Hadamard functions preserving nonnegative H-matrices

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ABSTRACT

For k nonnegative $n \times n$ matrices $A_l = (a_{ij}^l)$ and a function

$$f : \mathbf{R}_+^k \to \mathbf{R}_+$$

consider the matrix

$$C = f(A_1, \ldots, A_k) = (c_{ij}),$$

where

$$c_{ij} = f(a_{ij}^1, \ldots, a_{ij}^k), \qquad i, j = 1, \ldots, n.$$

Denote by $\rho(A)$ the spectral radius of a nonnegative square matrix A, and by $\sigma(A)$ the minimal real eigenvalue of its comparison matrix $M(A) = 2\mathrm{diag}(a_{ii}) - A$. It is known that the function

$$f(x_1, \ldots, x_k) = c x_1^{\alpha_1} \ldots x_k^{\alpha_k}$$

where $\alpha_i \in \mathbf{R}_+, \quad \Sigma_{i=1}^k \alpha_i \geq 1$ and $c > 0$, satisfies the inequalities

$$\rho(f(A_1, \ldots, A_k)) \leq f(\rho(A_1), \ldots, \rho(A_k)),$$

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as well as the inequalities

$$\sigma(f(A_1,\ldots,A_k)) \geq f(\sigma(A_1),\ldots,\sigma(A_k)).$$

whenever A_i are nonnegative H-matrices, i.e. $\sigma(A_i) \geq 0$. The last inequality implies that the above function f maps the set of nonnegative H-matrices into itself. In this note it is proven that these are the only continuous functions with this property.

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1 Introduction

Let $\mathbf{R}_+$ be the set of nonnegative real numbers. For k nonnegative $n \times n$ matrices $A_l = (a_{ij}^l), l = 1,\ldots,k$ and a function

$$f : \mathbf{R}_+^k \to \mathbf{R}_+$$

we consider the matrix

$$f(A_1,\ldots,A_k) = C = (c_{ij}),$$

where

$$c_{ij} = f(a_{ij}^1,\ldots,a_{ij}^k), \qquad i,j = 1,\ldots,n.$$

Such matrix functions are called Hadamard functions, e.g. [4, Section 6.3].

Denote by $\rho(A)$ the spectral radius of a nonnegative square matrix A, and by $\sigma(A)$ the minimal real eigenvalue of its comparison matrix $M(A) = 2\mathrm{diag}(a_{ii}) - A$. Denote by HP_n the set of all n-by-n nonnegative H-matrices, i.e. the nonnegative matrices A for which $\sigma(A) \geq 0$. These are the nonnegative matrices whose comparison matrices are M-matrices.

Let K be a set of nonnegative matrices. We say that f preserves K, if $A_1,\ldots,A_k \in K$ implies $f(A_1,\ldots,A_k) \in K$. In this note we mainly study the functions f preserving HP_n. First, we show that for $n \geq 3$ such a function satisfies the inequality

$$\rho(f(A_1,\ldots,A_k)) \leq f(\rho(A_1),\ldots,\rho(A_k)) \tag{1.1}$$

for all nonnegative $n \times n$ matrices $A_i, i = 1,\ldots,k$. We then characterize all *continuous* functions preserving nonnegative H-matrices. It turns out that these are exactly the functions f of the form

$$f(x_1,\ldots,x_k) = cx_1^{\alpha_1}\ldots x_k^{\alpha_k} \tag{1.2}$$

where $\alpha_i \geq 0, c \geq 0$ and $\alpha = \Sigma_{i=1}^k \alpha_i \geq 1$. (The latter condition is required whenever $n > 2$.) These functions are also characterized by the inequality

$$\sigma(f(A_1,\ldots,A_k)) \geq f(\sigma(A_1),\ldots,\sigma(A_k)), \tag{1.3}$$

for all $A_i \in HP_n$ for some $n,\ n > 2$.

Note that while the functions (1.2) are characterized by (1.3), they are not the only functions satisfying (1.1). The set of those is charcterized in [2] by simple functional inequalities and it contains more functions, e.g. the function $f(x) = \min(x, x^2)$ for $k = 1$.

