X) \subset \mathcal{M}(X),$$

where $\sum_{x \in \emptyset} \varepsilon_x :=$ zero measure and $\mathcal{M}(X)$ (resp. $\mathcal{M}_p(X)$) denotes the set of all positive (resp. positive integer-valued) Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X such that all maps

$$\Gamma_X \ni \gamma \ \mapsto \ \langle f, \gamma \rangle := \int_X f(x)\gamma(dx) \ = \ \sum_{x \in \gamma} f(x) \tag{2.1}$$

are continuous. Here $f \in C_0(X)$ (:= the set of all continuous functions on X with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ-algebra.

In order to give another description of $\mathcal{B}(\Gamma_X)$, for any $\Lambda \in \mathcal{O}_c(X)$ and $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ one defines

$$\Gamma_\Lambda^{(n)} \ := \ \{\gamma \in \Gamma_\Lambda \mid |\gamma| = n\}, \ \ \Gamma_\Lambda^{(0)} \ := \ \{\emptyset\}.$$

Note that we have a bijection

$$\tilde{\Lambda}^n / S_n \ \longrightarrow \ \Gamma_\Lambda^{(n)},$$

where

$$\tilde{\Lambda}^n \ := \ \{(x_1, \ldots, x_n) \in \Lambda^n | x_k \neq x_j, k \neq j\}$$

and S_n is the permutation group over $(1, \ldots, n)$. This bijection defines a locally compact metrizable Hausdorff topology on $\Gamma_\Lambda^{(n)}$. Let $s_\Lambda^n : \tilde{\Lambda}^n \longrightarrow \Gamma_\Lambda^{(n)}$ be such that $s_\Lambda^n : (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma_\Lambda^{(n)}$.

It is obvious that

$$\Gamma_\Lambda = \bigcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}.$$

This space is equipped with the usual topology of (disjoint) unions of topological spaces and corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$.

For $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X)$ with $\Lambda_1 \subset \Lambda_2$ there are natural maps

$$p_{\Lambda_2,\Lambda_1} : \Gamma_{\Lambda_2} \mapsto \Gamma_{\Lambda_1}$$

defined by

$$p_{\Lambda_2,\Lambda_1}(\gamma) := \gamma_{\Lambda_1}, \ \gamma \in \Gamma_{\Lambda_2}.$$

It can be shown (cf. [Sh]) that $(\Gamma_X, \mathcal{B}(\Gamma_X))$ coincides with the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)), \Lambda \in \mathcal{O}_c(X)$, i.e., $\mathcal{B}(\Gamma_X)$ is the smallest σ-algebra on Γ_X such that all restriction mappings

$$\Gamma_X \ni \gamma \mapsto p_\Lambda \gamma = \gamma_\Lambda \in \Gamma_\Lambda, \Lambda \in \mathcal{O}_c(X),$$

are $\mathcal{B}(\Gamma_X)/\mathcal{B}(\Gamma_\Lambda)$-measurable. We have that

$$\mathcal{B}(\Gamma_X) \cap \Gamma_\Lambda = \mathcal{B}(\Gamma_\Lambda)$$

(cf. e.g. [Ob]). Note that hence for any positive measure μ on $\mathcal{B}(\Gamma_X)$ the family $\{\mu^\Lambda := \mu \circ p_\Lambda^{-1} | \Lambda \in \mathcal{O}_c(X)\}$ is consistent, i.e., for all $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(X), \Lambda_1 \subset \Lambda_2$

$$p^*_{\Lambda_2,\Lambda_1}(\mu^{\Lambda_2}) = \mu^{\Lambda_2} \circ p^{-1}_{\Lambda_2,\Lambda_1} = \mu^{\Lambda_1}. \tag{2.2}$$

Conversely, by a version of Kolmogorov's theorem for the projective limit space Γ_X any such family determines uniquely a measure μ on Γ_X such that $\mu^\Lambda = p^*_\Lambda \, \mu, \Lambda \in \mathcal{B}_c(X)$, see e.g. [Pa].

$\mathcal{B}(\Gamma_X)$ is generated by the sets

$$C_{\Lambda,n} := \{\gamma \in \Gamma_X | \, |\gamma \cap \Lambda| = n\}, \tag{2.3}$$

where $\Lambda \in \mathcal{O}_c(X), n \in \mathbb{Z}_+$ [VGG, Sh]. Note that for any $\Lambda \in \mathcal{B}(X)$ and all $n \in \mathbb{Z}_+$ the set $C_{\Lambda,n}$ is, indeed, a Borel subset of Γ_X. Sets of the form (2.3) are called cylinder sets.

For any $B \subset X$ we introduce a function $N_B : \Gamma_X \longrightarrow \mathbb{Z}_+$ such that

$$N_B(\gamma) \ = \ |\gamma \cap B|, \ \gamma \in \Gamma_X. \tag{2.4}$$

Then $\mathcal{B}(\Gamma_X)$ is the smallest σ-algebra on Γ_X such that all the functions $N_B, B \in \mathcal{B}_c(X)$, are measurable.

For any subset $Y \subset X$ we introduce the σ-algebra $\mathcal{B}_Y(\Gamma_X)(\subset \mathcal{B}(\Gamma_X))$ which is generated by all functions $N_B, B \subset Y, B \in \mathcal{B}_c(X)$.

2.2 Poisson measures

For the construction of a Poisson measure on Γ_X first of all we need to fix an intensity measure σ on the underlying manifold X. We take a density $\rho > 0$ m-a.s. such that $\rho^{1/2} \in H^{1,2}_{loc}(X)$ and put $\sigma(dx) = \rho(x)m(dx)$. Here $H^{1,2}_{loc}(X)$ denotes the local Sobolev space of order 1 in $L^2_{loc}(X;m)$. Then σ is a non-atomic Radon measure on X, in particular, $\sigma(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_c(X)$.

For any $n \in \mathbb{N}$ we introduce the product-measure $\sigma^{\otimes n}$ on $(X^n, \mathcal{B}(X^n))$. Clearly, $\sigma^{\otimes n}(X^n \setminus \tilde{X}^n) = 0$.

Let us fix $\Lambda \in \mathcal{O}_c(X)$. The measure $\sigma^{\otimes n}$ can be considered as a (finite) measure on $\tilde{\Lambda}^n$, and by $\sigma_{\Lambda,n} := \sigma^{\otimes n} \circ (s^n_\Lambda)^{-1}$ we denote the corresponding image measure on $\Gamma^{(n)}_\Lambda$ under s^n_Λ.

The so-called *Lebesgue-Poisson measure* on $\mathcal{B}(\Gamma_\Lambda)$ with intensity measure σ on $\mathcal{B}(\Gamma_\Lambda)$ is defined by

$$\lambda^\Lambda_\sigma \ := \ \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_{\Lambda,n}, \tag{2.5}$$

where $\sigma_{\Lambda,0} := \varepsilon_\emptyset$ on $\Gamma^{(0)}_\Lambda = \{\emptyset\}$, see e.g. [Do]. The measure λ^Λ_σ is a finite measure on $\mathcal{B}(\Gamma_\Lambda)$ such that $\lambda^\Lambda_\sigma(\Gamma_\Lambda) = e^{\sigma(\Lambda)}$. Then the measure

$$\pi^\Lambda_\sigma \ := \ e^{-\sigma(\Lambda)} \lambda^\Lambda_\sigma \tag{2.6}$$

is a probability measure on $\mathcal{B}(\Gamma_\Lambda)$. The measure π^Λ_σ has the following property: for any collection $B_1, \ldots, B_M \in \mathcal{B}(X)$, $B_k \cap B_j = \emptyset, k \neq j$, $B_k \subset \Lambda$, $k = 1, \ldots, M$ and $n_k \in \mathbb{Z}_+$, $k = 1, \ldots, M$,

$$\pi^\Lambda_\sigma(\{\gamma | \, |\gamma_{B_k}| = n_k, \ k = 1, \ldots, M\}) \ = \ \prod_{k=1}^{M} \frac{\sigma(B_k)^{n_k} e^{-\sigma(B_k)}}{n_k!}. \tag{2.7}$$

In other words, the functions N_{B_k} form a family of independent Poissonian random variables with mean values $\sigma(B_k)$ on the probability space $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda), \pi_\sigma^\Lambda)$. Using (2.5) it is not hard to check the consistency property of the family $\{\pi_\sigma^\Lambda | \Lambda \in \mathcal{O}_c(X)\}$ and thus obtain the existence of a unique probability measure π_σ on $\mathcal{B}(\Gamma_X)$ such that

$$\pi_\sigma^\Lambda \;=\; p_\Lambda^* \pi_\sigma, \; \Lambda \in \mathcal{O}_c(X),$$

see e.g. [Sh]. The measure π_σ is called the *Poisson measure* with intensity measure σ. Note that for π_σ the relation (2.7) is still true (without the assumption $B_k \subset \Lambda$ as long as $B_k \in \mathcal{B}_c(X)$).

Let us compute the Laplace transform of the measure π_σ. For a given $f \in C_0(X)$ we have $\mathrm{supp} f \subset \Lambda$ for some $\Lambda \in \mathcal{O}_c(X)$. Then

$$\langle f, \gamma \rangle \;=\; \langle f, \gamma_\Lambda \rangle, \; \gamma \in \Gamma_X,$$

and

$$\int_{\Gamma_X} e^{\langle f, \gamma \rangle} \, \pi_\sigma(d\gamma) \;=\; \int_{\Gamma_\Lambda} e^{\langle f, \gamma_\Lambda \rangle} \, \pi_\sigma^\Lambda(d\gamma_\Lambda).$$

Using (2.6) the latter is equal to

$$e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \exp\left(\sum_{k=0}^n f(x_k) \right) \sigma(dx_1) \dots \sigma(dx_n)$$

$$= \; e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \left(\int_\Lambda e^{f(x)} \sigma(dx) \right)^n \;=\; \exp\left(\int_X (e^{f(x)} - 1) \sigma(dx) \right).$$

That is, for all $f \in C_0(X)$ the following formula holds:

$$l_{\pi_\sigma}(f) \;:=\; \int_{\Gamma_X} e^{\langle f, \gamma \rangle} \pi_\sigma(d\gamma) \;=\; \exp\left(\int_X (e^{f(x)} - 1) \sigma(dx) \right). \qquad (2.8)$$

This formula can be considered as an equivalent definition of the Poisson measure π_σ, see e.g. [GV]. Let us mention in this connection that such an approach gives (via Minlos theorem) a measure π_σ on a linear space $F(X)$ of generalized functions on X. An additional analysis shows that the support of the measure π_σ consists of generalized functions of the form $\sum_{x \in \gamma} \varepsilon_x, \; \gamma \in \Gamma_X$, and then π_σ can be considered as a measure on Γ_X.

Remark 2.1 *By the same argument* (2.8) *holds for any* $\mathcal{B}(X)$*-measurable function* f *with compact support such that* e^f *is* σ*-integrable on* $\mathrm{supp} f$. *A simple limit-argument then implies that* (2.8) *holds for all* f *such that* $e^f - 1 \in L^1(\sigma)$.

2.3 The group of diffeomorphisms and Poisson measures

Let us denote the group of all diffeomorphisms of X by $\mathrm{Diff}(X)$ and by $\mathrm{Diff}_0(X)$ the subgroup of all diffeomorphisms $\phi : X \longrightarrow X$ with compact support, i.e., which are equal to the identity outside of a compact set (depending on ϕ).

Any $\phi \in \mathrm{Diff}_0(X)$ defines (pointwisely) a transformation of any subset of X and, consequently, the diffeomorphism ϕ has the following "lifting" from X to Γ_X:

$$\Gamma_X \ni \gamma \mapsto \phi(\gamma) \;=\; \{\phi(x) | x \in \gamma\} \in \Gamma_X. \tag{2.9}$$

(Note that for a given $\gamma \in \Gamma$ the configuration $\phi(\gamma)$ coincides with γ for all but a finite number of points). This mapping is obviously measurable and we can define the image $\phi^*\pi_\sigma$ of the measure π_σ under ϕ as usually by $\phi^*\pi_\sigma = \pi_\sigma \circ \phi^{-1}$, i.e.,

$$(\phi^*\pi_\sigma)(A) \;=\; \pi_\sigma(\phi^{-1}(A)),\ A \in \mathcal{B}(\Gamma_X).$$

The following well-known proposition shows that this transformation is nothing but a change of the intensity measure.

Proposition 2.1 *For any $\phi \in \mathrm{Diff}_0(X)$ we have*

$$\phi^*\pi_\sigma \;=\; \pi_{\phi^*\sigma}. \tag{2.10}$$

Proof. Due to Subsection 2.2 it is sufficient to compute the Laplace transform of the measure $\phi^*\pi_\sigma$. For any $f \in C_0(X)$ we have

$$\int_{\Gamma_X} e^{\langle f,\gamma\rangle}(\phi^*\pi_\sigma)(d\gamma) \;=\; \int_{\Gamma_X} e^{\langle f,\phi(\gamma)\rangle}\pi_\sigma(d\gamma)$$
$$=\; \int_{\Gamma_X} \exp\Big(\sum_{x\in\gamma} f(\phi(x))\Big)\pi_\sigma(d\gamma) \;=\; \int_{\Gamma_X} e^{\langle f\circ\phi,\gamma\rangle}\pi_\sigma(d\gamma)$$
$$=\; \exp\Big(\int_X (e^{(f\circ\phi)(x)} - 1)\sigma(dx)\Big) \;=\; \exp\left(\int_X (e^{f(x)} - 1)(\phi^*\sigma)(dx)\right)$$
$$=\; \int_{\Gamma_X} e^{\langle f,\gamma\rangle}\pi_{\phi^*\sigma}(d\gamma). \qquad \square$$

For any $\phi \in \mathrm{Diff}_0(X)$ we introduce the Radon–Nikodym density

$$
\begin{aligned}
p^\sigma_\phi(x) &:= \tfrac{d(\phi^*\sigma)(x)}{d\sigma(x)} = \tfrac{\rho(\phi^{-1}(x))}{\rho(x)}\tfrac{dm(\phi^{-1}(x))}{dm(x)} = \tfrac{\rho(\phi^{-1}(x))}{\rho(x)} J_m(\phi)(x)\ , \\
&\qquad \text{if} \quad x \in \{0<\rho<\infty\} \cap \{0 < \rho\circ\phi^{-1} < \infty\}; \qquad (2.11)\\
p^\sigma_\phi(x) &:= 1 \text{ otherwise,}
\end{aligned}
$$

where $J_m(\phi)$ is the Jacobian determinant of ϕ (with respect to the Riemannian volume m), see e.g. [Bo]. Note that $p^\sigma_\phi \equiv 1$ outside a compact.

The following proposition is a direct consequence of the previous one and Skorokhod's theorem on absolute continuity of Poisson measures [Sk, Ta], see also [Sh]. We include the simple proof for the convenience of the reader.

Proposition 2.2 *The Poisson measure π_σ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$, and for any $\phi \in \mathrm{Diff}_0(X)$ we have*

$$
\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma) = \prod_{x\in\gamma} p^\sigma_\phi(x) \exp\left(\int_X (1-p^\sigma_\phi(x))\sigma(dx)\right). \qquad (2.12)
$$

where p^σ_ϕ is as in (2.11).

Proof. For any $f \in C_0(X)$ we have

$$
\begin{aligned}
&\int_{\Gamma_X} e^{\langle f,\gamma\rangle} \prod_{x\in\gamma} p^\sigma_\phi(x) \exp(\int_X (1-p^\sigma_\phi)d\sigma)\pi_\sigma(d\gamma) \\
&= \int_{\Gamma_X} \exp(\langle f + \ln p^\sigma_\phi, \gamma\rangle)\pi_\sigma(d\gamma)\exp(\int_X(1-p^\sigma_\phi)d\sigma) \\
&= \exp(\int (e^f-1)p^\sigma_\phi d\sigma) = \exp(\int (e^f - 1)d(\phi^*\sigma))
\end{aligned}
$$

where we used Remark 2.1. Now the assertion follows from Proposition 2.1. □

3 The differential geometry of configuration spaces

The differentiable structure of the underlying manifold X has a natural lifting to the configuration space Γ_X. As a result we obtain objects as the gradient, the tangent space etc. associated with Γ_X. We describe below the corresponding constructions in detail using the approach proposed in our paper [AKR 1].

3.1 The tangent bundle of Γ_X

Let us denote the set of all C^∞-vector fields on X (i. e., smooth sections of $T(X)$) by $V(X)$. We will also consider the subset $V_0(X) \subset V(X)$ consisting of all vector fields with compact support. $V_0(X)$ can be considered as an infinite dimensional Lie algebra which corresponds to the group $\mathrm{Diff}_0(X)$ in the following sense: for any $v \in V_0(X)$ we can construct the flow of this vector field as the collections of mappings $\phi_t^v : X \longrightarrow X, t \in I\!R$ obtained by integrating this vector field.

More precisely, for any $x \in X$ the curve

$$I\!R \ni t \ \mapsto \ \phi_t^v(x) \in X$$

is defined as the solution to the following Cauchy problem

$$\begin{cases} \frac{d}{dt}\phi_t^v(x) \ = \ v(\phi_t^v(x)) \\ \phi_0^v(x) \ = \ x. \end{cases} \tag{3.1}$$

That no explosion is possible and ϕ_t^v is well-defined for each $t \in I\!R$, is a consequence of $v \in V_0(X)$ (the latter implies that v is a complete vector field). The mappings $\{\phi_t^v, t \in I\!R\}$ form a one-parameter subgroup of diffeomorphisms in the group $\mathrm{Diff}_0(X)$ (see e.g. [Bo]), that is,

$$\begin{array}{l} 1)\ \forall t \in I\!R \quad \phi_t^v \in \mathrm{Diff}_0(X) \\ 2)\ \forall t, s \in I\!R \quad \phi_t^v \circ \phi_s^v \ = \ \phi_{t+s}^v. \end{array} \tag{3.2}$$

Let us fix $v \in V_0(X)$. Having the group $\phi_t^v, t \in I\!R$, we can consider for any $\gamma \in \Gamma_X$ the curve

$$I\!R \ni t \ \mapsto \ \phi_t^v(\gamma) \in \Gamma_X.$$

Definition 3.1 *For a function $F : \ \Gamma_X \longrightarrow I\!R$ we define the directional derivative along the vector field v as*

$$(\nabla_v^\Gamma F)(\gamma) \ := \ \frac{d}{dt} F(\phi_t^v(\gamma))|_{t=0},$$

provided the right hand side exists.

Note that this definition is closely related to the concept of the Lie derivative in differential geometry. In fact, $\nabla_v^\Gamma F$ is the Lie derivative corresponding to a special class of vector fields on Γ_X (see below).

Let us introduce a special class of "nice" functions on Γ_X which play an important role in our considerations below. We set $\mathcal{D} = C_0^\infty(X)$ (i.e., the set of all C^∞-functions on X with compact support) and introduce $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ as the set of all functions $F : \Gamma_X \longrightarrow I\!R$ of the form

$$F(\gamma) \; = \; g_F(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle), \;\; \gamma \in \Gamma_X, \tag{3.3}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and $g_F \in C_b^\infty(I\!R^N)$.

