Factorizations of primitive permutation groups.

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1 Introduction.

In this paper we attack the following problem in the theory of permutation groups

> Classify all finite primitive permutation groups (G, Ω) which admit a factorization $G = G_\omega G_\omega^\alpha$ in a point stabilizer G_ω and the image G_ω^α of G_ω under an automorphism α of G.

We will call such a factorization of a group a *primitive factorization*.

We are interested in primitive factorizations, since we want to determine the primitive flag–transitive $m \times m$ grids (Γ, X). Here primitive means that the stabilizer G in the group X of one of the two parallel classes of lines of Γ, say Π_1, acts primitively on Π_1.

A primitive flag–transitive grid gives rise to a primitive factorization and conversely: If (Γ, X) is a flag–transitive $m \times m$ grid, then X contains a subgroup G of index 2 that admits a factorization $G = G_l G_l^\alpha$, where G_l is the stabilizer of a line l and α an element of $X \setminus G$. The factorization $G = G_l G_l^\alpha$ is primitive if and only if (Γ, X) is a primitive grid.

On the other hand suppose that a group G admits a factorization $G = G_\omega G_\omega^\alpha$ for an $\alpha \in \mathrm{Aut}(G)$ with $\alpha^2 \in G$. Then the coset geometry Γ with set of points being the cosets of $(G_\omega \cap G_\omega^\alpha)\langle\alpha\rangle$ in $X = G\langle\alpha\rangle$ and set of lines being the cosets of G_ω in X supplies us with a $m \times m$ grid, where $m = |G_\omega : G_\omega \cap G_\omega^\alpha|$, with the group X acting flag–transitively.

For geometric purposes it is natural and usual to assume primitivity. Often a problem on flag-transitive geometries can be reduced to a problem where a group is acting primitively in some way. This happens for instance in the classification of linear spaces, see [BueDeDo], or in the classification of distance transitive-graphs, see [Sm]. In fact the classification of flag-transitive grids can be reduced to this one of primitive flag-transitive grids: Any imprimitive flag-transitive grid (Γ, X) gives rise to a primitive flag-transitive grid (Γ', X), see Lemma 1.5.

Recall that the group G is a primitive permutation group with point stabilizer G_ω if and only if G_ω is maximal in G and contains no normal subgroup of G. M. Liebeck, C. Praeger and J. Saxl [LPS2] determined the factorizations $G = G_\omega B$, for G any almost simple group and B any maximal subgroup of G which does not contain $E(G)$. P. P. Palfy and J. Saxl extracted from these those factorizations $E(G) = G_\omega B$ with G_ω and B being of the same order, [PaSa]. See also [BP], where some general results on factorizations of almost simple groups are presented.

Let G_i be a group and π_i a permutation representation of G_i on the set Ω_i, for $i = 1, 2$. Then π_1 and π_2 are said to be *isomorphic*, if there exists an isomorphism $\phi : G_1 \to G_2$ and a bijection $\theta : \Omega_1 \to \Omega_2$ such that

$$g^{\pi_1}\theta = \theta g^{\phi\pi_2} \text{ for all } g \in G_1.$$

Two permutation groups are *permutation equivalent*, if their permutation representations are isomorphic. Our results read as follows.

Theorem 1.1 *Let G be a primitive permutation group on a set Ω of size m. Suppose that $G = G_\omega G_\omega^\alpha$ for G_ω a point stabilizer and for $\alpha \in \mathrm{Aut}(G)$. Then one of the following holds. (Conversely each of the listed groups gives rise to such an example.)*

(i) *G is affine:*
$G = E_{2^3}L_3(2) \wr X$, *where X is a transitive subgroup of S_n and $m = 2^{3n}$.*

(ii) *G is almost simple:*

(a) *$E(G) \cong P\Omega_8^+(q)$ and G is an extension of $E(G)$ by field automorphisms. $E(G)_\omega \cong P\Omega_7(q)$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong G_2(q)$ and $m = q^4(q^4 - 1)$;*

(b) *$E(G) \cong Sp_4(q)$, $q > 2$ even, and G is an extension of $E(G)$ by field automorphisms. $E(G)_\omega \cong L_2(q^2)2$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong \mathrm{Frob}(q^2 + 1 : 4)$ and $m = q^2(q^2 - 1)/2$;*

(c) *$G \cong Sp_4(2) \cong S_6$, $G_\omega \cong P\Gamma L_2(4) \cong S_5$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5 : 4)$ and $m = 6$;*

(d) *$G \cong Sp_4(2)' \cong A_6$, $G_\omega \cong L_2(4) \cong A_5$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5 : 2)$ and $m = 6$;*

(e) *$G \cong M_{12}$, $G_\omega \cong M_{11}$, $G_\omega \cap G_\omega^\alpha \cong L_2(11)$ and $m = 12$;*

(iii) *G is permutation equivalent to an blow–up K of index n of an almost simple group A: $K \leq A \wr S_n$, where A is a permutation group listed in (ii) and $m = |A : A_\omega|^n$. Moreover $E(G)_\omega = E(A)_\omega^n$ and there exists $a \in \mathrm{Aut}(A)$ such that $A = A_\omega A_\omega^a$ and such that $E(G)_\omega \cap E(G)_\omega^\alpha \cong (E(A)_\omega \cap E(A)_\omega^a)^n$.*

Remark. Let G be a primitive permutation group such that $G = G_\omega G_\omega^\alpha$ so that G is listed in Theorem 1.1. Note that it is possible by inspection of each case to determine those G for which it is possible to choose α such that $G = G_\omega G_\omega^\alpha$ and such that $\alpha^2 \in G$. In fact this can be done in all cases except those satisfying (ii)(b) or (iii) with $E(G) \cong (Sp_4(q))^n$ with q even and $q > 2$ for some positive integer n: indeed, for such G a suitable α exists if and only if $G \cap Aut((Sp_4(q))^n)$ contains L^n where L satisfies $Sp_4(q) \leq L \leq Aut(Sp_4(q))$ and $L/Sp_4(q)$ is a Sylow 2-subgroup of the group of field automorphisms of $Sp_4(q)$.

An equivalent formulation of Theorem 1.1 is

Theorem 1.2 *Let G be a group which has two isomorphic primitive permutation representations π_1 and π_2 on the sets Ω_1 and Ω_2, respectively. Suppose that a point stabilizer of π_1 acts transitively on Ω_2. Then G and $m = |\Omega_1| = |\Omega_2|$ are as listed in Theorem 1.1 and each of the listed groups gives rise to such an example.*

The following two results are immediate consequences of Theorem 1.1.

Corollary 1.3 *Let (X, Γ) be a flag–transitive $m \times m$ grid such that the stabilizer of a parallel class of lines, G, acts faithfully and primitively on the two parallel classes of lines. Then G and m are as listed in Theorem 1.1.*

Proof. Let l be a line of Γ and X_l the stabilizer of l in X. Since X acts flag–transitively on Γ, the stabilizer X_l acts transitively on the points incident with l and therefore X_l also acts transitively on the parallel class of lines of Γ which does not contain l. Let p be a point on l and let $x \in X_p \setminus (X_p \cap X_l)$. Then $l^x \in \Pi$, $x^2 \in G$ and the Lemma of Frattini yields $G = G_l G_l^x$, which proves the corollary. □

Remark. Notice, if (X, Γ) is a flag-transitive grid, then $X = G\langle x \rangle$ for some $x \in X$ with $x^2 \in G$. Therefore, in fact G is as listed in the remark to Theorem 1.1. Moreover, each of the listed groups gives rise to such an example.

Corollary 1.4 *Let $(Aut(\Gamma), \Gamma)$ be a flag–transitive generalized two–gon such that $G = Aut(\Gamma)$ acts faithfully and primitively on the points of Γ. Suppose that Γ admits a duality. Then G and m, the number of points of Γ, are as listed in the remark to Theorem 1.1 and each of the listed groups gives rise to such an example.*

Proof. Again application of the Lemma of Frattini yields the assertion. □

The reduction of imprimitive grids to primitive grids can be established as follows. Suppose that (Γ, X) is a flag-transitive grid with two parallel classes of lines Π_1 and Π_2 and suppose that the stabilizer G of Π_1 in X acts imprimitivly on Π_1 with blocks of imprimitivity $B_{11}, \ldots, B_{1r}$. Let the blocks be such that they are maximal non–trivial blocks of imprimitivity. Then G acts imprimitivly on Π_2 with blocks of imprimitivity $B_{21} = B_{11}^x, \ldots, B_{2r} = B_{1r}^x$, where $x \in X \setminus G$. These systems of imprimitivity induces a system of imprimitivity $C_{i,j}, 1 \leq i, j \leq r$ on the points of Γ; namely $C_{i,j} = \{\sigma_0(l) \mid l \in B_{1i}\} \cap \{\sigma_0(l) \mid l \in B_{2j}\}$.

Let Γ' be the incidence structure whose lines are the blocks $B_{ki},\ k = 1, 2,\ i = 1, \ldots r$ and whose points are the blocks $C_{ij}, 1 \leq i, j \leq r$; a point C_{ij} being incident to a line B_{kl} if $k = 1$ and $i = l$ or $k = 2$ and $j = l$.

