Representation Varieties of the Fundamental Groups of Compact Non-Orientable Surfaces

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Abstract

We give a description of the varieties of n-dimensional representations and characters of fundamental groups of compact non-orientable surfaces.

1 Introduction

Let $\Gamma =< g_1, \ldots, g_m >$ be a finitely generated group and $G \subset GL_n(K)$ a connected linear algebraic group defined over a field K which throughout the paper will be assumed to be algebraically closed and of characteristic zero. For any homomorphism $\rho : \Gamma \to G(K)$ the set of elements

$$(\rho(g_1), \ldots, \rho(g_m)) \in G(K)^m = G(K) \times \cdots \times G(K)$$

satisfies evidently all the relations of Γ and thus the correspondence

$$\rho \to (\rho(g_1), \ldots, \rho(g_m))$$

gives a bijection between points of the set $\mathrm{Hom}(\Gamma, G(K))$ and K-points of some affine K-variety $R(\Gamma, G) \subset G^m$ whose geometric structure does not depend on the choice of generators $g_1, \ldots, g_m$ of Γ.

The variety $R(\Gamma, G)$ is usually called the *representation variety* of Γ into the algebraic group G. In the case $G = \mathrm{GL}_n(K)$ we will denote it simply by $R_n(\Gamma)$ and call it the variety of n-dimensional representations of Γ.

The group G acts on $R(\Gamma, G)$ by simultaneous conjugation and its orbits are in one-to-one correspondence with the equivalence classes of representations of Γ. Under this action orbits of G are not necessarily closed and so the set of orbits (also called the geometric quotient) is not an algebraic variety. However if G is a reductive group, then one can consider a categorical quotient $R(\Gamma, G)/G$ (see [11]), which is usually denoted by $X(\Gamma, G)$ and is called the *variety of characters* (for more details see [10]). By construction, its points parametrize closed G-orbits. For $G = \mathrm{GL}_n(K)$ an orbit of a representation ρ is closed iff ρ is fully reducible. It follows that the points of the variety $X_n(\Gamma) = X(\Gamma, \mathrm{GL}_n(K))$ are in one-to-one correspondence with the equivalence classes of fully reducible n-dimensional representations of Γ (see [10]).

If Γ is an arbitrary finitely generated group we know practically nothing about the structure of the varieties $R_n(\Gamma)$, $X_n(\Gamma)$. It has been studied for classes of infinite nilpotent and solvable groups (see [10], [15]) and in detail for the class of finite groups only. Recall that if $|\Gamma| < \infty$ then the description of $R_n(\Gamma)$, $X_n(\Gamma)$ is given by the classical representation theory of finite groups. Namely, every representation is fully reducible and up to equivalence there is only a finite number of irreducible representations and all of them are uniquely determined by their characters; in particular, $\dim X(\Gamma, G) = 0$, i.e. $X(\Gamma, G)$ is a finite set for all finite groups Γ.

For topological applications it is important to know the description of the varieties of n-dimensional representations and characters for those groups Γ which arise as fundamental groups of some natural classes of manifolds. At present the answer is known for fundamental groups Γ_g of compact orientable surfaces of genus g only. First Goldman [4] found the number of connected components of $R(\Gamma_g, \mathrm{SL}_2(\mathbb{C}))$ in real and complex topology and then Rapinchuk and the authors in [14] and independently Simpson in [16] established the more general fact that for any n and any $g > 1$ the variety $R_n(\Gamma)$ is irreducible and of dimension $(2g-1)n^2+1$; in particular, $R(\Gamma_g, \mathrm{GL}_n(\mathbb{C})$ is connected with respect to the complex topology. Furthermore, we also obtained in [14] that $R(\Gamma_g, \mathrm{SL}_n(K))$ is irreducible and proved that $R_n(\Gamma_g)$ is Q-rational and that $R(\Gamma_g, \mathrm{SL}_n(K))$ is Q-unirational.

Notice that later Serre (private communication) estimated the number of F_q-points of the representation variety of Γ_g into any semisimple simply connected algebraic group defined over a finite field F_q. He concluded from this and Deligne-Weyl's formula that the variety $R(\Gamma_g, G)$ is irreducible for all semisimple simply connected algebraic groups G.

The aim of the present paper is to study the varieties $R_n(\Gamma_g)$, $X_n(\Gamma_g)$, $R(\Gamma_g, \mathrm{SL}_n(K))$ for fundamental groups

$$\Gamma_g = < \gamma_1, \ldots, \gamma_g \mid \gamma_1^2 \cdots \gamma_g^2 = 1 >$$

of compact non-orientable surfaces of genus $g \geq 1$ and thus complete the description of the varieties of n-dimensional representations and characters for all surface groups. The main results are the following two theorems.

Theorem A. *Let Γ_g be the fundamental group of a compact non-orientable surface of genus g. Then the number of irreducible components of $R_n(\Gamma_g)$ and $X_n(\Gamma_g)$ equals*

1) $n+1$, if $g=1$;

2) $(n^2+4n+4)/4$, if n is even and $g=2$;

3) $(n^2+4n+3)/4$, if n is odd and $g=2$;

4) 3, if $g=3$ and $n=2$;

5) 2, if $g \geq 3$ and $(n,g) \neq (2,3)$.

Moreover, for all n and g all components of $R_n(\Gamma_g)$ are Q-rational varieties.

Theorem B. *Under the conditions of Theorem A the number of irreducible components of $R(\Gamma_g, \mathrm{SL}_n(K))$ equals*

1) $[n/2]+1$, if $g=1$;

2) $(n^2+6n+8)/8$, if n is even and $g=2$;

3) $(n^2+4n+3)/8$, if n is odd and $g=2$;

4)1, if $g \geq 3$.

Moreover, for all n and g all components of $R(\Gamma_g, \mathrm{SL}_n(K))$ are Q-unirational varieties.

Remark. *The question if the components of $R(\Gamma_g, \mathrm{SL}_n(K))$ are rational is still open.*

In fact, we give a direct description of all irreducible components of $R_n(\Gamma_g)$, $X_n(\Gamma_g)$, $R(\Gamma_g, \mathrm{SL}_n(K))$ which makes it possible to compute their dimensions and find the number of connected components of these varieties with respect to the complex topology.

The irreducible components of the representation varieties of Γ_g into SL_n can actually be described similarly as the components of $R_n(\Gamma_g)$. However the proof of the unirationality is different and based on the following idea which was first used in [14]. If we are given two elements $a, b \in \mathrm{GL}_n$ and multiply a by an element of the centralizer of b or, symmetrically, multiply b by an element of the centralizer of a (i.e. perform a so-called standard transformation) then the commutator $[a, b]$ doesn't change. Now suppose we are given elements $a, b, c, d \in \mathrm{GL}_n$ (resp. SL_n) such that $[a, b] = [c, d]$, then we can ask if it is possible to pass from the pair (a,b) to (c,d) by a chain of standard transformations. We proved in [14] that this is true "generically", i.e. for (a,b,c,d) in some Zariski open subset $U \subset (\mathrm{GL}_n)^4$, and this fact allows us to prove the Q-unirationality of $R\left(\Gamma_g, \mathrm{SL}_n(K)\right), g \geq 3$.

We would also like to mention that the methods developed both in [14] and the present paper allow to study in full the varieties of n-dimensional representations of one-relator groups of the form

$$\Gamma =< \gamma_1, \ldots, \gamma_n \mid r_1(\gamma_1, \ldots, \gamma_{n-4})[\gamma_{n-3}, \gamma_{n-2}][\gamma_{n-1}, \gamma_n] = 1 >$$

where $[x, y] = xyx^{-1}y^{-1}$ is a commutator of x and y, r_1 is an arbitrary word in the derived subgroup of the free group $F(\gamma_1, \ldots, \gamma_{n-4})$, as well as groups of the form

$$\Gamma =< \gamma_1, \ldots, \gamma_n \mid \gamma_{i_1}^{\varepsilon_1}\gamma_{i_2}^{\varepsilon_2} \cdots \gamma_{i_s}^{\varepsilon_s} = 1 >$$

where $\varepsilon_i = \pm 1$ and every generator γ_l occurs at most twice in the above relation.

Throughout this paper Γ_g denotes the fundamental group of a compact non-orientable surface of genus g, unless otherwise stated. By a point in "generic" position in an algebraic variety X we mean a point in some Zariski open subset $U \subset X$ which always can be easily specified from the argument.

A sketch of the proofs of Theorem A, B for small values of n and g was published in [1], [2].

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2 Some results on regular elements.

As a first step to describe the variety of n-dimensional representations $R_n(\Gamma_g)$ we will establish that any irreducible component V of $R_n(\Gamma_g)$ contains elements with regular components. Recall that an element x of a reductive group G is called regular if its centralizer $Z_G(x)$ has minimal possible dimension (which is always equal to the rank of G). It is well known that the set G_{reg} of regular elements is Zariski open in G (see [17]).

One can easily check that in the case $G = \mathrm{GL}_n(K)$ an element $X \in G$ is regular if and only if in its Jordan normal form every eigenvalue corresponds to a single Jordan block. Note that the latter condition can be reformulated as follows: for any λ in an algebraic closure of the field of definition the rank of the matrix $X - \lambda E_n$ is at least $n-1$ (here and in the sequel the unit matrix of order n will be denoted by E_n); in particular, a semisimple $X \in \mathrm{GL}_n(K)$ is regular iff each of its eigenvalues has multiplicity one.