Remark 3.1 *We will call a function $F : \Gamma_X \longrightarrow I\!R$ cylindrical if F is $\mathcal{B}_\Lambda(\Gamma_X)$-measurable for some $\Lambda \in \mathcal{B}_c(X)$. Such a function depends only on the value $\gamma \cap \Lambda, \gamma \in \Gamma_X$. Any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ is a cylinder function in this sense. $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ will be called the set of smooth cylinder functions on Γ_X.*

For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ of the form (3.3) and given $v \in V_0(X)$ we have

$$\begin{aligned} F(\phi_t^v(\gamma)) \; &= \; g_F(\langle\varphi_1, \phi_t^v(\gamma)\rangle, \ldots, \langle\varphi_N, \phi_t^v(\gamma)\rangle) \\ &= \; g_F(\langle\varphi_1 \circ \phi_t^v, \gamma\rangle, \ldots, \langle\varphi_N \circ \phi_t^v, \gamma\rangle) \end{aligned}$$

and, therefore, an application of Definition 3.1 gives

$$(\nabla_v^\Gamma F)(\gamma) \; = \; \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle) \cdot \langle\nabla_v^X \varphi_j, \gamma\rangle, \tag{3.4}$$

where $\nabla_v^X \varphi$ is the directional (or Lie) derivative of $\varphi : X \longrightarrow I\!R$ along the vector field $v \in V_0(X)$, i.e.,

$$(\nabla_v^X \varphi)(x) \; := \; \langle\nabla^X \varphi(x), v(x)\rangle_{T_x(X)},$$

where ∇^X denotes the gradient on X.

The expression of ∇_v^Γ on smooth cylinder functions given by (3.4) motivates the following definition.

Definition 3.2 *We introduce the tangent space $T_\gamma(\Gamma_X)$ to the configuration space Γ_X at the point $\gamma \in \Gamma_X$ as the Hilbert space of measurable γ-square-integrable sections (measurable vector fields) $V_\gamma : X \longrightarrow T(X)$ with the scalar product*

$$\langle V_\gamma^1, V_\gamma^2 \rangle_{T_\gamma(\Gamma_X)} \;=\; \int \langle V_\gamma^1(x), V_\gamma^2(x) \rangle_{T_x(X)} \gamma(dx) \tag{3.5}$$

$V_\gamma^1, V_\gamma^2 \in T_\gamma(\Gamma_X)$. *The corresponding tangent bundle is*

$$T(\Gamma_X) = \bigcup_{\gamma \in \Gamma_X} T_\gamma(\Gamma_X).$$

Let us stress that any $v \in V_0(X)$ can be considered as a "constant" vector field on Γ_X such that

$$\Gamma_X \ni \gamma \;\mapsto\; V_\gamma(\cdot) = v(\cdot) \in T_\gamma(\Gamma_X),$$
$$\langle v, v \rangle_{T_\gamma(\Gamma_X)} \;=\; \int_X |v(x)|^2_{T_x(X)} \gamma(dx) \;<\; \infty.$$

As usual in Riemannian geometry, having directional derivatives and a Hilbert space as the tangent space we can introduce the gradient.

Definition 3.3 *We define the intrinsic gradient of a function $F : \Gamma_X \longrightarrow \mathbb{R}$ as the mapping*

$$\Gamma_X \ni \gamma \;\mapsto\; (\nabla^\Gamma F)(\gamma) \in T_\gamma(\Gamma_X)$$

such that for any $v \in V_0(X)$

$$(\nabla_v^\Gamma F)(\gamma) \;=\; \langle \nabla^\Gamma F(\gamma), v \rangle_{T_\gamma(\Gamma_X)}. \tag{3.6}$$

Note that (3.6), in particular, implies that $\nabla_v^\Gamma F$ is the directional derivative along the "constant" vector field v on Γ_X. Furthermore, by (3.4) for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ of the form (3.3) we have

$$(\nabla^\Gamma F)(\gamma; x) \;=\; \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \nabla^X \varphi_j(x), \tag{3.7}$$
$$\gamma \in \Gamma_X, x \in X.$$

Let us clarify the terminology we have used. The point is that it is also possible to introduce another gradient on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$, related to an imbedding of Γ_X into a linear space. More precisely, we have the Schwartz type triplet

$$\mathcal{D} \subset L^2(X,m) \subset \mathcal{D}'$$

where $\mathcal{D} = C_0^\infty(X)$ is equipped with the usual topology (see e.g. [Au, Ch.2]). We have then an imbedding $\mathcal{M}(X) \subset \mathcal{D}'$, and for any function F on the linear space $\mathcal{D}'$ of type

$$F(\xi) = g_F(\langle\varphi_1,\xi\rangle_{L^2(m)},\dots,\langle\varphi_N,\xi\rangle_{L^2(m)}),\ \xi \in \mathcal{D}', \tag{3.8}$$

we obtain another "flat" gradient ∇^L considering the same Hilbert space $L^2(X,m)$ as the tangent space to $\mathcal{D}'$ at any point $\xi \in \mathcal{D}'$. More precisely, from (3.8) we have

$$(\nabla^L F)(\xi;x) = \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle\varphi_1,\xi\rangle_{L^2(m)},\dots,\langle\varphi_N,\xi\rangle_{L^2(m)})\varphi_j(x),\ \xi \in \mathcal{D}', x \in X.$$

This definition is easily checked to be independent of the representation for F in (3.8). Then (after restriction to $\Gamma_X \subset \mathcal{D}'$)

$$(\nabla^\Gamma F)(\gamma;x) = (\nabla^X \nabla^L F)(\gamma;x),$$

where ∇^X means the gradient on X of the function $x \mapsto (\nabla^L F)(\xi;x), x \in X$. This shows the analogy of the intrinsic gradient ∇^Γ and the so-called Malliavin gradient in stochastic analysis, see e.g. [Mal, Nu].

3.2 Integration by parts and divergence on Poisson space

.

Let the configuration space Γ_X be equipped with the Poisson measure π_σ (cf. Subsection 2.2). The set $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ is a dense subset in the space $L^2(\Gamma_X,\mathcal{B}(\Gamma_X),\pi_\sigma) := L^2(\pi_\sigma)$. For any $v \in V_0(X)$ we have a differential operator in $L^2(\pi_\sigma)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ given by

$$\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X) \ni F \mapsto \nabla_v^\Gamma F \in L^2(\pi_\sigma).$$

Our aim now is to compute the adjoint operator $\nabla_v^{\Gamma *}$ in $L^2(\pi_\sigma)$. It corresponds, of course, to an integration by parts formula with respect to the measure π_σ.

To this end we recall first of all the integration by parts formula for the measure σ. The logarithmic derivative of σ is given by the vector field

$$X \ni x \mapsto \beta^\sigma(x) := \frac{\nabla^X \rho(x)}{\rho(x)} \in T_x(X).$$

(where as usual $\beta^\sigma := 0$ on $\{\rho = 0\}$). For all $\varphi_1, \varphi_2 \in \mathcal{D}$ we have

$$\int_X (\nabla_v^X \varphi_1)(x)\varphi_2(x)\sigma(dx) =$$
$$-\int_X \varphi_1(x)(\nabla_v^X \varphi_2)(x)\sigma(dx) - \int_X \varphi_1(x)\varphi_2(x)\beta_v^\sigma(x)\sigma(dx), \tag{3.9}$$

where

$$\beta_v^\sigma(x) := \langle \beta^\sigma(x), v(x)\rangle_{T_x(X)} + \operatorname{div}^X v(x) \tag{3.10}$$

is the so-called logarithmic derivative of the measure σ along the vector field v and $\operatorname{div}^X := \operatorname{div}_m^X$ is the divergence on X w.r.t. m. Analogously, we define $\operatorname{div}_\sigma^X$ as the divergence on X w.r.t. σ, i.e., $\operatorname{div}_\sigma^X$ is the dual operator on $L^2(X,\sigma) =: L^2(\sigma)$ of ∇^X. Then on the one hand we can rewrite (3.9) as an operator equality on the domain $\mathcal{D} \subset L^2(\sigma)$:

$$\nabla_v^{X*} = -\nabla_v^X - \beta_v^\sigma, \tag{3.11}$$

where the adjoint operator is considered w.r.t. $L^2(\sigma)$. Note here that, obviously, $\beta_v^\sigma \in L^2(\sigma)$ for all $v \in V_0(X)$. On the other hand we have

$$\operatorname{div}_\sigma^X = \beta^\sigma. \tag{3.12}$$

Having the logarithmic derivative β_v^σ we introduce an analogous object for the Poisson measure.

Definition 3.4 *For any $v \in V_0(X)$ we define the logarithmic derivative of the Poisson measure π_σ along v as the following function on Γ_X:*

$$\Gamma_X \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) := \langle \beta_v^\sigma, \gamma\rangle$$
$$= \int_X [\langle \beta^\sigma(x), v(x)\rangle_{T_x(X)} + \operatorname{div}^X v(x)]\gamma(dx). \tag{3.13}$$

A motivation for this definition is given by the following integration by parts formula.

Theorem 3.1 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ and any $v \in V_0(X)$ we have*

$$\int_{\Gamma_X} (\nabla_v^\Gamma F)(\gamma) G(\gamma) \pi_\sigma(d\gamma) \;=\;$$

$$- \int_{\Gamma_X} F(\gamma)(\nabla_v^\Gamma G)(\gamma)\pi_\sigma(d\gamma) \;-\; \int_{\Gamma_X} F(\gamma)G(\gamma)B_v^{\pi_\sigma}(\gamma)\pi_\sigma(d\gamma), \qquad (3.14)$$

or

$$\nabla_v^{\Gamma *} \;=\; -\nabla_v^\Gamma - B_v^{\pi_\sigma}(\gamma), \qquad (3.15)$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ in $L^2(\pi_\sigma)$.

Proof. By Proposition 2.1 we have that

$$\int_{\Gamma_X} F(\phi_t^v(\gamma))G(\gamma)\pi_\sigma(d\gamma) \;=\; \int_{\Gamma_X} F(\gamma)G(\phi_{-t}^v(\gamma))\pi_{\phi_t^{v*}\sigma}(d\gamma).$$

Differentiating this equation w.r.t. t and interchanging d/dt with the integrals, by Definition 3.1 the l.h.s. becomes the l.h.s. of (3.14). To see that the r.h.s. then also coincides with the r.h.s. of (3.14) we note that

$$\frac{d}{dt} G(\phi_{-t}^v(\gamma))|_{t=0} \;=\; -(\nabla_v^\Gamma G)(\gamma)$$

and (by Proposition 2.2)

$$\frac{d}{dt}\left[\frac{d\pi_{\phi_t^{v*}\sigma}}{d\pi_\sigma}(\gamma)\right]\Bigg|_{t=0} \;=\; \frac{d}{dt}\left[\prod_{x\in\gamma} \frac{\rho((\phi_t^v)^{-1}(x))}{\rho(x)} J_m(\phi_t^v)(x)\right]\Bigg|_{t=0}$$
$$+\; \frac{d}{dt}\left[\exp\left\{\int_X \left(1 - \frac{\rho((\phi_t^v)^{-1}(x))}{\rho(x)} J_m(\phi_t^v)(x)\right)\sigma(dx)\right\}\right]\Bigg|_{t=0}.$$

Using (3.10) and the formula $\frac{d}{dt}[J_m(\phi_t^v)]|_{t=0} = -\mathrm{div}^X v(x)$, the latter expressions becomes equal to

$$-\sum_{x\in\gamma}[\langle \beta^\sigma(x), v(x)\rangle_{T_x(X)} + \mathrm{div}^X v(x)]$$
$$+\; \int_X [\langle \beta^\sigma(x), v(x)\rangle_{T_x(X)} + \mathrm{div}^X v(x)]\sigma(dx)$$

$$= -\sum_{x\in\gamma}\beta_v^\sigma(x) + \int_X \beta_v^\sigma(x)\sigma(dx) \;=\; -B_v^{\pi_\sigma}(\gamma),$$

where we have used the equality

$$\int_X \beta_v^\sigma(x)\sigma(dx) \;=\; -\int_X (\nabla_v^{X*}1)(x)\sigma(dx) \;=\; 0.$$

This completes the proof.

$\square$

Remark 3.2 (i) *Let us distinguish the case of the so-called canonical Poisson measure on Γ_X which corresponds to $\rho \equiv 1$, i.e., $\sigma = m$. This case is geometrically of special interest because all the objects we have introduced are generated by the original Riemannian structure of X. In particular, for π_m formula* (3.15) *becomes*

$$\nabla_v^{\Gamma *} \;=\; -\nabla_v^\Gamma - \langle \mathrm{div}^X v, \gamma\rangle. \tag{3.16}$$

Recall that in statistical physics Poissonian white noise measures for $X = \mathbb{R}^d$ and $\sigma = zm$ (m = Lebesgue measure), describe the Gibbs state of the free gas in $\mathbb{R}^d$ with activity $z > 0$.

(ii) As it follows from (3.14)*, the logarithmic derivative $B_v^{\pi_\sigma} : \Gamma_X \longrightarrow \mathbb{R}$ of π_σ is obtained via a "lifting" of the corresponding object on the underlying manifold X (namely $\beta_v^\sigma : X \longrightarrow \mathbb{R}$) to the configuration space Γ_X since*

$$B_v^{\pi_\sigma}(\gamma) \;=\; \langle \beta_v^\sigma, \gamma\rangle, \quad \gamma\in\Gamma_X.$$

We shall comment on this more in Subsection 3.4 *below.*

Definition 3.5 *For a vector field*

$$V : \Gamma_X \ni \gamma \;\mapsto\; V_\gamma \in T_\gamma(\Gamma_X)$$

the divergence $\mathrm{div}_{\pi_\sigma}^\Gamma V$ *is defined via the duality relation*

$$\int \langle V_\gamma, \nabla^\Gamma F(\gamma)\rangle_{T_\gamma(\Gamma_X)}\pi_\sigma(d\gamma) \;=\; -\int_{\Gamma_X} F(\gamma)(\mathrm{div}_{\pi_\sigma}^\Gamma V)(\gamma)\pi_\sigma(d\gamma) \tag{3.17}$$

for all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$, provided it exists (i.e., provided

$$F \mapsto \int \langle V_\gamma, \nabla^\Gamma F(\gamma)\rangle_{T_\gamma(\Gamma_X)}\pi_\sigma(d\gamma)$$

is continuous on $L^2(\pi_\sigma)$).

The existence of the divergence, of course, requires some smoothness of the vector field. A class of smooth vector fields on Γ_X for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.1 *For any vector field*

$$V_\gamma(x) \;=\; \sum_{j=1}^{N} G_j(\gamma)v_j(x),\;\; \gamma\in\Gamma_X, x\in X, \tag{3.18}$$

with $G_j\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X), v_j\in V_0(X), j=1,\dots,N,$ *we have*

$$\begin{aligned}(\mathrm{div}^\Gamma_{\pi_\sigma}V)(\gamma) \;&=\; \sum_{j=1}^{N}(\nabla^\Gamma_{v_j}G_j)(\gamma)+\sum_{j=1}^{N}B^{\pi_\sigma}_{v_j}(\gamma)G_j(\gamma)\\ &=\; \sum_{j=1}^{N}\langle\nabla^\Gamma G_j(\gamma),v_j\rangle_{T_\gamma(\Gamma_X)}+\sum_{j=1}^{N}\langle\beta^\sigma_{v_j},\gamma\rangle G_j(\gamma). \end{aligned}\tag{3.19}$$

Proof. Due to the linearity of ∇^Γ it is sufficient to consider the case $N=1$, i.e., $V_\gamma(x)=G(\gamma)v(x)$. Then for all $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$

$$\begin{aligned}-\int_{\Gamma_X}\langle V_\gamma,\nabla^\Gamma F(\gamma)\rangle_{T_\gamma(\Gamma_X)}\pi_\sigma(d\gamma) \;&=\; -\int_{\Gamma_X}G(\gamma)\langle v,\nabla^\Gamma F(\gamma)\rangle_{T_\gamma(\Gamma_X)}\pi_\sigma(d\gamma)\\ =\; -\int_{\Gamma_X}G(\gamma)\nabla^\Gamma_v F(\gamma)\pi_\sigma(d\gamma) \;&=\; -\int(\nabla^{\Gamma*}_v G)(\gamma)F(\gamma)\pi_\sigma(d\gamma)\\ =\; \int_{\Gamma_X}(\nabla^\Gamma_v G)(\gamma)F(\gamma)\pi_\sigma(d\gamma)&+\int_{\Gamma_X}B^{\pi_\sigma}_v(\gamma)G(\gamma)F(\gamma)\pi_\sigma(d\gamma),\end{aligned}$$

where we have used (3.15). Hence

$$\begin{aligned}\mathrm{div}^\Gamma_{\pi_\sigma}(V)(\gamma) \;&=\; \nabla^\Gamma_v G(\gamma)+B^{\pi_\sigma}_v(\gamma)G(\gamma)\\ &=\; \langle\nabla^\Gamma G(\gamma),v\rangle_{T_\gamma(\Gamma_X)}+\langle\beta^\sigma_v,\gamma\rangle G(\gamma).\end{aligned}\tag{3.20}$$

□

Remark 3.3 *Note that we have determined* $\mathrm{div}^\Gamma_{\pi_\sigma}$ *only for the class of vector fields* V *in* (3.18). *Nevertheless, extending the definition of* B^{π_σ} *in* (3.13) *analogously to* (3.10) *to such* $V=\sum_{j=1}^n G_j\otimes v_j$ *by*

$$B^{\pi_\sigma}_V(\gamma) \;:=\; \sum_{j=1}^{n}G_j(\gamma)\langle\mathrm{div}^X_\sigma v,\gamma\rangle+\sum_{j=1}^{n}\nabla^\Gamma_{v_j}G_j(\gamma) \tag{3.21}$$

we obtain the analogue of (3.12), *i.e.,*

$$\operatorname{div}^{\Gamma}_{\pi_\sigma} \ = \ B^{\pi_\sigma}_{\cdot} \ .$$

In particular, if $v \in V_0(X)$*, it follows for the "constant" vector field* $V_\gamma := v, \gamma \in \Gamma$*, on* Γ_X *that*

$$\operatorname{div}^{\Gamma}_{\pi_\sigma} v(\gamma) \ = < \operatorname{div}^X_\sigma v, \gamma > . \tag{3.22}$$

3.3 Integration by parts characterization

It is well-known that the volume element m of our Riemannian manifold X is the up to a constant unique (positive) Radon measure μ on X such that div^X_m is the dual operator on $L^2(\mu)$ of ∇^X when considered on $V_0(X)$ resp. $C_0^\infty(X)$ (cf. e.g. [Ch]). We now want to investigate the analogous question for the Poisson measure π_m or more generally π_σ on Γ_X. Before we can formulate and prove the corresponding theorem we need to recall the following notion (cf. e.g. [Ge, NZ]): let λ be a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. The measure

$$\mu_{\lambda,\sigma} \ := \ \int_{\mathbb{R}_+} \pi_{s\sigma} \ \lambda(ds) \tag{3.23}$$

on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ is called *mixed Poisson measure* (or *process*). Here $\pi_{0\sigma}$ denotes the Dirac measure on Γ_X with mass in $\gamma = \emptyset$.