Lemma 1.5 (Γ', X) *is a flag–transitive primitive grid.*

Proof. It is an straightforward calculation that Γ' defines a grid on which X acts flag–transitively. □

This paper is organized as follows. In the next section we give without proofs facts on groups that will be of great use in the proof of our main result, Theorem 1.1. Section 3 contains a few preliminaries on factorizations of groups. In Section 4 we introduce notation of wreath products. In Section 5 we discuss the O'Nan–Scott theorem on finite primitive permutation groups. We will follow [LPS1] in partitioning these groups in five different types. By easy counting arguments we immediately rule out one of these types. In Sections 6 (resp. 8,9 and 10) we determine the factorization of groups of affine type (resp. almost simple type, diagonal action and product action type). To obtain the factorizations of the almost simple groups we discuss in Section 7 some known results on maximal subgroups of the classical groups, including an important theorem of Aschbacher [As].

2 Preliminaries.

In this section we provide the necessary group-theoretic background for the proof of Theorem 1.1.

In order to determine the factorizations of the primitive permutation groups of diagonal type we want to know the groups fulfilling the following hypothesis:

(Fix) The simple non–abelian group T possesses an automorphism $x \in \mathrm{Aut}(T)$ such that $|C_{\mathrm{Aut}(T)}(tx)| \le 2|\mathrm{Out}(T)|$, for all $t \in T$.

Clearly, if T satisfy hypothesis (Fix), then x is an outer automorphism of T. These groups are classifyed in [Ba2] (the proof may also be found in [Ba1, Proposition 2.4.1]).

Proposition 2.1 *[Ba1, Proposition 2.4.1] Assume hypothesis (Fix). Then T is a group of Lie type and one of the following holds.*

(i) $T \cong L_2(3^r)$, *$t \cdot x$ is the product of a diagonal automorphism and a field automorphism of order r or $r/2$ for some $t \in T$;*

(ii) $T \cong U_3(2^r)$, *$r \ge 3$ odd and $t \cdot x$ is the product of a diagonal automorphism and a field automorphism of order $2r$ for some $t \in T$.*

In all cases there exists $t \in T$ such that $|C_{\mathrm{Aut}(T)}(tx)| = 2|\mathrm{Out}(T)|$.

For the factorization of the groups of diagonal type we will also need to know the following fact.

Lemma 2.2 *Let T be a group of Lie type $\mathcal{L}(q)$. Then there exists a prime s which divides $|T|$, but not $|Out(T)|$.*

Proof. Assume that there is a group of Lie type T such that $\Pi(|T|) \subseteq \Pi(|Out(T)|)$. Let $q = p^r$. By checking the orders of outer automorphisms given in Table 5 of [CCNPW], we see that any prime, which divides $|Out(T)|$ is either $2, 3$ or divides $q-1, q+1$ or r.

According to Table 6 of [CCNPW], there exists $m \geq 2$ such that $q^m - 1$ divides $|T|$.

Assume $m \geq 3$ and $q^m \neq 64$. Then there exists a prime s which divides $q^m - 1$, but not $p^i - 1$ for $i < mr$ (see [Zsig]). As $m \geq 3$, we have $s \neq 2, 3$ and, as $s \mid p^{s-1} - 1$, we have $(s, r) = 1$. Hence in this case the assertion of the lemma follows in contradiction with our assumption. If m is even, then $s \mid q^{m/2} + 1$. Using these two observations and consulting Tables 5 and 6 of [CCNPW], we obtain that s does not divide $|Out(T)|$ for all the groups, except possibly $G_2(2)'$, ${}^3D_4(2)$ and $L_2(q)$.

Assume first $T \cong L_2(q)$. Then $|T| = q(q^2-1)/d$ and $|Out(T)| = dr$, where $d = (2, p)$ (see Tables 1 and 2). Clearly, $r \neq 1$ and $T \ncong L_2(8)$. Therefore, there exists a Zsigmondy prime s for $q^2 - 1$. Since $(s, r) = 1$, it follows $T \ncong L_2(q)$.

Hence $T \cong G_2(2)'$ or $T \cong {}^3D_4(2)$. In both cases $|Out(T)|$ divides 18, but $s = 7$ is a divisor of $|T|$ in contradiction with our assumption. This proves the lemma. $\square$

In section 6 we will be interested in the groups G that fulfil the following properties:

(A) The Fitting subgroup $F(G)$ of G is trivial;

(B) G has an irreducible and faithful $GF(p)$–module V of dimension d and G contains a subgroup K with $|G : K| = p^a$ for an $a \geq d$.

The classification of these groups has been done in [Ba3], but can also be found in [Ba1, Proposition 2.7.1]. The result reads as follows:

Proposition 2.3 *[Ba1, Proposition 2.7.1] Suppose that G is a group with properties (A) and (B). Then the following holds:*

(i) *$p = 2$ and $d = 3n$, for an $n \in \mathbb{N}$;*

(ii) *$E(G) = T_1 \times \ldots \times T_n$, where $T_i \cong L_3(2)$, for $i = 1, \ldots, n$;*

(iii) *$G \cong T_1 \wr X$, where X is a transitive subgroup of S_n;*

(iv) *$E(G) \nleq K$ and either $T_i \leq K$ or $K \cap T_i \cong \mathrm{Frob}(7:3)$, for $i = 1, \ldots, n$;*

(v) *$V = [V, T_1] \oplus \ldots \oplus [V, T_n]$, where $[V, T_i]$ is a natural module for $GF(2)T_i$.*

3 Simple facts about factorizations.

Lemma 3.1 *Let G be a group and A and B two subgroups. The following statements are equivalent*

(i) *$G = AB$.*

(ii) *$G = AB^g$ for any $g \in G$.*

(iii) *$G = AB^n$ for any $n \in N_{\mathrm{Aut}(G)}(A)$.*

(iv) $|G : A| = |A : A \cap B|$.

(v) $|G : B| = |B : A \cap B|$.

Proof. The equivalence of (i),(iii), (iv) and (v) is obvious. Assume (i). Then B acts transitively on the cosets of A in G, so B^g acts transitively on these cosets, as well. Thus (ii) holds. By the same argument (ii) implies (i). $\square$

Lemma 3.2 *Let G be a group, which acts transitively on a set Ω and suppose $G = G_\omega G_\omega^\alpha$ for a point stabilizer G_ω and some $\alpha \in \mathrm{Aut}(G)$. If $U \geq G$ acts on Ω, as well, and $[U, \alpha] \leq U$, then $U = U_\omega U_\omega^\alpha$.*

Proof. Since G acts transitively, we have $U = U_\omega G$. Hence $U = U_\omega G = U_\omega G_\omega G_\omega^\alpha = U_\omega U_\omega^\alpha$. $\square$

Moreover, the following holds.

Lemma 3.3 *Suppose that G is a group and A a subgroup of G such that $G = AA^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. If p^b divides $|G|$, then $p^{\lceil b/2 \rceil}$ divides $|A|$.*

Proof. The factorization $G = AA^\alpha$ yields that $|G|$ divides $|A|^2$, which shows the assertion. $\square$

The above results will be of great use in the proof of Theorem 1.1.

4 Notation of wreath products.

For A a group, the *direct power* A^n will be thought of as the group of all functions

$$\Sigma \to A,\ \Sigma = \{1, ..., n\},$$

multiplication being defined pointwise, this means

$$i(a_1 a_2) = (ia_1)(ia_2) \text{ for all } i \in \Sigma \text{ and all } a_1, a_2 \in A^n.$$

By a *wreath product* $A \wr S_n$ we mean the usual semidirect product of A^n with S_n: if $s \in S_n$ and $a \in A^n$, then $ia^s = i^{s^{-1}}a$ for all $i \in \Sigma$. Further for $U \leq A$ we denote by U_i the subgroup

$$U_i = \{a \in U^n \mid ja = 1 \text{ for } j \neq i\}$$

of A^n. Then in particular $A^n = A_1 \times ... \times A_n$. Since $U \cong U_i$ for $i = 1, \ldots, n$, we will sometimes identify U_i with U.

Let π be the permutation representation of $W = A \wr S_n$ on $\{A_1, \ldots, A_n\}$ and for $i = 1, \ldots, n$ let W_i be the stabilizer of A_i in W, then $W_i = N_W(A_i) \cong A \times A \wr S_{n-1}$. Denote by π_i the *projection* of W_i onto the ith direct factor A_i, $1 \leq i \leq n$.

5 The Theorem of O'Nan–Scott.

The primitive permutation groups are classified by O'Nan and Scott, see [ONSc]. Unfortunately one case was missing in their version. Meanwhile there are different proofs of the corrected version of this Theorem, cf. [AsSc, Bue] or [LPS1]. F. Buekenhout gives in [Bue] nice geometric descriptions of the different cases. For our purpose the version of M. Liebeck, C. Praeger and J. Saxl, [LPS1], turned out to be most suitable. We will use their partition of the primitive permutation groups into different types. Only in the product action case we follow the more detailed description of L. G. Kovaćs in [Ko]. Let G be a primitive permutation group on a set Ω of size m, $m \in \mathbb{N}$.

There are up to permutational equivalence five classes of primitive permutation groups, cf. [LPS1] and [Ko]. Let $S = E(G)$. Then $S \cong T^n$, $n \geq 1$, where T is a simple group.

Affine groups.