Proposition 2.1 *Let $g \geq 3$ and $(n, g) \neq (2, 3)$. Then for any irreducible component V of $R_n(\Gamma_g)$ the set*

$$V' = \left\{(X_1, \ldots, X_g) \in V \mid X_i^2 \textit{ is regular semisimple for all } i = 1, \ldots, g\right\}$$

is non-empty.

To begin with we show that V contains points with regular components. The proof rests on the following statement.

Proposition 2.2 *For any $X, Y \in \mathrm{GL}_n(K)$ the set $XZ_{\mathrm{GL}_n(K)}(Y)$ contains a regular element.*

Remark 2.1 *This statement was proved by Rapinchuk and by the authors [14] and independently by Djoković [5]. In the sequel we will use it many times and so for the sake of expository completeness we include here a simple proof due to the second author (see [14]).*

Proof. Replacing X and Y by suitable conjugates we may assume Y to be a Jordan matrix, i.e. $Y = \mathrm{diag}(J_{k_1}(\alpha_1), \ldots, J_{k_m}(\alpha_m))$, where

$$J_k(\alpha) = \begin{pmatrix} \alpha & 1 & 0 & \ldots & 0 \\ 0 & \alpha & 1 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & \alpha \end{pmatrix}$$

is a Jordan block of order k and α, α_1,..., α_m belong to the fixed algebraically closed field K. It is easy to check that the centralizer of $J_k(\alpha)$ in the full matrix algebra $M_k(K)$ consists of all matrices of the form

$$X_k(a_1, \ldots, a_k) = \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ 0 & a_1 & \ldots & a_{k-1} \\ & & \ddots & \\ 0 & 0 & \ldots & a_1 \end{pmatrix}.$$

Now let $x_1^{(1)}, \ldots, x_{k_1}^{(1)}, \ldots, x_1^{(m)}, \ldots, x_{k_m}^{(m)}$ be algebraically independent over K and

$$C(k_1, \ldots, k_m) = \mathrm{diag}(X_{k_1}(x_1^{(1)}, \ldots, x_{k_1}^{(1)}), \ldots, X_{k_m}(x_1^{(m)}, \ldots, x_{k_m}^{(m)})).$$

Then $C = C(k_1, \ldots, k_m)$ is a non-degenerate matrix over a field

$$L = K(x_1^{(1)}, \ldots, x_{k_1}^{(1)}, \ldots, x_1^{(m)}, \ldots, x_{k_m}^{(m)}),$$

commuting with Y. Furthermore, since the set of regular elements is open with respect to the Zariski K-topology it suffices to show that XC^{-1} is regular. Thus we need to prove that for any λ in the algebraic closure $\overline{L}$ we have the following inequality for the ranks of matrices:

$$\operatorname{rank}(XC^{-1} - \lambda E_n) = \operatorname{rank}(X - \lambda C) \geq n - 1.$$

It turns out that the latter fact is true for arbitrary $X \in M_n(K)$ (even degenerate).

Lemma 2.1 *Let $X \in M_n(K)$, C be as introduced above. Then for any $\lambda \in \overline{L}$ we have* $\operatorname{rank}(X - \lambda C) \geq n - 1$.

Proof. Put

$$F_n(k_1, \ldots, k_m) = \left\{ X - \lambda C(k_1, \ldots, k_m) \mid X \in M_n(K), \lambda \in \overline{L} \right\}.$$

We are going to show by induction on n that for any choice of $k_1, \ldots, k_m$ the rank of any element from $F_n(k_1, \ldots, k_m)$ is at least $n - 1$. For $n = 2$ this is easily verified by direct computations. Assume that for some $n > 2$ there are some $k_1, \ldots, k_m$ such that $k_1 + \ldots + k_m = n$ and the rank of $B = X - \lambda C \in F_n(k_1, \ldots, k_m)$ is strictly less than $n - 1$.

Let X_1 (resp. B_1) be the matrix obtained from X (resp. B) by deleting the last row and the last column. Then $B_1 = X_1 - \lambda C_1$, where

$$C_1 = \operatorname{diag}(X_{k_1}(x_1^{(1)}, \ldots, x_{k_1}^{(1)}), \ldots, X_{k_m - 1}(x_1^{(m)}, \ldots, x_{k_m - 1}^{(m)})).$$

Since $\operatorname{rank} B_1 \leq \operatorname{rank} B < n - 1$, we have $\det B_1 = \det C_1 \det(X_1 C_1^{-1} - \lambda E_{n-1}) = 0$. Therefore, λ is an eigenvalue of the matrix $X_1 C_1^{-1}$; in particular, λ belongs to the algebraic closure of the field $L_1 = K(x_1^{(1)}, \ldots, x_{k_m - 1}^{(m)})$, i.e. λ is independent of the last variable $x_{k_m}^{(m)}$. Then $B_1 \in F_{n-1}(k_1, \ldots, k_m - 1)$ (under the convention that if $k_m = 1$ we put $F_{n-1}(k_1, \ldots, k_{m-1}, 0) = F_{n-1}(k_1, \ldots, k_{m-1})$), and by the inductive hypothesis $\operatorname{rank} B_1 \geq n - 2$. Consequently, $\operatorname{rank} B = \operatorname{rank} B_1 = n - 2$. Moreover, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of B then a basis of the subspace $\langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle \subset \overline{L}^n$ generated over $\overline{L}$ by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ can be chosen among $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, i.e. $\mathbf{e}_n$ is in fact a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$.

Now consider the projection $\overline{L}^n \to \overline{L}^{n-1}$, $v \mapsto \widetilde{v}$, omitting the $(n - k_m + 1)$-th component. We claim that

$$\dim \langle \widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_n \rangle = n - 2. \tag{2.1}$$

Indeed, if X_2 (resp. B_2) is the matrix obtained from X (resp. B) by deleting the row and the column with the number $n - k_m + 1$ then clearly $B_2 = X_2 - \lambda C_1$ with the same C_1 as above, implying that $B_2 \in F_{n-1}(k_1, \ldots, k_{m-1})$. Arguing as above we obtain that in our setting $\operatorname{rank} B = \operatorname{rank} B_2 = n - 2$ yielding (2.1).

Since $\mathbf{e}_n$ is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ then $\widetilde{\mathbf{e}}_n$ is a linear combination of $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_{n-1}$. So one can find a basis $\widetilde{\mathbf{e}}_{i_1}, \ldots, \widetilde{\mathbf{e}}_{i_{n-2}}$ of the space $\langle \widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_n \rangle$ such that all i_j are different from n. The corresponding vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-2}}$ are linearly independent and therefore form a basis of the space $\langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle$.

Consider a presentation

$$\mathbf{e}_n = \beta_1 \mathbf{e}_{i_1} + \ldots + \beta_{n-2} \mathbf{e}_{i_{n-2}}, \qquad \beta_i \in \overline{L}, \tag{2.2}$$

We claim that all β_i in (2.2) in fact belong to $\overline{L}_1$. Indeed, from (2.2) one obtains

$$\widetilde{\mathbf{e}}_n = \beta_1 \widetilde{\mathbf{e}}_{i_1} + \ldots + \beta_{n-2} \widetilde{\mathbf{e}}_{i_{n-2}}.$$

However all the coordinates of $\widetilde{\mathbf{e}}_n$, $\widetilde{\mathbf{e}}_{i_1}, \ldots$, $\widetilde{\mathbf{e}}_{i_{n-2}}$ belong to $\overline{L}_1$, consequently all β_i belong to $\overline{L}_1$ too, as required. Now matching the $(n - k_m + 1)$-th coordinates in (2.2) and bearing in mind that these coordinates of all $\mathbf{e}_{i_1}, \ldots$, $\mathbf{e}_{i_{n-2}}$ are in $\overline{L}_1$ we derive that $x^m_{k_m} \in \overline{L}_1$. The contradiction proves Proposition 2.2. $\square$

Proposition 2.3 *For all $g \geq 2$ and any irreducible component V of the variety $R_n(\Gamma_g)$ the set*

$$U = \{(X_1, \ldots, X_g) \in V \mid X_1, \ldots, X_g \textit{ are regular } \}$$

is non-empty.

Proof. Consider any point $v = (X_1, \ldots, X_g) \in V$ which does not belong to any other component of $R_n(\Gamma_g)$. It is easy to check that for any $i = 1, \ldots, g-1$ and any $\alpha \in Z_{\mathrm{GL}_n(K)}(X_i X_{i+1})$ one has

$$X_1^2 \ldots X_{i-1}^2 (X_i \alpha)^2 (\alpha^{-1} X_{i+1})^2 X_{i+2}^2 \ldots X_g^2 = 1.$$

This means that the irreducible set

$$C = \left\{ (X_1, \ldots, X_{i-1}, X_i \alpha, \alpha^{-1} X_{i+1}, X_{i+2}, \ldots, X_g) \mid \alpha \in Z_{\mathrm{GL}_n(K)}(X_i X_{i+1}) \right\}$$

is contained in $R_n(\Gamma_g)$. Furthermore, since v belongs only to the component V and clearly belongs to C, we have $C \subseteq V$. Then applying Proposition 2.2 we obtain that the set

$$U_i = \{(X_1, \ldots, X_g) \in V \mid X_i \text{ is regular } \}$$

is non-empty, forcing $U = \cap_{i=1}^g U_i$ to be non-empty, too. Thus, proposition 2.3 is proved. $\square$

Let us also consider the set

$$S_t = \left\{ X A X^{-1} \mid X \in \mathrm{GL}_n(K), \quad A = \mathrm{diag}(a_1, -a_1, \ldots, a_t, -a_t, b_{t+1}, \ldots, b_{n-2t}) \right\}.$$

Clearly, $\dim \overline{S}_t = n^2 - t$ and one has the chain

$$\mathrm{GL}_n(K) = \overline{S}_0 \supset \overline{S}_1 \supset \ldots \supset \overline{S}_{[n/2]}.$$

Denote by π_i the projection

$$\pi_i : V \to \mathrm{GL}_n(K), \qquad (X_1, \ldots, X_g) \mapsto X_i,$$

onto the i-th component. Obviously, for any $i \in \{1, \ldots, g\}$ there is a number $p_i \in \{0, \ldots, [n/2]\}$ such that $\overline{\pi_i(V)} \subset \overline{S}_{p_i}$, but $\overline{\pi_i(V)} \not\subset \overline{S}_{p_i+1}$ (under the convention that if $\overline{\pi_i(V)} \subset \overline{S}_{[n/2]}$ we put $p_i = [n/2]$).