For $l \in \mathbb{N}$ let $\mathcal{M}_l(\Gamma_X)$ denote the set of all probability measures μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ such that

$$\int |\langle \varphi, \gamma \rangle|^l \mu(d\gamma) \ < \ \infty \quad \text{for all} \quad \varphi \in \mathcal{D}. \tag{3.24}$$

Obviously, $\mu_{\lambda,\sigma} \in \mathcal{M}_l(\Gamma_X)$ if and only if

$$\int_{\mathbb{R}_+} s^l \ \lambda(ds) \ < \ \infty. \tag{3.25}$$

Remark 3.4 *Clearly,* $\mu_{\lambda_1,\sigma} = \mu_{\lambda_2,\sigma}$ *implies* $\lambda_1 = \lambda_2$*. Hence for every* $l \in \mathbb{N}$ *trivially* $\pi_{s\sigma}, s \geq 0$*, are exactly the extreme points of the convex set of all measures in* $\mathcal{M}_l(\Gamma_X)$ *of type* (3.23).

For abbreviation we define $(IbP)^\sigma$ to be the set of all $\mu \in \mathcal{M}_1(\Gamma_X)$ which satisfy (3.14) with μ replacing π_σ for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X), v \in V_0(X)$. Obviously, since ∇_v^Γ obeys the product rule for all $v \in V_0$, we may always take $G \equiv 1$. Furthermore, $(IbP)^\sigma$ is convex.

Theorem 3.2 *Suppose that $\rho(= d\sigma/dm)$ is C^∞, strictly positive and $\sigma(X) = \infty$. Then the following assertions are equivalent:*

(i) $\mu \in (IbP)^\sigma$.

(ii) $\mu = \mu_{\lambda,\sigma}$ *for some probability measure* λ *on* $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *satisfying* (3.25) *with* $l = 1$.

Remark 3.5 *As we shall see below the proof of* "(ii) $\Rightarrow$ (i)" *does not require the smoothness resp. positivity assumption on* ρ. *The proof of* "(i) $\Rightarrow$ (ii)" *is a modification and simplication of the proof of* [GGPS, Theorem 7]. *The latter is the analogue of Theorem* 3.2, "(i) $\Rightarrow$ (ii)", *in the special case* $X = \mathbb{R}^3, \sigma = m$, *and provided* $\int \langle \varphi, \gamma \rangle^2 \mu(d\gamma) < \infty$ *for all* $\varphi \in \mathcal{D}$ *(stated, however, in a "different language"). This proof depends in an essential way on the fact that we work with the Poisson measures* $\pi_{s\sigma}, s \geq 0$, *i.e., we know their Laplace resp. Fourier transforms explicitly.*

We shall describe the said modifications and simplifications below. We only mention here that the way we deduce Theorem 3.2 "(i) $\Rightarrow$ (ii)" *from Proposition* 3.2 *is considerably easier than the corresponding step in* [GGPS].

In our forthcoming paper [AKR 5] *a much more general result is proved. More precisely, we prove an analogue of Theorem* 3.2 *above for Gibbs measures with interaction (whereas the mixed Poisson measures above correspond to Gibbs measures without interaction, cf.* [NZ]). *The proof is completely different and yields e.g. a generalization of Theorem* 3.2 *above and Corollary* 3.1 *below. More precisely, the two additional conditions on* ρ *can be relaxed to the much weaker assumption that* $\beta_v^\sigma \in L^1_{loc}(m)$ *for all* $v \in V_0(X)$.

Proof of Theorem 3.2, (ii) $\Rightarrow$ (i). Note first that $\pi_{0\sigma} \in (IbP)^\sigma$. Furthermore, since $\beta^{s\sigma} = \beta^\sigma$, it follows from (3.13) that $B^{\pi_\sigma} = B^{\pi_{s\sigma}}$. Hence $\pi_{s\sigma} \in (IbP)^\sigma$ for all $s \in]0, \infty[$ and, therefore, so does any $\mu_{\lambda,\sigma}$ of type (3.23) by Fubini's theorem.

For the proof of the converse we need some preparations. So, fix $\mu \in (IbP)^\sigma$. We consider its Fourier transform

$$L_\mu(\varphi) := \int_\Gamma e^{i\langle\varphi,\gamma\rangle}\mu(d\gamma), \quad \varphi \in \mathcal{D},$$

and also the functional

$$K_\sigma(\varphi) := \int_X (e^{i\varphi} - 1)d\sigma, \quad \varphi \in \mathcal{D}.$$

We shall use the following modification of a beautiful result in [GGPS] (cf. Theorem 6 ibidem).

Proposition 3.2 *There exist a function $F_\mu : \mathbb{C} \to \mathbb{C}$ holomorphic on the interior of $\{K_\sigma(\varphi)|\varphi \in \mathcal{D}\}$, i.e., on $\mathcal{K}_- := \{z \in \mathbb{C}| Re\, z < 0\}$, such that*

$$L_\mu(\varphi) = F_\mu(K_\sigma(\varphi)) \quad \textit{for all} \quad \varphi \in \mathcal{D}.$$

We come back to the proof of Proposition 3.2 below.

Proof of Theorem 3.2, (i) $\Rightarrow$ (ii). By Proposition 3.2

$$|F_\mu(K_\sigma(\varphi))| = |L_\mu(\varphi)| \leq 1.$$

Hence $|F_\mu| \leq 1$ on $\mathcal{K}_-$, and $F_\mu(0) = L(0) = 1$. We want to show that for all $z_1, \dots, z_n \in \mathcal{K}_-$,

$$\sum_{k,j=1;k\neq j}^{N} F_\mu(z_k + \bar{z}_j)\xi_k\bar{\xi}_j + \sum_{k=1}^{N} F_\mu(0)|\xi_k|^2 \geq 0 \quad \text{for all} \quad \xi_1, \dots, \xi_N \in \mathbb{C}. \tag{3.26}$$

Let $\varphi_1, \dots, \varphi_N \in \mathcal{D}$ with pairwise disjoint supports and set

$$z_k := K_\sigma(\varphi_k), \quad 1 \leq k \leq N.$$

Since there exist N pairwise disjoint subsets of X each single one having arbitrary large mass under σ, points of this type are dense in $\mathcal{K}_-$. Because of the continuity of K_σ on $\mathcal{K}_-$, it suffices to prove (3.26) for such points.

Since for $1 \leq j, k \leq N, j \neq k$,

$$K_\sigma(\varphi_k) + \bar{K}_\sigma(\varphi_j) = K_\sigma(\varphi_k - \varphi_j)$$

(because $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset$ if $j \neq k$), we obtain that the left hand side of (3.26) is equal to

$$\sum_{k,j=1}^{N} L_\mu(\varphi_k - \varphi_j)\xi_k\bar{\xi}_j \;\geq\; 0,$$

and (3.26) is proved.

For any $\varepsilon > 0$ we introduce the function

$$G_\varepsilon(y) \;:=\; F_\mu(-2\varepsilon + iy),\; y \in \mathbb{R}.$$

Then (3.26) implies that for all $y_1, \ldots, y_N \in \mathbb{R}$ and $\xi_1, \ldots, \xi_N \in \mathbb{C}$:

$$\sum_{k,j=1;k\neq j}^{N} G_\varepsilon(y_k - y_j)\xi_k\bar{\xi}_j + \sum_{k=1}^{N} F_\mu(0)|\xi_k|^2 \geq 0.$$

Now we take any $\xi \in C_0(\mathbb{R})$. Then $\operatorname{supp}\xi \subset [-a, a]$ for some $a > 0$. We set $\xi_k := \xi(y_k)$ in the latter formula. Integrating with respect to $y_k \in [-a, a]$, $k = 1, \ldots, N$, each term of the second sum transforms into

$$(2a)^{N-1}F_\mu(0)\int_{-a}^{a}|\xi(t)|^2 dt,$$

while each term of the first sum becomes

$$(2a)^{N-2}\int_{-a}^{a}\int_{-a}^{a} G_\varepsilon(t-s)\xi(t)\bar{\xi}(s)dtds\;.$$

Hence we have

$$N(2a)^{N-1}F_\mu(0)\int_{\mathbb{R}}|\xi(t)|^2 dt + N(N-1)(2a)^{N-2}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\varepsilon(t-s)\xi(t)\bar{\xi}(s)dtds \geq 0.$$

Dividing by $N(N-1)(2a)^{N-2}$, and then letting $N \to \infty$, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} G_\varepsilon(t-s)\xi(t)\bar{\xi}(s)dtds \geq 0,$$

for any $\xi \in C_0(\mathbb{R})$. I.e., the bounded continuous function G_ε is positive definite in the integral sense. Therefore, there exists a finite measure λ_ε on $\mathcal{B}(\mathbb{R})$ such that for all $y \in \mathbb{R}$

$$G_\varepsilon(y) = \int_{\mathbb{R}} e^{isy}\lambda_\varepsilon(ds).$$

This gives

$$F_\mu(-2\varepsilon + iy) = \int_{\mathbb{R}} e^{isy}\lambda_\varepsilon(ds),\ \ y \in \mathbb{R}.$$

Since F_μ is a bounded holomorphic function in the domain $\{z \in \mathbb{C} | \mathrm{Re} z < 2\varepsilon\}$ extending G_ε, it follows that $\mathrm{supp}\lambda_\varepsilon \subset \mathbb{R}_+$ and that

$$F_\mu(-2\varepsilon - x + iy) = \int_{\mathbb{R}_+} e^{-sx+isy}\lambda_\varepsilon(ds), x > -2\varepsilon$$

(see e.g. [Ra, Theorem 2.2.2]). Now, let us consider the points $z = -x + iy$, $x > 2\varepsilon > 2\varepsilon' > 0$. Using the above representation we get

$$F_\mu(-x + iy) = \int_{\mathbb{R}_+} e^{s(2\varepsilon - x)+isy}\lambda_\varepsilon(ds) = \int_{\mathbb{R}_+} e^{s(2\varepsilon' - x)+isy}\lambda_{\varepsilon'}(ds).$$

The latter shows that the measure $e^{2\varepsilon s}\lambda_\varepsilon(ds)$ does not depend on $\varepsilon > 0$. We denote this measure by λ. Then obviously

$$F_\mu(z) = \int_{\mathbb{R}_+} e^{sz}\lambda(ds)$$

for all $z \in \mathcal{K}_-$. The measure λ is a finite measure, since

$$\lambda(\mathbb{R}_+) \ = \ \lim_{\varepsilon \nearrow 0} \int_{\mathbb{R}_+} e^{-\varepsilon s}\lambda(ds) \ \le \ \sup_{z \in \mathcal{K}_-} |F(z)| \ \le \ 1.$$

By Lebesgue's dominated convergence theorem the function F_μ then has a continuous boundary value on the imaginary axis given by

$$F_\mu(iy) = \int_{\mathbb{R}_+} e^{isy}\lambda(ds),\ \ \ y \in \mathbb{R}.$$

The condition $F_\mu(0) = 1$ shows $\lambda(\mathbb{R}_+) = 1$.

Therefore,

$$L_\mu(\varphi) \ = \ F_\mu(K_\sigma(\varphi)) \ = \ \int_0^\infty \exp(s \int (e^{i\varphi} - 1)d\sigma)\lambda(ds)$$

for all $\varphi \in \mathcal{D}$.

$\square$

Now let us return to the proof of Proposition 3.2. It can be reduced exactly as in [GGPS] to proving the following two lemmas.

Lemma 3.1 *Let*

$$[0,1] \ni t \ \mapsto \ g_t \in \mathcal{D}$$

be a continuous map, differentiable on $(0,1)$*, such that* $t \mapsto K_\sigma(g_t)$ *is constant on* $[0,1]$*, then so is* $t \mapsto L_\mu(g_t)$*.*

Lemma 3.2 *Let* $\varphi_1, \varphi_2 \in \mathcal{D}$ *such that*

$$K_\sigma(\varphi_1) \ = \ K_\sigma(\varphi_2).$$

Then for all neighbourhoods $U_j \subset \mathcal{D}$ *of* φ_j *there exist* $h_j \in U_j, j = 1,2,$ *and a continuous map*

$$[0,1] \ni t \ \mapsto \ g_t \in \mathcal{D}$$

differentiable on $(0,1)$ *such that* $g_0 = h_1, g_1 = h_2$ *and* $t \mapsto K_\sigma(g_t)$ *is constant on* $[0,1]$*.*

The proof of Lemma 3.2 is again exactly as in [GGPS], the one of Lemma 3.1 requires some modifications.

Proof of Lemma 3.1. Since $\mu \in (IbP)^\sigma$, it follows that

$$\int e^{i\langle\varphi,\gamma\rangle}[\langle \mathrm{div}^X v, \gamma\rangle + \langle \beta_v^\sigma, \gamma\rangle + i\langle \nabla_v^X \varphi, \gamma\rangle]\mu(d\gamma) \ = \ 0 \tag{3.27}$$

for all $\varphi \in \mathcal{D}, v \in V_0(X)$. We can realize the dual space of $\mathcal{D}$ via

$$\mathcal{D} \ \subset \ L^2(\sigma) \ \subset \ \mathcal{D}'_\sigma.$$

Fix $g \in \mathcal{D}$ and define $T_g \in \mathcal{D}'$ by

$$T_g(\varphi) \ := \ \int e^{i\langle g,\gamma\rangle}\langle\varphi,\gamma\rangle\mu(d\gamma), \ \ \varphi \in \mathcal{D}.$$

Then by (3.27)

$$T_g(\mathrm{div}^X v + \langle \beta^\sigma, v\rangle_{T(X)} + i\langle \nabla^X g, v\rangle_{T(X)}) \ = \ 0 \tag{3.28}$$

for all $v \in V_0(X)$.

Define $\tilde{T}_g \in \mathcal{D}'_\sigma$ by

$$\tilde{T}_g(\varphi) \ = \ T_g(e^{-ig}\varphi), \ \ \varphi \in \mathcal{D}.$$

Then for all $w \in V_0(X)$

$$\tilde{T}_g(\mathrm{div}_\sigma^X w) \;=\; T_g(e^{-ig}\mathrm{div}^X w + e^{-ig}\beta_w^\sigma)$$
$$=\; T_g(\mathrm{div}^X(we^{-ig}) + i\langle\nabla^X g, e^{-ig}w\rangle_{T(X)} + \langle\beta^\sigma, e^{-ig}w\rangle_{T(X)} \;=\; 0,$$

where the last equality follows from (3.28). Hence there exists $c_g \in \mathbb{C}$ such that

$$T_g(\varphi) \;=\; c_g \int_X e^{ig}\varphi d\sigma \quad \text{for all} \quad \varphi \in \mathcal{D}.$$

Consequently,

$$\frac{dL_\mu(g_t)}{dt} \;=\; i\int e^{i\langle g_t,\gamma\rangle}\langle\frac{d}{dt}g_t,\gamma\rangle\mu(d\gamma)$$
$$=\; iT_{g_t}(\frac{d}{dt}g_t) \;=\; ic_{g_t}\int_X e^{ig_t}\frac{d}{dt}g_t\,d\sigma = ic_{g_t}\frac{d}{dt}K_\sigma(g_t) = 0$$

by the assumption on K_σ.

□

Corollary 3.1 *Suppose that ρ $(= d\sigma/dm)$ is C^∞, strictly positive and $\sigma(X) = \infty$. Then:*

(i) *The extreme points of $(IbP)^\sigma$ are exactly $\pi_{s\sigma}, s \geq 0$.*

(ii) *Suppose $\mu \in (IbP)^\sigma$ and let $\mathcal{M}_\mu$ be the set of all $\mathcal{B}(\Gamma_X)$-measurable, μ-integrable functions $g : \Gamma_X \to \mathbb{R}_+$ such that $(\int g d\mu)^{-1}g\mu \in (IbP)^\sigma$. Then $\mathcal{M}_\mu$ is a convex cone stable under taking finite infima resp. suprema.*

(iii) *Let $\mu \in (IbP)^\sigma$. Then μ is extreme if and only if*

$$\{\nu \in (IbP)^\sigma | \nu \text{ is absolutely continuous w.r.t. } \mu\} \;=\; \{\mu\}.$$

Proof. (i): The assertion is an immediate consequence of Theorem 3.2 and Remark 3.4.

To prove (ii), (iii) we note that by Theorem 3.2, it follows from [NZ, Theorem 1] (see also [Ge, (4.29)]) that $(IbP)^\sigma$ is a convex subset of the convex set of Gibbs measures $\mathcal{G}$ corresponding to a specification such that the extreme points of $(IbP)^\sigma$ coincide with the extreme points of $\mathcal{G}$. Then

(ii) is an immediate consequence of [P, Lemma 2.4]. Likewise (iii) is then a consequence of (ii) and [P, Lemma 2.4 and Theorem 2.1 (1)].

□

3.4 A lifting of the geometry

We have constructed in the consideration above some objects related to the differential geometry of the configuration space Γ_X. Now we would like to present an interpretation of the formulas obtained in Subsections 3.1–3.3 via a simple "lifting rule".

Any function $\varphi \in \mathcal{D}$ generates a (cylinder) function on Γ_X by the formula

$$L_\varphi(\gamma) \;:=\; \langle \varphi, \gamma \rangle, \;\; \gamma \in \Gamma_X. \tag{3.29}$$

We will call L_φ the lifting of φ. As before any vector field $v \in V_0(X)$ can be considered as a vector field on Γ_X (the lifting of v) which we denote by L_v, see Definition 3.2. For $v, w \in V_0(X)$ formula (3.5) can be written as

$$\langle L_v, L_w \rangle_{T_\gamma(\Gamma_X)} \;=\; L_{\langle v,w \rangle_{T(X)}}(\gamma), \tag{3.30}$$

i.e., the scalar product of lifted vector fields is computed as the lifting of the scalar product $\langle v, w \rangle_{T_x(X)} = \varphi(x)$. This rule can be used as a definition of the tangent space $T_\gamma(\Gamma_X)$.

Formula (3.4) has now the following interpretation:

$$(\nabla_v^\Gamma L_\varphi)(\gamma) \;=\; L_{\nabla_v^X \varphi}(\gamma), \; \gamma \in \Gamma_X, \tag{3.31}$$

and the gradient of L_φ is nothing but the lifting of the corresponding underlying gradient:

$$(\nabla^\Gamma L_\varphi)(\gamma) \;=\; L_{\nabla_\bullet^X \varphi}(\gamma) \; (= \nabla^X \varphi). \tag{3.32}$$

As follows from (3.13) the logarithmic derivative $B_v^{\pi_\sigma} : \Gamma_X \longrightarrow I\!R$ is obtained via the same lifting procedure of the corresponding logarithmic derivative $\beta_v^\sigma : X \longrightarrow I\!R$, namely,

$$B_v^{\pi_\sigma}(\gamma) \;=\; L_{\beta_v^\sigma}(\gamma).$$

Or, equivalently, one has for the divergence of a lifted vector field:

$$\mathrm{div}_{\pi_\sigma}^\Gamma(L_v) \;=\; L_{div_\sigma^X v}. \tag{3.33}$$

We underline that by (3.31) and (3.32) one recovers the action of ∇_v^Γ and ∇^Γ on all of $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ algebraically from requiring the product or the chain rule to hold. Also the action of $\mathrm{div}_{\pi_\sigma}^\Gamma$ on more general cylindrical vector fields follows as in Remark 3.3 if one assumes the usual product rule for $\mathrm{div}_{\pi_\sigma}^\Gamma$ to hold.