Here $T \cong \mathbb{Z}_p$ for some p, and S is the unique minimal normal subgroup of G. Moreover S acts regularly on Ω of degree $m = p^n$. Thus G_ω acts as an irreducible subgroup of $GL_n(p)$ on $S \cong E_{p^n}$.

Almost simple groups.

In this case $n = 1$, T is a nonabelian simple group and $T \leq G \leq \mathrm{Aut}(T)$. Further $T_\omega \neq 1$.

Diagonal action.

Let T be a non–abelian simple group, $n \geq 2$ and let D be the set of functions

$$D = \{a \in \mathrm{Aut}(T)^n \mid ia = d \text{ for } i = 1, \ldots, n,\ d \in \mathrm{Aut}(T)\}.$$

Set $W = (T \wr S_n)D$. Hence we have $W \cong T^n(S_n \times Out(T))$. Define an action of W by setting $W_\omega = D \times S_n$. We say that $G \leq W$ is of diagonal action type, if $S \leq G$ and either

(i) G acts primitively on $\{T_1, \ldots, T_n\}$ or

(ii) $n = 2$ and G acts trivially on $\{T_1, T_2\}$.

Product action and blow–ups.

Let H be a primitive permutation group on a set Γ which is either almost simple or of diagonal action type. For $r > 1$, let $W = H \wr S_r$, and take W to act on $\Omega = \Gamma^r$ in its natural product action. Then for $\gamma \in \Gamma$ and $\alpha = (\gamma, \ldots, \gamma) \in \Omega$ we have $W_\alpha = H_\gamma \wr S_r$, and $m = |\Gamma|^r$. We say that the subgroup G of W is of product action type if $E(W) \leq G$ and G acts transitively on these r factors. Moreover, according to Kovaćs such a group G is primitive if and only if the projection $(G \cap W_1)\pi_1$ on H_1 is a primitive subgroup of H_1. (Recall the definition of H_1 and W_1 given in Section 4).

We now introduce some terminology (due to Kovaćs; see [Ko]) relevant to the product action. Let H be a primitive group which is almost simple or of diagonal action type and let $W = H \wr S_r$ be a wreath product. A subgroup U of W is called *large* if $U\pi$ is transitive (as a subgroup of S_r) and $(U \cap W_1)\pi_1 = H_1$. Note that once $U\pi$ is transitive, $(U \cap W_i)\pi_i = H_i$ holds for $i = 1, \ldots, r$. We say that the subgroup G of W is a *blow–up* of H of index r if G is a large subgroup of $H \wr S_r$ and $E(H)^r \leq G$.

Up to permutational equivalence the primitive permutation groups of product action type are the blow ups of primitive groups of almost simple or diagonal action type, as the following Proposition by Kovaćs demonstrates.

Proposition 5.1 *[Ko, 2.4] Let H be a primitive permutation group on a set Γ and let $W = H \wr S_r$ act naturally on $\Omega = \Gamma^r$. If $G \leq W$ is a primitive group of product action type, then G is conjugate in W to a blow up of a primitive subgroup B of H of index r.*

The converse is shown in the next proposition.

Proposition 5.2 *[Ko, Theorem 1, (2.1)] Let H be a primitive group, which is almost simple or of diagonal type. Then each blow up $G \leq W$, $W = H \wr S_r$, of index r is a primitive group of product action type and $E(G) = E(H)^r = E(W)$.*

Twisted product action.

Here G is a twisted wreath product $T\ \mathrm{twr}_{\varphi} P$, defined as follows. Let P be a transitive permutation group on $\{1, \ldots, n\}$ and let Q be the stabilizer P_1. We suppose that there is a homomorphism $\varphi : Q \to \mathrm{Aut}\ (T)$ such that Im φ contains Inn T. Define

$$B = \{f : P \to T \mid f(pq) = f(p)^{\varphi(q)} \text{ for all } p \in P, q \in Q\}.$$

Then B is a group under pointwise multiplication, and $B \cong T^n$. Let P act on B by

$$f^p(x) = f(px) \quad \text{for } p, x \in P.$$

We define $G = T\ \mathrm{twr}_{\varphi} P$ to be the semidirect product of B by P with this action, and define an action of G on Ω by setting $G_\omega = P$. We then have $m = |T|^n$, and B is the unique minimal normal subgroup of G and acts regularly on Ω.

Theorem 5.3 *[ONSc,AsSc,LPS1,Ko] Let G be a primitive permutation group. Then one of the following holds:*

(i) *G is an affine group;*

(ii) *G is almost simple;*

(iii) *G is of diagonal type;*

(iv) *G is permutation equivalent to a blow–up of an almost simple group or of a group of diagonal type;*

(v) *G is of twisted product action type.*

In fact, we can reduce the problem to classify the factorizations of primitive permutation groups to the four cases, the primitive groups of affine, almost simple or diagonal action type and the blow–ups, as is shown in the following corollary.

Corollary 5.4 *A group G of twisted wreath action type admits no factorization $G = G_\omega G_\omega^\alpha$ with $\alpha \in \mathrm{Aut}(G)$.*

Proof. Assume $G = G_\omega G_\omega^\alpha$. Then $|G|$ divides $|G_\omega|^2$, Lemma 3.3. As $G_\omega \cong P$ and $G \cong T^n P$ we obtain $|T|^n$ divides $|P|$, and, as $P \leq S_n$, we conclude that $|T|^n$ divides $|S_n|$. Since T is a non–abelian simple group $|T|_2 \geq 2^2$. So 2^{2n} divides $|T|^n$ and $|S_n|$ in contradiction to $|S_n|_2 \leq 2^{n-1}$, see [Hu, III. 15]. □

6 Factorizations of groups of affine type.

In this section we study the factorizations of a primitive affine permutation groups. Let G be a primitive affine group. Then $G = O_p(G)G_0$, where $O_p(G) \cong E_{p^d}$ is a faithful and irreducible $GF(p)$–module for G_0.

Example. Notice that $G = (E_{2^3}L_3(2)) \wr S_r$ admits a primitive factorization: Let $r = 1$. Then there exists an automorphism α of order two of $G = O_2(G)G_0 \cong E_{2^3}L_3(2)$ such that G_0 and G_0^α are not conjugate in G. We may assume that α normalizes a Sylow 7-subgroup of G_0. Therefore $G_0 \cap G_0^\alpha \cong Frob(7:3)$ and $G = G_0 G_0^\alpha$. This example can be generalized. Let H be the wreath product $H = G \wr S_n$. We can extend α in the natural way to an automorphism β of H:
Let β act on G as α does and let β centralizes S_r. Then we have $H = H_0 H_0^\beta$.

In this section we show that there are no other primitive factorizations of affine groups than those described in the example.

Proposition 6.1 *Suppose that the affine group $G = O_p(G)G_0$, $O_p(G) \cong E_{p^d}$ admits a factorization $G = G_0 G_0^\alpha$ for some $\alpha \in Aut(G)$. Then the following holds.*

(i) $p = 2$ *and* $d = 3n$ *for some* $n \in \mathbb{N}$;

(ii) $G_0 \cong L_3(2) \wr X$, *where X is a transitive subgroup of S_n;*

(iii) $G \cong E_{2^{3n}}(L_3(2) \wr X) \cong (E_{2^3}L_3(2)) \wr X$;

(iv) $G_0 \cap G_0^\alpha \cong Frob(7:3)^n : X^g$ *for some $g \in E(G_0)$.*

(v) We may choose α such that $\alpha^2 = 1$.

Moreover, there is such an example for each n and each X.

Proof. Set $H = G_0$ and $V = O_p(G)$.

In order to apply Proposition 2.3 we first show that H fulfils the properties (A) and (B) introduced in the preliminaries.

From $G = HH^\alpha$ follows $|V| = |G : H| = |H : H \cap H^\alpha|$. Therefore and since G is a primitive permutation group, V is an irreducible and faithful $GF(p)H$–module with $|V| = |H : K|$, where $K = H \cap H^\alpha$. Thus H has property (B).

Now we want to show that H has property (A). Assume $F(H) \neq 1$. Then there exists a prime r, such that $O_r(H) \neq 1$. As V is a faithful and irreducible G–module, we have $r \neq p$. Since $VO_r(H)$ is a characteristic subgroup in G, the automorphism α normalizes $VO_r(H)$. According to the Theorem of Schur–Zassenhaus, [Su, Chapter 2, (8.10)], and Lemma 3.1 we may assume $[O_r(H), \alpha] \leq O_r(H)$. Since V is a faithful and irreducible H–module $C_V(O_r(H)) = 1$ which yields $N_G(O_r(H)) = H$. Hence α normalizes $H = N_G(O_r(H))$ in contradiction to $G = HH^\alpha$. This proves that H has property (A). (Notice, that therefore H is not a soluble group.)

Thus we may apply Proposition 2.3 to H:
Proposition 2.3(i) implies the statement (i); Proposition 2.3(ii),(iii) imply (ii) and (ii) together with Proposition 2.3(v) imply (iii).

It remains to show (iv) and (v).

As the subgroups H and H^α are two complements of V in G, according to [Su, Chapter 2, Corollary to 8.7] there is an automorphism $a \in Aut(G)$ such that $H^a = H^\alpha$, $[V, a] = 1$ and $[H, a] \leq V$. In particular, a is of order 2 which implies (v).

