Some Existentially Closed Graphs

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Abstract

We give a characterization of the existentially closed models and pregeneric models of the theory of planar graphs and graphs without $K_{3,3}$-minor. We show that there are $2^{\aleph_0}$ countable existentially closed models in these classes and that all are elementary equivalent, though there are no model-companions.

1 Introduction

We deal with planar graphs and related classes. Since planarity is a notion from topology which is not first order, we say -according to the theorem of Kuratowski [1]- a graph is planar if it contains no subdivision of K_5 or $K_{3,3}$. It is then easy to show that the theory of planar graphs is axiomatizable but not finitely axiomatizable and that the class of planar graphs is an inductive class. We then characterize the pregeneric elements which are the star-closed graphs. A planar graph is star-closed if it is not possible to add a star, three edges with a common vertex, without leaving the class of planar graphs.

In contrast to the whole class of planar graphs the class of star-closed planar graphs has the amalgamation property AP in a natural way. The existentially closed planar graphs are the star-closed planar graphs which are finitely universal over pseudo-edges. A pseudo-edge is just a pair of vertices which could be joined by an edge leaving the graph planar. Finitely universal over pseudo-edges means that every finite planar graph is embeddable over every pseudo-edge. This describes the class of existentially closed planar graph. The existentially closed graphs without $K_{3,3}$-minor behave very similar but there are differences concerning the pregeneric ones.

The theory of existentially closed planar graphs is not the model-companion of the theory of planar graphs, because it is not axiomatizable. Existentially closed planar graphs are connected. This fact can not be described in a first order language. We will show that there exist $2^{\aleph_0}$ non isomorphic countable existentially closed planar graphs. Though there is no model-companion these graphs are all

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elementary equivalent. In [4] this was shown for a special subclass using an Ehrenfeucht-Fraïssé game. Later Reker [7] gave a proof for all existentially closed planar graphs by extending the previous technique. We give a much shorter proof for the whole class. This paper is a condensed and extented version of a part of [8]. The last section is included to give a feeling for what will happen for classes of graphs embeddable in other surfaces.

2 Planar Graphs and Graphs without $K_{3,3}$-Minor

We start with a remark on our use of *planar*. Planarity is a notion from topology which has a graph-theoretical characterization by the famous theorem of Kuratowski [1]. He considered finite graphs only, but his theorem carries over to the countable case, as was shown by Dirac and Schuster [2]. Unfortunately this is not true for the power of the continuum and of course not for even greater cardinalities. So some cardinality-considerations are necessary which are not expressible in the first order language of graph-theory. The theory of planar graphs is not first order. Nevertheless, we call a graph planar if it meets the conditions in Kuratowski's theorem (rather then to restrict ourselves to the countable case). The theory of those graphs is first order.

We identify a graph (G, E) with a structure $< G, R >$, where R is a binary, irreflexive and symmetric relation on G. A *subgraph* of a given graph G is a submodel of $< G, E >$, it is always an induced subgraph. A not necessary induced subgraph of a given graph G is called a *part* of G.

The language of our theory of planar graphs has exactly one binary relation symbol R. The axioms are as follows:
R is an irreflexive and symmetric relation and *There is no part homeomorphic to K_5 and $K_{3,3}$.*
The latter is expressible by an infinite set of sentences each requiering that a certain finite graph homeomorphic to K_5 or $K_{3,3}$ is not a part of the model of that theory.
The class of planar graphs, of models of the above theory, is denoted by $\mathcal{P}_0$. Another theory of graphs we are interested in, is the theory of graphs without $K_{3,3}$-minor. The second axiom is changed to *There is no part homeomorphic to $K_{3,3}$.* The class of models of this theory, is denoted by $\mathcal{P}_1$.

We study these two theories simultaniously to see which role the forbidden subdivisions of K_5 are playing in the theory of planar graphs.

Now we prove:

Theorem 2.1 The theory of $\mathcal{P}_i$ is not finitely axiomatizable for $i = 0, 1$.

Proof: Let G_0 be isomorphic to $K_{3,3}$. Fix an edge and let G_j be the graph where this edge is subdivided by exactly i vertices. For all natural numbers j G_j is not in $\mathcal{P}_i$. Let U be a non-principal ultrafilter on the natural numbers and

$$G := \prod_{j \in \omega} G_j / U$$

Clearly G is in $\mathcal{P}_i$. The complements of the classes $\mathcal{P}_i$ are not closed under ultraproducts, hence the theories are not finitely axiomatizable.