Lemma 2.2 $p_1 = \ldots = p_g$.

Proof. It suffices to show that for any $i < g$ we have $p_i = p_{i+1}$. Assume the contrary. For example, let $p_i < p_{i+1}$. Then $\overline{\pi_{i+1}(V)} \subset \overline{S}_{p_{i+1}} \subset \overline{S}_{p_i}$ and hence there exists a point $u = (X_1, \ldots, X_g) \in V$ which does not belong to any other irreducible component of $R_n(\Gamma_g)$ and such that $X_i \in \overline{S}_{p_i}$, but $X_i \notin \overline{S}_{p_{i+1}}$. We know (see the proof of Proposition 2.3) that for any $\alpha \in Z_{\mathrm{GL}_n(K)}(X_i X_{i+1})$

$$(X_1, \ldots, X_{i-1}, X_i\alpha, \alpha^{-1}X_{i+1}, X_{i+2}, \ldots, X_g) \in V;$$

in particular, putting $\alpha = X_i X_{i+1}$ we have

$$(X_1, \ldots, X_{i-1}, X_i^2 X_{i+1}, X_{i+1}^{-1} X_i^{-1} X_{i+1}, X_{i+2}, \ldots, X_g) \in V. \tag{2.3}$$

It follows that $X_{i+1}^{-1} X_i^{-1} X_{i+1} \in \overline{S}_{p_{i+1}}$ and hence $X_i^{-1} \in \overline{S}_{p_{i+1}}$. But it is obvious that $X_i^{-1} \in \overline{S}_{p_{i+1}}$ iff $X_i \in \overline{S}_{p_{i+1}}$, whence $X_i \in \overline{S}_{p_{i+1}}$, which contradicts the assumption that $\pi_i(u) \notin \overline{S}_{p_{i+1}}$. $\square$

Let us proceed with the proof of Proposition 2.1. Put $p = p_1 = \ldots = p_g$. Then our statement is equivalent to the equality $p = 0$. Assuming that $p \neq 0$ we will find lower and upper bounds for $\dim V$ which contradict each other. To this end, consider

$$\pi : V \to \mathrm{GL}_n(K)^{g-1}, \qquad (X_1, \ldots, X_g) \mapsto (X_2, \ldots, X_g).$$

Since $\mathrm{Im}\,\pi \subseteq \overline{S}_p^{g-1}$, we have

$$\dim \overline{\mathrm{Im}\,\pi} \leq \dim \overline{S}_p^{g-1} = (g-1)n^2 - (g-1)p.$$

To estimate the dimension of fibres of π, for any point $X \in \mathrm{GL}_n(K)$ we put

$$G(X) = \left\{ Y \in \mathrm{GL}_n(K) \mid Y^2 = X^2 \right\}.$$

Lemma 2.3 *Let $X \in \mathrm{GL}_n(K)$ be a regular element conjugate to $\widetilde{X} = \mathrm{diag}(X_1, \ldots, X_k)$, where $X_i = \mathrm{diag}(J_{s_i}(a_i), -J_{t_i}(a_i))$, $s_i \geq t_i \geq 0$, $a_i \in K^*$. Then $\dim G(X) = \dim G(\widetilde{X}) = 2\sum_{i=1}^{k} t_i$.*

Proof. If $X = F\widetilde{X}F^{-1}$, then clearly $G(X) = FG(\widetilde{X})F^{-1}$ and hence $\dim G(X) = \dim G(\widetilde{X})$. Further, in the next section we will prove that for any $Y \in \mathrm{GL}_n(K)$ such that $Y^2 = \widetilde{X}^2$ the matrix Y is of the form $Y = \mathrm{diag}(Y_1, \dots, Y_k)$, where

$$Y_i = Z_i \,\mathrm{diag}(\varepsilon_i J_{s_i}(a_i), \delta_i J_{t_i}(a_i)) Z_i^{-1}, \qquad Z_i \in Z_{\mathrm{GL}_{s_i+t_i}(K)}(X_i^2),\ \varepsilon_i = \pm 1,\ \delta_i = \pm 1.$$

Then $G(\widetilde{X}) \simeq \prod_{i=1}^k G(X_i)$, implying that $\dim G(\widetilde{X}) = \sum_{i=1}^k \dim G(X_i)$, and it remains to observe that

$$\dim G(X_i) = \dim Z_{\mathrm{GL}_{s_i+t_i}(K)}(X_i^2) - \dim Z_{\mathrm{GL}_{s_i+t_i}(K)}(X_i) = (s_i + 3t_i) - (s_i + t_i) = 2t_i.$$

Thus $\dim G(\widetilde{X}) = 2\sum_{i=1}^k t_i$, as claimed. □

Lemma 2.4 *Let $s \geq [n/2]$ and $A = \mathrm{diag}(J_s(a), -J_{n-s}(a))$. Then $A \in \overline{S}_{n-s}$.*

The proof follows from the fact that A lies in the closure of the set consisting of all semisimple matrices of the form $B = \mathrm{diag}(B_1, B_2)$, where

$$B_1 = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ 0 & a_2 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & a_s \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -a_1 & -1 & 0 & \dots & 0 \\ 0 & -a_2 & -1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & -a_{n-s} \end{pmatrix},$$

$a_i \neq a_j$ for $i \neq j$, which, in turn, lies in S_{n-s}.

Lemma 2.5 *For any point $v \in V$ in "generic" position one has $\dim \pi^{-1}(\pi(v)) \leq 2p$.*

Proof. It suffices to find a point $\widetilde{v} \in V$ such that $\dim \pi^{-1}(\pi(\widetilde{v})) \leq 2p$. Since $\pi_1(V) \subseteq \overline{S}_p$, but $\pi_1(V) \not\subseteq \overline{S}_{p+1}$, the intersection $\pi_1(V) \cap (\overline{S}_p \setminus \overline{S}_{p+1})$ is non-empty. By Proposition 2.3, we can pick a point $\widetilde{v} = (A_1, \dots, A_g)$ with regular components A_i and such that $\pi_1(\widetilde{v}) \in \overline{S}_p \setminus \overline{S}_{p+1}$.

We claim that $\widetilde{v}$ is as required. Indeed,

$$\pi^{-1}(\pi(\widetilde{v})) = \{(Y, A_2, \dots, A_p) \mid Y^2 = A_1^2\}$$

and hence $\pi^{-1}(\pi(\widetilde{v})) \simeq G(A_1)$. Let A_1 be conjugate to

$$\mathrm{diag}(J_{s_1}(a_1), -J_{t_1}(a_1), \dots, J_{s_k}(a_k), -J_{t_k}(a_k))$$

for some $s_i \geq t_i \geq 0$, $a_1, \dots, a_k \in K^*$. Then by virtue of Lemma 2.4 we have $A_1 \in \overline{S}_{\sum_{i=1}^k t_i}$. But $A_1 \notin \overline{S}_{p+1}$, forcing $\sum_{i=1}^k t_i \leq p$, and hence

$$\dim \pi^{-1}(\pi(\widetilde{v})) = \dim G(A_1) = 2\sum_{i=1}^k t_i \leq 2p.$$

yielding the required result. □

¿From what has been said above we get the following upper estimate:

$$\dim V \leq (g-1)n^2 - (g-1)n + 2p = (g-1)n^2 - (g-3)p. \tag{2.4}$$

To find a lower bound, recall that $R_n(\Gamma_g)$ is defined by a single matrix equation. Therefore,

$$\dim V \geq gn^2 - n^2 = (g-1)n^2. \tag{2.5}$$

If $g > 3$, then the upper bound (2.4) for $\dim V$ is clearly less than the lower bound (2.5), which is impossible. Now let $g = 3$. Then it follows from (2.4) and (2.5) that $\dim V = (g-1)n^2$, implying $\overline{\operatorname{Im}\pi_i} = \overline{S}_p$ for all $i \in \{1, \ldots, g\}$ and $\overline{\operatorname{Im}\pi} = \overline{S}_p^{g-1} = \overline{S}_p^2$. Consider the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi} & \overline{\operatorname{Im}\pi} = \overline{S}_p^2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow {\scriptstyle \varphi_2} \\ \overline{S}_p & \xrightarrow{\varphi_1} & \mathrm{GL}_n(K) \end{array}$$

where $\varphi_1(X) = X^{-2}$, $\varphi_2(Y,Z) = Y^2Z^2$. Put $T = \overline{\varphi_1(\overline{S}_p)}$. Since π_1, π are dominant morphisms the above diagram yields $\overline{\varphi_2(\overline{S}_p^2)} = T$. By assumption, we have $p \neq 0$ and hence $\operatorname{Im}\varphi_1$ and its closure T consist of non-regular matrices. On the other hand, $\operatorname{Im}\varphi_2$ always contains regular elements. Indeed, if $p = 1$ then for $(X,Y) \in \overline{S}_p^2$, where

$$\begin{aligned} Y &= \operatorname{diag}(a_1, -a_1, b_1, \ldots, b_{n-2}) \\ Z &= \operatorname{diag}(\widetilde{a}_1, \widetilde{b}_1, -\widetilde{a}_1, \widetilde{b}_2, \ldots, \widetilde{b}_{n-2}) \end{aligned}$$

the image $\varphi_2(Y,Z) = Y^2Z^2$ is a regular semisimple element if a_i, $\widetilde{a}_i$, b_j, $\widetilde{b}_j$ are in "generic" position. Analogously, if $p \geq 2$ then for

$$\begin{aligned} Y &= \operatorname{diag}(a_1, \ldots, a_p, -a_1, \ldots, -a_p, b_1, \ldots, b_{n-2p}), \\ Z &= \operatorname{diag}(\widetilde{a}_1, \ldots, \widetilde{a}_p, -\widetilde{a}_2, \ldots, -\widetilde{a}_p, -\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{b}_{n-2}). \end{aligned}$$

the element Y^2Z^2 is again regular semisimple. The contradiction proves Proposition 2.1. □