Let $G_i = [V, T_i]T_i$, for $i = 1, \dots, n$. As $[T_i^a, (\prod_{j \neq i} T_j)^a] = 1$, we deduce $[T_i, a] \leq C_V(\prod_{j \neq i} T_j) = [V, T_i]$. Thus a normalizes G_i, for $i = 1, \dots, n$. This implies $E(H) \cap E(H)^a = (T_1 \times \cdots \times T_n) \cap (T_1 \times \cdots \times T_n)^a = (T_1 \cap T_1^a) \times \cdots \times (T_n \cap T_n^a)$.

According to Proposition 2.3(iv) $E(H) \cap E(H)^\alpha$ contains a subgroup U isomorphic to $\mathrm{Frob}(7:3)^r$, which implies $U \leq E(H) \cap E(H)^\alpha = E(H) \cap E(H)^a = \prod_{i=1}^n (T_i \cap T_i^a)$. Therefore, for $i \in \{1, \cdots, n\}$, either $T_i \cap T_i^a \cong \mathrm{Frob}(7:3)$ or $T_i = T_i^a$.

We claim that $T_i \neq T_i^a$, for $i = 1, \dots, n$. Assume $T_1 = T_1^a$. Since H is transitive on $\{T_1, \dots, T_n\}$, there exists $x_i \in H$ such that $T_1^{x_i} = T_i$, for $i = 2, \dots, n$. As $[G/V, a] = 1$, it follows $x_i^a = x_i v_i$ with $v_i \in V$. Further there exists $w_i \in [V, T_i]$ such that $v_i w_i \in C_V(T_i)$; namely w_i is the image of v_i under the natural projection of V onto $[V, T_i]$.

Set $w = w_2 \cdots w_n$. Then $w \in G$, $T_1^{aw} = T_1$ and, for $i = 2, \dots, n$,

$$T_i^{aw} = T_1^{x_i aw} = T_1^{a x_i v_i w_2 \cdots w_n} = T_i^{v_i w_i} = T_i,$$

which yields that $E(H)^{aw} = E(H)$. Therefore aw normalizes $N_G(E(H)) = H$. Thus H and $H^\alpha = H^a$ are conjugate in G in contradiction to Lemma 3.1(ii). This proves that $E(H) \cap E(H)^\alpha \cong \mathrm{Frob}(7:3)^n$.

According to (iii) we have $H = E(H)X$. Since

$$|H \cap H^\alpha| = |H|^2/(|V||H|) = |E(H)||X|/|V| = |E(H) \cap E(H)^\alpha||X|,$$

$E(H) \cap E(H)^\alpha$ is of index $|X|$ in $H \cap H^\alpha$. Since $E(H) \cap E(H)^\alpha = H \cap H^\alpha \cap E(H)$ is normal in $H \cap H^\alpha$ and since $N_H(E(H) \cap E(H)^\alpha) = (E(H) \cap E(H)^\alpha) : X^g$ for some

$g \in E(G)$ ($E(H) \cap E(H)^\alpha$ is the normalizer in $E(H)$ of a Sylow 7-subgroup of $E(H)$), it follows $H \cap H^\alpha = (E(H) \cap E(H)^\alpha) X^g$, as claimed in (iv). This proves the proposition. $\square$

7 On subgroups of classical groups.

In this section we discuss a result due to M. Aschbacher, see [As], on the subgroups of groups whose socles are finite simple classical groups. Let T be a finite classical group, this means T is either a linear group $L_n(q)$ or an orthogonal group $P\Omega_n(q)$ or a symplectic group $PSp_n(q)$ or an unitary group $U_n(q)$. Let G be a group such that $T \trianglelefteq G \leq Aut(T)$ and let V be the natural projective module for T. Then V is defined over a field k, where we take $k = GF(q)$ if T is not the unitary group, and $k = GF(q^2)$ if $T = U_n(q)$. Then Aschbacher defines a collection $\mathcal{C}$ of *geometric subgroups* of $\overline{G} = G \cap P\Gamma L(V)$. See [As] for a more detailed description of the various cases. Roughly speaking, they are

C_1: stabilizers in $\overline{G}$ of nonsingular or totally singular subspaces of V,

C_2: stabilizers in $\overline{G}$ of decompositions $V = \oplus V_i$ with dim V_i constant,

C_3: stabilizers in $\overline{G}$ of extension fields of k of prime index,

C_4: stabilizers in $\overline{G}$ of tensor product decompositions $V = V_1 \otimes V_2$,

C_5: stabilizers in $\overline{G}$ of subfields of k of prime index,

C_6: normalizers in $\overline{G}$ of extraspecial groups in irreducible representations,

C_7: stabilizers in $\overline{G}$ of decompositions $V = \otimes V_i$ with dim V_i constant,

C_8: normalizers in $\overline{G}$ of classical groups on V contained in $\overline{G}$,

For the case $T \cong L_n(q)$ Aschbacher defines a further family of geometric subgroups, see [A1].

C_1': stabilizers in $\overline{G}$ of pairs U, W of subspaces of V with dim $U+$ dim $W =$ dim V and either $U \leq W$ or $U \cap W = 0$,

Finally there is the family $\mathcal{S}$ of non geometric subgroups, see [As].

$\mathcal{S}$: A is a subgroup of $\overline{G}$ which is almost simple, so $S \trianglelefteq A \leq Aut(S)$ for a non abelian simple group S, and V is an absolutely irreducible kS–module, which is not defined over any proper subfield of k; moreover A is not a geometric subgroup.

Aschbacher proves:

Theorem 7.1 *[As, Theorem] Let T be a finite simple classical group defined over the field k and let V be the natural projective module for T. Furthermore let G be a subgroup of $P\Gamma L(V)$ such that $T \trianglelefteq G \leq Aut(T)$. If A is a proper maximal subgroup of G such that $G = AT$, then $A \in \mathcal{C} \cup \mathcal{S}$.*

In the case the socle of G is a linear group and G contains a graph automorphism, so $G \not\leq P\Gamma L(V)$, Aschbacher obtained the following result.

Theorem 7.2 *Let G be a finite group whose socle is $L_n(q)$, $n \geq 3$, with natural projective module V such that G contains a graph automorphism. Let A be a proper maximal subgroup of G such that $G = L_n(q)A$. Then A is contained in $C_1' \cup C_2 \cup \ldots \cup C_8 \cup \mathcal{S}$.*

By the above results we have descriptions of all the maximal subgroups of a classical group G, except for the case where $T \cong Sp_4(2^r)$ or $T \cong P\Omega_8^+(q)$ and G can not be embedded in $P\Gamma L(V)$. For these groups P. Kleidman determined the maximal subgroups, see [Kl1] and [Kl2]. Aschbacher also showed

Theorem 7.3 *[As, Theorem BΓ] The orbits of $Aut(T) \cap PGL(V)$ and $Aut(T) \cap P\Gamma L(V)$ on $\mathcal{C}$ are the same.*

We collect some further facts on the action of T on V.

Lemma 7.4 *Let T be a simple finite classical group.*

(i) *If T is not an orthogonal group in odd characteristic, then T acts transitively on the non–singular 1–spaces.*

(ii) *If T is an orthogonal group in odd characteristic and odd dimension, then T and $Aut(T) = \Gamma$ have two orbits on the non–singular 1–spaces; the 1–spaces in these orbits have an orthogonal complement of type O^+ or O^-, respectively.*

(iii) *If T be an orthogonal group, then T acts transitively on the planes of type O_2^+ and O_2^-, respectively.*

(iv) *Let T be a symplectic group, then T acts transitively on the hyperbolic planes.*

Proof. This follows immediately from the Lemma of Witt. □

8 Factorization of almost simple groups.

Throughout this section let G be an almost simple group, that means $T \leq G \leq \mathrm{Aut}(T)$ for some simple group T. We show

Proposition 8.1 *If G is almost simple and $G = G_\omega G_\omega^\alpha$ for some maximal subgroup G_ω of G and for some $\alpha \in \mathrm{Aut}(G)$, then one of the following holds.*

(i) *$E(G) \cong P\Omega_8^+(q)$ and G is an extension of $E(G)$ by field automorphisms. $E(G)_\omega \cong P\Omega_7(q)$ and $E(G)_\omega \cap E(G)_\omega^\alpha \cong G_2(q)$;*

(ii) (a) *$E(G) \cong Sp_4(q)$, $q > 2$ even, and G is an extension of $E(G)$ by field automorphisms $E(G)_\omega \cong L_2(q^2)2$ and $E(G)_\omega \cap E(G)_\omega^\alpha \cong \mathrm{Frob}(q^2+1:4)$;*

(b) *$G \cong Sp_4(2) \cong S_6$, $G_\omega \cong P\Gamma L_2(4) \cong S_5$ and $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5:4)$;*

(c) *$G \cong Sp_4(2)' \cong A_6$, $G_\omega \cong L_2(4) \cong A_5$ and $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5:2)$;*

(iii) *$G \cong M_{12}$, $G_\omega \cong M_{11}$ and $G_\omega \cap G_\omega^\alpha \cong L_2(11)$.*

Thus we assume in this section that G is an almost simple group and that $G = G_\omega G_\omega^\alpha$ holds for some maximal subgroup G_ω of G and for some $\alpha \in \mathrm{Aut}(G)$.