We stress that formulas (3.30)–(3.33) are just simple consequences of the formulas we have obtained before. But we can also use them as an *independent* definition of a differentiable structure on Γ_X which uses the differential geometry of X and the liffting rule we have proposed.

We remark that these formulas appear directly from related underlying formulas on X without refering to the Poisson measure π_σ.

The results of the previous subsection identify the mixed Poisson measures $\mu_{\lambda,\sigma}$ (cf. (3.23), (3.24)) as exactly those probability measures on Γ_X so that ∇^Γ and $\mathrm{div}_{\pi_\sigma}^\Gamma$ (the latter now being defined independently of π_σ) are dual operator on $L^2(\mu_{\lambda,\sigma})$. In view of the characterization result on the volume element m on X mentioned in the beginning of Subsection 3.3, this means that the mixed Poisson measures $\mu_{\lambda,\sigma}$ can be considered as exactly the "volume elements" of our Riemannian structure on Γ_X.

3.5 Representations of the Lie algebra of vector fields

Using the property of quasi-invariance of the Poisson measure π_σ we can define a unitary representation of the diffeomorpism group $\mathrm{Diff}_0(X)$ in the space $L^2(\pi_\sigma)$, see [VGG]. Namely, for $\phi\in\mathrm{Diff}_0(X)$ we define a unitary operator

$$(V_{\pi_\sigma}(\phi)F)(\gamma) \ := \ F(\phi(\gamma))\sqrt{\frac{d\pi_\sigma(\phi(\gamma))}{d\pi_\sigma(\gamma)}}, \quad F\in L^2(\pi_\sigma).$$

Then we have

$$V_{\pi_\sigma}(\phi_1)V_{\pi_\sigma}(\phi_2)=V_{\pi_\sigma}(\phi_2\circ\phi_1),\ \phi_1,\phi_2\in\mathrm{Diff}_0(X). \tag{3.34}$$

As in Subsection 3.1, to any vector field $v\in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms $\phi_t^v, t\in I\!R$. It generates a one-parameter unitary group

$$V_{\pi_\sigma}(\phi_t^v):=\exp[itJ_{\pi_\sigma}(v)],\ t\in I\!R,$$

where $J_{\pi_\sigma}(v)$ denotes the self-adjoint generator of this group.

Proposition 3.3 *For any $v \in V_0(X)$ the following operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ holds:*

$$J_{\pi_\sigma}(v) \;=\; \frac{1}{i}\nabla_v^\Gamma \;+\; \frac{1}{2i}B_v^{\pi_\sigma}. \tag{3.35}$$

Proof. This equality follows immediately from the definition of the directional derivative ∇_v^Γ, Theorem 3.1 and the form of the operator $V_{\pi_\sigma}(\phi_t^v)$.
□

Remark 3.6 *In non-relativistic quantum field theory the operators $J_{\pi_\sigma}(v)$, $v \in V_0(X)$, are known as momentum density operators, see e.g.* [GGPS]. *More generally, one can study a family of self-adjoint operators $J(v)$, $v \in V_0(X)$, in a Hilbert space $\mathcal{H}$ which gives a representation of the Lie algebra $V_0(X)$ in the sense of the following commutation relation:*

$$[J(v_1), J(v_2)] = -iJ([v_1, v_2]) \tag{3.36}$$

(*on a dense domain in $\mathcal{H}$*), *where $[v_1, v_2]$ is the Lie-bracket of the vector fields $v_1, v_2 \in V_0(X)$.*

In the case discussed, this relation is a direct consequence of (3.35). *Thus, we have constructed a Poisson space representation of the Lie algebra $V_0(X)$.*

Let us define, in addition, a unitary representation of the additive group $\mathcal{D}$ given by the formula

$$(U_{\pi_\sigma}(f)F)(\gamma) \;:=\; \exp\,[i\langle f,\gamma\rangle]F(\gamma), \quad F \in L^2(\pi_\sigma), \;\; \gamma \in \Gamma_X,$$

for any $f \in \mathcal{D}$. As usual, the semi-direct product $\mathcal{G} := \mathcal{D} \wedge \mathrm{Diff}_0(X)$ of the groups $\mathcal{D}$ and $\mathrm{Diff}_0(X)$ is defined as the set of pairs (f,ϕ) with multiplication operation

$$(f_1,\phi_1)(f_2,\phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1),$$

see e.g. [VGG]. Let us introduce for any element $(f,\phi) \in \mathcal{G}$ the following operator on $L^2(\pi_\sigma)$:

$$W_{\pi_\sigma}(f,\phi) \;:=\; U_{\pi_\sigma}(f)V_{\pi_\sigma}(\phi).$$

These operators are unitary and form a representation of the group $\mathcal{G}$. If we introduce multiplication operators $\rho_{\pi_\sigma}(f)$, $f \in \mathcal{D}$, as self-adjoint operators on $L^2(\pi_\sigma)$ which are defined for $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$ by the formula

$$(\rho_{\pi_\sigma}(f)F)(\gamma) \;:=\; \langle f,\gamma\rangle F(\gamma), \quad \gamma \in \Gamma_X,$$

then $U_{\pi_\sigma}(f) = \exp[i\rho_{\pi_\sigma}(f)]$ and the form of the multiplication in $\mathcal{G}$ implies

$$[\rho_{\pi_\sigma}(f), J_{\pi_\sigma}(v)] = i\rho_{\pi_\sigma}(\nabla_v^X f) \tag{3.37}$$

(on a dense domain in $L^2(\pi_\sigma)$) for all $f \in \mathcal{D}, v \in \mathrm{Diff}_0(X)$. We also have the relation $[\rho_{\pi_\sigma}(f_1), \rho_{\pi_\sigma}(f_2)] = 0$. The family of operators $J_{\pi_\sigma}(v)$, $\rho_{\pi_\sigma}(f)$, $v \in V_0(X)$, $f \in \mathcal{D}$, thus forms a Poisson representation of an infinite-dimensional Lie algebra which is known as the *Lie algebra of currents* in nonrelativistic quantum field theory, see [GGPS].

4 Intrinsic Dirichlet forms on Poissonian spaces

4.1 Definition of the intrinsic Dirichlet form

We start with introducing some useful spaces of smooth cylinder functions on Γ_X in addition to $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$. By $\mathcal{FP}(\mathcal{D}, \Gamma_X)$ we denote the set of all cylinder functions of the form (3.3) in which the generating function g_F is a polynomial on $I\!R^N$, i.e., $g_F \in \mathcal{P}(I\!R^N)$. Analogously we define $\mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X)$ where now $g_F \in C_p^\infty(I\!R^N)$ (:= the set of all C^∞-functions f on $I\!R^N$ such that f and all its partial derivatives of any order are polynomially bounded).

We have, obviously,

$$\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X) \;\subset\; \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X),$$

$$\mathcal{FP}(\mathcal{D}, \Gamma) \;\subset\; \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X)$$

and these spaces are algebras with respect to the usual operations. The existence of the Laplace transform $l_{\pi_\sigma}(f), f \in \mathcal{D}$, implies $\mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X) \subset L^2(\pi_\sigma)$.

Note, that after embedding Γ_X into $\mathcal{D}'$ (see Subsection 3.1) and a natural extension to $\mathcal{D}'$ the space $\mathcal{FP}(\mathcal{D}, \mathcal{D}')$ is nothing but the well-known space of cylinder polynomials on $\mathcal{D}'$, see [BK, Ch.2].

Definition 4.1 *For $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X)$ we introduce a pre-Dirichlet form as*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F, G) \;=\; \int_{\Gamma_X} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma_X)} \pi_\sigma(d\gamma). \tag{4.1}$$

Note that for $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X)$ formula (3.7) is still valid and therefore

$$\langle \nabla^\Gamma F, \nabla^\Gamma G \rangle_{T(\Gamma_X)} \ \in \ \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X),$$

such that (4.1) is well-defined.

We will call $\mathcal{E}_{\pi_\sigma}^\Gamma$ the intrinsic pre-Dirichlet form corresponding to the Poisson measure π_σ on Γ_X. The name "intrinsic" means that $\mathcal{E}_{\pi_\sigma}^\Gamma$ is associated with the geometry of Γ_X generated by the original Riemannian structure of X, in particular, by the intrinsic gradient ∇^Γ. In the next subsection we shall prove the closability of $\mathcal{E}_{\pi_\sigma}^\Gamma$.

4.2 Intrinsic Dirichlet operators

Let us introduce a differential operator $H_{\pi_\sigma}^\Gamma$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ which is given on any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ of the form

$$\begin{gathered} F(\gamma) \ = \ g_F(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle), \\ \gamma \in \Gamma_X, \ \varphi_1, \dots, \varphi_N \in \mathcal{D}, \ g_F \in C_b^\infty(\mathbb{R}^N), \end{gathered} \tag{4.2}$$

by the formula

$$\begin{aligned} &(H_{\pi_\sigma}^\Gamma F)(\gamma) \ := \\ &- \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle) \int_X \langle \nabla^X \varphi_i(x), \nabla^X \varphi_j(x)\rangle_{T_x(X)} \gamma(dx) \\ &- \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle) \int_X \triangle^X \varphi_j(x) \gamma(dx) \\ &- \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle) \int_X \langle \nabla^X \varphi_j(x), \beta^\sigma(x)\rangle_{T_x(X)} \gamma(dx), \end{aligned} \tag{4.3}$$

where $\triangle^X$ denotes the Laplace–Beltrami operator on X. In this formula all expressions are from $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ or have the form $\langle \psi, \gamma\rangle, \gamma \in \Gamma_X, \psi \in \mathcal{D}$, except for the functions $\langle h_j, \gamma\rangle$, $\gamma \in \Gamma_X$, with

$$h_j(x) \ := \ \langle \nabla^X \varphi_j(x), \beta^\sigma(x)\rangle_{T_x(X)}, \ x \in X, \ j = 1, \dots, N.$$

To clarify the situation with these fuctions note that due to the assumption on σ we have $\rho^{1/2} \in H^{1,2}_{loc}(X)$ that gives $h_j \in L^2(\sigma)$ and these functions have compact supports. Therefore $h_j \in L^1(\sigma), j = 1, \ldots, N$. On the other hand we know that a function $\langle f, \gamma \rangle, \gamma \in \Gamma_X$, is from $L^2(\pi_\sigma)$ if $f \in L^1(\sigma) \cap L^2(\sigma)$. The latter follows from the formula for the second moment of the measure π_σ, namely

$$\int_{\Gamma_X} \langle f, \gamma \rangle^2 \pi_\sigma(d\gamma) \ = \ \int_X f^2 \sigma(dx) + \Big(\int_X f(x) \sigma(dx) \Big)^2 \tag{4.4}$$

which is a direct consequence of (2.8). As a result the r.h.s. of (4.3) is well-defined. To show that the operator $H^\Gamma_{\pi_\sigma}$ is well-defined we still have to show that its definition does not depend on the representation of F in (4.2), which will be done below.

Remark 4.1 *In the applications to the study of unitary representations of the group* $\mathrm{Diff}_0(X)$ *given by Poisson measures, there is usually an additional assumption on the smoothness of the density* $\rho := d\sigma/dm$, *namely* $\rho \in C^\infty(X), \rho(x) > 0, x \in X$, *see e.g.* [VGG]. *In this case it is obvious that the operator* $H^\Gamma_{\pi_\sigma}$ *preserves the spaces* $\mathcal{F}C^\infty_p(\mathcal{D}, \Gamma_X)$ *and* $\mathcal{FP}(\mathcal{D}, \Gamma_X)$.

Let us also consider the classical pre-Dirichlet form corresponding to the measure σ on X:

$$\mathcal{E}^X_\sigma(\varphi, \psi) \ := \ \int_X \langle \nabla^X \varphi(x), \nabla^X \psi(x) \rangle_{T_x(X)} \sigma(dx), \tag{4.5}$$

where $\varphi, \psi \in \mathcal{D}$. This form is associated with the Dirichlet operator H^X_σ which is given on $\mathcal{D}$ by

$$H^X_\sigma \varphi(x) \ := \ -\triangle^X \varphi(x) - \langle \beta^\sigma(x), \nabla^X \varphi(x) \rangle_{T_x(X)}, \tag{4.6}$$

and which satisfies

$$\mathcal{E}^X_\sigma(\varphi, \psi) \ = \ (H^X_\sigma \varphi, \psi)_{L^2(\sigma)}, \ \varphi, \psi \in \mathcal{D}.$$

The closure of this form on $L^2(\sigma)$ is defined by $(\mathcal{E}^X_\sigma, D(\mathcal{E}^X_\sigma))$. Note that $D(\mathcal{E}^X_\sigma)$ is nothing but the Sobolev space of order 1 in $L^2(\sigma)$ (sometimes also denoted by $H^{1,2}_0(X, \sigma)$). $(\mathcal{E}^X_\sigma, D(\mathcal{E}^X_\sigma))$ generates a positive self-adjoint operator in $L^2(\sigma)$ (the so-called Friedrich's extension of H^X_σ, see e.g. [BKR]).

For this extension we preserve the previous notation H_σ^X and denote the domain by $D(H_\sigma^X)$. Note that for $\sigma = m$ we have $H_m^X = -\triangle^X$. Using the underlying Dirichlet operator we obtain the representation

$$(H_{\pi_\sigma}^\Gamma F)(\gamma) = -\sum_{i,j=1}^N \frac{\partial^2 F}{\partial s_i \partial s_j}(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\langle\nabla^X\varphi_i,\nabla^X\varphi_j\rangle_{T_\gamma(\Gamma_X)}$$

$$+\sum_{i=1}^N \frac{\partial F}{\partial s_i}(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\langle H_\sigma^X\varphi_i,\gamma\rangle. \tag{4.7}$$

Let us define

$$(\nabla^\Gamma\nabla^\Gamma F)(\gamma,x,y)$$

$$:= \sum_{i,j=1}^N \frac{\partial^2 F}{\partial s_i \partial s_j}(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\ \nabla^X\varphi_i(x)\otimes\nabla^X\varphi_j(y)$$

$$\in T_\gamma(\Gamma_X)\otimes T_\gamma(\Gamma_X).$$

Then

$$\triangle^\Gamma F(\gamma) := \mathrm{Tr}(\nabla^\Gamma\nabla^\Gamma F)(\gamma)$$

$$= \sum_{i,j=1}^N \frac{\partial^2 F}{\partial s_i \partial s_j}(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\langle\nabla^X\varphi_i,\nabla^X\varphi_j\rangle_{T_\gamma(\Gamma_X)}.$$

Hence the operator $H_{\pi_\sigma}^\Gamma$ can be written as

$$(H_{\pi_\sigma}^\Gamma F)(\gamma) = -\triangle^\Gamma F(\gamma) - \langle \mathrm{div}_\sigma^X \nabla^\Gamma F(\gamma,\cdot),\gamma\rangle. \tag{4.8}$$

The following theorem implies that both $H_{\pi_\sigma}^\Gamma$ and $\triangle^\Gamma$ are well-defined as linear operators on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$, i.e., independently of the representation of F as in (4.2).

Theorem 4.1 *The operator $H_{\pi_\sigma}^\Gamma$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_\sigma}^\Gamma$, i.e., for all $F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(F,G) = (H_{\pi_\sigma}^\Gamma F,G)_{L^2(\pi_\sigma)},$$

or

$$H_{\pi_\sigma}^\Gamma = -\mathrm{div}_{\pi_\sigma}^\Gamma\nabla^\Gamma \quad on\ \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X).$$

We call $H_{\pi_\sigma}^\Gamma$ the intrinsic Dirichlet operator of the measure π_σ.

Proof. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ of the form (4.2) we have

$$(\nabla^\Gamma F)(\gamma, x) \;=\; \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle)(\nabla^X \varphi_j)(x).$$

By (3.19) we conclude that

$$\mathrm{div}^\Gamma_{\pi_\sigma}(\nabla^\Gamma F) \;=\; -H^\Gamma_{\pi_\sigma} F,$$

which by (3.17) for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ gives

$$\begin{aligned}(H^\Gamma_{\pi_\sigma} F, G)_{L^2(\pi_\sigma)} \;&=\; -\int_{\Gamma_X} \mathrm{div}^\Gamma_{\pi_\sigma}(\nabla^\Gamma F)(\gamma) G(\gamma) \pi_\sigma(d\gamma) \\ &=\; \int_{\Gamma_X} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma_X)} \pi_\sigma(d\gamma).\end{aligned}$$

□

Remark 4.2 (i) *In the case $\sigma = m$ we will call the Dirichlet form $\mathcal{E}^\Gamma_{\pi_\sigma}$ the canonical Dirichlet form on Γ_X. The canonical Dirichlet form and canonical Dirichlet operator $H^\Gamma_{\pi_m}$ are defined directly in terms of the Riemannian geometry of X.*

(ii) *The operator $H^\Gamma_{\pi_\sigma}$ can be naturally extended to cylinder functions of the form*

$$F(\gamma) \;:=\; e^{\langle \varphi, \gamma\rangle},\; \varphi \in \mathcal{D},\; \gamma \in \Gamma_X,$$

since such F belong to $L^2(\pi_\sigma)$. We then have

$$H^\Gamma_{\pi_\sigma} e^{<\varphi,\gamma>} \;=\; \langle H^X_\sigma \varphi - |\nabla^X \varphi|^2_{T(X)}, \gamma\rangle e^{\langle \varphi, \gamma\rangle} \tag{4.9}$$

and for $\sigma = m$

$$H^\Gamma_{\pi_m} e^{<\varphi,\gamma>} \;=\; -\langle \triangle^X \varphi + |\nabla^X \varphi|^2_{T(X)}, \gamma\rangle e^{\langle \varphi, \gamma\rangle}.$$

As an immediate consequence of Theorem 4.1 we obtain

Corollary 4.1 $(\mathcal{E}^\Gamma_{\pi_\sigma}, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X))$ *is closable on $L^2(\pi_\sigma)$. Its closure $(\mathcal{E}^\Gamma_{\pi_\sigma}, D(\mathcal{E}^\Gamma_{\pi_\sigma}))$ is associated with a positive definite self-adjoint operator, the Friedrichs' extension of $H^\Gamma_{\pi_\sigma}$, which we also denote by $H^\Gamma_{\pi_\sigma}$ (and its domain by $D(H^\Gamma_{\pi_\sigma})$).*

Clearly, ∇^Γ also extends to $D(\mathcal{E}^\Gamma_{\pi_\sigma})$. We denote this extension again by ∇^Γ.