Lemma 2.6 *Let Γ be a finitely generated group, V any irreducible component of the variety $R_n(\Gamma_g)$ and $\rho \in V$. Then any representation $\rho' : \Gamma \to \mathrm{GL}_n(K)$ equivalent to ρ also belongs to V.*

The proof follows from the fact that for

$$\varphi : V \times \mathrm{GL}_n(K) \to R_n(\Gamma_g), \qquad (\rho, X) \mapsto X\rho X^{-1},$$

the image $\operatorname{Im}\varphi$ is irreducible, contains V and hence must coincide with V.

3 Cases of the projective plane and the Klein bottle.

The case $g=1$ is simple. Indeed, Γ_1 is a cyclic group of order two and hence

$$R_n(\Gamma_1) = \left\{X \in \mathrm{GL}_n(K) \mid X^2 = E_n\right\}.$$

In other words, $R_n(\Gamma_1)$ consists of the semisimple elements having eigenvalues ± 1.

For any integer s, $0 \le s \le n$, let $A_s = \mathrm{diag}(E_s, -E_{n-s})$ and consider the set

$$F_s = \left\{XA_sX^{-1} \mid X \in \mathrm{GL}_n(K)\right\}.$$

It is well known (see [17] or [7]) that conjugacy classes of semisimple elements are Zariski closed in $\mathrm{GL}_n(K)$. Therefore, F_s is an irreducible closed algebraic variety in $\mathrm{GL}_n(K)$.

Theorem 3.1 *The varieties $F_0, \ldots, F_n$ are the irreducible components of $R_n(\Gamma_1)$. Moreover, every F_s is a $\mathbb{Q}$-rational variety of dimension $2s(n-s)$.*

Proof. Obviously, for all $s_1 \neq s_2$ the sets F_{s_1}, F_{s_2} are pairwise disjoint and $R_n(\Gamma_1) = \bigcup_{s=0}^n F_s$, yielding the first statement. Furthermore, it is easy to see that

$$\dim F_s = \dim \mathrm{GL}_n(K) - \dim Z_{\mathrm{GL}_n(K)}(A_s) = n^2 - (s^2 + (n-s)^2) = 2s(n-s).$$

Thus, it remains only to prove the $\mathbb{Q}$-rationality of F_s. To this end, consider the $\mathbb{Q}$-rational subvariety $W_s \subset \mathrm{GL}_n(K)$ consisting of all matrices of the form $X = \begin{pmatrix} E_s & X_1 \\ X_2 & E_{n-s} \end{pmatrix}$, where X_1, X_2 are arbitrary matrices of orders $s \times (n-s)$ and $(n-s) \times s$ respectively, and the $\mathbb{Q}$-morphism

$$\alpha_s : W_s \to F_s, \qquad X \mapsto XA_sX^{-1}.$$

We claim that α_s is a birational isomorphism. Indeed, a dimension consideration shows that it is enough to prove the injectivity of α_s on some Zariski open subset $U \subset W_s$. Let $X, \widetilde{X} \in W_s$ and $\alpha_s(X) = \alpha_s(\widetilde{X})$. Then $XA_sX^{-1} = \widetilde{X}A_s\widetilde{X}^{-1}$ or, equivalently, $X^{-1}\widetilde{X} \in Z_{\mathrm{GL}_n(K)}(A_s)$. Therefore, there exist $B_1 \in \mathrm{GL}_s(K)$, $B_2 \in \mathrm{GL}_{n-s}(K)$ such that $\widetilde{X} = X\,\mathrm{diag}(B_1, B_2)$. Now matching the upper left and lower right hand corners of orders s and $n-s$ respectively, we derive that $B_1 = E_s$, $B_2 = E_{n-s}$, yielding the required result. □

Corollary 3.1 *If $K = \mathbb{C}$, then $R_n(\Gamma_1)_{\mathbb{C}}$ has $n+1$ connected components with respect to the complex topology.*

The proof follows from the fact that the irreducible and hence connected (with respect to the complex topology) components $F_0, \ldots, F_n$ are pairwise disjoint. □

We now turn to the more difficult case of the surface of genus $g=2$. Clearly, Γ_2 can be represented in the form $\Gamma_2 = \langle g_1, g_2 \mid g_1^2 = g_2^2 \rangle$ and so $R_n(\Gamma_2)$ can be identified with the variety

$$R_n(\Gamma_2) = \left\{(X, Y) \in \mathrm{GL}_n(K) \times \mathrm{GL}_n(K) \mid X^2 = Y^2\right\}.$$

For any two non-negative integers s, t such that $0 \le s \le n/2, 0 \le t \le n-2s$, let $\varphi_{s,t}$ be a $\mathbb{Q}$-defined rational morphism

$$\varphi_{s,t} : \mathbb{A}^s \times \mathbb{A}^{n-2s} \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K) \to \mathrm{GL}_n(K) \times \mathrm{GL}_n(K)$$

mapping the point

$$\alpha = (a_1, \dots, a_s, b_1, \dots, b_{n-2s}, Z_1, \dots, Z_s, A),$$

where $a_i, b_j \in K^*$, $Z_i \in \mathrm{GL}_2(K)$, $A \in \mathrm{GL}_n(K)$, to the point (X, Y), where

$$\begin{aligned} X &= A\,\mathrm{diag}(a_1, -a_1, \dots, a_s, -a_s, b_1, \dots, b_{n-2s})A^{-1}, \\ Y &= A\,\mathrm{diag}(Z_1\,\mathrm{diag}(a_1, -a_1)Z_1^{-1}, \dots, Z_s\,\mathrm{diag}(a_s, -a_s)Z_s^{-1}, \\ &\qquad - b_1, \dots, -b_t, b_{t+1}, \dots, b_{n-2s})A^{-1}. \end{aligned}$$

It is easy to make sure that for the above matrices X, Y one has $X^2 = Y^2$ and hence $\mathrm{Im}\,\varphi_{s,t} \subset R_n(\Gamma_2)$. Denote the closure of $\mathrm{Im}\,\varphi_{s,t}$ with respect to the Zariski topology by $V_{s,t}$. The following two statements describe the structure of the variety $R_n(\Gamma_2)$.

Theorem 3.2 *The varieties $V_{s,t}$ are the irreducible components of $R_n(\Gamma_2)$.*

Theorem 3.3 *$V_{s,t}$ is a $\mathbb{Q}$-rational variety of dimension $n^2 + s$.*

Corollary 3.2 *The number of irreducible components of $R_n(\Gamma_2)$ equals $(n^2 + 4n + 4)/4$, if n is even, and $(n^2 + 4n + 3)/4$, if n is odd.*

Corollary 3.3 *The set $\pi_0(R_n(\Gamma_2)_{\mathbb{C}})$ consists of two elements. More exactly, the sets $W_1 = \bigcup V_{s,2t}$, $W_2 = \bigcup V_{s,2t+1}$ are the connected components of $R_n(\Gamma_2)_{\mathbb{C}}$.*

The following statement will play a crucial role in the proof of Theorem 3.2.

Proposition 3.1 *Let V be any irreducible component of the variety $R_n(\Gamma_2)$. Then*

$$U = \{(X, Y) \in V \mid X, Y \text{ regular semisimple } \}$$

is a non-empty Zariski open subset.

Proof. Since the set of regular semisimple elements in $\mathrm{GL}_n(K)$ is Zariski open, it suffices to find a point $(X, Y) \in V$ with regular semisimple components X, Y.

By Proposition 2.3, we can pick a point $(X_0, Y_0) \in V$ which does not lie in any other irreducible component $W \ne V$ of $R_n(\Gamma_2)$ and such that both X_0 and Y_0 are regular matrices. Since all irreducible components are invariant under conjugation (see Lemma 2.6) we may in addition assume that

$$X_0 = \mathrm{diag}(X_1, \dots, X_k) \tag{3.1}$$

where $X_i = \mathrm{diag}(J_{r_i}(a_i), -J_{s_i}(a_i))$, $a_i^2 \neq a_j^2$ and $r_i \geq s_i \geq 0$ for all $i \in \{1, \ldots, k\}$.