Next we state a fact which we use again and again in the sequel.

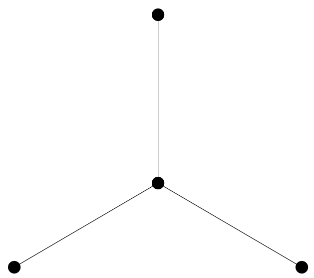
Lemma 2.1 let G and H be graphs in $\mathcal{P}_i$ and $\mid G \mid \cap \mid H \mid = \{a, b\}$ and $\{a, b\}$ be an edge. Then $G \cup H$ is in $\mathcal{P}_i$ $(i = 0, 1)$.

Proof: The proof follows from the easy observation, that two distinct vertices of K_5 and $K_{3,3}$ are joined by three disjoint paths.

2.1 Star-closed Graphs

We start with some definitions.

Definition 2.1 Three edges joined at a common vertex are a *simple star.*

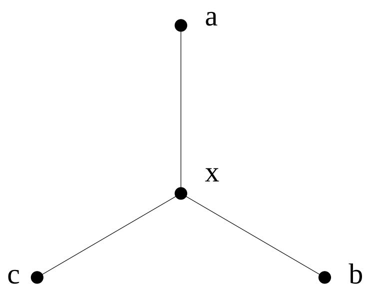


A graph homeomorphic to a simple star is called a *star.* The vertices having valency one in a star are called the *basic vertices* or the *base* of the star.

Definition 2.2 A graph $G \in \mathcal{P}_i$, $i = 0, 1$ is *star-closed* iff

1. G is the empty graph, the one-vertex graph or consists of exactly one edge (and two vertices). These graphs are called *trivial star-closed* graphs.

2. G has no supergraph $H \in \mathcal{P}_i$ having a part



such that $a, b, c \in G$ and $x \notin G$.

Lemma 2.2 Every non-trivial star-closed graph is 2-connected.

Proof: Let G be a star-closed graph.
Suppose G is not connected. Choose vertices a, b from a component A and a vertex c from $G \setminus A$. It is possible to add a star with base a, b, c. It is easy to see, that the graph obtained has no part homeomorphic to $K_{3,3}$ or K_5.
Suppose $\{c\}$ is a cut and a, b are vertices from dinstinct components. We add a simple star with base a, b, c. Two vertices in different components of G are joined by at most two disjoint paths. The new graph has no part homeomorphic to $K_{3,3}$ or K_5.
The graph G is not star-closed. This is a contradiction.

Lemma 2.3 Let G be a 2-connected graph and H be a star-closed graph, H a subgraph of G. Then there are two vertices a, b seperating H and $G \setminus H$.

Proof: Let $c \in G \setminus H$. H is star-closed, hence there is an edge $\{e, f\}$ of H. Since G is 2-connected c and $\{e, f\}$ lie on a common cycle C.
Walking along C from c to $\{e, f\}$ we find two points a, b which are the first points of $C \cap H$.
Let $d \in H$ be a vertex distinct from a, b. Assume there is a path joining d and c not containing a or b, then we can choose d such that the path contains no further vertices from H. So c is the center of a star s with base a, b, d. $H \cup S$ as apart of G contains no subdivision of $K_{3,3}$ or K_5. Hence we could add a simple star with base a, b, d to H without getting a forbidden subdivision. So H is not star-closed. This is a contradiction.

Theorem 2.2

1. If G, H and $G \cap H$ are star-closed and $\mid G \cap H \mid \geq 2$, then $G \cup H$ is star-closed.

2. The star-closed graphs are the pregeneric graphs in $\mathcal{P}_i$, $i = 0, 1$.

Proof: (1) Suppose $G \cup H$ has a part homeomorphic to one of the forbidden minors. Then there are main-vertices of such a subdivision in $G \setminus (G \cup H)$ and $H \setminus (G \cup H)$. These vertices are pairwise joined by three disjoint paths. This is a contradiction to the lemma above, since $G \cup H$ is 2-connected.
Suppose $G \cup H$ is not star-closed. Then there are without loss of generality vertices $a \in G \setminus H$ and $b, c \in H \setminus G$ such that we could add a simple star S with base a, b, c having center z and do not end up with one of the forbidden subdivisions as a part of the new graph. There are two vertices e, f which seperate $H \setminus G$ and G. There is a path from b to e and a disjoint one from c to f. The union of this two paths with S and G is a graph without any forbidden minor. As above we could exchange the star with center z and base a, e, f with a simple star and get a contradiction to the fact that G is star-closed.
(2) Every graph G in $\mathcal{P}_i$ is a subgraph of a star-closed graph in $\mathcal{P}_i$,$i = 0, 1$. The subclasses of star-closed graphs are amalgamation bases. Since star-closed graphs are 2-connected, this amalgamation bases are maximal. This implies (2).