In order to prove the Proposition we use the work of Liebeck, Praeger and Saxl [LPS2] on the factorizations of almost simple groups. They classified the factorizations $G = AB$, where A and B are maximal subgroups of G which do not contain T.

Since we assume G to be a permutation group on the set of cosets of G_ω in G, the socle, T, of G is not a subgroup of G_ω. Therefore we are in the situation as considered in [LPS2]. Recall $T_\omega = T \cap G_\omega$.

Lemma 8.2 *Let T be a sporadic group, then $G \cong M_{12}$, $G_\omega \cong M_{11}$ and $G_\omega \cap G_\omega^\alpha \cong L_2(11)$.*

Proof. If p^b divides $|G|$, then $p^{\lceil b/2 \rceil}$ divides $|G_\omega|$ (see Lemma 3.3). Since, moreover, $|G_\omega| = |G_\omega^\alpha|$, we obtain the assertion considering the Table 6 in [LPS2]. □

Lemma 8.3 *Let T be an alternating group A_n, then $G \cong A_6$ or S_6, $G_\omega \cong A_5$ or S_5 and $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5:2)$ or $\mathrm{Frob}(5:4)$, respectively.*

Proof. Let $T \cong A_n$ and $n > 6$. Then $\mathrm{Aut}(G) \le S_n$ and according to [LPS2, Theorem D] either

(i) $n = 8, 10$ or

(ii) $A_{n-k} \trianglelefteq G_\omega \le S_{n-k} \times S_k$ for some k with $1 \le k \le 5$, and G_ω^α is k–homogeneous on $\{1, \ldots, n\}$

holds.

In the latter case G_ω is the stabilizer of a k–subset of $\{1, \ldots n\}$. Hence G_ω and G_ω^α are stabilizers of a k–subset in contradiction with the fact that G_ω^α is k–homogeneous.

Thus $n = 8$ or 10 and $G_\omega \cong L_2(8), L_2(8)3$ or $AGL_3(2)$, respectively, see [LPS2, Theorem D]. But 5 divides $|G|$ and not G_ω in contradiction with Lemma 3.3.

Thus $n = 6$. According to Lemma 3.3 the number $3 \cdot 5$ divides $|T_\omega|$, which yields $T_\omega \cong A_5$ and $G_\omega \cong A_5$ or S_5, as claimed. □

Lemma 8.4 *Let T be a group of Lie–type, then T is a classical group.*

Proof. Assume that T is not classical. Then $T \cong G_2(q)$, $q = 3^r$ and $T'_\omega \cong SL_3(q)$ or $T \cong F_4(q)$, $q = 2^r$ and $T_\omega \cong Sp_8(q)$ or $T \cong G_2(4)$ and $T'_\omega \cong J_2$ or $G_2(2)$, respectively, cf. [LPS2, Table 5].

Let $T \cong G_2(q)$ or $F_4(q)$ and let t be a Zsigmondy prime for $q^6 - 1$ or $q^{12} - 1$ (see [Zsig]), respectively. Then t divides $|G|$, but not $|G_\omega|$ in contrary Lemma 3.3. Hence $T \cong G_2(4)$ and $T_\omega \cong J_2$ or $G_2(2)$.

Then 13 divides $|G|$, but neither $|J_2|$ nor $|G_2(2)|$ which is not possible by Lemma 3.3. This shows the assertion. $\square$

Hence T is a classical group. Set $A = G_\omega$ and recall the subdivision of Aschbacher in geometric groups and members of the family $\mathcal{S}$ described in section 7.2.

Lemma 8.5 *The subgroup A is geometric.*

Proof. Assume that A is a group in $\mathcal{S}$. Then according to [LPS2, Theorem A] A^α is a geometric subgroup. This is not possible, since $\mathrm{Aut}(T)$ maps geometric subgroups onto geometric subgroups. $\square$

In order to rule out non–possible factorizations we will often use the following fact:

Lemma 8.6 *If $\mathrm{Aut}(T)$ acts on a set Ω and $A = G_\omega$ for some $\omega \in \Omega$, then T does not act transitively on Ω.*

Proof. Indeed, if T acts transitively on Ω, then $A^\alpha = A^t$ for some $t \in T$ and by Lemma 3.1 $G = AA^\alpha = AA^{\alpha \cdot t^{-1}} = A$ which is obviously false. $\square$

Lemma 8.7 *Let V be the natural projective module of T. If T admits no diagonal automorphism, then $\mathrm{Aut}(T)$ is not a subgroup of $P\Gamma L(V)$.*

Proof. Assume $\mathrm{Aut}(T) \leq P\Gamma L(V)$. Then Theorem 7.3 yields T and $\mathrm{Aut}(T)$ have the same orbits on the set of geometric subgroups C. As by Lemma 8.5 A is a geometric subgroup, we obtain a contradiction to Lemma 8.6. $\square$

Lemma 8.8 (i) *A is not the stabilizer of a totally singular subspace.*

(ii) *If $T \cong L_n(q)$, then A is not of type C'_1.*

(iii) *Let $T \ncong PSp_4(2^r), P\Omega_8^+(q), P\Omega_{2m+1}(q)$ and let q be even, if $T \cong P\Omega_{2m}^+(q)$. Then A is not the stabilizer of a non–singular 1–space.*

(iv) *If $T \cong P\Omega_{2m}^+(q)$, $P\Omega_{2m}^-(q)$, $m > 4$, then A is not the stabilizer of a subspace which is of type O_2^+ or O_2^-.*

(v) *If* $T \cong PSp_{2m}(q), m > 2$, *then* A *is not the stabilizer of a hyperbolic plane.*

Proof. We begin by supposing that A is the stabilizer of a totally singular subspace W of dimension i. Then A^α stabilizes a totally isotropic subspace of dimension j. We may assume $j \leq i$. According to [LPS2, Tables 1 and 4], if $T \cong P\Omega^+_{2m}(q)$, then $i \neq m$ and $j \neq m$. As $G = AA^\alpha$, also A acts transitively on the totally isotropic subspaces of dimension j [KlLi, (3.5)]. Thus each totally isotropic subspace of dimension j is contained in W which is not possible. This proves (i).

According to [As, (13.2)] the orbits of $T \cong L_n(q)$ and $\mathrm{Aut}(T)$ on C'_1 are the same. Hence (ii) follows from Lemma 8.6.

To prove (iii) assume $T \ncong PSp_4(2^r), P\Omega^+_8(q), P\Omega_{2m+1}(q)$ and let q be even, if $T \cong P\Omega^+_{2m}(q)$ or if $T \cong P\Omega^-_{2m}(q)$ and assume that A is the stabilizer of a non–singular 1–space. Since by (i) the claim holds for $T \cong L_n(q)$, we may (and do) assume $T \ncong L_n(q)$. This implies $\mathrm{Aut}(T) \leq P\Gamma(V)$. Since T acts transitively on the non–singular 1–spaces, Lemma 7.4(i), we obtain a contradiction to Lemma 8.6.

Now let $T \cong P\Omega^-_{2m}(q)$, q odd. Let $q = p^r$, p prime. Then there exists a Zsigmondy prime t dividing $p^{r\cdot 2m} - 1$ but not $p^i - 1$ for $i < r \cdot 2m$. Hence t divides the order of G, but not the order of A in contradiction to Lemma 3.3. This proves (iii).

If T is as assumed in (iv) or (v), then again $\mathrm{Aut}(T) \leq P\Gamma(V)$. Hence from Lemma 7.4(iii),(iv) and Lemma 8.6 we conclude (iv) and (v). This proves the Lemma. $\square$

Proof of Proposition 8.1. Our first goal is to show that $T \cong P\Omega^+_8(q)$ or $Sp_4(2^r)$. Therefore we investigate the lists of [LPS2]. Recall that A is a geometric subgroup of G, Lemma 8.5. Moreover by Lemma 8.8(i) A is not a parabolic subgroup of G.

Assume $T \ncong P\Omega^+_8(q)$ or $Sp_4(2^r)$.

Assume $T \cong L_n(q)$. Then Lemma 8.8 (i) and (ii) implies $A \notin C_1 \cup C'_1$. Thus according to [LPS2, Tables 1 and 3] we have $(n, q) = (2, 16)$ or $(3, 4)$ and $T_\omega \cong D_{34}$ or $L_2(7)$, respectivly. In both cases 5 divides $|G|$, but not $|A|$ contrary to Lemma 3.3. Hence $T \ncong L_n(q)$

Assume $T \cong PSp_{2m}(q)$. Let q be even. Then by assumption $m > 2$. Hence $\mathrm{Aut}(T) \cap PGL(V) = T$ and $\mathrm{Aut}(T)$ is contained in $P\Gamma L(V)$ in contradiction to Lemma 8.7.

Hence q is odd. Then according to [LPS2, Tables 1 and 3] A is a maximal parabolic subgroup of H in contradiction to Lemma 8.8(i). Thus $T \ncong PSp_{2m}(q)$.

Assume $T \cong U_n(q)$. Then, according to [LPS2, Tables 1 and 3], either A is the stabilizer of a non-singular 1–space or $(n, q) = (4, 3)$ and $T_\omega \cong PSp_4(3)$. The first case is not possible because of Lemma 8.8(iii) In the latter case 7 divides $|T|$, but not $|A|$ in contradiction to Lemma 3.3. Hence $T \ncong U_n(q)$.