First let $k = 1$, i.e. let $X_0 = \mathrm{diag}(J_r(a), -J_s(a))$, $r \geq s \geq 0$. Obviously, both $J_r(a)^2$ and $J_r(a^2)$ have the same Jordan normal form and hence for any $X \in \mathrm{GL}_n(K)$ both X and X^2 have the same Jordan structure, i.e. the number and lengths of Jordan blocks. ¿From this and $X_0^2 = Y_0^2$ we conclude that Y_0 is conjugate to a matrix of the form $\widetilde{Y}_0 = \mathrm{diag}(\varepsilon J_r(a), \delta J_s(a))$, where $\varepsilon = \pm 1$, $\delta = \pm 1$, and $\delta = -\varepsilon$ since Y_0 (and hence $\widetilde{Y}_0$) is regular. Thus putting $\Omega = \mathrm{diag}(\varepsilon E_r, -\varepsilon E_s)$ we have that the matrices $\widetilde{Y}_0 = \Omega X_0$ and Y_0 are conjugate.

We are going to construct an irreducible variety F and a rational map $\psi : F \to R_n(\Gamma_2)$ such that $\mathrm{Im}\, \psi$ contains both a point (X, Y) with regular semisimple components X, Y and (X_0, Y_0). It then follows from the latter condition that $\mathrm{Im}\, \psi \subset V$, implying that V contains points with regular semisimple components, as required.

Let $T_1 = \{\mathrm{diag}(A, B)\}$, where

$$A = \begin{pmatrix} a_1 & 1 & 0 & \ldots & 0 \\ 0 & a_2 & 1 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & a_r \end{pmatrix}, \qquad B = \begin{pmatrix} -a_1 & -1 & 0 & \ldots & 0 \\ 0 & -a_2 & -1 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & -a_s \end{pmatrix}$$

and $a_i \in K^*$, $a_i^2 \neq a_j^2$, and let

$$T_2 = \left\{ X \,\mathrm{diag}(A, B) X^{-1} \mid X \in \mathrm{GL}_n(K), \quad \mathrm{diag}(A, B) \in T_1 \right\}.$$

Clearly, T_2 is an irreducible set consisting of semisimple elements. Since $X_0 \in \overline{T}_2$ and for any $X \in T_2$

$$\dim Z_{\mathrm{GL}_n(K)}(X^2) = \dim Z_{\mathrm{GL}_n(K)}(X_0^2) = n + 2s.$$

the set

$$T_3 = \left\{ X \in \overline{T}_2 \mid \dim Z_{\mathrm{GL}_n(K)}(X^2) = n + 2s \right\}$$

contains X_0 and is Zariski open in $\overline{T}_2$; in particular, T_3 is irreducible. Put

$$F = \left\{ (X, Z) \in T_3 \times \mathrm{GL}_n(K) \mid X^2 Z = Z X^2 \right\}$$

and

$$\varphi : F \to R_n(\Gamma_2), \qquad (X, Z) \mapsto (X, Z \Omega X Z^{-1}).$$

The variety F is defined by a system of equations which are linear with respect to coefficients $z_{i,j}$ of Z and its rank is constant on T_3.

Lemma 3.1 *Let U be an irreducible K-defined algebraic variety. Fix an integer $n > 0$ and consider a subvariety $X \subset U \times \mathbb{A}^n$ defined by the system of linear equations*

$$\sum_{j=1}^{n} f_{i,j}(u) t_j = g_i(u), \qquad i = 1, \ldots, m,$$

where $f_{i,j}(u)$, $g_i(u) \in K[U]$ are regular functions and $t_1, \dots, t_n$ are the coordinates in $\mathbb{A}^n$. Put

$$F(u) = \begin{pmatrix} f_{11}(u) & \dots & f_{1n}(u) \\ & \ddots & \\ f_{m1}(u) & \dots & f_{mn}(u) \end{pmatrix}, \qquad \widetilde{F}(u) = \begin{pmatrix} f_{11}(u) & \dots & f_{1n}(u) & g_1(u) \\ & \ddots & & \\ f_{m1}(u) & \dots & f_{mn}(u) & g_m(u) \end{pmatrix},$$

and assume that everywhere on U we have $\operatorname{rank} F(u) = \operatorname{rank} \widetilde{F}(u) = r$ *for some constant r. Then X is irreducible and the field of rational functions $K(X)$ is isomorphic to $K(U)(s_1, \dots, s_{n-r})$ for some algebraically independent parameters $s_1, \dots, s_{n-r}$.*

Proof. Let $m_1(u), \dots, m_l(u)$ be all $r \times r$-minors of $F(u)$. Put

$$U_i = \{u \in U \mid m_i(u) \neq 0\}, \qquad X_i = \{(u, a) \in X \mid u \in U_i\}$$

and let $I = \{i \mid U_i \neq \varnothing\}$. It follows from Cramer's Rule for linear systems and our assumptions that for every $i \in I$ we have $X_i \simeq U_i \times \mathbb{A}^{n-r}$. In particular, X_i is an irreducible variety of dimension $d = \dim U + n - r$. Since U is irreducible, for any $i, j \in I$ the intersection $X_i \cap X_j$ is non-empty, and therefore is, in fact, dense in both X_i and X_j. Furthermore, we have $X = \cup_{i=1}^{l} X_i$, implying, in particular, that $\dim X = d$. Fix some $i \in I$ and let X' be an irreducible component of X containing X_i. If we assume that X is reducible then for some $j \in I$ we shall have $X_j \not\subset X'$. Let X'' be a component containing X_j. Since $\dim X' = \dim X_i = d$, X_i is dense in X'. For the same reason, X_j is dense in X''. Therefore $X_i \cap X_j$ is dense in both X' and X'' forcing $X' = X''$. This contradiction proves the irreducibility of X. Now if we fix some $i \in I$ then

$$K(X) = K(X_i) = K(U_i \times \mathbb{A}^{n-r}) = K(U_i)(s_1, \dots, s_{n-r}) = K(U)(s_1, \dots, s_{n-r}),$$

as required. The lemma is proved. $\square$

Applying Lemma 3.1 we conclude that the variety F constructed above is irreducible and obviously, $\operatorname{Im} \varphi$ contains points (X, Y) with regular semisimple components X, Y. Thus, it remains to show that $(X_0, Y_0) \in \varphi(F)$. In other words, we have to check that Y_0 is of the form $Y_0 = Z \Omega X_0 Z^{-1}$ for some Z which commutes with X_0^2.

It was proved above that Y_0 and $\widetilde{Y}_0 = \Omega X_0$ are conjugate. So if $Y_0 = Z \Omega X_0 Z^{-1}$ for some $Z \in \mathrm{GL}_n(K)$ then it follows from $Y_0^2 = X_0^2$ and $\Omega X_0 = X_0 \Omega$ that $X_0^2 = Y_0^2 = Z X_0^2 \Omega^2 Z^{-1} = Z X_0^2 Z^{-1}$, yielding $X_0^2 Z = Z X_0^2$.

We now turn to the case of arbitrary k. We claim that the Jordan normal form of Y_0 looks like (3.1). Indeed, since $X_0^2 = Y_0^2$, the Jordan normal form of Y_0 contains Jordan blocks $J_{r_1}(\varepsilon_1 a_1), J_{s_1}(\delta_1 a_1), \dots, J_{r_k}(\varepsilon_k a_k), J_{s_k}(\delta_k a_k)$, where $\varepsilon_i = \pm 1$, $\delta_i = \pm 1$ for all i, and $\delta_i = -\varepsilon_i$ for Y_0 is regular. Put

$$\widetilde{Y}_i = \operatorname{diag}(\varepsilon_i J_{r_i}(a_i), -\varepsilon_i J_{s_i}(a_i)) = \varepsilon_i X_i, \qquad \widetilde{Y}_0 = \operatorname{diag}(\widetilde{Y}_1, \dots, \widetilde{Y}_l).$$

Then $Y_0 = Z\widetilde{Y}_0 Z^{-1}$ and as above one can show that $ZX_0^2 = X_0^2 Z$. Since in addition $a_i^2 \neq a_j^2$ for all $i \neq j$, we have $Z = \operatorname{diag}(Z_1, \ldots, Z_k)$ for some $Z_i \in Z_{\mathrm{GL}_{r_i+s_i}(K)}(X_i^2)$ and hence putting $Y_i = Z_i \widetilde{Y}_i Z_i^{-1}$ we have finally $Y_0 = \operatorname{diag}(Y_1, \ldots, Y_k)$, as claimed.

Consider any irreducible component $V_i \subset R_{r_i+s_i}(\Gamma_2)$ which contains (X_i, Y_i) and put $W = V_1 \times \ldots \times V_k$. The variety W is clearly irreducible and contains (X_0, Y_0), implying $W \subset V$. Then an inductive argument shows that W contain points (X, Y) with regular semisimple components X, Y and hence V is so. Proposition 3.1 is proved. □

Lemma 3.2 $R_n(\Gamma_2) = \bigcup_{s,t} V_{s,t}$.

Proof. Let V be any irreducible component of the variety $R_n(\Gamma_2)$ and $U \subset V$ be the Zariski open subset specified in Proposition 3.1. Since V (and hence U) is invariant under conjugation, any representation $\rho = (X, Y) \in U$ is equivalent to a representation $\rho' = (X', Y') \in U$, where X' is a diagonal regular matrix of the form

$$X' = \operatorname{diag}(a_1, -a_1, \ldots, a_s, -a_s, b_1, \ldots, b_{n-2s}).$$

Then the other component Y' is of the form

$$Y' = \operatorname{diag}(Z_1 \operatorname{diag}(a_1, -a_1) Z_1^{-1}, \ldots, Z_s \operatorname{diag}(a_s, -a_s) Z_s^{-1}, \varepsilon_1 b_1, \ldots, \varepsilon_{n-2s} b_{n-2s}),$$

where $\varepsilon_i = \pm 1$, $Z_i \in \mathrm{GL}_2(K)$. Without loss of generality we may also assume that

$$\varepsilon_1 = \ldots = \varepsilon_t = -1, \qquad \varepsilon_{t+1} = \ldots = \varepsilon_{n-2s} = 1.$$

Then $\rho' \in \operatorname{Im} \varphi_{s,t}$, forcing $U \subset \cup_{s,t} \operatorname{Im} \varphi_{s,t}$ and hence $V = \overline{U} \subset \cup_{s,t} \overline{\operatorname{Im} \varphi_{s,t}} = \cup_{s,t} V_{s,t}$. □

Lemma 3.3 $\dim V_{s,t} = n^2 + s$.