Corollary 4.2 *Let*

$$F(\gamma) \ := \ g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle), \ \ \gamma\in\Gamma_X,$$
$$\varphi_1,\ldots,\varphi_N\in D(\mathcal{E}^X_\sigma), \ \ g_F\in C_b^\infty(I\!\!R^N). \tag{4.10}$$

Then
$$F \ \in \ D(\mathcal{E}^\Gamma_{\pi_\sigma})$$
and
$$\nabla^\Gamma F \ = \ \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle\varphi_1,\cdot\rangle,\ldots,\langle\varphi_N,\cdot\rangle)\nabla^X\varphi_j.$$

Proof. By approximation this is an immediate consequence of (3.7) and the fact that for all $1\le i\le N$

$$\int\langle|\nabla^X\varphi_i|^2_{T(X)},\gamma\rangle\pi_\sigma(d\gamma) \ = \ \mathcal{E}^X_\sigma(\varphi_i,\varphi_i). \tag{4.11}$$

$\square$

Remark 4.3 *Let* $\mu_{\lambda,\sigma}\in\mathcal{M}_2(\Gamma_X)$ *be given as in* (3.23). *Then by Theorem* 3.2, (ii) $\Rightarrow$ (i), *all results above are valid with* $\mu_{\lambda,\sigma}$ *replacing* π_σ. *By* (4.7) *we have*
$$H^\Gamma_{\pi_\sigma} \ = \ H^\Gamma_{\mu_{\lambda,\sigma}} \text{ on } \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X).$$
We note that the right hand side of (4.7) only depends on σ *and the Riemannian structure of* X. *The respective Friedrichs' extensions on* $L^2(\mu_{\lambda,\sigma})$, *again denoted by* $H^\Gamma_{\mu_{\lambda,\sigma}}$, *however, do not coincide. They depend on* $\mu_{\lambda,\sigma}$ *in an essential way. Their asociated semigroups* $(\exp(-tH^\Gamma_{\mu_{\lambda,\sigma}}))_{t>0}$ *and the corresponding Dirichlet forms* $(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}))$ *will be studied in the following subsection in more detail. We only point out here that if* $\lambda=\varepsilon_0$ *on* $\mathcal{B}(I\!\!R_+)$, *then* $\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}\equiv 0$. *Clearly,* $D(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}})$ *is nothing but the Sobolev space* $H_0^{1,2}(\Gamma_X;\mu_{\lambda,\sigma})$ *on* Γ_X *of order 1 in* $L^2(\Gamma_X;\mu_{\lambda,\sigma})$.

4.3 The heat semigroup and ergodicity

In this subsection we assume that $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$, or equivalently $(H_\sigma^X, D(H_\sigma^X))$ is conservative, i.e.,

$$\int_X H_\sigma^X \varphi \, d\sigma \;=\; 0 \;\text{ for all } \varphi \in D(H_\sigma^X) \cap L^1(\sigma) \text{ such that } H_\sigma^X \varphi \in L^1(\sigma).$$

For sufficient condition we refer e.g. to [T]. Probabilistically, this condition means that the corresponding distorted Brownian motion on X (see the introduction) does not explode.

For $\mu_{\lambda,\sigma} \in \mathcal{M}_2(\Gamma_X)$ as in (3.23) let $T^\Gamma_{\mu_{\lambda,\sigma}}(t) := \exp(-tH^\Gamma_{\mu_{\lambda,\sigma}}), t > 0$. We first want to calculate these semigroups (whose generators $H^\Gamma_{\mu_{\lambda,\sigma}}$ on $L^2(\mu_{\lambda,\sigma})$ all coincide on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$) explicitly on the following set of functions

$$E(\mathcal{D}_1, \Gamma) \;=\; l.h.\{\exp(\langle \log(1+\varphi), \cdot \rangle) \mid \varphi \in \mathcal{D}_1\}, \tag{4.12}$$

where l.h. means linear hull and

$$\begin{aligned} \mathcal{D}_1 \;:=\; & \{\varphi \in D(H_\sigma^X) \cap L^1(\sigma) \mid H_\sigma^X \varphi \in L^1(\sigma) \\ & \text{and } -\delta \le \varphi \le 0 \;\text{ for some }\; \delta \in (0,1)\}. \end{aligned} \tag{4.13}$$

Note that since $(\exp(-tH_\sigma^X))_{t>0}$ is sub-Markovian (i.e., $0 \le \exp(-tH_\sigma^X)\varphi \le 1$ for all $t > 0$ and $\varphi \in L^2(\sigma), 0 \le \varphi \le 1$), because $(\mathcal{E}_\sigma^X, \mathcal{D}(\mathcal{E}_\sigma^X))$ is a Dirichlet form, $(\exp(-tH_\sigma^X))_{t>0}$ is a C_0-semigroup on all $L^p(\sigma), p \in [1, \infty)$. Fix $\mu_{\lambda,\sigma} \in \mathcal{M}_2(\Gamma_X)$ as in (3.23). We point out that all functions in $E(\mathcal{D}_1, \Gamma)$ are $\mu_{\lambda,\sigma}$-a.e. well-defined. Furthermore, since $\{\varphi \in \mathcal{D} | -\delta \le \varphi \le 0$ for some $\delta \in (0,1)\} \subset \mathcal{D}_1$, $E(\mathcal{D}_1, \Gamma)$ generates $\mathcal{B}(\Gamma_X)$ and is dense in $L^2(\mu_{\lambda,\sigma})$.

For $t > 0$ define $P(t) : E(\mathcal{D}_1, \Gamma) \to E(\mathcal{D}_1, \Gamma)$ by

$$P(t) \exp\left(\langle \log(1+\varphi), \cdot \rangle\right) \;:=\; \exp\left(\langle \log(1 + e^{-tH_\sigma^X}\varphi), \cdot \rangle\right) \tag{4.14}$$

and by extending $P(t)$ to $E(\mathcal{D}_1, \Gamma)$ by linearity.

Since $\exp(-tH_\sigma^X)\varphi \in D(H_\sigma^X) \cap L^1(\sigma)$ for all $\varphi \in \mathcal{D}_1$ and since $\exp(-tH_\sigma^X)$ is sub-Markovian, we have that $\exp(-tH_\sigma^X)\mathcal{D}_1 \subset \mathcal{D}_1$ for all $t > 0$. Therefore, indeed $P(t)$ maps $E(\mathcal{D}_1, \Gamma)$ into itself.

Proposition 4.1 *Let $\mu_{\lambda,\sigma} \in \mathcal{M}_2(\Gamma_X)$ be as in* (3.23). *Assume that the following condition holds:*

(C) $(H^X_\sigma, \mathcal{D})$ *is essentially self-adjoint on* $L^2(\sigma)$.
Then:
(i) $T^\Gamma_{\mu_{\lambda,\sigma}}(t) = P(t)$ *on* $E(\mathcal{D}_1, \Gamma)$ *for all* $t > 0$.
(ii) $E(\mathcal{D}_1, \Gamma) \subset D(H^\Gamma_{\mu_{\lambda,\sigma}})$ *and*

$$H^\Gamma_{\mu_{\lambda,\sigma}} \exp(\langle \log(1+\varphi), \cdot\rangle) = \langle \frac{H^X_\sigma \varphi}{1+\varphi}, \cdot\rangle \exp(\langle \log(1+\varphi), \cdot\rangle) \ \textit{for all} \ \varphi \in \mathcal{D}_1.$$

Remark 4.4 (i) *Condition* (C) *is always fulfilled if* X *is complete and if* $|\beta^\sigma|_{T(X)} \in L^p_{loc}(m)$*, for some* $p > \dim(X)$*. This immediately follows from the proof of Theorem* 1 *in* [BKR] (*see also* [BKR, Remark 4 (iii)]) *and Gaffney's lemma* (*cf. e.g.* [Ba]).

(ii) *By a simple approximation argument Proposition* 4.1 (i) *implies that for* $t > 0$ *and all* $\varphi \in L^1(\sigma), -1 < \varphi \leq 0$,

$$T^\Gamma_{\mu_{\lambda,\sigma}}(t) \exp(\langle \log(1+\varphi), \cdot\rangle) = \exp(\langle \log(1 + e^{-tH^X_\sigma}\varphi), \cdot\rangle).$$

Before proving Proposition 4.1 let us first do some heuristics and consider the following equation for the unknown function φ_t:

$$T^\Gamma_{\mu_{\lambda,\sigma}}(t) e^{\langle \log(1+\varphi), \gamma\rangle} = e^{\langle \log(1+\varphi_t), \gamma\rangle}.$$

Differentiating both parts w.r.t. $t \geq 0$ we obtain

$$\langle \frac{\partial \varphi_t}{\partial t}(1+\varphi_t)^{-1}, \gamma\rangle e^{\langle \log(1+\varphi_t), \gamma\rangle}$$

$$= -H^\Gamma_{\mu_{\lambda,\sigma}} e^{\langle \log(1+\varphi_t), \gamma\rangle} = \langle \frac{H^X_\sigma \varphi_t}{1+\varphi_t}, \gamma\rangle e^{\langle \log(1+\varphi_t), \gamma\rangle}$$

(cf. (4.9) and Remark 4.3). Then φ_t is a solution to the equation

$$\begin{cases} \frac{\partial}{\partial t}\varphi_t = -H^X_\sigma \varphi_t \\ \varphi_0 =: \varphi, \end{cases}$$

i.e., $\varphi_t = e^{-tH^X_\sigma}\varphi$, $t \geq 0$.

The rigorous implementation of these heuristics to prove Proposition 4.1, is technically quite complicated. It consists in part of showing that above $\partial/\partial t$ can be taken in the sense of L^2 w.r.t. $\mu_{\lambda,\sigma}$ and that (4.9) holds with $\mu_{\lambda,\sigma}$ replacing π_σ and for $\varphi \in \mathcal{D}_1$. The details are given in the Appendix.

Theorem 4.2 *Assume that condition* (C) *holds. Then* $E(\mathcal{D}_1, \Gamma)$ *is an operator core for the Friedrich's extension* $H^{\Gamma}_{\mu_{\lambda,\sigma}}$ *on* $L^2(\mu_{\lambda,\sigma})$ *for any* $\mu_{\lambda,\sigma} \in \mathcal{M}_2(\Gamma_X)$ *as in* (3.23). *(In other words:* $(H^{\Gamma}_{\mu_{\lambda,\sigma}}, E(\mathcal{D}_1, \Gamma))$ *is essentially self-adjoint on* $L^2(\mu_{\lambda,\sigma})$*).*

Proof. This is an immediate consequence of Proposition 4.1 and a standard theorem on operator semigroups (cf. e.g. [RS, Theorem X.49]).

□

Remark 4.5 *In the special case where* $\mu_{\lambda,\sigma} = \pi_\sigma$ *we shall give an alternative proof of this result in Theorem* 5.3 *below.*

After we have calculated $T^{\Gamma}_{\mu_{\lambda,\sigma}}(t), t > 0$, and also $H^{\Gamma}_{\mu_{\lambda,\sigma}}$ explicitly on a core (i.e., a set, dense in the domains w.r.t. the respective graph norms) we can characterize for which $\mu_{\lambda,\sigma}$ we have that $(T^{\Gamma}_{\mu_{\lambda,\sigma}}(t))_{t>0}$ is ergodic.

Theorem 4.3 *Suppose that* ρ $(= d\sigma/dm)$ *is* C^∞*, strictly positive and* $\sigma(X) = \infty$*. Let* $\mu_{\lambda,\sigma} \in \mathcal{M}_2(\Gamma_X)$ *as in* (3.23). *Then the following assertions are equivalent:*

(i) $\mu_{\lambda,\sigma} = \pi_{s\sigma}$ *for some* $s \geq 0$.

(ii) $(\mathcal{E}_{\mu_{\lambda,\sigma}}, D(\mathcal{E}_{\mu_{\lambda,\sigma}}))$ *is irreducible (i.e., for* $F \in D(\mathcal{E}_{\mu_{\lambda,\sigma}})$, $\mathcal{E}_{\mu_{\lambda,\sigma}}(F,F) = 0$ *implies that* $F = \text{const}$ *).*

(iii) $(T^{\Gamma}_{\mu_{\lambda,\sigma}}(t))_{t>0}$ *is irreducible (i.e., if* $G \in L^2(\mu_{\lambda,\sigma})$ *such that* $T^{\Gamma}_{\mu_{\lambda,\sigma}}(t)(GF) = GT^{\Gamma}_{\mu_{\lambda,\sigma}}(t)F$ *for all* $F \in L^\infty(\mu_{\lambda,\sigma}), t > 0$*, then* $G = \text{const}$ *).*

(iv) *If* $F \in L^2(\mu_{\lambda,\sigma})$ *such that* $T^{\Gamma}_{\mu_{\lambda,\sigma}}(t)F = F$ *for all* $t > 0$*, then* $F = \text{const}$ *.*

(v) $(T^{\Gamma}_{\mu_{\lambda,\sigma}}(t))_{t>0}$ *is ergodic (i.e.,*

$$\int (T^{\Gamma}_{\mu_{\lambda,\sigma}}(t)F - \int F d\mu_{\lambda,\sigma})^2 d\mu_{\lambda,\sigma} \to 0 \ \ as \ \ t \to \infty \ \ for \ \ all \ \ F \in L^2(\mu_{\lambda,\sigma})).$$

(vi) *If* $F \in D(H^{\Gamma}_{\mu_{\lambda,\sigma}})$ *with* $H^{\Gamma}_{\mu_{\lambda,\sigma}} F = 0$*, then* $F = \text{const}$ *.*

Remark 4.6 *For the proof of Theorem* 4.3 *we only use that Theorem* 3.2 *and Corollary* 3.1 *resp. Theorem* 4.2 *hold, rather than the additional assumption on* ρ. *According to Remarks* 3.5, 4.4 (i) *it is, therefore, e.g. enough to assume that* X *is complete and that* $\beta_v^\sigma \in L^p(X, m), p > dim(X)$, *for all* $v \in V_0(X)$. *Furthermore, Theorem* 4.3 *generalizes to the Gibbs measures mentioned in Remark* 3.5 (*cf.* [AKR 5] *for details*).

Proof of Theorem 4.3. The equivalence of (ii)–(vi) follows by standard theory (the proof can e.g. be carried out entirely analogously to that of [AKR 3, Proposition 2.3]). So, we only have to prove (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii): Assume (i). If $s = 0$, then $\mathcal{E}^\Gamma_{\pi_{0\sigma}} \equiv 0$ and $H^\Gamma_{\pi_{0\sigma}} \equiv 0$, hence (ii) trivially holds. So, let $s > 0$ and set $\mathcal{E} := \mathcal{E}^\Gamma_{\pi_{s\sigma}}$. Let $G \in D(\mathcal{E})$ such that $\mathcal{E}(G, G) = 0$. Let $n \in I\!N$. Then $G_n := G \vee (-n) \wedge n \in D(\mathcal{E})$ and $\mathcal{E}(G_n, G_n) = 0$ (cf. e.g. [MR1, Theorem 4.17]) and $G_n \to G$ as $n \to \infty$ in $L^2(\pi_{s\sigma})$. Hence replacing G by $G_n + n$ we may assume that $G \geq 0$ and, of course, that $\int G d\pi_{s\sigma} = 1$. Define $\mu := G\ \pi_{s\sigma}$. Then, since $\nabla^\Gamma G = 0$ and since $\pi_{s\sigma} \in (IbP)^\sigma$, it follows that for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ and all $v \in V_0(X)$

$$\int \nabla_v^\Gamma F d\mu \ = \ - \int F B_v^{\pi_\sigma} d\mu,$$

i.e., $\mu \in (IbP)^\sigma$. By Corollary 3.1 (i), (iii), this implies $G \equiv 1$.

(ii) $\Rightarrow$ (i): Assume (ii) and let $\mathcal{E} := \mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}$. By Corollary 3.1 (i) we have to prove that $\mu_{\lambda,\sigma}$ is extreme in $(IbP)^\sigma$. To prove this we shall apply Corollary 3.1 (iii). So, let $\mu := G\ \mu_{\lambda,\sigma}$ for some $G \in L^1(\mu_{\lambda,\sigma}), G \geq 0, \int G d\mu_{\lambda,\sigma} = 1$, and assume that $\mu \in (IbP)^\sigma$. We have to show that G is constant. By Corollary 3.1 (ii) we may replace G by $(\int (G \wedge n) d\mu_{\lambda,\sigma})^{-1} (G \wedge n)$, $n \in I\!N$, hence may assume that $G \in L^\infty(\mu_{\lambda,\sigma})$. Since $\mu \in (IbP)^\sigma$ we have for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X), v \in V_0(X)$,

$$- \int F B_v^{\pi_\sigma} G d\mu_{\lambda,\sigma} \ = \ \int \nabla_v^\Gamma F G d\mu_{\lambda,\sigma}$$

Since by Remark 4.4 (i) condition (C) holds in our situation, we can apply (the proof of) Theorem 4.2 and [Eb 95, Theorem 2.2 and Remark 2.4 (ii)] to conclude that $G \in D(\mathcal{E})$ and that $\mathcal{E}(G, G) = 0$. Consequently, by assumption G is constant.

$\square$

5 Intrinsic Dirichlet operator and second quantization

In this section we want to describe the Fock space realization of Poisson spaces over our manifold X and show that $H^{\Gamma}_{\pi_\sigma}$ is the second quantization of the operator H^X_σ.

5.1 Poissonian gradient and chaos decomposition

Let us define another "gradient" on functions $F : \Gamma_X \longrightarrow \mathbb{R}$. This gradient ∇^P has specific useful properties on Poissonian spaces. Having this in mind we will call ∇^P the Poissonian gradient.

Definition 5.1 *For any $F \in \mathcal{F}C^\infty_P(\mathcal{D}, \Gamma_X)$ we define the Poissonian gradient ∇^P as*

$$(\nabla^P F)(\gamma, x) = F(\gamma + \varepsilon_x) - F(\gamma), \tag{5.1}$$
$$\gamma \in \Gamma_X, \; x \in X.$$

Let us mention that the operation

$$\Gamma_X \ni \gamma \mapsto \gamma + \varepsilon_x \in \Gamma_X$$

is a π_σ-a.e. well-defined map because of the property:

$$\pi_\sigma(\{\gamma \in \Gamma_X | x \in \gamma\}) = 0$$

for any $x \in X$. We consider ∇^P as a mapping

$$\nabla^P : \mathcal{F}C^\infty_P \ni F \mapsto \nabla^P F \in L^2(\sigma) \otimes L^2(\pi_\sigma)$$

that corresponds to using the Hilbert space $L^2(\sigma)$ as a tangent space at any point $\gamma \in \Gamma_X$. Note that the Poissonian gradient appears from different points of view in many papers on Poissonian analysis, see e.g. [IK, NV, KSS] and references therein.

The most important feature of the Poissonian gradient is that it produces (via a corresponding "integration by parts" formula) the orthogonal system of Charlier polynomials on $(\Gamma_X, \mathcal{B}(\Gamma_X), \pi_\sigma)$. Below we describe this construction in detail using the general scheme of [KSS].