2 Results

Lemma 2.1 *Let $f : \mathbf{R}_+^k \to \mathbf{R}_+$ preserve HP_n where $n \geq 3$. Then $f(0) = 0$ and for all $a, b \in \mathbf{R}_+^k$ we have*

$$f(a+b) \geq f(a) + f(b) \tag{2.2}$$

Proof. Consider first the case $n = 3$. For given $a = (a_i)$ and $b = (b_i)$ let

$$A_i = \begin{pmatrix} a_i + b_i & a_i & b_i \\ a_i & a_i + b_i & b_i \\ a_i & b_i & a_i + b_i \end{pmatrix} \qquad i = 1, \ldots, k. \tag{2.3}$$

Since $A_i \in HP_3,\ i = 1, \ldots, k$, we have that

$$C = f(A_1, \ldots, A_k) = \begin{pmatrix} f(a+b) & f(a) & f(b) \\ f(a) & f(a+b) & f(b) \\ f(a) & f(b) & f(a+b) \end{pmatrix} \in HP_3$$

and thus (2.2) follows. Setting $a = b = 0$ in (2.2) gives $f(0) = 0$.

Now let $n > 3$. A similar construction using $n-1$ nonnegative vectors $a^{(i)},\ i = 1 \ldots, n-1$ yields

$$f(a^{(1)} + \ldots + a^{(n-1)}) \geq f(a(1)) + \ldots + f(a^{(n-1)}).$$

From this we get $f(0) = 0$. Hence, by setting $a^{(1)} = a,\ a^{(2)} = b,\ a^{(i)} = 0,\ i \geq 3$, we obtain (2.2). $\square$.

Lemma 2.4 *Let $f : \mathbf{R}_+^k \to \mathbf{R}_+$ preserve HP_n where $n \geq 3$. Then*

$$\rho(f(A_1, \ldots, A_k)) \leq f(\rho(A_1), \ldots, \rho(A_k)), \tag{2.5}$$

for all nonnegative $n \times n$ matrices $A_i,\ i = 1, \ldots, k$

Proof. Let $A_r = (a_{st}^r),\ r = 1, \ldots, k$ and let $\rho_r = \rho(A_r)$. Since $\rho_r I - A_r$ is an M-matrix, the matrix $B_r = (b_{st}^r)$ defined by

$$b_{st}^r = \begin{cases} a_{st}^r\,, & s \neq t \\ \rho_r - a_{ss}^r\,, & s = t \end{cases}$$

is in HP_n. Hence $B = (b_{st}) = f(B_1, \ldots, B_k) \in HP_n$. By (2.2), the diagonal entries of B satisfy

$$b_{ss} = f(\rho_1 - a_{ss}^1, \ldots, \rho_k - a_{ss}^k) \leq f(\rho_1, \ldots, \rho_k) - f(a_{ss}^1, \ldots, a_{ss}^k),\ \ s = 1, \ldots, n.$$

Since increasing of diagonal elements leaves a matrix in HP_n, it now follows that the matrix $\tilde{B}_r = (\tilde{b}_{st}^r)$ defined by

$$\tilde{b}_{st}^r = \begin{cases} b_{st} = f(a_{st}^1, \ldots, a_{st}^k)\,, & s \neq t \\ f(\rho_1, \ldots, \rho_k) - f(a_{ss}^1, \ldots, a_{ss}^k)\,, & s = t \end{cases}$$

is also in HP_n. Therefore, its comparison matrix $f(\rho_1, \ldots, \rho_k)I - f(A_1, \ldots, A_k)$ is an M-matrix, and (2.5) follows. $\square$

Lemma 2.6 *Let $f : \mathbf{R}_+^k \to \mathbf{R}_+$ preserve HP_n where $n \geq 2$. Then*

$$f(a)f(b) = f(a \circ b)f(1, \ldots, 1) \qquad a, b \in \mathbf{R}_+^k. \tag{2.7}$$