Proof. Consider the projection onto the first component

$$\pi : R_n(\Gamma_2) \to \mathrm{GL}_n(K), \qquad (X, Y) \mapsto X.$$

Obviously, $\dim \overline{\pi(V_{s.t})} = n^2 - s$ and we need only to prove that $\dim(\pi^{-1}(X) \cap V_{s,t}) = 2s$ for all points X in a dense subset U' of $\overline{\pi(V_{s,t})}$, where

$$U' = \{A \operatorname{diag}(a_1, -a_1, \ldots, a_s, -a_s, b_1, \ldots, b_{n-2s}) A^{-1} \mid \\ a_i^2 \neq a_j^2, i \neq j, b_k \neq b_l, k \neq l, a_i^2 \neq b_k^2\}.$$

Denote

$$X_0 = \operatorname{diag}(a_1, -a_1, \ldots, a_s, -a_s, b_1, \ldots, b_{n-2s}),$$
$$X = A X_0 A^{-1}, \quad T = \{Y \in \mathrm{GL}_n(K) \mid Y^2 = X^2\}.$$

Clearly, $T \simeq \pi^{-1}(X)$ and we proved above that any $Y \in T$ is of the form

$$Y = A \operatorname{diag}(Z_1 \operatorname{diag}(a_1, -a_1) Z_1^{-1}, \ldots, Z_s \operatorname{diag}(a_s, -a_s) Z_s^{-1}, \\ \varepsilon_1 b_1, \ldots, \varepsilon_{n-2s} b_{n-2s}) A^{-1}, \tag{3.2}$$

where $\varepsilon_i = \pm 1$ for all $i \in \{1, \dots, n-2s\}$ and $Z_1, \dots, Z_s \in \mathrm{GL}_2(K)$.

For a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-2s})$ consisting of $\varepsilon_i = \pm 1$ let $T_\varepsilon \subset T$ be the irreducible subvariety consisting of all $Y \in T$ which have the same signs in (3.2). We have $T = \bigcup_\varepsilon T_\varepsilon$ and $\pi^{-1}(X) = \cup_\varepsilon \{(X, Y) \mid Y \in T_\varepsilon\}$. Let $W_i = \{Z_i \operatorname{diag}(a_i, -a_i) Z_i^{-1} \mid Z_i \in \mathrm{GL}_2(K)\}$. Then $\dim W_i = 2$ and $T_\varepsilon \simeq W_1 \times \ldots \times W_s$, implying that $\dim T_\varepsilon = 2s$ for all sequences ε. Therefore, T has also dimension $2s$ and it remains to observe that $\pi^{-1}(X) \cap V_{s,t}$ is the union of all sets (X, T_ε) such that the number -1 occurs t times in $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-2s})$. □

The next lemma completes the proof of Theorem 3.2.

Lemma 3.4 *If $(s,t) \neq (s_1, t_1)$ then $V_{s,t} \not\subseteq V_{s_1,t_1}$.*

Proof. Assume the contrary. Let first $s = s_1$. Then Lemma 3.3 yields the equality $\dim V_{s,t} = \dim V_{s_1,t_1}$, forcing in fact $V_{s,t} = V_{s_1,t_1}$. Denote

$$U = \left\{(a_1, \dots, a_s, b_1, \dots, b_{n-2s}) \mid a_i^2 \neq a_j^2, b_k \neq b_l, a_i^2 \neq b_k^2\right\}.$$

Since the sets $\varphi_{s,t}(U \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K))$ and $\varphi_{s_1,t_1}(U \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K))$ are dense in $V_{s,t} = V_{s_1,t_1}$, their intersection is non-empty. However, if

$$(X, Y) \in \varphi_{s,t}(U \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K)) \cap \varphi_{s_1,t_1}(U \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K))$$

then the matrices X, Y have exactly $s+t = s_1+t_1$ common eigenvalues which is impossible in view of our assumptions $s = s_1$ and $t \neq t_1$.

Next, let $s \neq s_1$. By assumption, we have $V_{s,t} \subseteq V_{s_1,t_1}$ and by Lemma 3.3, one has $\dim V_{s,t} = n^2 + s$, $\dim V_{s_1,t_1} = n^2 + s_1$, implying $s < s_1$. Furthermore, the assumption $V_{s,t} \subseteq V_{s_1,t_1}$ implies $\pi(V_{s,t}) \subseteq \pi(V_{s_1,t_1})$, where π is as above the projection onto the first component. Since $\dim \overline{\pi(V_{s_1,t_1})} = n^2 - s_1$, $\dim \overline{\pi(V_{s,t})} = n^2 - s$, the latter forces $s_1 \leq s$, which is impossible. □

Proof of Theorem 3.3. We are going to show first that the field of rational functions $\mathbb{Q}(V_{s,t})$ can be realized as the field of invariants $\mathbb{Q}(u_1, \dots, u_{n^2+s})^G$ where G is a finite group acting on a purely transcendental extension $\mathbb{Q}(u_1, \dots, u_{n^2+s})$ of $\mathbb{Q}$ and then by straightforward computations we will show that $\mathbb{Q}(u_1, \dots, u_{n^2+s})^G$ is also a purely transcendental extension of $\mathbb{Q}$.

Consider the (n^2+s)-dimensional subvariety $W \subset \mathbb{A}^s \times \mathbb{A}^{n-2s} \times \mathrm{GL}_2(K)^s \times \mathrm{GL}_n(K)$ consisting of all elements of the form

$$w = (a_1, \dots, a_s, b_1, \dots, b_{n-2s}, Z_1, \dots, Z_s, A), \tag{3.3}$$

where a_i, $b_i \in K^*$, $Z_i = \begin{pmatrix} u_i & 1 \\ 1 & v_i \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\det Z_i \neq 0$, $\det A \neq 0$.

Lemma 3.5 *The restriction $\varphi_{s,t}|_W : W \to V_{s,t}$ is a morphism of finite degree.*

Proof. Since $\dim W = \dim V_{s,t}$, we need only to show that for any point (3.3) in "generic" position the fibre $\varphi_{s,t}^{-1}(\varphi_{s,t}(w))$ is finite. Pick $\widetilde{w} \in W$ such that $\varphi_{s,t}(w) = \varphi_{s,t}(\widetilde{w})$. Let

$$\widetilde{w} = (\widetilde{a}_1, \dots, \widetilde{a}_s, \widetilde{b}_1, \dots, \widetilde{b}_{n-2s}, \widetilde{Z}_1, \dots, \widetilde{Z}_s, \widetilde{A}),$$

where

$$\widetilde{Z}_i = \begin{pmatrix} \widetilde{u}_i & 1 \\ 1 & \widetilde{v}_i \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \dots & \widetilde{a}_{2n} \\ & & \ddots & \\ \widetilde{a}_{n1} & \widetilde{a}_{n2} & \dots & \widetilde{a}_{nn} \end{pmatrix}.$$

Then we have

$$\begin{aligned} &A \operatorname{diag}(a_1, -a_1, \dots, a_s, -a_s, b_1, \dots, b_{n-2s}) A^{-1} = \\ &\widetilde{A} \operatorname{diag}(\widetilde{a}_1, -\widetilde{a}_1, \dots, \widetilde{a}_s, -\widetilde{a}_s, \widetilde{b}_1, \dots, \widetilde{b}_{n-2s}) \widetilde{A}^{-1}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &A \operatorname{diag}(Z_1 \operatorname{diag}(a_1, -a_1) Z_1^{-1}, \dots, Z_s, \operatorname{diag}(a_s, -a_s) Z_s^{-1}, \\ &-b_1, \dots, -b_t, b_{t+1}, \dots, b_{n-2s}) A^{-1} = \\ &\widetilde{A} \operatorname{diag}(\widetilde{Z}_1 \operatorname{diag}(\widetilde{a}_1, -\widetilde{a}_1) \widetilde{Z}_1^{-1}, \dots, \widetilde{Z}_s \operatorname{diag}(\widetilde{a}_s, -\widetilde{a}_s) \widetilde{Z}_s^{-1}, \\ &-\widetilde{b}_1, \dots, -\widetilde{b}_t, \widetilde{b}_{t+1}, \dots, \widetilde{b}_{n-2s}) \widetilde{A}^{-1}. \end{aligned} \tag{3.5}$$

It follows from (3.4) that the diagonal matrices

$$\operatorname{diag}(a_1, -a_1, \dots, a_s, -a_s, b_1, \dots, b_{n-2s}), \tag{3.6}$$

$$\operatorname{diag}(\widetilde{a}_1, -\widetilde{a}_1, \dots, \widetilde{a}_s, -\widetilde{a}_s, \widetilde{b}_1, \dots, \widetilde{b}_{n-2s}) \tag{3.7}$$

are conjugate and hence their eigenvalues coincide.