For any $\varphi \in \mathcal{D}$ we introduce the directional derivative

$$\begin{aligned}(\nabla^P_\varphi F)(\gamma) \;&=\; \langle \nabla^P F(\gamma), \varphi\rangle_{L^2(\sigma)}\\ &= \int_X [F(\gamma+\varepsilon_x) - F(\gamma)]\varphi(x)\sigma(dx).\end{aligned} \tag{5.2}$$

Then for all $F \in \mathcal{F}C^\infty_p(\mathcal{D}, \Gamma_X)$ the following integration by parts formula holds:

$$\int_{\Gamma_X} (\nabla^P_\varphi F)(\gamma)\pi_\sigma(d\gamma) \;=\; \int_{\Gamma_X} F(\gamma)\langle\varphi,\gamma\rangle\pi_\sigma(d\gamma) \;-\; \langle F\rangle_{\pi_\sigma}\langle\varphi\rangle_\sigma, \tag{5.3}$$

(where $\langle\cdot\rangle_\mu$ denotes the mean with respect to a measure μ), i.e.,

$$(\nabla^{P*}_\varphi 1)(\gamma) \;=\; \langle\varphi,\gamma\rangle - \langle\varphi\rangle_\sigma. \tag{5.4}$$

We set

$$(\nabla^{P*}_\varphi 1)(\gamma) \;=:\; Q^{\pi_\sigma}_1(\varphi,\gamma),\ \gamma\in\Gamma_X,$$

and inductively for all $n \in I\!N$ introduce functions in $L^2(\pi_\sigma)$ by

$$Q^{\pi_\sigma}_n(\varphi^{\otimes n},\gamma) \;:=\; ((\nabla^{P*}_\varphi)^n 1)(\gamma),\ \ \gamma\in\Gamma_X. \tag{5.5}$$

Due to the kernel theorem [BK, Ch.1] the functions $Q^{\pi_\sigma}_n(\varphi^{\otimes n},\gamma)$ have the following representation:

$$Q^{\pi_\sigma}_n(\varphi^{\otimes n},\gamma) \;=\; \langle\varphi^{\otimes n}, Q^{\pi_\sigma}_n(\gamma)\rangle, \tag{5.6}$$

where $Q^{\pi_\sigma}_n(\gamma), n\in I\!N$, are generalized symmetric kernels

$$\Gamma_X \ni \gamma \;\mapsto Q^{\pi_\sigma}_n(\gamma) \in (\mathcal{D}')^{\hat\otimes n}.$$

Here $(\mathcal{D}')^{\hat\otimes n}$ denotes the n-th symmetric tensor power of the dual to the nuclear space $\mathcal{D}$, see e.g. [BK, Ch.2]. Using these mappings for smooth symmetric kernels $\varphi^{(n)}\in\mathcal{D}^{\hat\otimes n}$ we define

$$Q^{\pi_\sigma}_n(\varphi^{(n)},\gamma) \;:=\; \langle\varphi^{(n)}, Q^{\pi_\sigma}_n(\gamma)\rangle,\ \gamma\in\Gamma_X. \tag{5.7}$$

We also set $Q^{\pi_\sigma}_0 :\equiv 1$.

The main properties of these functions are described by the following proposition, see e.g. [IK].

Proposition 5.1 *For any $n, m \in \mathbb{Z}_+$, and all smooth kernels $\varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$, $\psi^{(m)} \in \mathcal{D}^{\hat{\otimes} m}$*

$$\int_{\Gamma_X} Q_n^{\pi_\sigma}(\varphi^{(n)}, \gamma) Q_m^{\pi_\sigma}(\psi^{(m)}, \gamma) \pi_\sigma(d\gamma) \; = \; \delta_{nm} (\varphi^{(n)}, \psi^{(n)})_{L^2(\sigma^{\otimes n})}. \tag{5.8}$$

This proposition gives a possibility to extend the Q-functions (as measurable ones) to kernels from the so-called n-particle Fock spaces over $L^2(\sigma)$. We set

$$\mathrm{Exp}_n L^2(\sigma) \; := \; (L^2(\sigma))^{\hat{\otimes} n} \; = \; \hat{L}^2(X^n, \sigma^{\otimes n}), \; n \in \mathbb{N},$$

and $\mathrm{Exp}_0 L^2(\sigma) \; := \; \mathbb{R}$. The Fock space $\mathrm{Exp}\, L^2(\sigma)$ is then defined as the Hilbert direct sum

$$\mathrm{Exp} L^2(\sigma) \; = \; \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n L^2(\sigma).$$

For any $F \in L^2(\pi_\sigma)$ there exists a sequence $(f^{(n)})_{n=0}^{\infty} \in \mathrm{Exp}\, L^2(\sigma)$ such that

$$F(\gamma) \; = \; \sum_{n=0}^{\infty} Q_n^{\pi_\sigma}(f^{(n)}, \gamma) \tag{5.9}$$

and, moreover,

$$||F||^2_{L^2(\pi_\sigma)} \; = \; \sum_{n=0}^{\infty} n! |f^{(n)}|^2_{L^2(\sigma^{\otimes n})}, \tag{5.10}$$

where the r.h.s. coincides with the square of the norm in $\mathrm{Exp} L^2(\sigma)$. And vice versa, any series (5.9) with coefficients $(f^{(n)})_{n=0}^{\infty} \in \mathrm{Exp} L^2(\sigma)$ gives a function from $L^2(\pi_\sigma)$. As a result, we have the well-known chaos decomposition (5.9) which produces an isomorphism between $L^2(\pi_\sigma)$ and $\mathrm{Exp} L^2(\sigma)$. In probability theory the functions $Q_n^{\pi_\sigma}(f^{(n)}, \gamma)$ are called the n-tuple stochastic integrals of $f^{(n)}$ with respect to the Poisson process (generated by the Poisson measure π_σ), see e.g. [NV].

5.2 Poisson exponentials

Let us describe an alternative approach to the chaos decomposition on the Poisson space which uses the concept of generalized Appel polynomials and was proposed in [KSS].

As before, we will have in mind the embeddings $\Gamma_X \subset \mathcal{M}_p(X) \subset \mathcal{M}(X) \subset \mathcal{D}'$. For any $\varphi \in \mathcal{D}$, $\varphi > -1$, we define the Poissonian exponential

$$e_{\pi_\sigma}(\varphi, \gamma) \;=\; \exp(\langle \log(1+\varphi), \gamma\rangle - \langle \varphi\rangle_\sigma),\; \gamma \in \Gamma_X. \tag{5.11}$$

This function has a natural extension to $e_{\pi_\sigma}(\varphi, \xi), \xi \in \mathcal{D}'$. As we have mentioned in Section 1 the Poisson measure π_σ can be considered also as a probability measure on $\mathcal{D}'$ (equipped with the σ-algebra $\mathcal{C}_\sigma(\mathcal{D}')$ generated by cylinder subsets of this linear space). Then we have a linear realization of the Poisson space:

$$(\Gamma_X, \mathcal{B}(\Gamma_X), \pi_\sigma) \;\sim\; (\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'), \pi_\sigma).$$

The function $e_{\pi_\sigma}(\varphi, \xi)$ (for given $\xi \in \mathcal{D}'$) is a real holomorphic function of φ on a neighbourhood of $O \in \mathcal{D}$. Its Taylor decomposition has the form (cf. [KSS])

$$e_{\pi_\sigma}(\varphi, \xi) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \xi) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, Q_n^{\pi_\sigma}(\xi)\rangle. \tag{5.12}$$

Here

$$Q_n^{\pi_\sigma} :\; \mathcal{D}' \;\longrightarrow\; (\mathcal{D}')^{\hat{\otimes} n},\; n \in \mathbb{N},$$

are natural extensions of the Q-functions defined above. From (5.12) it follows immediately that for any $\varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$ the function

$$\mathcal{D}' \ni \xi \;\mapsto\; \langle \varphi^{(n)}, Q_n^{\pi_\sigma}(\xi)\rangle \;=:\; Q_n^{\pi_\sigma}(\varphi^{(n)}, \xi)$$

is a polynomial of order n on $\mathcal{D}'$.

The system of functions

$$Q^{\pi_\sigma} \;:=\; \{Q_n^{\pi_\sigma}(\varphi^{(n)}, \xi) \mid \varphi^{(n)} \in \mathcal{D}^{\hat{\otimes} n},\; \xi \in \mathcal{D}',\; n \in \mathbb{Z}_+\}$$

is called the system of Charlier polynomials for the Poisson measure π_σ. The Poisson exponential $e_{\pi_\sigma}(\varphi, \xi), \xi \in \mathcal{D}', \varphi \in \mathcal{D}$, plays the role of a generating function for the system of Charlier polynomials. Note that using (2.8) we obtain the following property of the Poisson exponentials:

$$\int e_{\pi_\sigma}(\varphi_1, \gamma) e_{\pi_\sigma}(\varphi_2, \gamma) \pi_\sigma(d\gamma) \;=\; \exp\{(\varphi_1, \varphi_2)_{L^2(\sigma)}\}. \tag{5.13}$$

Having this relation and decomposition (5.12) we obtain once more the orthogonality property (5.8).

For any $\varphi \in L^2(\sigma)$ we introduce the vector

$$\operatorname{Exp}\varphi := \Big(\frac{1}{n!}\varphi^{\otimes n}\Big)_{n=0}^{\infty} \in \operatorname{Exp} L^2(\sigma)$$

which is called the coherent state corresponding to the one-particle state φ. For any set $\mathcal{L} \subset L^2(\sigma)$ which is total in $L^2(\sigma)$ the set of coherent states $\{\operatorname{Exp}\varphi | \varphi \in \mathcal{L}\} \subset \operatorname{Exp} L^2(\sigma)$ is also total in $\operatorname{Exp} L^2(\sigma)$, see e.g. [BK]. Note that $e_{\pi_\sigma}(\varphi, \cdot)$ is nothing but the coherent state in the Fock space picture, more precisely, for any $\varphi \in \mathcal{D}$, $\varphi > -1$, we have (see (5.9))

$$L^2(\pi_\sigma) \ni e_{\pi_\sigma}(\varphi, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \cdot) \mapsto \operatorname{Exp}\varphi \in \operatorname{Exp} L^2(\sigma).$$

The mapping

$$e^{\langle \varphi, \gamma \rangle} \mapsto e_{\pi_\sigma}(\varphi, \gamma) = \exp(\langle \log(1+\varphi), \gamma \rangle - \langle \varphi \rangle_\sigma) := :e^{\langle \varphi, \gamma \rangle}:_{\pi_\sigma}$$

is called Wick regularization with respect to the measure π_σ. This mapping can be extended by linearity to the dense domain

$$l.h.\, \{e^{\langle \varphi, \cdot \rangle} \mid \varphi \in \mathcal{D},\ \varphi > -1\}.$$

The following proposition shows that the operators ∇_ψ^P and ∇_ψ^{P*} play the role of annihilation and creation operators resp. in the Poisson realization of the Fock space $\operatorname{Exp} L^2(\sigma)$.

Proposition 5.2 *For all $\psi, \varphi \in \mathcal{D}, n \in I\!N_+$, the following formulas hold:*

$$\nabla_\psi^P Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \gamma) = n(\varphi, \psi)_{L^2(\sigma)} Q_{n-1}^{\pi_\sigma}(\varphi^{\otimes(n-1)}, \gamma), \tag{5.14}$$

$$\nabla_\psi^{P*} Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \gamma) = Q_{n+1}^{\pi_\sigma}(\varphi^{\otimes n} \hat{\otimes} \psi, \gamma),\ \gamma \in \Gamma_X, \tag{5.15}$$

where $\varphi^{\otimes n} \hat{\otimes} \psi$ means the symmetric tensor product, i.e., the projection of $\varphi^{\otimes n} \otimes \psi \in L^2(\sigma^{\otimes(n+1)})$ onto $\operatorname{Exp}_n L^2(\sigma)$.

Proof. By the definition of the Poisson derivative

$$\nabla_\psi^P e_{\pi_\sigma}(\varphi, \gamma) = (\varphi, \psi)_{L^2(\sigma)} e_{\pi_\sigma}(\varphi, \gamma)$$

and then (5.14) is a direct consequence of this formula and decomposition (5.12). Having (5.14) we deduce (5.15) via the orthogonality relation (5.8). Note that (5.15) is also a consequence of the definition of the Q-functions:

$$Q_m^{\pi_\sigma}(\varphi_1 \hat{\otimes} \cdots \hat{\otimes} \varphi_m, \gamma) \;=\; (\nabla_{\varphi_1}^{P*} \cdots \nabla_{\varphi_m}^{P*} 1)(\gamma) \tag{5.16}$$

(see Subsection 5.1).

□

5.3 Second quantization on Poisson spaces

Let B be a contraction on $L^2(\sigma)$, i.e., $B \in \mathcal{L}(L^2(\sigma), L^2(\sigma)), \|B\| \le 1$. Then we can define the operator $\operatorname{Exp} B$ as the contraction on $\operatorname{Exp} L^2(\sigma)$ given by

$$\operatorname{Exp} B \upharpoonright \operatorname{Exp}_n L^2(\sigma) \;:=\; B \otimes \cdots \otimes B \;\; (n \;\text{ times}), \;\; n \in I\!N,$$

$$\exp B \upharpoonright \operatorname{Exp}_0 L^2(\sigma) \;=\; 1.$$

For any self-adjoint positive operator A in $L^2(\sigma)$ we have a contraction semigroup e^{-tA}, $t \ge 0$, and it is possible to introduce a positive self-adjoint operator $d \operatorname{Exp} A$ as the generator of the semigroup $\operatorname{Exp}(e^{-tA}), t \ge 0$:

$$\operatorname{Exp}(e^{-tA}) \;=\; \exp(-t \; d \operatorname{Exp} A). \tag{5.17}$$

The operator $d\mathrm{Exp}A$ is called the second quantization of the one-particle operator A, see e.g. [RS, BK]. We denote by H_A^P the image of the operator $d \operatorname{Exp} A$ in the Poisson space $L^2(\pi_\sigma)$.

Theorem 5.1 *Let $\mathcal{D} \subset \mathrm{Dom}A$. Then the symmetric bilinear form corresponding to the operator H_A^P has the following representation*

$$(H_A^P F, G)_{L^2(\pi_\sigma)} \;=\; \int_{\Gamma_X} (\nabla^P F, A \nabla^P G)_{L^2(\sigma)} \pi_\sigma(d\gamma) \tag{5.18}$$

for all $F, G \in \mathcal{FP}(\mathcal{D}, \Gamma_X)$.

Remark 5.1 *The bilinear form (5.18) uses the Poissonian gradient ∇^P and a coefficient operator $A > 0$. We will call*

$$\mathcal{E}_{\pi_\sigma, A}^P(F, G) \;=\; \int_{\Gamma_X} (\nabla^P F, A \nabla^P G)_{L^2(\sigma)} \pi_\sigma(d\gamma)$$

the Poissonian pre-Dirichlet form with coefficient A.

Proof of Theorem 5.1. Note that for any $F \in \mathcal{FP}(\mathcal{D}, \Gamma_X)$ we have

$$F(\gamma) \;=\; g_F(\langle \varphi_1, \gamma\rangle, \dots, \langle \varphi_N, \gamma\rangle),$$

$$g_F \in \mathcal{P}(\mathbb{R}^N),\ \varphi_1, \dots, \varphi_N \in \mathcal{D},\ \gamma \in \Gamma_X,$$

and this function is a linear combination of monomials $\langle \varphi_1, \gamma\rangle^{\alpha_1} \cdots \langle \varphi_N, \gamma\rangle^{\alpha_N}$. Then for any $\gamma \in \Gamma_X$ the gradient $\nabla^P F(\gamma, x) = F(\gamma + \varepsilon_x) - F(\gamma)$ is a function in $\mathcal{D}$ which follows from the explicit form of ∇^P on monomials. Hence

$$(\nabla^P F, A\nabla^P G)_{L^2(\sigma)} \;\in\; \mathcal{FP}(\mathcal{D}, \Gamma_X)$$

and the form (5.18) is well-defined. We first prove (5.18) for polynomials

$$F(\gamma) \;:=\; Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \gamma),\ G(\gamma) \;:=\; Q_m(\psi^{\otimes m}, \gamma), \tag{5.19}$$

$$\varphi, \psi \in \mathcal{D},\ \gamma \in \Gamma_X,\ m, n \in \mathbb{Z}_+$$

which are images of vectors $\varphi^{\otimes n} \in \mathrm{Exp}_n L^2(\sigma)$ and $\psi^{\otimes m} \in \mathrm{Exp}_m L^2(\sigma)$ correspondingly. If $m \neq n$ both sides in (5.18) are equal to zero. For $m = n$ we have

$$(H_A^P F, G)_{L^2(\pi_\sigma)} \;=\; ((d\,\mathrm{Exp}\,A)F, G)_{\mathrm{Exp}\,L^2(\sigma)} \;=\; n(A\varphi, \psi)_{L^2(\sigma)}(\varphi, \psi)_{L^2(\sigma)}^{n-1} n!.$$

On the other hand equality (5.14) gives

$$\nabla^P Q_n^{\pi_\sigma}(\varphi^{\otimes n}, \gamma) \;=\; n\varphi(x) Q_{n-1}^{\pi_\sigma}(\varphi^{\otimes(n-1)}, \gamma)$$

and the r.h.s. in (5.18) has the form

$$n^2 \int_{\Gamma_X} (\varphi, A\psi)_{L^2(\sigma)} Q_{n-1}^{\pi_\sigma}(\varphi^{\otimes(n-1)}, \gamma) Q_{n-1}^{\pi_\sigma}(\psi^{\otimes(n-1)}, \gamma)\pi_\sigma(d\gamma)$$

$$=\; n(\varphi, A\psi)_{L^2(\sigma)} n!,$$

where we have used the orthogonality relation for Charlier polynomials. Hence (5.18) holds for all F, G as in (5.19).

For any generalized Appell system (and, in particular, for Charlier polynomials) the following lemma holds (see [KSS]):

Lemma 5.1 *Any cylinder polynomial $F \in \mathcal{FP}(\mathcal{D}, \Gamma_X)$ can be represented as a finite linear combination of polynomials of the form* (5.19).

As a result (5.18) is fulfilled on $\mathcal{FP}(\mathcal{D}, \Gamma_X)$.

□

Let us introduce also a Gaussian measure μ_σ on $\mathcal{D}'$ which is canonically associated with $L^2(\sigma)$ in the following sense

$$\int_{\mathcal{D}'} e^{\langle\varphi,\xi\rangle}\mu_\sigma(d\xi) \;=\; \exp\Big[\frac{1}{2}(\varphi,\varphi)_{L^2(\sigma)}\Big],\ \varphi\in\mathcal{D}.$$

Then $\operatorname{Exp} L^2(\sigma)$ can be realized via an isometric isomorphism as the Gaussian space $L^2(\mu_\sigma)$. Such an isomorphism is called the Itô–Segal isomorphism, see e.g. [BK]. The image of the operator $d\operatorname{Exp} A$ under this isomorphism is denoted by H^G_A. The symmetric bilinear form corresponding to the operator H^G_A has the representation (cf. [BK])

$$(H^G_A F, G)_{L^2(\mu_\sigma)} \;=\; \int_{\mathcal{D}'}(\nabla^L F(\xi), A\nabla^L G(\xi))_{L^2(\sigma)}\mu_\sigma(d\xi) \;=\; \mathcal{E}^G_{\mu_\sigma,A}(F,G),$$

$F, G\in\mathcal{FP}(\mathcal{D},\mathcal{D}')$. Here (as before) ∇^L means the “flat” gradient on functions $F:\mathcal{D}'\longrightarrow I\!R$, as introduced in Subsection 3.1. As a result we have an isomorphism between Poissonian and Gaussian spaces (via the Fock space and corresponding chaos decomposition). In this picture the “flat” gradient ∇^L (related to the linear geometry of $\mathcal{D}'$) is transformed into the Poissonian gradient ∇^P on Γ_X.