Assume $T \cong P\Omega_{2m+1}(q)$. Then according to [LPS2, Tables 1 and 3] $(2m + 1, q) = (7, 3)$ and either A is the stabilizer of a non–singular 1–space of $+$ type or $T_\omega \cong G_2(3)$.

In the first case $T_\omega \cong L_4(3)2$ and thereby 7 divides $|G|$, but not $|A|$. In the second case 5 divides $|G|$ but not $|A|$. Thus for both possiblities we obtain a contradiction to Lemma 3.3. Therefore $T \not\cong P\Omega_{2m+1}(q)$.

Assume $T \cong P\Omega^-_{2m}(q)$. Then either A is the stabilizer of a non–singular 1–space or of a O_2^+ plane, cf. [LPS2, Tables 1 and 3] in contradiction to Lemma 8.8. Thus $T \not\cong P\Omega^-_{2m}(q)$.

Assume $T \cong P\Omega^+_{2m}(q)$. By assumption $m > 4$. Then either A is the stabilizer of a non–singular 1–space or of a O_2^+, O_2^- plane, cf. [LPS2, Tables 1 and 3]. According to Lemma 8.8 A is the stabilizer of a non–singular 1–space. As $\mathrm{Aut}(T) \leq P\Gamma L(V)$ the automorphism group $\mathrm{Aut}(T)$ acts on the non–singular 1–spaces of V. Hence A^α is as well the stabilizer of a non–singular 1–space in contradiction to [LPS2, Tables 1 and 3]. Thus $T \not\cong P\Omega^+_{2m}(q)$.

By Lemma 8.8 and [LPS2, Tables 1 and 3] $T \not\cong PSp_4(q)$, q odd. Hence our assumption was false and $T \cong P\Omega^+_8(q)$ or $T \cong Sp_4(2^r)$.

Let $T \cong Sp_4(2^r)$. Since the assertion is already shown for $Sp_4(2) \cong S_6$, Lemma 8.3, we may assume $r > 1$. According to Lemma 8.8(i) and [LPS2, Tables 1 and 3] either

(i) $T_\omega \cong T_\omega^\alpha \cong SL_2(q^2)2$ or

(ii) $T_\omega \cong L_2(q^2)2$ and $T_\omega^\alpha = L_2(q) \wr S_2$ or

(iii) $q = 4$, $T_\omega \cong L_2(16)2$ and $T_\omega^\alpha \cong (L_2(4) \times L_2(4))2$ or

(iii) $q = 4$, $T_\omega \cong L_2(16)2$ and $T_\omega^\alpha \cong S_6$

(iv) $q = 16$, $T_\omega \cong L_2(2^8)2$ and $T_A^\alpha \cong Sp_4(4)$.

From $T_\omega \cong T_\omega^\alpha$ follows $T_\omega \cong SL_2(q^2)2$. This implies $A \cap A^\alpha \cong \mathrm{Frob}(q^2+1:4)$, cf. [LPS2, (3.2.1d)], as claimed in (ii)(a) of the statement of the proposition.

As $Out(T)$ is a cyclic group and as T_ω extends to a maximal subgroup in $T\langle f\rangle$ where $\langle f\rangle \cong \mathbb{Z}_r$ is the set of field automorphisms of T, G is a subgroup of $T\langle f\rangle$.

Now let $T \cong P\Omega^+_8(q)$. Again using the fact that $T_\omega \cong T_\omega^\alpha$, we obtain from [LPS2, Table 4] that $A \cong P\Omega_7(q)$. Therefore $A \cap A^\alpha \cong G_2(q)$, see [LPS2, (5.1.15)].

According to [Kl2, Table I] G is a subgroup of $T\langle f\rangle$ where $\langle f\rangle \cong \mathbb{Z}_r$ is the set of field automorphisms of T. This proves the proposition. $\square$

9 Factorizations of groups of diagonal type.

Let G be a primitive permutation group of diagonal type. Recall

$$D = \{a \in \mathrm{Aut}(T)^n \mid ia = d \text{ for } i = 1, \ldots, n,\ d \in \mathrm{Aut}(T)\}.$$

and $W = (T \wr S_n)D$. As $[D, S_n] = 1$ we have $W \cong T^n(S_n \times \mathrm{Out}(T))$. The group W acts primitively on the cosets of $W_\omega = D \times S_n$ in W, (see Section 5). The group G is of diagonal type, if G is a subgroup of W such that $E(G) = soc(W) = T_1 \times \cdots \times T_n$ and either

(i) G acts primitively on $\{T_1, \ldots, T_n\}$ or

(ii) $n = 2$ and G acts trivially on $\{T_1, T_2\}$.

In this section we prove the following:

Proposition 9.1 *A group G of diagonal action type on a set Ω admits no factorization $G = G_\omega G_\omega^\alpha$ with G_ω a point stabilizer and $\alpha \in \mathrm{Aut}(G)$.*

The strategy of the proof of Proposition 9.1 is as follows. We assume that G is a group of diagonal action type which admits a factorization $G = G_\omega G_\omega^\alpha$. First we show that $n = 2$ and that T fulfils the hypothesis (Fix) (see the Preliminaries). Then Proposition 2.1 implies $T \cong L_2(3^r)$ or $T \cong U_3(2^r)$. These groups we rule out case by case.

After the paper was finished we learned that there is a result by R. W. Baddeley [Bad, Theorem 1.4] which implies that a primitive permutation group of diagonal type of type (ii) admits no factorization $G = G_\omega G_\omega^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. His result says the following. Let G be a primitive permutation group of diagonal type of type (ii), so that $E(G) = T_1 \times T_2$. If K is a transitive subgroup of G with a minimal normal subgroup N that is isomorphic to T_1, then either N equals T_1 or T_2.

Lemma 9.2 *We have $n = 2$.*

Proof. Assume $n \geq 3$. Then G acts primitively on $\{T_1, \ldots, T_n\}$ and $G \cong T^n.A.S$ with A isomorphic to a subgroup of $\mathrm{Out}(T)$ and S isomorphic to a subgroup of S_n. Since $G = G_\omega G_\omega^\alpha$, the order of G divides $|G_\omega|^2$. Hence $|G| = |T|^n \cdot |A| \cdot |S|$ divides $|T|^2 \cdot |A|^2 \cdot |S|^2 = |G_\omega|^2$. This forces $|T|^{n-2}$ to divide $|A| \cdot |S|$, so $|T|^{n-2}$ divides $|\mathrm{Out}(T)| \cdot |S_n|$.

Form the facts that for T a sporadic group $\mathrm{Out}(T) \leq \mathbb{Z}_2$ and for T an alternating group $|\mathrm{Out}(T)| \leq 4$ and from Lemma 2.2, we conclude that there exists prime $p > 3$ such that p divides $|T|$, but not $|\mathrm{Out}(T)|$. Thus p^{n-2} has to divide the order of S_n in contradiction to $|S_n|_p \leq p^{n-4}$, see [Hu, III. 15]. This proves the lemma. $\square$

By the previous lemma $W = (T \wr S_2)D$. Let $\langle \varphi \rangle = S_2 \leq W$. Then $W_\omega = C_W(\varphi)$. Thus $G_\omega = C_G(\varphi)$ and there exists a subgroup A of $\mathrm{Out}(T)$ such that either

(i) $G \cong (T \times T^\varphi).A$ or

(ii) $G \cong (T \times T^\varphi).A\langle \tilde{\varphi} \rangle$, where $\tilde{\varphi} = aa^\varphi \varphi$ with $a = 1$ or $a \in \mathrm{Aut}(T) \setminus (TA)$.

Notice that φ can be extended to an automorphism of $\mathrm{Aut}(T) \times \mathrm{Aut}(T^\varphi)$. Since φ is an automorphism of G that centralizes G_ω, the equivalence between (i) and (iii) in Lemma 3.1 yields that $G = G_\omega G_\omega^{\alpha\varphi}$. Thus we may assume $\alpha = a_1 a_2^\varphi$ for some $a_1, a_2 \in \mathrm{Aut}(T)$. Let $w = a_1^{-1} a_2$. Then, as φ is an involution,

$$\varphi^\alpha = a_1^{-1}(a_2^\varphi)^{-1}\varphi a_1 a_2^\varphi = (a_1^{-1}a_2)(a_2^{-1}a_1)^\varphi \varphi = w(w^{-1})^\varphi \varphi.$$

Lemma 9.3 $G_\omega \cap G_\omega^\alpha = C_G(\varphi) \cap C_G(\varphi)^\alpha = C_{C_G(\varphi)}(w(w^{-1})^\varphi)$.

Proof. The fact that $G_\omega = C_G(\varphi)$ yields that

$$G_\omega \cap G_\omega^\alpha = C_G(\varphi) \cap C_G(\varphi)^\alpha = C_G(\varphi) \cap C_G(\varphi^\alpha).$$

Since $\varphi^\alpha = w(w^{-1})^\varphi \varphi$, the assertion follows. □

Lemma 9.4 *In case (i)* $|G_\omega \cap G_\omega^\alpha| = |A|$ *and in case (ii)* $|G_\omega \cap G_\omega^\alpha| = 2 \cdot |A|$

Proof. As $|G_\omega \cap G_\omega^\alpha| = |G_\omega|^2 \cdot |G|^{-1}$, we obtain in case (i) $|G_\omega \cap G_\omega^\alpha| = |T|^2 \cdot |A|^2 \cdot (|T|^2 \cdot |A|)^{-1} = |A|$ and in case (ii) $|G_\omega \cap G_\omega^\alpha| = 4 \cdot |T|^2 \cdot |A|^2 \cdot (2 \cdot |T|^2 \cdot |A|)^{-1} = 2 \cdot |A|$, as claimed. □

We need some further notation.