Since w is in "generic" position we may assume without loss of generality that the eigenvalues of (3.6) are distinct and moreover, $a_i^2 \neq a_j^2$, $a_i^2 \neq b_j^2$, $b_i^2 \neq b_j^2$. Then we get

$$\{\pm a_1, \dots, \pm a_s\} = \{\pm \widetilde{a}_1, \dots, \pm \widetilde{a}_s\}, \qquad \{b_1, \dots, b_{n-2s}\} = \left\{\widetilde{b}_1, \dots, \widetilde{b}_{n-2s}\right\} \tag{3.8}$$

Using (3.5) and arguing as above one can show that the latter equality in (3.8) falls apart

$$\{b_1, \dots, b_t\} = \left\{\widetilde{b}_1, \dots, \widetilde{b}_t\right\}, \qquad \{b_{t+1}, \dots, b_{n-2s}\} = \left\{\widetilde{b}_{t+1}, \dots, \widetilde{b}_{n-2s}\right\}.$$

Thus, there exist numbers $\varepsilon_1, \dots, \varepsilon_s$ equal ± 1 and permutations

$$\tau_1 = \begin{pmatrix} 1 & \dots & s \\ i_1 & \dots & i_s \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 1 & \dots & t \\ j_1 & \dots & j_t \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} t+1 & \dots & n-2s \\ k_{t+1} & \dots & k_{n-2s} \end{pmatrix}$$

of the sets $\{1, \dots, s\}$, $\{1, \dots, t\}$, $\{t+1, \dots, n-2s\}$ respectively such that

$$\begin{aligned} \widetilde{a}_i &= \varepsilon_i a_{\tau_1(i)}, & i \in \{1, \dots, s\} \\ \widetilde{b}_j &= b_{\tau_2(j)}, & j \in \{1, \dots, t\} \\ \widetilde{b}_k &= b_{\tau_3(k)}, & k \in \{t+1, \dots, n-2s\} \end{aligned} \tag{3.9}$$

Now let us analyze the structure of the matrices A and $\widetilde{A}$. Since (3.6) and (3.7) are regular semisimple matrices it is easy to see that $A^{-1}\widetilde{A} = \omega d$, where d is a diagonal matrix, $\omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ and ω_1, ω_2, ω_3 are monomial matrices of orders $2s$, t, $n-2s-t$ respectively. Matching the first row in the equality $\widetilde{A} = A\omega d$ we conclude that $d = 1$ and hence $\widetilde{A}$ is obtained from A by permuting its columns. Furthermore, it follows from (3.4), (3.5), (3.9) that the monomial matrices ω_2, ω_3 correspond to the permutations τ_2, τ_3 and for ω_1 one has

$$\omega_1 \,\mathrm{diag}(\widetilde{a}_1, -\widetilde{a}_1, \dots, \widetilde{a}_s, -\widetilde{a}_s)\omega_1^{-1} = \mathrm{diag}(a_1, -a_1, \dots, a_s, -a_s) \tag{3.10}$$

In the sequel we need to have more information about the structure of ω_1 and $\widetilde{A}$. Namely, it follows from (3.9) and (3.10) that if we conjugate (3.7) by ω_1 then the (2×2)-block $d_i = \mathrm{diag}(\widetilde{a}_i, -\widetilde{a}_i)$ maps into either $\widetilde{d}_{\tau_1^{-1}(i)}$ or $-\widetilde{d}_{\tau_1^{-1}(i)}$. Therefore, the monomial matrix ω_1 is of the form $\omega_1 = (w_{i,j})$, where $w_{i,j}$ are the blocks of order $(2\times\ 2)$ defined as follows:

$$w_{i,j} = \begin{cases} 0, & \text{if } i \neq \tau_1(j), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i = \tau_1(j) \text{ and } \varepsilon_i = 1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } i = \tau_1(j) \text{ and } \varepsilon_i = -1. \end{cases} \tag{3.11}$$

Let A_j (resp. $\widetilde{A}_j$) be the j-th column of A (resp. $\widetilde{A}$). Then bearing in mind (3.11), we derive from $\widetilde{A} = A\omega$ that

$$(\widetilde{A}_{2j-1}, \widetilde{A}_{2j}) = \begin{cases} (A_{2\tau_1(j)-1}, A_{2\tau_1(j)}), & \text{if } \varepsilon_i = 1, \\ (A_{2\tau_1(j)}, A_{2\tau_1(j)-1}), & \text{if } \varepsilon_i = -1 \end{cases} \tag{3.12}$$

for all $j \in \{1, \dots, s\}$ and moreover

$$(\widetilde{A}_{2s+1}, \dots, \widetilde{A}_{2s+t}) = (A_{2s+\tau_2(1)}, \dots, A_{2s+\tau_2(t)}), \tag{3.13}$$

$$(\widetilde{A}_{2s+t+1}, \dots, \widetilde{A}_{n}) = (A_{2s+\tau_3(t+1)}, \dots, A_{2s+\tau_3(n-2s)}). \tag{3.14}$$

To complete the proof it remains to show that for the matrices $\widetilde{Z}_1, \dots, \widetilde{Z}_s$ there are also a finite number of possibilities. To this end, substitute the equality $\widetilde{A} = A\omega$ into (3.5) and consider the upper left corner of order $2s$. Cancelling A, we have

$$\begin{gathered} \mathrm{diag}(Z_1 \,\mathrm{diag}(a_1, -a_1)Z_1^{-1}, \dots, Z_s \,\mathrm{diag}(a_s, -a_s)Z_s^{-1}) = \\ \omega_1 \,\mathrm{diag}(\widetilde{Z}_1 \,\mathrm{diag}(\widetilde{a}_1, -\widetilde{a}_1)\widetilde{Z}_1^{-1}, \dots, \widetilde{Z}_s \,\mathrm{diag}(\widetilde{a}_s, -\widetilde{a}_s)\widetilde{Z}_s^{-1})\omega_1^{-1}. \end{gathered} \tag{3.15}$$

Let $Z = \mathrm{diag}(Z_1, \dots, Z_s)$, $\widetilde{Z} = \mathrm{diag}(\widetilde{Z}_1, \dots, \widetilde{Z}_s)$. Then from (3.10) and (3.15) we get

$$\mathrm{diag}(a_1, -a_1, \dots, a_s, -a_s) = (Z^{-1}\omega_1\widetilde{Z}\omega_1^{-1})\,\mathrm{diag}(a_1, -a_1, \dots, a_s, -a_s)(Z^{-1}\omega_1\widetilde{Z}\omega_1^{-1})^{-1},$$

and thus $Z^{-1}\omega_1\widetilde{Z}\omega_1^{-1} = t$ is a diagonal matrix. But we have already known (see (3.11)) that under conjugation by means of ω_1 the block $\widetilde{Z}_i$ of the matrix $\widetilde{Z}$ maps into either

$\widetilde{Z}_{\tau_1^{-1}(i)}$ (if $\varepsilon_i = 1$) or

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{Z}_{\tau_1^{-1}(i)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \widetilde{v}_{\tau_1^{-1}(i)} & 1 \\ 1 & \widetilde{u}_{\tau_1^{-1}(i)} \end{pmatrix}$$

which is obtained from $\widetilde{Z}_{\tau_1^{-1}(i)}$ by permuting its diagonal elements.

It follows immediately that $t = 1$ and for $\widetilde{u}_i$, $\widetilde{v}_i$ one has

$$(\widetilde{u}_i, \widetilde{v}_i) = \begin{cases} (u_{\tau_1(i)}, v_{\tau_1(i)}), & \text{if } \varepsilon_i = 1, \\ (v_{\tau_1(i)}, u_{\tau_1(i)}), & \text{if } \varepsilon_i = -1. \end{cases} \tag{3.16}$$

So there exists only a finite number of points $\widetilde{w} \in W$ such that $\varphi_{s,t}(w) = \varphi_{s,t}(\widetilde{w})$ and all of them are given by (3.9),(3.12),(3.13),(3.14),(3.16). □

Let us proceed with the proof of Theorem 3.3. Consider a finite group $G = S_s \times S_t \times S_{n-2s-t} \times \prod_{i=1}^{s} H_i$, where $H_i = \langle \sigma_i \mid \sigma_i^2 = 1 \rangle$ and S_s, S_t, S_{n-2s-t} are the permutation groups. Define the action of G on the field $Q(W)$ as follows. Let

$$Q(W) = Q(a_1, \dots, a_s, b_1, \dots, b_{n-2s}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}, u_1, \dots, u_s, v_1, \dots, v_s),$$

where a_i, b_i, a_{ij}, u_i, v_i are the corresponding coordinate functions. Then by definition, the elements of the group S_s permute the columns of the matrices

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ a_{21} & a_{23} & \dots & a_{2\,2s-1} \\ & & \ddots & \\ a_{n1} & a_{n3} & \dots & a_{n\,2s-1} \end{pmatrix}, \qquad \begin{pmatrix} v_1 & v_2 & \dots & v_s \\ a_{22} & a_{24} & \dots & a_{2\,2s} \\ & & \ddots & \\ a_{n2} & a_{n4} & \dots & a_{n\,2s} \end{pmatrix}$$

and the elements of the set $\{a_1, \dots, a_s\}$; the elements of the groups S_t, S_{n-2s-t} permute the columns of the matrices

$$\begin{pmatrix} b_1 & \dots & b_t \\ a_{2\,2s+1} & \dots & a_{2\,2s+t} \\ & \ddots & \\ a_{n\,2s+1} & \dots & a_{n\,2s+t} \end{pmatrix}, \qquad \begin{pmatrix} b_{t+1} & \dots & b_{n-2s} \\ a_{2\,2s+t+1} & \dots & a_{2n} \\ & \ddots & \\ a_{n\,2s+t+1} & \dots & a_{nn} \end{pmatrix}$$

respectively; the generator σ_i of H_i maps a_i, u_i, v_i, $a_{k\,2i-1}$, $a_{k\,2i}$ $(k \in \{2, \dots, n\})$ into $-a_i$, v_i, u_i, $a_{k\,2i}$, $a_{k\,2i-1}$ respectively.