5.4 The intrinsic Dirichlet operator as a second quantization

Let us consider the special case of the second quantization operator $d\operatorname{Exp} A$ where the one-particle operator A coincides with the Dirichlet operator H^X_σ generated by the measure σ on X.

Theorem 5.2 *We have the equality*

$$H^P_{H^X_\sigma} \;=\; H^\Gamma_{\pi_\sigma} \tag{5.20}$$

on the dense domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$. In particular, for all $F, G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_X)$

$$\int_{\Gamma_X}\langle\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma_X)}\pi_\sigma(d\gamma)$$

$$= \int_{\Gamma_X} (\nabla^P F(\gamma), H_\sigma^X \nabla^P G(\gamma))_{L^2(\sigma)} \pi_\sigma(d\gamma), \tag{5.21}$$

or

$$\nabla^{\Gamma *} \nabla^\Gamma = \nabla^{P*} H_\sigma^X \nabla^P \tag{5.22}$$

as an equality on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$.

Proof. Let us take any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ of the form (4.2). Then using $H_\sigma^X = -\mathrm{div}_\sigma^X \nabla^X$ we obtain

$$\mathcal{E}^P_{\pi_\sigma, H_\sigma^X}(F, F)$$

$$= \int_{\Gamma_X} \left(\int_X \langle \nabla^X [F(\gamma + \varepsilon_x)], \nabla^X [F(\gamma + \varepsilon^x)] \rangle_{T_x(X)} \sigma(dx) \right) \pi_\sigma(d\gamma) \tag{5.23}$$

$$= - \int_{\Gamma_X} \int_X F(\gamma + \varepsilon_x) \mathrm{div}_\sigma^X \{\nabla^X [F(\gamma + \varepsilon_x)]\} \sigma(dx) \pi_\sigma(d\gamma)$$

An application of (3.7) and (3.12) gives

$$\mathrm{div}_\sigma^X \{\nabla^X [F(\gamma + \varepsilon_x)]\}$$

$$= \sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j} (\gamma + \varepsilon_x) \langle \nabla^X \varphi_i(x), \nabla^X \varphi_j(x) \rangle_{T_x(X)} \tag{5.24}$$

$$- \sum_{j=1}^N \frac{\partial g_F}{\partial s_j} (\gamma + \varepsilon_x)(H_\sigma^X \varphi_j)(x).$$

To finish the computation we need the following useful lemma (see e.g. [Ka]):

Lemma 5.2

$$\int_{\Gamma_X} \left(\int_X G(\gamma + \varepsilon_x) \psi(x) \sigma(dx) \right) \pi_\sigma(d\gamma) = \int_{\Gamma_X} G(\gamma) < \psi, \gamma > \pi_\sigma(d\gamma) \tag{5.25}$$

for $G : \Gamma_X \longrightarrow I\!R, \psi : X \longrightarrow I\!R$ *provided the integrals in* (5.25) *make sense. In particular,* (5.25) *holds for* $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ *and* $\psi \in L^2(\sigma)$.

Using (5.24) in the r.h.s. of (5.23) and applying Lemma 5.1 we transform (5.23) into

$$- \int_{\Gamma_X} F(\gamma) \left(\sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\gamma) \langle \langle \nabla^X_{\varphi_i}, \nabla^X_{\varphi_j} \rangle_{T(X)}, \gamma \rangle \right.$$

$$- \sum_{j=1}^{N} \frac{\partial g_F}{\partial s_j}(\gamma)\langle H_\sigma^X \varphi_j, \gamma\rangle \Bigg) \pi_\sigma(d\gamma)$$
$$= \int_{\Gamma_X} F(\gamma)(H_{\pi_\sigma}^\Gamma F)(\gamma)\pi_\sigma(d\gamma),$$

where we have used (4.7).

□

Remark 5.2 *Equality* (5.20) *can be proven in the same way on the set* $\mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_X)$.

Theorem 5.3 *Suppose that the* H_σ^X *is essentially self-adjoint on the domain* $\mathcal{D} \subset \mathrm{Dom}(H_\sigma^X)$, *i.e., the closure of* H_σ^X *in* $L^2(\sigma)$ *from the domain* $\mathcal{D}$ *is a self-adjoint operator. Then the intrinsic Dirichlet operator* $H_{\pi_\sigma}^\Gamma$ *is an essentially self-adjoint operator in* $L^2(\pi_\sigma)$ *on the domain* $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$.

Proof. Any second quantization operator H_A^P is essentially self-adjoint on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ if the coefficient operator A has $\mathcal{D}$ as an essential domain. This immediately follows from the essential self-adjointness of any restriction (see e.g. [BK])

$$d\mathrm{Exp} A \upharpoonright a.\mathcal{D}^{\hat{\otimes} n} \ \subset \ \mathrm{Exp}_n L^2(\sigma), \ n \in \mathbb{Z}_+,$$

where $a.\mathcal{D}^{\hat{\otimes} n}$, i.e., the n-th symmetric algebraic tensor power of $\mathcal{D}$, is equal to

$$l.h.\{\varphi_1 \hat{\otimes} \cdots \hat{\otimes} \varphi_n | \varphi_j \in \mathcal{D}\},$$

and the fact that the image of $\bigoplus_{n=0}^\infty a.\mathcal{D}^{\hat{\otimes} n}$ in $L^2(\pi_\sigma)$ coincides with $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ [KSS].

Equality (5.20) then means that $H_{\pi_\sigma}^\Gamma$ is essentially self-adjoint on $\mathcal{F}\mathcal{P}(\mathcal{D}, \Gamma_X)$. Having $\mathcal{F}\mathcal{P}(\mathcal{D}, \Gamma_X)$ as an essential domain for $H_{\pi_\sigma}^\Gamma$ we obtain the essential self-adjointness on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ using standard approximation of any $F \in \mathcal{F}\mathcal{P}(\mathcal{D}, \Gamma_X)$ by a sequence $(F_j)_{j=1}^\infty \ \subset \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_X)$ such that $F_j \longrightarrow F$, $H_{\pi_\sigma}^\Gamma F_j \longrightarrow H_{\pi_\sigma}^\Gamma F$, $j \longrightarrow \infty$ in $L^2(\pi_\sigma)$.

□

Remark 5.3 *Note that in the case of a general second quantization operator* H_A^P *instead of* $\mathcal{D}$ *we can use any linear subset* $\mathcal{L} \subset L^2(\sigma)$ *which is an essential domain for A. This implies the essential self-adjointness of* H_A^P *on the* *set $\mathcal{FP}(\mathcal{L},\Gamma_X)$ which is defined as before with $\mathcal{L}$ replacing $\mathcal{D}$. In particular, the intrinsic Dirichlet operator $H^\Gamma_{\pi_\sigma}$ is essentially self-adjoint on $\mathcal{FP}(\mathcal{L},\Gamma_X)$ where $\mathcal{L}\subset L^2(\sigma)$ is any essential domain for the underlying Dirichlet operator H^X_σ.*

6 Distorted Brownian motion on the configuration space

In this section we shall prove the existence of diffusion processes corresponding to our Dirichlet forms $(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}},\mathcal{D}(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}))$ and identify them as the well-known independent infinite particle processes on X, at least if $(\mathcal{E}^X_\sigma, D(\mathcal{E}^X_\sigma))$ is conservative. After all our preparations the proofs are quite standard using the results in [MR1]. So, we are quite brief here. In this section we fix $\mu_{\lambda,\sigma}\in\mathcal{M}_2(\Gamma_X)$ as in (3.23) and assume X to be complete.

6.1 Existence and continuity of sample paths.

The desired processes in general will live on the bigger state space $\ddot{\Gamma}_X$ consisting of all $\mathbb{Z}_+$-valued Radon measures on X (which is Polish, see e.g. [Ka]). To state the corresponding existence result precisely we need Definition 6.1 below. Set $(\mathcal{E},D(\mathcal{E})) := (\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}))$. Since $\Gamma_X\subset\ddot{\Gamma}_X$ and $\mathcal{B}(\ddot{\Gamma}_X)\cap\Gamma_X=\mathcal{B}(\Gamma_X)$, we can consider $\mu_{\lambda,\sigma}$ as a measure on $(\ddot{\Gamma}_X,\mathcal{B}(\ddot{\Gamma}_X))$ and correspondingly $(\mathcal{E},D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}_X;\mu_{\lambda,\sigma})$.

Definition 6.1 (*cf.* [MR1, Ch.III, Definitions 2.1 and 3.2]).

(i) *A sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $\ddot{\Gamma}_X$ is called an $\mathcal{E}$-nest, if*

$$\bigcup_{n\in\mathbb{N}}\{F\in D(\mathcal{E})|F=0\ \mu_{\lambda,\sigma}\text{-}a.e.\ on\ \ddot{\Gamma}_X\setminus F_n\}$$

is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_1^{1/2}(:=[\mathcal{E}+(\ ,\ )_{L^2(\mu_{\lambda,\sigma})}]^{1/2})$.

(ii) *A set $N\subset\ddot{\Gamma}_X$ is called $\mathcal{E}$-exceptional, if $N\subset\ddot{\Gamma}_X\setminus\bigcup_{n\in N}F_n$ for some $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$. We say that a property of points in $\ddot{\Gamma}_X$ holds $\mathcal{E}$-quasi-everywhere (abbeviated $\mathcal{E}$-q.e.) if it holds outside some $\mathcal{E}$-exceptional set.*

(iii) *A function* $f : \ddot{\Gamma}_X \to I\!R$ *is called* $\mathcal{E}$*-quasi-continuous if there exists an* $\mathcal{E}$*-nest* $(F_n)_{n\in I\!N}$ *such that the restriction* $f_{|F_n}$ *of* f *to* F_n *is continuous for all* $n \in I\!N$.

We recall that the notion "$\mathcal{E}$-q.e." is much finer than "$\mu_{\lambda,\sigma}$-a.e." (cf. [MR1, Exercise 2.6 (ii)]).

Theorem 6.1 *There exists a conservative diffusion process*

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F})_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_X})$$

on $\ddot{\Gamma}_X$ (*cf.* [Dy]) *which is properly associated with* $(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}))$, *i.e., for all* ($\mu_{\lambda,\sigma}$*-versions of*) $F \in L^2(\ddot{\Gamma}_X, \mu_{\lambda,\sigma})$ *and all* $t > 0$ *the function*

$$\gamma \ \mapsto \ p_t F(\gamma) \ := \ \int_\Omega F(X_t) d\mathbf{P}_\gamma, \ \ \gamma \in \ddot{\Gamma}_X, \tag{6.1}$$

is an $\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}$*-quasi-continuous version of* $\exp(-tH^{\Gamma}_{\mu_{\lambda,\sigma}})F$. $\mathbf{M}$ *is up to* $\mu_{\lambda,\sigma}$*-equivalence unique* (*cf.* [MR1, Ch. IV, Sect. 6]). *In particular,* $\mathbf{M}$ *is* $\mu_{\lambda,\sigma}$ *symmetric* (*i.e.,* $\int G\, p_t F d\mu_{\lambda,\sigma} = \int F\, p_t G\, d\mu_{\lambda,\sigma}$ *for all* $F, G : \ddot{\Gamma}_X \to I\!R_+, \mathcal{B}(\ddot{\Gamma}_X)$*-measurable*) *and has* $\mu_{\lambda,\sigma}$ *as an invariant measure.*

Proof. The proof follows directly from [MR1, Ch.V, Theorem 1.11] since by [MR2], [AKMR] (see also [Yo, Definition 3, Remark 2, and Theorem 1] for the case $X = I\!R^d$, $\sigma = m$) $(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}))$ is a quasi-regular and local Dirichlet form.

□

In the above theorem $\mathbf{M}$ is canonical, i.e., $\mathbf{\Omega} = C([0,\infty) \to \ddot{\Gamma}_X)$, $\mathbf{X}_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, $(\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [F80, Sect.4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

6.2 Identification with the independent infinite particle process

Let $(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}))$, $\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_X})$ be as in Theorem 6.1. As usual we define

$$\mathbf{P}_{\mu_{\lambda,\sigma}} \ := \ \int \mathbf{P}_\gamma \ \mu_{\lambda,\sigma}(d\gamma). \tag{6.2}$$

Let $M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in X})$ be the diffusion process on X associated with the Dirichlet form $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ (cf. e.g. [MR1, Ch.IV, Subsect.4 a) resp. Ch. II, Subsect. 2 a)]). Set

$$P_\sigma \; := \; \int_X P_x \; \sigma(dx). \tag{6.3}$$

In order to identify $\mathbf{M}$ as the independent infinite particle process, we have to show that $\mathbf{P}_{\mu_{\lambda,\sigma}}$ has the same finite dimensional distributions as $\int_{\mathbb{R}_+} \pi_{sP_\sigma}\, \lambda(ds)$, where π_{sP_σ} is the Poisson measure on $\mathbf{\Omega}$ with intensity $sP_\sigma, s \geq 0$ (cf. e.g. [ST, Ml]). For this it suffices to prove the following proposition whose proof is straight-forward by Proposition 4.1 (see also [ST, Proposition 1.5]).

Proposition 6.1 *Assume that condition* (C) *in Proposition* 4.1 *holds and that* $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ *is conservative. Then for all* $\varphi_1, \ldots, \varphi_N \in \mathcal{D}_1$, $0 \leq t_1 < \cdots < t_N < \infty$

$$\begin{aligned} &\int_{\mathbf{\Omega}} \exp(\langle \log(1+\varphi_1), \mathbf{X}_{t_1}\rangle) \cdots \exp(\langle \log(1+\varphi_N), \mathbf{X}_{t_N}\rangle) d\mathbf{P}_{\mu_{\lambda,\sigma}} \\ &= \int \exp\left(s \int_\Omega [(1+\varphi_1(X_{t_1}))\cdots(1+\varphi_N(X_{t_N})) - 1] dP_\sigma\right) \lambda(ds). \end{aligned} \tag{6.4}$$

Proof. If $N=1$, then by Proposition 4.1

$$\begin{aligned} \int_{\mathbf{\Omega}} \exp(\langle \log(1+\varphi_1), \mathbf{X}_{t_1}\rangle) d\mathbf{P}_{\mu_{\lambda,\sigma}} \; &= \; \int_{\Gamma_X} \exp(\langle \log(1+e^{-t_1 H_\sigma^X}\varphi_1), \gamma\rangle) \mu_{\lambda,\sigma}(d\gamma) \\ = \int \exp[s\int_X e^{-t_1 H_\sigma^X}\varphi_1 \; d\sigma]\lambda(ds) \; &= \; \int \exp[s \int_\Omega \varphi_1(X_{t_1}) dP_\sigma]\lambda(ds). \end{aligned}$$

Suppose (6.4) holds for $N-1$. Then by Proposition 4.1 and the Markov property of $\mathbf{P}_{\mu_{\lambda,\sigma}}$ the left hand side of (6.4) is equal to

$$\begin{aligned} &\int_{\mathbf{\Omega}} \exp(\langle \log(1+\varphi_1), \mathbf{X}_{t_1}\rangle) \cdots \exp(\langle \log(1+\varphi_{N-1}), \mathbf{X}_{t_{N-1}}\rangle) \\ &\qquad \times \; \exp(\langle \log(1+e^{-t_N H_\sigma^X}\varphi_N), \mathbf{X}_{t_{N-1}}\rangle) d\mathbf{P}_{\mu_{\lambda,\sigma}}. \end{aligned}$$

This in turn by induction hypothesis is equal to

$$\begin{aligned} &\int \exp\left(s \int_\Omega [(1+\varphi_1(X_{t_1}))\cdots(1+\varphi_{N-1}(X_{t_{N-1}})) \right. \\ &\qquad \left. \times \; (1 + e^{-t_N H_\sigma^X}\varphi_N(X_{t_{N-1}})) - 1] dP_\sigma\right) \lambda(ds) \end{aligned}$$

which by the Markov property of P_σ is just the right hand side of (6.4).

$\square$

Remark 6.1 *The diffusion process M associated to $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ is usually called distorted Brownian motion on X. Therefore, we call the diffusion associated to $(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}))$ distorted Brownian motion on Γ_X because of the complete analogy. Likewise, since M is just ordinary Brownian motion on X if $\sigma = m$, it is justified to call* $\mathbf{M}$ *Brownian motion on Γ_X in this case. Therefore, by Proposition* 6.1 *we have shown that the usual independent particle process known for more than* 40 *years (cf.* [D, Do 1]) *is nothing but (distorted) Brownian motion on Γ_X.*

6.3 Time-ergodicity

In this subsection we shall use Theorem 4.3. Hence we assume that $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ is conservative and that $\rho(:= d\sigma/dm)$ is C^∞ and strictly positive resp. that the weaker assumptions indicated in Remark 4.6 hold. Let $(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}))$, $\mathbf{M}$ be as in Theorem 6.1. Recall that $\mathbf{P}_{\mu_{\lambda,\sigma}}$ (cf. (6.2)) is called (time) ergodic if any bounded $\mathbf{F}$–measurable function $G : \mathbf{\Omega} \to I\!R$, which is $\mathbf{\Theta}_t$-invariant for all $t \geq 0$, is constant $\mathbf{P}_{\mu_{\lambda,s}}$-a.e.

Theorem 6.2 $\mathbf{P}_{\mu_{\lambda,\sigma}}$ *is (time) ergodic if and only if $\mu_{\lambda,\sigma} = \pi_{s\sigma}$ for some $s \geq 0$ (resp. one of the other equivalent assertions in Theorem* 4.3 *holds). In this case, if $(p_t)_{t>0}$ is as defined in* (6.1), *then $\lim_{t\to\infty} p_t F = \int F d\pi_{s\sigma}$ $\mathcal{E}^\Gamma_{\mu_{\lambda,\sigma}}$-q.e. for all bounded $\mathcal{B}(\ddot{\Gamma}_X)$ - measurable functions $F : \ddot{\Gamma}_X \to I\!R$.*

Proof. This follows by Theorem 4.3 and the regularization method in [MR1, Ch. VI, Sect. 1] directly from the main results in [F 1, F 2].

□

Remark 6.2 *It is immediate from Theorem* 6.2 *and Remark* 4.4 (i) *that at least if $\exp(-tH_\sigma^X)\varphi$ has a continuous σ-version for all $t > 0$, $\varphi \in \mathcal{D}$ (which is e.g. the case if $\sigma = m$), then $(\mathcal{E}^\Gamma_{\pi_{s_i\sigma}}, D(\mathcal{E}^\Gamma_{\pi_{s_i\sigma}}))$, $i = 1, 2$, have disjoint $\mathcal{E}$-nests provided $s_1 \neq s_2$. More precisely, in this case there exist $\mathcal{E}_{\pi_{s_i\sigma}}$-nests $(F_n^{(i)})_{n\in I\!N}, i = 1, 2$ such that $\bigcup_{n\in I\!N} F_n^{(1)} \cap \bigcup_{n\in I\!N} F_n^{(2)} = \emptyset$.*

7 Appendix: Proof of Proposition 4.1

We first recall that for all $\varphi \in L^2(\sigma) \cap L^1(\sigma)$ by (3.23) and (4.4) we have

$$\int \langle\varphi, \gamma\rangle^2 \mu_{\lambda,\sigma}(d\gamma) \;=\; \int s\,\lambda(ds) \int \varphi^2 d\sigma + \int s^2 \lambda(ds) \left(\int \varphi\, d\sigma\right)^2. \qquad (7.1)$$

We also recall the following well-known result:

Lemma 7.1 *Let μ be a σ-finite measure on a measurable space $(E, \mathcal{B})$ and $p \in [1, \infty)$. Let $f \in C_b^1(I\!R)$ and let $I\!R_+ \ni t \mapsto \psi_t \in L^p(E;\mu)$ be a continuously differentiable map. Then so is $I\!R_+ \ni t \mapsto f \circ \psi_t \in L^p(E;\mu)$ and $d(f \circ \psi_t)/dt = f'(\psi_t)(d\psi_t/dt)$.*

Proof. Since $L^p(E, \mu) \ni g \mapsto \int h\ f(g) d\mu$ is differentiable for all $h \in L^q(E, \mu), q \in (1, \infty], p^{-1} + q^{-1} = 1$, the assertion follows immediately by the fundamental theorem of calculus.