Proof. Consider first the case $n = 2$. Let a, b, c and d be vectors in $\mathbf{R}_+^k$ such that $a_i b_i = c_i d_i,\ i = 1, \ldots, k$. The matrices

$$A_i = \begin{pmatrix} a_i & c_i \\ d_i & b_i \end{pmatrix} \qquad i = 1, \ldots, k$$

are in HP_2, and hence the matrix

$$C = f(A_1, \ldots, A_k) = \begin{pmatrix} f(a) & f(c) \\ f(d) & f(b) \end{pmatrix}.$$

is in HP_2, implying that $f(a)f(b) \geq f(c)f(d)$. Since the matrices

$$\begin{pmatrix} c_i & a_i \\ b_i & d_i \end{pmatrix} \qquad i = 1, \ldots, k$$

are in HP_2 as well, we similarly obtain that $f(a)f(b) \leq f(c)f(d)$, implying that $f(a)f(b) = f(c)f(d)$. By choosing $c = a \circ b, d = (1, \ldots, 1)$ we now obtain (2.7).

If $n \geq 3$ then by Lemma 2.1 we have $f(0) = 0$. Hence by extending A_i by zeros to an H-matrix of dimension n, we can proceed as above to get (2.7). □

Theorem 2.8 *Let $f : \mathbf{R}_+^k \to \mathbf{R}_+$ be a continuous function. The following are equivalent.*
(i) *f preserves HP_n for some $n > 2$.*
(ii) *for some $n > 2$ and all matrices $A_i \in HP_n,\ i = 1, \ldots, k$ we have*

$$\sigma(f(A_1, \ldots, A_k)) \geq f(\sigma(A_1), \ldots, \sigma(A_k)). \tag{2.9}$$

(iii) *there exist $c \geq 0$ and $\alpha_i \geq 0,\ i = 1, \ldots, k,\ \sum_{i=1}^k \alpha_i \geq 1$, such that*

$$f(x_1, \ldots, x_k) = c x_1^{\alpha_1} \ldots x_k^{\alpha_k}. \tag{2.10}$$

Proof. (iii)$\Longrightarrow$(ii) is proven in [3].

(ii)$\Longrightarrow$(i) is a trivial consequence of the fact that $A \in HP_n$ if and only if $\sigma(A) \geq 0$.

(i)$\Longrightarrow$(iii) By Lemma 2.4 we have

$$f(a)f(b) = f(a \circ b)f(1, \ldots, 1). \tag{2.11}$$

Let $\gamma = f(1, \ldots, 1)$. If $\gamma = 0$ then by (2.11) $f(a) = 0$ for all $a \in \mathbf{R}_+^k$. Hence we have (iii) with $c = 0$. If $\gamma > 0$ then define

$$\tilde{f}(a) = \gamma^{-1} f(a).$$

By (2.11) we have

$$\tilde{f}(a)\tilde{f}(b) = \tilde{f}(a \circ b). \tag{2.12}$$

Define for $s = 1, \dots, k$ the function $f_s : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$f_s(t) = \tilde{f}(1, \dots, 1, t, 1, \dots, 1)$$

where t is in position s. Then, by (2.12) we have $f_s(t)f_s(u) = f_s(tu)$. Also, f_s is continuous on $\mathbf{R}_+$ and is positive for positive arguments. By a well known result (e.g. [1], p.39 or Cauchy 1821) it follows that

$$f_s(t) = t^{\alpha_s}$$

for some nonnegative α_s. Therefore, for $x = (x_1, \dots, x_k)$ we have

$$f(x) = \gamma\tilde{f}(x) = \gamma f_1(x_1) \dots f_k(x_k) = \gamma x_1^{\alpha_1} \dots x_k^{\alpha_k},$$

and so (2.10) holds with $c = \gamma$. Now, by applying Lemma 2.1 to the vectors $a = b = (1, \dots, 1)$ we obtain $2\gamma \leq \gamma 2^{\alpha}$ where $\alpha = \sum_{i=1}^{k} \alpha_i$, implying that $\alpha \geq 1$. □

Remark 2.13 Since Statement (iii) in Theorem 2.8 is independent of n, we can replace in the other conditions of Theorem 2.8 the expression "*for some* $n > 2$" by "*for all* $n > 2$".