Our next aim is it to characterize star-closed graphs without referring to supergraphs.

Definition 2.3 Let G be a graph in $\mathcal{P}_i$, $i = 0, 1$, and $a, b, c \in G$. We say that a, b, c *accept n stars*, if n is maximal with the property that there are n stars $S_1, S_2, \ldots, S_n$ in G with base a, b, c and $\bigcap_{j=1}^{n} S_j = \{a, b, c\}$.

Lemma 2.4 Let G be a star-closed graph.

1. Every three vertices accept at least one star.
2. There are three vertices in G which accept two stars.
3. If G is planar, then every three vertices accept exactly two stars and G is infinite or G is a trivial star-closed graph.
4. There are star-closed graphs in $\mathcal{P}_1$ with three vertices accepting only one star.
5. There are non-trivial finite star-closed graphs in $\mathcal{P}_1$.

Proof: (1) If three vertices a, b, c do not accept any star, then it is impossible to produce a forbidden subdivision by adding a simple star.
(2) Suppose every 3-tupel of vertices accepts at most one star, then there is no part of G isomorphic to a subdivision of $K_{3,3}$. So it is possible to add a new star.
(3) This is more or less trivial.
(4),(5) This graph is star-closed and the vertices a, b, c accept only one star.

A non-trivial planar star-closed graph is infinite and 2-connected, every three vertices accept exactly two stars. It is built up from 3-connected star-closed planar graph, which are minimal in the sense that they contain no star-closed subgraphs, which are glued together over a vertex or an edge or two vertices a, b such that adding the edge $\{a, b\}$ leaves the graph planar. This leads to the next definition.

Definition 2.4 Let (G, E) be a graph in $\mathcal{P}_i$, $i = 0, 1$. A pair $\{a, b\}$ is a *pseudo-edge*, if $(G, E \cup \{a, b\}) \in \mathcal{P}_i$.

We could not say much about infinite star-closed graphs, but it is possible to classify all finite ones.

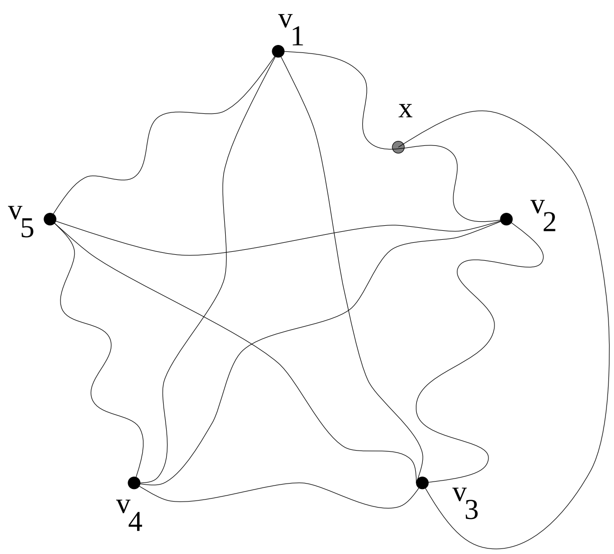
Lemma 2.5 Let $G \in \mathcal{P}_i$ and $a, b \in G$.

1. If H_1 and H_2 are subgraphs of G, which are separated by a, b and $\{a, b\} \subseteq H_1 \cap H_2$ and $G = H_1 \cup H_2$, then $\{a, b\}$ is an pseudo-edge.
2. If further H_1, H_2 are star-closed, then the supergraphs with the edge $\{a, b\}$ added are starclosed,too.

Proof: (1) Suppose not, we add an edge $\{a, b\}$ to G. This new graph G' has a part F isomorphic to a subdivision of $K_{3,3}$. Without loss we assume that the main vertices of this subdivision are in H_1. $F \cap H_2 \setminus \{a, b\} = \emptyset$ is impossible, hence a and b are not main vertices of F. There is a path joining a and b in H_2, so there is a subdivision of $K_{3,3}$ in G.
(2) Every star which base is in H_1 has a path from a basic vertex to the center containing a and b. So there is a shorter one using the edge $\{a, b\}$.