□

For $\varphi \in \mathcal{D}_1$ (cf. (4.13)) define

$$\varphi_t \ := \ e^{-tH_\sigma^X} \varphi, \quad t \geq 0.$$

Lemma 7.2 *Let $\varphi \in \mathcal{D}_1$. Then $I\!R_+ \ni t \mapsto \langle \log(1 + \varphi_t), \cdot \rangle \in L^2(\mu_{\lambda,\sigma})$ is differentiable with derivative equal to*

$$-\langle (1 + \varphi_t)^{-1} H_\sigma^X \varphi_t, \cdot \rangle, \ t \in I\!R_+.$$

Proof. Let $t > 0$ and $\varepsilon > 0$. Then by (7.1) (since $H_\sigma^X \varphi_t \in L^1(\sigma)$)

$$\int \left(\frac{1}{\varepsilon} (\langle \log(1 + \varphi_{t+\varepsilon}), \gamma \rangle - \langle \log(1 + \varphi_t), \gamma \rangle) + \langle (1 + \varphi_t)^{-1} H_\sigma^X \varphi_t, \gamma \rangle \right)^2 \mu_{\lambda,\sigma}(d\gamma)$$

$$= \int s\ \lambda(ds) \int A(\varepsilon, t)^2 d\sigma + \int s^2\ \lambda(ds) \left(\int A(\varepsilon, t)\ d\sigma \right)^2,$$

where

$$A(\varepsilon, t) \ := \ \frac{1}{\varepsilon} (\log(1 + \varphi_{t+\varepsilon}) - \log(1 + \varphi_t)) + (1 + \varphi_t)^{-1} H_\sigma^X \varphi_t.$$

But since $I\!R_+ \ni s \mapsto H_\sigma^X \varphi_s = e^{-sH_\sigma^X} H_\sigma^X \varphi \in L^1(\sigma)$ is continuous it follows that

$$\int A(\varepsilon, t)\ d\sigma \ = \ -\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int \frac{H_\sigma^X \varphi_s}{1 + \varphi_s} d\sigma\ ds \ + \ \int \frac{H_\sigma^X \varphi_t}{1 + \varphi_t} d\sigma \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Now the assertion follows by Lemma 7.1.

□

Lemma 7.3 *Let $\varphi \in \mathcal{D}_1$. Then*

$$\exp(\langle \log(1+\varphi), \cdot\rangle) \;\in\; D(H^{\Gamma}_{\mu_{\sigma,\lambda}})$$

and

$$H^{\Gamma}_{\mu_{\sigma,\lambda}}(\exp(\langle \log(1+\varphi), \cdot\rangle) \;=\; \langle \frac{H^X_\sigma \varphi}{1+\varphi}, \cdot\rangle \exp(\langle \log(1+\varphi), \cdot\rangle). \qquad (7.2)$$

Proof. Since $(T^{\Gamma}_{\mu_{\lambda,\sigma}}(t))_{t>0}$ is symmetric and sub-Markov (because $(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}, D(\mathcal{E}^{\Gamma}_{\mu_{\lambda,\sigma}}))$ is a Dirichlet form), it extends to a strongly continuous contraction semigroup on $L^1(\mu_{\lambda,\sigma})$. Therefore, since $\exp(\langle \log(1+\varphi), \cdot\rangle)$ and the right hand side of (7.2) are in $L^2(\mu_{\lambda,\sigma})$, it suffices to prove that $\exp(\langle \log(1+\varphi), \cdot\rangle) \in D(H^{\Gamma}_{\mu_{\lambda,\sigma},1})$, where $H^{\Gamma}_{\mu_{\lambda,\sigma},1}$ is the $L^1(\mu_{\lambda,\sigma})$-generator of $(T^{\Gamma}_{\mu_{\lambda,\sigma}}(t))_{t>0}$, and that (7.2) holds with $H^{\Gamma}_{\mu_{\lambda,\sigma},1}$ replacing $H^{\Gamma}_{\mu_{\lambda,\sigma}}$.

By condition (C) there exist $\psi_n \in \mathcal{D}, n \in I\!N$, such that

$$\psi_n \to \varphi,\; H^X_\sigma \psi_n \to H^X_\sigma \varphi \text{ in } L^2(\sigma) \text{ as } n \to \infty. \qquad (7.3)$$

Let $\delta, \delta' \in (0,1)$ such that $-1 < -\delta' < -\delta \leq \varphi$, and let $\chi \in C_0^\infty(I\!R)$ such that $-\delta' \leq \chi$ and $\chi(s) = s$ for all $s \in [-\delta, 1]$. Then for $n \in I\!N$ we have that $\varphi_n := \chi \circ \psi_n \in \mathcal{D}$, hence $\log(1+\varphi_n) \in \mathcal{D}$. Clearly,

$$\int \langle (\log(1+\varphi_n) - \log(1+\varphi))^2, \gamma\rangle \mu_{\lambda,\sigma}(d\gamma)$$

$$= \int (\log(1+\varphi_n) - \log(1+\varphi))^2 d\sigma \int s\; \lambda(ds) \;\longrightarrow_{n\to\infty}\; 0.$$

Hence for any $\xi \in C^\infty(I\!R)$ such that $\xi(s) = s$ for all $s \in]-\infty, 0[$ and $\operatorname{supp}\xi \cap [0,\infty[$ compact it easily follows that

$$\exp(\langle \log(1+\varphi_n), \cdot\rangle) \;\to\; \exp\circ\xi(\langle \log(1+\varphi), \cdot\rangle)$$

in $L^1(\mu_{\lambda,\sigma})$.

Furthermore, by (4.9) resp. Remark 4.3 (since $H^{\Gamma}_{\mu_{\lambda,\sigma},1}$ extends $H^{\Gamma}_{\mu_{\lambda,\sigma}}$)

$$\begin{aligned} H^{\Gamma}_{\mu_{\lambda,\sigma},1}(\exp\circ\xi(\langle \log(1+\varphi_n), \cdot\rangle)) \;=\; & \Big(\langle \frac{H^X_\sigma \varphi_n}{1+\varphi_n}, \cdot\rangle \cdot \xi'(\langle \log(1+\varphi_n), \cdot\rangle) \\ & + \Big[\xi'(\langle \log(1+\varphi_n), \cdot\rangle) - \xi''(\langle \log(1+\varphi_n), \cdot\rangle) \\ & -(\xi')^2(\langle \log(1+\varphi_n), \cdot\rangle)\Big] \langle \frac{|\nabla^X \varphi_n|^2_{T(X)}}{(1+\varphi_n)^2}, \cdot\rangle\Big) \\ & \cdot \exp\circ\xi(\langle \log(1+\varphi_n), \cdot\rangle). \qquad (7.4) \end{aligned}$$

We have that for all $n \in I\!N$

$$H_\sigma^X \varphi_n = \chi'(\psi_n) H_\sigma^X \psi_n + \chi''(\psi_n) |\nabla^X \psi_n|^2_{T(X)}. \tag{7.5}$$

Since $\chi''(\varphi) = 0$, (7.3) implies that

$$\frac{\chi''(\psi_n)}{1+\varphi_n} |\nabla^X \psi_n|^2_{T(X)} \;\to\; 0 \text{ in } L^1(\sigma) \text{ as } n \to \infty.$$

Hence

$$\langle \frac{\chi''(\psi_n)}{1+\varphi_n} |\nabla^X \psi_n|^2_{T(X)}, \cdot \rangle \;\to 0 \;\text{ in } L^1(\mu_{\lambda,\sigma}) \text{ as } n \to \infty. \tag{7.6}$$

Furthermore, by (7.1)

$$\int \langle \frac{H_\sigma^X \varphi}{1+\varphi} - \frac{\chi'(\psi_n)}{1+\varphi_n} H_\sigma^X \psi_n, \gamma \rangle^2 \mu_{\lambda,\sigma}(d\gamma)$$

$$= \int s \; \lambda(ds) \; \int (\frac{H_\sigma^X \varphi}{1+\varphi} - \frac{\chi'(\psi_n)}{1+\chi(\psi_n)} H_\sigma^X \psi_n)^2 d\sigma$$

$$+ \int s^2 \; \lambda(ds) (\int \frac{H_\sigma^X \varphi}{1+\varphi} \; d\sigma - \int \frac{\chi'(\psi_n)}{1+\chi(\psi_n)} H_\sigma^X \psi_n \; d\sigma)^2$$

which converges to zero as $n \to \infty$, since

$$\int \frac{\chi'(\psi_n)}{1+\chi(\psi_n)} H_\sigma^X \psi_n \; d\sigma$$

$$= \int \frac{\chi''(\psi_n)(1+\chi(\psi_n)) - \chi'(\psi_n)^2}{(1+\chi(\psi_n))^2} |\nabla^X \psi_n|^2_{T(X)} \; d\sigma$$

$$\to_{n\to\infty} - \int |\nabla^X \log(1+\varphi)|^2_{T(X)} \; d\sigma = \int \frac{H_\sigma^X \varphi}{1+\varphi} \; d\sigma$$

(because $(\mathcal{E}_\sigma^X, D(\mathcal{E}_\sigma^X))$ is conservative and $\chi(\varphi) = \varphi, \; \chi'(\varphi) = 1$). Therefore, since $\mu_{\lambda,\sigma}$ is a probability measure,

$$\langle \frac{\chi'(\psi_n)}{1+\varphi_n} H_\sigma^X \psi_n, \cdot \rangle \;\to\; \langle \frac{H_\sigma^X \varphi}{1+\varphi}, \cdot \rangle \;\text{ in } L^1(\mu_{\lambda,\sigma}) \text{ as } n \to \infty. \tag{7.7}$$

(7.4) – (7.7) now imply the assertion.

$\square$

Proof of Proposition 4.1. Assertion (ii) is exactly Lemma 7.3. So, it remains to prove (i). Let $\varphi \in \mathcal{D}_1$. Then by Lemmas 7.1, 7.2

$$I\!R_+ \ni t \;\mapsto\; P(t)\exp(\langle \log(1+\varphi), \cdot \rangle) \in L^2(\mu_{\lambda,\sigma})$$

is differentiable with derivative equal to

$$-\langle \frac{H_\sigma^X \varphi_t}{1+\varphi_t}, \cdot \rangle \exp(\langle \log(1+\varphi_t), \cdot \rangle), \quad t \in I\!R.$$

By Lemma 7.3 this is equal to

$$-H^\Gamma_{\mu_{\lambda,\sigma}} \exp(\langle \log(1+\varphi_t), \cdot \rangle).$$

Since $P(0)$ is the identity, the assertion follows.

$\square$

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References

[AKR 1] S. Albeverio, Yu. G. Kondratiev, M. Röckner, Differential geometry of Poisson spaces, *C. R. Acad. Sci. Paris*, **323**, 1129–1134, 1996.

[AKR 2] S. Albeverio, Yu. G. Kondratiev, M. Röckner, Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces, *C. R. Acad. Sci. Paris*, **323**, 1179–1184, 1996.

[AKR 3] S. Albeverio, Yu. G. Kondratiev, M. Röckner, Ergodicity of L^2-semigroups and extremality of Gibbs states, to appear in *J. Funct. Anal.*, 1997.

[AKR 4] S. Albeverio, Yu. G. Kondratiev, M. Röckner, Stochastic dymamics for classical continuous systems, in preparation, 1997.

[AKR 5] S. Albeverio, Yu. G. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces: The Gibbsian case, SFB–343–Preprint, 1997.

[AKMR] S. Albeverio, Yu. G. Kondratiev, Z.-M. Ma, M. Röckner, Stochastic dynamics for quantum continuous system, in preparation, 1997.

[Au] T. Aubin *Nonlinear analysis on manifolds. Monge-Ampère equations.*, Springer-Verlag, 1982.

[Ba] D. Bakry, Étude des transformations de Riesz dans les variétés a courbure minorée, Lect. Notes Math. 1247, 137–172, Springer-Verlag, 1987.

[BK] Yu. M. Berezansky, Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, Kluwer Academic Publishers, Holland, 1995.

[BKR] V. I. Bogachev, N. V. Krylov, M. Röckner, Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^d$, to appear in *Ann. Scuola Norm. di Pisa*, 1997.

[Bo] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Academic Press, 1975.

[Ch] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, 1984.

[CP] E. Carlen, E. Pardoux, Differential calculus and integration by parts on Poisson space, in *Stochastics, algebra and analysis in classical and quantum dynamics*, S. Albeverio et al. eds., Kluwer, 1990.

[D] J. L. Doob, *Stochastic processes*, New York – London, 1953.

[Do 1] R. L. Dobrushin, On Poisson distribution of particles in space, *Ukr. Math. J.*, **8**, 127–134, 1956.

[Do 2] R. L. Dobrushin, Gibbs random fields for particles without hard core, *Theor. Math. Phys.*, **4**, 101–118, 1970.

[Dy] E. B. Dynkin, *Markov processes*, Springer Verlag, 1965.

[Eb] A. Eberle, Weak Sobolev spaces and Markov uniqueness of operators, *C. R. Acad. Sci. Paris*, **320**, 1249–1254, 1995.

[F 1] M. Fukushima, A note on irreducibility and ergodicity of symmetric Markov processes, in Lect. Notes Phys. **173**, 200–207, Springer-Verlag, 1982.

[F 2] M.Fukushima, Capacitary maximal inequalities and an ergodic theorem, in Lect. Notes Math. 1021, Springer-Verlag, 1983.

[Ge] H. O. Georgii, *Canonical Gibbs measures*, LNM 760, Springer-Verlag, 1979

[GGPS] G. Goldin, K. J. Grodnik, R. T. Powers and D. H. Sharp, Nonrelativistic current algebra in the N/V limit, *J. Math. Phys.*, **15**, 88–100, 1974.

[GV] I. M. Gelfand, N. Ya. Vilenkin, *Some applications of harmonic analysis. Rigged Hilbert spaces*, Academic Press, 1964.

[IK] Y. Ito, I. Kubo, Calculus on Gaussian and Poisson white noises, *Nagoya Math. J.*, 111, 41-84, 1988.

[Ka] O. Kallenberg, *Random measures*, Academic Press, 1976.

[KLRRS] Yu. G. Kondratiev, E. W. Lytvynov, A. L. Rebenko, M. Röckner, G. V. Schepan'uk , Euclidean Gibbs states for continuous quantum systems with Boltzmann statistics, in preparation, 1997.

[KSS] Yu. G. Kondratiev, L. Silva, L. Streit, Generalized Appell systems, to appear in *Methods Funct. Anal. and Topology*, 1997.

[Mal] P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, in *Proc. Int. Sympos. on Stoch. Diff. Eqs.*, Kyoto, 1976, 195–263, Wiley, 1978.

[MKM] K. Matthes, J. Kerstan, J. Mecke, *Infinite divisible point processes*, Akademie-Verlag, Berlin, 1978.

[M-L] A. Martin-Löf, Limit theorems for the motion of a Poisson system of independent Markovian particles with high density, *Z. Wahrsch. verw, Gebiete*, **34**, 205-223, 1976.

[MR1] Z.-M. Ma, M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag, 1992.

[MR2] Z.-M. Ma, M. Röckner, *Construction of diffusion processes on configuration spaces*, Preprint 1997.

[Nu] D. NUALART *The Malliavin calculus and related topics*, Springer, 1995.

[NV] D. NUALART, J. VIVES, A duality formula on the Poisson space and some applications, Barcelona University preprint, 1995.

[NZ] X. X. NGUYEN, H. ZESSIN, Martin-Dynkin boundary of mixed Poisson processes, *Z. Wahrsch. verw. Gebiete*, **37**, 191–200, 1977.

[Ob] N. OBATA, Configuration space and unitary representations of the group of diffeomorphisms, *RIMS Kokyoku*, **615**, 129–153, 1987.

[Os] H. OSADA, Dirichlet form approach to infinite–dimensional Wiener process with singular interactions, *Comm. Math. Phys.*, **176**, 117–131, 1996.

[Pa] K .R. PARTHASARATHY, *Probability measures on metric spaces*, Academic Press, 1967.

[Pr] C. J. PRESTON, *Random fields*, LNM 534, Springer, 1976.

[Pri] N. PRIVAULT, A transfer principle from Wiener to Poisson space and applications, *J. Funct. Anal.*, **132**, 335–360, 1995.

[R] M. RÖCKNER, Dirichlet forms on infinite-dimensional "manifold-like" state spaces: a survey of recent results and some prospects for the future, SFB 343, Preprint 96–026, to appear in Proceedings to "Probability towards 2000," New York, October 1995.

[Ra] B. RAMACHANDRAN, *Advanced theory of characteristic functions*, Stat. Publ. Soc., 1967.

[RS] M. REED, B. SIMON, *Methods of modern mathematical physics: Functional Analysis*, vol. 1, Academic Press, 1977.

[Sh] H. SHIMOMURA, Poisson measures on the configuration space and unitary representations of the group of diffeomorphism, *J. Math. Kyoto Univ.*, **34**, 599–614, 1994.

[Sk] A. V. SKOROKHOD, On the differentiability of measures which correspond to stochastic processes I. Processes with independent increments, *Theria Verojat. Primen.*, **2**, 418–444, 1957.

[St] W. Stannat, The theory of generalized Dirichlet forms and its applications in analysis and stochastics. Doktorarbeit Universität Bielefeld, 1996.

[ST] T. Shiga, Y. Takahashi, Ergodic properties of the equilibrium process associated with infinitely many Markovian particles, *Publ. RIMS, Kyoto Univ.* **9**, 505–516 1974.

[T] M. Takeda, On a martingale method for symmetric diffusion processes and its applications, *Osaka J. Math.*, **26**, 605–623, 1989

[Ta] Y. Takahashi, Absolute continuity of Poisson random fields, *Publ. RIMS Kyoto Univ.*, **26**, 629–647, 1990.

[VGG] A. M. Vershik, I. M. Gelfand and M. I. Graev, Representations of the group of diffeomorphisms, *Russ. Math. Surv.*, **30**, 1–50, 1975.

[Yo] M. W. Yoshida, Construction of infinite-dimensional interacting diffusion processes through Dirichlet forms, *Probab. Th. Rel. Fields*, **106**, 265–297, 1996.