The analog of Theorem 2.8 for the case $n = 2$ is

Theorem 2.14 *Let* $f : \mathbf{R}_+^k \to \mathbf{R}_+$ *be a continuous function. Then the following are equivalent.*
(i) f *preserves* HP_2.
(ii) *there exist* $c \geq 0$ *and* $\alpha_i \geq 0,\ i = 1, \dots, k,$ *such that*

$$f(x_1, \dots, x_k) = c x_1^{\alpha_1} \dots x_k^{\alpha_k}.$$

Proof. (i)$\Longrightarrow$(ii). The proof of this implication is identical to the proof of the implication (i)$\Rightarrow$(iii) in Theorem 2.8, omitting the last part which uses Lemma 2.1.

(ii)$\Longrightarrow$(i) follows by a simple calculation. □

Remark 2.15 The functions preserving HP_2 do not necessarily satisfy (2.9), as is shown by $f(x) = \sqrt{(x)}$. So, in Theorem 2.14 we cannot add an equivalent condition which is the analog of (ii) of Theorem 2.8.

Remark 2.16 If we replace the statement "$c \geq 0$" in Condition (iii) of Theorem 2.8 and Condition (ii) of Theorem 2.14 by the statement "$c > 0$" then we obtain a characterization for continuous functions f preserving the interior of HP_n.

3 Some final remarks

One can consider other sets of nonnegative matrices and study the question of its preservation by Hadamard functions.

For example, one can consider the set $\tilde{H}P_n$ of all nonnegative H-matrices with *constant diagonal*. Here we have

Theorem 3.1 *Let $f : \mathbf{R}_+^k \to \mathbf{R}_+$ be a continuous function and let $n \geq 3$. The following are equivalent.*
(i) *f satisfies (1.1).*
(ii) *f preserves $\tilde{H}P_n$.*

Proof. In our proof we use the fact, proven in [2, Theorem 2.1], that the continuous functions $f : \mathbf{R}_+^k \to \mathbf{R}_+$ satisfying (1.1) are characterized by

$$f(a) + f(b) \leq f(a+b) \qquad \text{for all } a, b \in \mathbf{R}_+^k \tag{3.2}$$

and

$$\sqrt{f(a)f(b)} \leq f(\sqrt{a \circ b}) \qquad \text{for all } a, b \in \mathbf{R}_+^k. \tag{3.3}$$

We now prove the equivalence of the two conditions.
(ii)$\Longrightarrow$(i). Observe that the matrices in (2.3) are in $\tilde{H}P_3$, and hence the inequality (3.2) follows. Similarly, the inequality (3.3) follows by applying (ii) for the matrices

$$A_i = \begin{pmatrix} \sqrt{a_i b_i} & a_i \\ b_i & \sqrt{a_i b_i} \end{pmatrix}.$$

Therefore, the function f satisfies (1.1).
(i)$\Longrightarrow$(ii). Let $A_i \in \tilde{H}P_n$, $i = 1, \ldots, n$. It follows that $A_i = a_{11}^j I + N_i$ where N_i is nonnegative with zero diagonal and $\rho(N_i) \leq a_{11}^i$. It follows from (3.2) that $f(0) = 0$. Therefore, we now have

$$f(A_1, \ldots, A_k) = f(a_{11}^1, \ldots, a_{11}^k)I + f(N_1, \ldots, N_k).$$

Since f satisfies (1.1), we have

$$\rho(f(N_1, \ldots, N_k) \leq f(\rho(N_1), \ldots, \rho(N_k)).$$

Since

$$f(\rho(N_1), \ldots, \rho(N_k)) \leq f(\rho(N_1), \ldots, \rho(N_k)) + f(a_{11}^1 - \rho(N_1), \ldots, a_{11}^k - \rho(N_k)),$$

and since, by (3.2), we have

$$f(\rho(N_1), \ldots, \rho(N_k)) + f(a_{11}^1 - \rho(N_1), \ldots, a_{11}^k - \rho(N_k)) \leq f(a_{11}^1, \ldots, a_{11}^k),$$

it now follows that $f(A_1, \ldots, A_k) \in \tilde{H}P_n$. $\square$

Another interesting case is the set of the symmetric nonnegative H-matrices. This case will be studied in a subsequent paper.

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