Notation. Let B be the projection of $N_G(T)$ on the first direct factor of $\mathrm{Aut}(T) \times \mathrm{Aut}(T)^\varphi$. Set

$$C = \{gg^\varphi \mid g \in C_B(w)\}.$$

Clearly, $|C| = |C_B(w)|$ and $|B/T| = |A|$.

Finally, let $K \leq G$ be the kernel of the action of G on $\{T_1, T_2\}$. So $K \leq Aut(T_1)^2$ and $|G : K| \leq 2$.

Recall, that $a \in \mathrm{Aut}(T)$ such that $\widetilde{\varphi} = aa^\varphi \varphi$.

Lemma 9.5 *Either*

(1) $K = K_\omega K_\omega^\alpha$ *and* $|C_B(w)| = |A|$
or

(2) $K \neq K_\omega K_\omega^\alpha$, $G_\omega \cap G_\omega^\alpha = C \leq K$, $|C_B(w)| = 2 \cdot |A|$ *and there exists no* $b \in B$ *such that* $b \cdot a$ *inverts* w.

Proof. Notice that $K_\omega \cap K_\omega^\alpha = G_\omega \cap G_\omega^\alpha \cap K = C$.

Suppose that $K = K_\omega K_\omega^\alpha$. Then by Lemma 9.4(i) $|K_\omega \cap K_\omega^\alpha| = |A|$ which implies

$$|C_B(w)| = |C| = |K_\omega \cap K_\omega^\alpha| = |A|$$

as asserted in (1).

Now suppose $K \neq K_\omega K_\omega^\alpha$. Then case (ii) holds and

$$|K| = \frac{1}{2}|G| = \frac{4}{2}\frac{|K_\omega|^2}{|G_\omega \cap G_\omega^\alpha|} = 2\frac{|K_\omega|^2}{|G_\omega \cap G_\omega^\alpha|}.$$

The latter implies $G_\omega \cap G_\omega^\alpha = K_\omega \cap K_\omega^\alpha$. Thus $G_\omega \cap G_\omega^\alpha = C \leq K$ and because of Lemma 9.4 $|C_B(w)| = 2|A|$. Now Lemma 9.3 implies $C_{C_G(\varphi)}(w(w^{-1})^\varphi) = C$. If there exists an $b \in B$ such that $b \cdot a$ inverts w, then $x = (b \cdot a)(b \cdot a)^\varphi \varphi \in C_{C_G(\varphi)}(w(w^{-1})^\varphi)$, but $x \notin C$, a contradiction. This shows (2). □

Corollary 9.6 *Either*

(1) $K = K_\omega K_\omega^\alpha$ *and* $|C_{\mathrm{Aut}(T)}(w)| \leq |\mathrm{Out}(T)|$
or

(2) $K \neq K_\omega K_\omega^\alpha$ *and* $|C_{\mathrm{Aut}(T)}(w)| \leq 2 \cdot |\mathrm{Out}(T)|$.

Proof. Notice that $|\mathrm{Aut}(T) : B| = |\mathrm{Out}(T) : A|$. As $|C_{\mathrm{Aut}(T)}(w) : C_B(w)| \leq |\mathrm{Aut}(T) : B|$ we obtain

$$|C_{\mathrm{Aut}(T)}(w)| \leq |\mathrm{Aut}(T) : B| \cdot |C_B(w)| = |\mathrm{Out}(T) : A| \cdot |C_B(w)|$$

and the assertion follows from Lemma 9.5. □

By Lemma 3.1 $G = G_\omega G_\omega^\alpha$ (resp. $K = K_\omega K_\omega^\alpha$) implies $G = G_\omega G_\omega^{\alpha g}$ (resp. $K = K_\omega K_\omega^{\alpha g}$) for any $g \in G$. Thus in particular, $G = G_\omega G_\omega^{\alpha t^\varphi}$ for any $t \in T$. Therefore we may replace in all previous statements $\alpha = a_1 a_2^\varphi$ by $\alpha t^\varphi = a_1(a_2 t)^\varphi$ and $w = a_1^{-1}a_2$ by $wt = a_1^{-1}a_2 t$. Thereby we obtain the following generalization of Corollary 9.6.

Corollary 9.7 *Either*

(1) $K = K_\omega K_\omega^\alpha$ *and* $|C_{\mathrm{Aut}(T)}(wt)| \leq |\mathrm{Out}(T)|$ *for all* $t \in T$
or

(2) $K \neq K_\omega K_\omega^\alpha$ *and* $|C_{\mathrm{Aut}(T)}(wt)| \leq 2 \cdot |\mathrm{Out}(T)|$ *for all* $t \in T$.

In particular T fulfilles the hypothesis (Fix) (see for the definition of (Fix) the Preliminaries). Therefore we can apply Proposition 2.1 to T and obtain

Lemma 9.8 *K admits no factorization, $K \neq K_\omega K_\omega^\alpha$, and there exists no $b \in B$ such that $b \cdot a$ inverts w. Moreover, $T \cong L_2(3^r)$ or $T \cong U_3(2^r)$.* □

Let, as in [Ca, p. 200], $\hat{H} \setminus H$ be the set of diagonal automorphism of T.

Lemma 9.9 *$T \not\cong L_2(3^r)$.*

Proof. Assume the contrary. Then according to Proposition 2.1 we may assume $w = h \cdot f^i$, $h \in \hat{H} \setminus H, f$ the Frobenius automorphism, with $o(f^i) = r/2$ or r.

Notice, that $\mathrm{Out}(T)$ is an abelian group. Hence $[D, \alpha] \leq T \times T^\varphi$ and α is an automorphism of $GD = W$ and Lemma 3.2 yields $W = W_\omega W_\omega^\alpha$. Therefore, we may assume $G = W$. Then, in particular, φ and $[\alpha, \varphi] = w(w^{-1})^\varphi$ are in G. Thus $w(w^{-1})^\varphi \in G \cap (\mathrm{Aut}(T) \times \mathrm{Aut}(T)^\varphi) = (T \times T^\varphi)D$ which implies that there are $t_1, t_2 \in T$ and an $d \in \mathrm{Aut}(T)$ such that

$$w(w^{-1})^\varphi = t_1 t_2^\varphi d d^\varphi.$$

Hence $w(w^{-1})^\varphi = (t_1 d)(t_2 d)^\varphi$ and $(t_2 d)^{-1} = t_1 d$. This shows $d^2 \in T$, so $w^2 = (t_1 d)^2$ is in T, as well.

This forces $r = 2$ or $r = 4$. Hence it remains to consider $T \cong L_2(9)$ and $T \cong L_2(3^4)$. In the first case $w = h \cdot f$ with $o(h) = 8$ and in the second case $w = hf^2$ with $o(h) = 16$.

We claim that in both cases there exists $g \in \mathrm{Aut}(T)$ that inverts w in contradiction with Lemma 9.8. If $T \cong L_2(9)$, then set $M = \langle h, f \rangle$ and if $T \cong L_2(3^4)$, then set $M = \langle h, f^2 \rangle$. Then $M \cong SD_{2^4}$ or Mod_{2^5}, respectively. In both cases M is a subgroup of index 2 of $N_{\mathrm{Aut}(T)}(\langle h \rangle)$.

Let i be an involution in $N_{T\hat{H}}(\hat{H}) \cong D_{2(q-1)}$ which is not in $\hat{H}$. Then i inverts $w^2 \in H$. Since in both cases $\langle h \rangle$ acts transitively on the elements in $M \setminus \langle h \rangle$ that have the same square, there exists $t \in \langle h \rangle$ such that $g = i \cdot t$ inverts w, as claimed. This contradicts with Lemma 9.8, which proves the lemma. $\square$

Lemma 9.10 *$T \not\cong U_3(2^r)$.*

Proof. Assume $T \cong U_3(2^r)$. Then according to Proposition 2.1 r is odd and $w = hf^i$ with $h \in \hat{H}$ and $o(f^i) = 2r$.

Since $\mathrm{Out}(T) \cong S_3 \times \mathbb{Z}_r$, there exists $g \in \mathrm{Aut}(T)$, $t \in T$ and $j \in \{1, \ldots, 2r-1\}$ such that $w^g = f^j t$ with $o(f^j) = 2r$. Observe, that $G^{gg^\varphi} = G_\omega^{gg^\varphi} G_\omega^{gg^\varphi \beta}$ with $\beta = (gg^\varphi)^{-1}\alpha(gg^\varphi)$, and that $(g^{-1}a_1 g)^{-1}(g^{-1}a_2 g) = w^g$.

Therefore we may switch from G to G^{gg^φ} and we may assume $w = f^j t$. Moreover by Lemma 3.1 we may (and do) assume $t = 1$.

We have $C_T(w) \cong SL_2(2)$ and $|B : C_B(w)T| = m$ for some $m \in \{1, 2, 3\}$. Thus $|C_B(w)| = 6/m \cdot |B/T| = 6/m \cdot |A|$. Since by Lemma 9.5 $|C_B(w)| = 2 \cdot |A|$, we obtain $m = 3$, which yields $\hat{H} \leq B$.