Lemma 2.6 If $G \in \mathcal{P}_1$ is star-closed and has a part isomorphic to a subdivision of K_5, then G is not 3-connected or isomorphic to K_5.

Proof: We consider several cases:
1. Case: $K_5 \subseteq G$. K_5 is star-closed, hence there are two vertices a and b in K_5 separating K_5 and $G \setminus K_5$. G is not 3-connected.
2.Case: Suppose there is a proper subdividion of K_5 as in the picture.



Since G is 3-connected there are three disjoint paths from x to v_3. At least one of these paths does not contain the subpaths $x, \ldots, v_1$ and $x, \ldots, v_2$. Let p be such a path. There has to be a vertex which is in p and one of the other paths indicated in the picture, since otherwise x is the center of a third star, which is accepted by v_1, v_2, v_3.
Claim: If v is not a main vertex of the subdivision, then $v \notin p$.
Proof: Let y be the first vertex on p while travelling from x to v_3 which is in the subdivision of K_5.
Case a: If y is in the subdivision of the edge $\{v_2, v_3\}$, then y is the center of a third star accepted by v_1, v_2, v_3.
Case b: If y is in the subdivision of the edge $\{v_3, v_4\}$, then y, v_1, v_2 is a base for three stars with centers v_3, v_4, x.
Case c: If y is in the subdivision of the edge $\{v_2, v_4\}$, then y is the center of a third star accepted by v_1, v_2, v_4.
Case d: If y is in the subdivision of the edge $\{v_1, v_3\}$, then y is the center of a third star accepted by v_1, v_2, v_3.
Case e: If y is in the subdivision of the edge $\{v_3, v_5\}$, then y, v_1, v_2 is a base for three stars with centers v_3, v_5, x.
The other cases follow by symmetry.

Hence there are only main vertices on p. v_1 and v_2 are not on p and v_4 and v_5 are not on p, since then x is the center of a third star accepted by v_1, v_2, v_3 respectively v_1, v_2, v_4. We proved that G contains no proper subdivision of K_5.

Corollary 2.1 If $G \in \mathcal{P}_1$ is star-closed and 3-connected, then G is planar and infinite or G is isomorphic to K_5.

The finite star-closed graphs which contain no K_5-minor are trivial or distinct K_5 glued together over trivial ones. Glueing over less than two vertices does not end up in a star-closed graph; we had to add other $K_5's$ in such a way that we get a graph in which the K_5-components are glued together over edges in a tree-like way. Then we can remove edges belonging to two components, the graph is still star-closed.

Theorem 2.3 The set S is the set of finite star-closed graphs without K_5-minor.

$$S_0 := \{\emptyset, K_1, K_2\} \qquad \textit{and} \quad S_1 := \{K_5\}$$

For all $i > 1$ define:

$$\begin{aligned} S_{i+1} \; := \; & \{H : H \text{ is an amalgam of two graphs from } S_i \text{over an edge}\} \\ \cup \; & \{H : H \text{ is anamalgam of two graphs from } S_i \\ & \qquad\qquad \text{over an edge which is removed then}\} \\ \cup \; & S_i \end{aligned}$$

$$S := \bigcup_{i \in \omega} S_i$$

Proof: It is easy to see that the graphs in S are finite star-closed graphs without K_5-minor. We show:
Every graph G of the above kind is in S.
If G is not in $S_0 \cup S_1$, then there is a subgraph of G isomorphic to K_5 orK_5^- (adding the edge leaves the graph starclosed). There are two vertices a, b in this subgraph seperating it from the remaining part of G. We cut this subgraph of and add if necessary the edge $\{a, b\}$. This graph is still star-closed, so we can cut off another K_5-like component. Since G is finite this procedure must stop. S is defined just the other way round.

2.2 Existentially closed Graphs

A graph $G \in \mathcal{P}_i$, $i = 0, 1$, is *existentially closed* iff every formula with existantial quantifiers only and parameters in G which holds for a supergraph H of G, $H \in \mathcal{P}_i$, is already true in G. To describe the existentially closed graphs we first give another definition.