Let $h \in \hat{H} \setminus H$. Then f^r inverts h modulo T. Since $hh^\varphi \in G$ and since $\alpha = a_1 a_2^\varphi \in \mathrm{Aut}(G)$, it follows that both a_1 and a_2 either centralize or invert h modulo T. As $w = a_1^{-1} a_2$, it follows $[w, h] \leq T$. This contradicts with the fact that $w = f^j$, where $o(f^j) = 2r$. Thus $T \not\cong U_3(2^r)$, as claimed. $\square$

Lemmas 9.2, ..., 9.10 prove Proposition 9.1.

10 Factorizations of blow–ups.

In this section let H be a primitive group of almost simple or of diagonal action type on a set Γ and let G be a blow up of H of index n. This means G is a large subgroup of $W = H \wr S_n$ (see the previous section). Let $\gamma \in \Gamma$, and $W_\omega = H_\gamma \wr S_n$. Then we have $G_\omega = G \cap W_\omega$ and $H_\omega = H_\gamma$. Further $E(G) = E(H)^n = E(W)$.

If H admits a factorization $H = H_\gamma H_\gamma^a$ for some $a \in \mathrm{Aut}(H)$, then, clearly, $E(H) \wr S_n$ (and W) also admits a factorization. We show in this section that any primitive factorization of a blow–up arises in this way. We prove

Proposition 10.1 *Let G be a blow–up of a primitive group H of index n. If $G = G_\omega G_\omega^\alpha$ for a point stabilizer G_ω and an $\alpha \in \mathrm{Aut}(G)$, then H itself admits a factorization $H = H_\omega H_\omega^a$ with $a \in \mathrm{Aut}(H)$. Moreover, $H_i = (H_i)_\omega (H_i)_\omega^{a_i}$ for $i = 1, \ldots, n$ and $E(G)_\omega \cap E(G)_\omega^\alpha = \prod_{i=1}^n E(H)_\omega \cap E(H)_\omega^{a_i}$, where $\alpha = s \cdot a_1 \cdot \ldots \cdot a_n$ with $s \in S_n$ and $a_i \in \mathrm{Aut}(E(H))_i$ ($i = 1, \ldots, n$).*

Corollary 10.2 *Let G be a blow–up of a primitive group H of index n. If $G = G_\omega G_\omega^\alpha$ for a point stabilizer G_ω and an $\alpha \in \mathrm{Aut}(G)$, then $E(H)$ is isomorphic to a group listed in (ii) of Theorem 1.1, that is $E(H) \cong U$, $U \in \Omega$ with $\Omega = \{P\Omega_8^+(q);\ Sp_4(q),\ q > 2;\ A_6; M_{12}\}$, and $E(G)_\omega \cap E(G)_\omega^\alpha \cong (E(H)_\omega \cap E(H)_\omega^a)^n$ for some $a \in Aut(H)$.*

Thus in this section we assume $G = G_\omega G_\omega^\alpha$ for a point stabilizer G_ω and an $\alpha \in \mathrm{Aut}(G)$.

Lemma 10.3 *The normalizer of G in $\mathrm{Aut}(E(H)) \wr S_n$ induces the full automorphism group $\mathrm{Aut}(G)$ on G.*

Proof. Set $S = E(G)$. Then $S = E(W) = E(H)^n$. Since $C_G(S) \le C_W(S) \le Z(S) = 1$ we may and do identify G with a subgroup of $\mathrm{Aut}(S)$.

Since S is a characteristic subgroup of G, the group $\mathrm{Aut}(G)$ acts on S. Moreover, if $[S, g] = 1$ for $g \in \mathrm{Aut}(G)$, then the Three–Subgroup Lemma yields $[G, g] \le C_G(S) = 1$. This shows that $\mathrm{Aut}(G)$ acts faithfully on S, which proves the lemma. $\square$

Lemma 10.3 yields that we may (and do) assume

$$\alpha = s \cdot a_1 \cdot \ldots \cdot a_n$$

with $s \in S_n$ and $a_i \in \mathrm{Aut}(E(H))_i$, for $i = 1, \ldots, n$.

Lemma 10.4 $[H^n, \alpha] \le H^n$.

Proof. Since $[H^n, s] \le H^n$, it remains to show $[H^n, a_i] \le H^n$, for $i = 1, \ldots, n$.

Assume $H_i^{a_i} \not\le H_i$ for some $i \in \{1, \ldots, n\}$. Then there is some $h \in H_i$ such that $h^{a_i} \notin H_i$. Since G is a large subgroup of W, there exists $k \in G \cap K$, $K = H \times H \wr S_{n-1}$ such that $\pi_i(k^s) = h$. It follows that $\pi_i(k^\alpha) = \pi_i(h)^{a_i} = h^{a_i}$ is not contained in H_i which contradicts the fact that α normalizes G. Thus $[H_i, a_i] \le H_i$, for $i = 1, \ldots, n$. This shows the assertion. $\square$

Corollary 10.5 *There exists a transitive subgroup X of S_n such that α normalizes $U = H \wr X$ and such that $U = U_\omega U_\omega^\alpha$.*

Proof. Since α normalizes both H^n, Lemma 10.4, and G, it normalizes $U = GH^n$. Now Lemma 3.2 yields $U = U_\omega U_\omega^\alpha$.

As $H^n \leq U \leq H \wr S_n$, there is a subgroup X of S_n, such that $U = H \wr X$. The fact that G is a primitive subgroup of W implies that X is a transitive subgroup of S_n, which proves the lemma. □

The next lemma is the cruical step in the proof of Proposition 10.1.

Lemma 10.6 *For any $j \in \{1, \ldots, n\}$, there is an $h = h(j) \in H^n$ such that $\alpha h = s \cdot b_1 \cdot \ldots \cdot b_n$, where $ib_i = ja_j$, for $i = 1, \ldots, n$.*

Proof. To establish the lemma we show that $(ia_i)^{-1}(ja_j) \in H$, for $1 \leq i, j \leq n$. Let $x \in X$, then $[\alpha, x] = (a_1^{-1}(a_{1^y})^{y^{-1}}) \cdot \ldots \cdot (a_n^{-1}(a_{n^y})^{y^{-1}}) \cdot [s, x]$ with $y = (x^s)^{-1}$.

From the facts, that $[X, \alpha] \leq U$, that $\prod_{i=1}^n (a_i^{-1}(a_{i^y})^{y^{-1}}) \in \mathrm{Aut}(E(H))^n$ and that $[s, x] \in S_n$, we conclude that $[s, x] \in S_n \cap U = X$ and $\prod_{i=1}^n (a_i^{-1}(a_{i^y})^{y^{-1}}) \in H^n$. Therefore $(a_i^{-1}(a_{i^y})^{y^{-1}}) \in H_i$, for $i = 1, \ldots, n$. As X is a transitive subgroup of S_n, we obtain $(ia_i)^{-1}(ja_j) \in H$ for $1 \leq i, j \leq n$, as claimed.

Hence there exists $h_i \in H_i$ such that $ih_i = (ia_i)^{-1}(ja_j)$. The assertion follows with $h = h_1 \cdot \ldots \cdot h_n$. □

Proof of Proposition 10.1.
Let $j \in \{1, \ldots, n\}$ and set $\beta = \alpha h$, where h = h(j) is as in Lemma 10.6. By the choice β normalizes S_n (see Lemma 10.6), therefore $[X, \beta] \leq S_n \cap U = X$. Hence according to the Modular Property of groups

$$\begin{aligned} U_\omega \cap U_\omega^\beta &= (H_\omega^n : X) \cap (H_\omega^n : X)^\beta = (H_\omega^n : X) \cap (H_\omega^{n\beta} : X) \\ &= (H_\omega^n \cap H_\omega^{n\beta} : X)X = (H_\omega^n \cap H_\omega^{n\beta})X = (H_\omega \cap H_\omega^{a_j})^n X. \end{aligned}$$

As $h \in H^n \leq U$, the equivalence between (i) and (ii) in Lemma 3.1 implies $U = U_\omega U_\omega^{\alpha h}$. Therefore

$$|H|^n |X| = |U| = |U_\omega U_\omega^\beta| = \frac{|U_\omega|^2}{|U_\omega \cap U_\omega^\beta|} = \frac{|H_\omega|^{2n} |X|^2}{|H_\omega \cap H_\omega^{a_j}|^n |X|}$$

and $|H| = |H_\omega|^2 / |H_\omega \cap H_\omega^{a_j}|$, which shows $H = H_\omega H_\omega^{a_j}$. Finally,

$$E(G)_\omega \cap E(G)_\omega^\alpha == (E(H)_\omega)^n \cap (E(H)_\omega)^{n\alpha} = \prod_{i=1}^n E(H)_\omega \cap E(H)_\omega^{a_i}.$$

□

Proof of Corollary 10.2.
Since H is a primitive group of almost simple or of diagonal action type which admits by Proposition 10.1 a factorization $H = H_\omega H_\omega^a$ for some $a \in Aut(E(H))$, Propositions 8.1 and 9.1 yield that $E(H)$ is isomorphic to a group in Ω. The second part of the assertion follows with Propositions 10.1 and 8.1. □

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