Definition 2.5 A graph $G \in \mathcal{P}_i$, $i = 0, 1$, is called *finitely universal over pseudo-edges* if the following is true:
For every pseudo-edge $\{a, b\}$ of G and any finite graph $F \in \mathcal{P}_i$ which contains a pseudo-edge $\{c, d\}$ such that $\{a, b\}$ is an edge in G if and only if $\{c, d\}$ is an edge in F, there is a monomorphism $f : F \to G$ such that $f(c) = a$ and $f(d) = b$.

Theorem 2.4 A graph $G \in \mathcal{P}_i$, $i = 0, 1$, is existentially closed iff (1) G is star-closed and (2) finitely universal over pseudo-edges.

Proof: It is easy to see that existentially closed graphs are star-closed and finitely universal over pseude-edges.
Let $\phi(\bar{a})$ be in $\exists_1$, $\quad \bar{a} \in G$. Without loss $\phi(\bar{a}) = \exists \bar{x} \psi(\bar{x}, \bar{a})$. Let G' be a $\mathcal{P}_i$-supergraph of G such that $G' \models \exists \bar{x} \psi(\bar{x}, \bar{a})$. ψ describes a finite graph with vertices $\bar{a}, \bar{x}$, i.e. says which ones are connected by an edge and which are not. There is a finite subgraph H of G' such that

$$H \models \exists \bar{x} \psi(\bar{x}, \bar{a})$$

Let $F_0 = H_0 = H \cap G$ and let $H_1, \ldots, H_n$ be the set of all maximal connected subgraphs of G which contain no edge of G. Then $H = H_0 \cup H_1 \cup \ldots \cup H_n$.
Claim: For all $j > 0$, H_j has one of the following properties:

$$\begin{array}{llll} (1) & H_j \cap H_0 & = & \emptyset \\ (2) & H_j \cap H_0 & = & \{a\} \\ (3) & H_j \cap H_0 & = & \{a, b\} \text{ is an pseudo-edge in } G \end{array}$$

Suppose there are three elements a, b, c in $H_j \cap H_0$, then $H_j \setminus H_0$ contains a star with base a, b, c. This is a contradiction to the fact that G is star-closed. Hence there are at most two elements in $H_j \cap H_0$.
If there are two elements in $H_j \cap H_0$, then there is a path in $H_j \setminus H_0$ joining a and b. Therefore $\{a, b\}$ is a pseudo-edge.
Since every pseudo-edge of G is in a copy of any finite graph we can find such graphs $F_1, \ldots, F_n$ of G such that

1. F_j and H_j are isomorphic.

2. $H_j \cap H_0 = F_j \cap F_0$

3. If $a, b \in \bigcup_{j=1}^{n} F_i$ are joined by an edge, then there is an j such that $a, b \in F_j$.

Let $F = \bigcup_{j=0}^{n}$ then (1)-(3) imply that F and H are isomorphic. From (2) we get that the isomorphism f can be choosen in such a way that $f(a) = a$ for all $a \in F_0$. So we get

$$F \models \exists \bar{x} \psi(\bar{x}, \bar{a})$$

Since F is a subgraph of G we have

$$G \models \exists \bar{x} \psi(\bar{x}, \bar{a})$$

We proofed that G is existentially closed.

Theorem 2.5 The theory of planar graphs and the theory of graphs without $K_{3,3}$-minor have no model companion. The existentially closed models of these theories are elementary equivalent.

Proof: Since the existentially closed graphs are 2-connected but an ultrapower by an nonprincipal ultrafilter is not 2-connected the class of existentially closed models of this theories are not closed under ultrapowers. Hence the theories have no model companions.
The existentially closed models are elementary equivalent since the theories have the joint embedding property, the empty graph is pregeneric.

Theorem 2.6 There are $2^{\aleph_0}$ countable existentially closed graphs in both of the classes $\mathcal{P}_0$ and $\mathcal{P}_1$.

Proof: First we show that there is a 3-connected countable existentially closed planar graph. Let $(T_i)_{i\in\omega}$ be an enumeration of all plane triangulations. We define a planar graph on the sphere in ω steps.
$G_0 := D_0$
G_{n+1} the graph that is obtained, if every triangle of every face of G_n is identified with the outermost triangle of a copy of T_{n+1}.
$G := \bigcup_{n\in\omega} G_n$.
G is planar and existentially closed by construction. It is 3-connected, since every triangulation with more then three vertices is 3-connected.
We show that there are $2^{\aleph_0}$ countable existentially closed planar graphs. The other part of the theorem can then be proofed by a slight modification of the construction.

Let G now be a 3-connected existentailly closed planar graph with exactly two disjoint pseudo-edges $\{a,b\}$ and $\{c,d\}$ which are no edges. For every $n\in\omega$ we take an isomorphic copy G_n of G in which $\{a_n,b_n\}$ and $\{c_n,d_n\}$ are the two proper pseudo-edges. We glue together every pair G_n, G_{n+1} of graphs by identifying c_n with a_{n+1} and d_n with b_{n+1}. We get a new planar graph H which is existentially closed. For every $A\subseteq\omega$ we define a graph H_A:

$$\begin{aligned} H_\emptyset &:= (H, E(H)) = H \\ H_A &:= (H, E(H)\cup\{\{c_n,d_n\} : n\in A\}) \end{aligned}$$

It is easy to see that for any two subsets A, B of ω it is true that $A \ncong B$ implies H_A is not isomorphic to H_B, since $(\omega,<)$ is rigid.

3 Graphs embeddable in other Surfaces

To illustrate the differences between the theory of planar graphs and the theories of graphs embeddable into a surface of another genus it will be sufficient to look at the class $\mathcal{T}$ of countable graphs, that are embeddable into the torus. We restrict ourselves to the countable case to get a first order theory. There is a Kuratowski-like characterization of the graphs in $\mathcal{T}$ using about 800 minors [6], so the theory of $\mathcal{T}$ is axiomatizable but not finitley axiomatizable, too.

We define then nontrivial star-closed graphs as we did before, but there will not be any trivial ones. Glueing together two graphs from $\mathcal{T}$ over an edge or a vertex, or even the disjoint union, leads not always to a graph in $\mathcal{T}$. Recognizing the fact, that we don't have JEP we immediately get:

Theorem 3.1 The existentially closed graphs in $\mathcal{T}$ are not all elementary equivalent.

Proof: For example different embedding of a K_4 can have star-closures which could not be amalgameted.

The star-closed graphs are once again the pregeneric ones. We get a desription of the existentially closed graphs in $\mathcal{T}$ by modifying the contruction. We could amalgamate graphs from $\mathcal{T}$ with planar ones.

Lemma 3.1 Let $G \in \mathcal{T}$ and F be a planar graph, $\{a_1, b_1\} \in E(G)$ and $\{a_2, b_2\} \in E(F)$. If H is the graphs in which a_1, a_2 and b_1, b_2 are identified,then $H \in \mathcal{T}$.

Proof: Embedd H into the torus. Choose a (topological) circle C dividing the torus into two parts, such that $\{a_1, b_1\}$ and no other vertex of H is in one part. Say the intersection points of this circle with the edges of H are new vertices. Let S be the graph induced by a_1, b_1 and the new vertices. S is a plane graph. We embedd S in a sphere in such a way, that all new vertices belong to the same region. We identify all new vertices and get a graph S'. Now we amalgamate S' and H over $\{a_1, b_1\}$ resp. $\{a_2, b_2\}$. Next we forget about the identification of the new points and put the circle back to the torus. H is now embedded into the torus.

We say a pair $\{a, b\}$ of vertices of $G \in \mathcal{T}$ is a *pseudo-edge* if $(G, E \cup \{a, b\}) \in \mathcal{T}$.

Definition 3.1 $G \in \mathcal{T}$ is *planar-finitely-universal over pseudo-edges* iff the following is true:
For every pseudo-edge $\{a, b\}$ of G and any finite graph $F \in \mathcal{P}_i$ which contains a pseudo-edge (in the planar sense) $\{c, d\}$ such that $\{a, b\}$ is an edge in G if and only if $\{c, d\}$ is an edge in F, there is a monomorphism $f : F \to G$ such that $f(c) = a$ and $f(d) = b$.

Theorem 3.2 A graph $G \in \mathcal{T}$ is existentially closed iff G is star-closed and planar-finitely-universal over pseudo-edges.

Proof: We could mimic the proof for the planar graphs. Then we have to show, that the components $H_1, \ldots, H_n$ of the graph H described by the formula $\phi \in \exists_1$ are plane graphs. Since every H_i and G have at most two vertices a, b in common and $\{a, b\}$ is a pseudo-edge, there are infinitly many paths of every length joining a and b in G, because G is planar-finitely-universal over pseudo-edges. H_i is then in the interior of a circle described by an a, b-path and the pseudo-edge $\{a, b\}$ or two a, b-paths. So each H_i is plane.

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