As a by-product of the argument above we obtain (compare (3.9), (3.12), (3.13), (3.14), (3.16) and definition above) that the comorphism $\varphi^*_{s,t}$ allows to identify the field of rational functions $Q(V_{s,t})$ with the invariant subfield $Q(W)^G$. Furthermore, it is obvious that $Q(W)^G$ coincides with the compositum of the fields

$$\begin{aligned} K_1 &= Q(a_1, \dots, a_s, u_1, \dots, u_s, v_1, \dots, v_s, a_{21}, \dots, a_{2\,2s}, \dots, a_{n1}, \dots, a_{n\,2s})^{S_s \times \prod H_i}, \\ K_2 &= Q(b_1, \dots, b_t, a_{2\,2s+1}, \dots, a_{2\,2s+t}, a_{n\,2s+1}, \dots, a_{n\,2s+t})^{S_t}, \\ K_3 &= Q(b_{t+1}, .., b_{n-2s}, a_{2\,2s+t+1}, \dots, a_{2n}, a_{n\,2s+t+1}, \dots, a_{nn})^{S_{n-2s-t}}. \end{aligned}$$

Since K_2 and K_3 are clearly purely transcendental extensions of Q it remains to prove the Q-rationality of K_1. To this end, we first recall the following well known statement.

Lemma 3.6 *Let G be any finite automorphism group of a field L and let $E = L^G$. Then for any faithful linear representation $G \to \mathrm{GL}(V)$, where V is a vector space over L, the field of invariants $L(V)^G$ is a purely transcendental extension of E.*

The proof follows from the fact that the vector space V contains a G-invariant basis. $\square$

Let

$$\begin{aligned}&\widetilde{u}_i = a_i(u_i - v_i), \quad \widetilde{v}_i = u_i + v_i,\\ &\widetilde{a}_i = a_i,\ i \in \{1, \ldots, s\},\ \widetilde{a}_{i\,2j-1} = a_j(a_{i\,2j-1} - a_{i\,2j}),\\ &\widetilde{a}_{i\,2j} = a_{i\,2j+1} + a_{i\,2j},\ i \in \{2, \ldots, n\},\ j \in \{1, \ldots, s\}\end{aligned}$$

Then elements of the group $\prod_{i=1}^{s} H_i$ act identically onto $\widetilde{u}_1, \ldots, \widetilde{u}_s, \widetilde{v}_1, \ldots, \widetilde{v}_s, \widetilde{a}_{21}, \ldots, \widetilde{a}_{n\,2s}$ and hence

$$\begin{aligned}K_1 = Q(&\widetilde{u}_1, \ldots, \widetilde{u}_s, \widetilde{v}_1, \ldots, \widetilde{v}_s, \widetilde{a}_{21}, \ldots, \widetilde{a}_{n\,2s}, \widetilde{a}_1, \ldots, \widetilde{a}_s)^{S_s \times \prod_{i=1}^{s} H_i} =\\ &Q(\widetilde{u}_1, \ldots, \widetilde{u}_s, \widetilde{v}_1, \ldots, \widetilde{v}_s, \widetilde{a}_{21}, \ldots, \widetilde{a}_{n\,2s}, \widetilde{a}_1^2, \ldots, \widetilde{a}_s^2)^{S_s}\end{aligned}$$

Note also that the elements of the group S_s permute the columns of matrices

$$\begin{pmatrix} \widetilde{u}_1 & \widetilde{u}_2 & \ldots & \widetilde{u}_s \\ \widetilde{a}_{21} & \widetilde{a}_{23} & \ldots & \widetilde{a}_{2\,2s-1} \\ & & \ddots & \\ \widetilde{a}_{n1} & \widetilde{a}_{n3} & \ldots & \widetilde{a}_{n\,2s-1} \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{v}_1 & \widetilde{v}_2 & \ldots & \widetilde{v}_s \\ \widetilde{a}_{22} & \widetilde{a}_{24} & \ldots & \widetilde{a}_{2\,2s} \\ & & \ddots & \\ \widetilde{a}_{n2} & \widetilde{a}_{n4} & \ldots & \widetilde{a}_{n\,2s} \end{pmatrix}$$

and the elements of the set $\{\widetilde{a}_1^2, \ldots, \widetilde{a}_s^2\}$. So putting $E = Q(\widetilde{a}_1^2, \ldots, \widetilde{a}_s^2)^{S_s} = Q(\sigma_1, \ldots, \sigma_s)$, and $L = Q(\widetilde{a}_1^2, \ldots, \widetilde{a}_s^2)$ where $\sigma_1, \ldots, \sigma_s$ are the elementary symmetric polynomials of $\widetilde{a}_1^2, \ldots, \widetilde{a}_s^2$, and applying Lemma 3.6 we finally get that K_1 (and hence $Q(V_{s,t})$) is a purely transcendental extension of Q. Theorem 3.3 is proved completely. $\square$

Proof of corollary 3.2. To compute the number of the irreducible components of the variety $R_n(\Gamma_2)$ it is enough to observe that for a given s we have $n-2s+1$ components $V_{s,t}$ since $t \in \{0, \ldots, n-2s\}$. Therefore, if n is even we get

$$\sum_{s=0}^{n/2}(n - 2s + 1) = 1 + 3 + \ldots + (n + 1) = \sum_{k=0}^{n/2}(2k + 1) = (n/2 + 1)^2 = (n^2 + 4n + 4)/4$$

components, and if n is odd then we have

$$\sum_{s=0}^{(n-1)/2}(n - 2s + 1) = 2 + 4 + \ldots + (n + 1) = 2\sum_{i=1}^{(n+1)/2} i = (n + 1)(n + 3)/4 = (n^2 + 4n + 3)/4$$

components. $\square$

Proof of Corollary 3.3. Denote

$$W_1 = \bigcup V_{s,2t}, \qquad W_2 = \bigcup V_{s,2t+1}.$$

It is obvious that $W_1 \cup W_2 = R_n(\Gamma_2)$. Since the regular function $\det(XY^{-1})$ equals 1 on W_1 and -1 on W_2 we have $W_1 \cap W_2 = \varnothing$ and so it remains to show that W_i $(i = 1, 2)$ is connected with respect to the complex topology. Take, for example, W_1 (the variety W_2 is handled similarly). Let

$$\begin{aligned} X_0 &= \mathrm{diag}(a_1, -a_1, \ldots, a_s, -a_s, b_1, \ldots, b_{n-2s}), \\ Y_0 &= \mathrm{diag}(a_1, -a_1, \ldots, a_s, -a_s, -b_1, \ldots, -b_{2k}, b_{2k+1}, \ldots, b_{n-2s}), \\ \widetilde{Y}_0 &= \mathrm{diag}(-a_1, a_1, \ldots, -a_s, a_s, b_1, \ldots, b_{n-2s}). \end{aligned}$$

Then $(X_0, Y_0) \in V_{s,2k} \cap V_{0,2k}$ and hence $\bigcup_{s=0}^{[n/2]} V_{s,2k}$ is connected. Analogously, since

$$(X_0, \widetilde{Y}_0) \in V_{0,2k} \cap V_{k,0} \quad \textit{and} \quad (X_0, X_0) \in V_{k,0} \cap V_{0,0}$$

the union $U_k = (\bigcup_{s=0}^{[n/2]} V_{s,2k}) \bigcup V_{k,0} \bigcup V_{0,0}$ is connected. But $V_{0,0} \subset \bigcap_k U_k$, implying $W_1 = \bigcup_k U_k$ is also connected. $\square$

4 The case $g \geq 3$.

Let us begin with the case when the pairs (n, g) and $(2, 3)$ are distinct. Denote

$$\begin{aligned} W_1 &= \{(X_1, \ldots, X_g) \in R_n(\Gamma_g) \mid \det(X_1 \ldots X_g) = 1\}, \\ W_2 &= \{(X_1, \ldots, X_g) \in R_n(\Gamma_g) \mid \det(X_1 \ldots X_g) = -1\}. \end{aligned}$$

Then $R_n(\Gamma_g) = W_1 \cup W_2$ and $W_1 \cap W_2 = \varnothing$. Our goal is to show that in fact W_1, W_2 are the irreducible components of $R_n(\Gamma_g)$. To begin with, let us outline the main idea of the proof. Picking any irreducible component $V \subset W_i$ we will construct a commutative diagram of the form

$$\begin{tikzcd} V \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & W \arrow[dl, "\varphi"] \\ & \mathrm{GL}_n(K)^{g-1} & \end{tikzcd} \tag{4.1}$$

where W is an irreducible variety of dimension $(g-1)n^2$, φ, ψ are regular maps and π is the projection defined by the formula $\pi(X_1, \ldots, X_g) = (X_2, \ldots, X_g)$. Moreover, we will establish that π, φ, ψ are dominant morphisms of finite degree, the degree of φ being 2^{n-1}. It follows then that for any point $\alpha \in \mathrm{GL}_n(K)^{g-1}$ in "generic" position the order of the fibre $\pi^{-1}(\alpha)$ is at least 2^{n-1}. On the other hand, it is easy to see that for the analogous projection $\widetilde{\pi} : W_i \to \mathrm{GL}_n(K)^{g-1}$ the fibre $\widetilde{\pi}^{-1}(\alpha)$ consists of exactly 2^{n-1} elements and hence V must coincide with W_i, yielding the irreducibility of W_i.

Proposition 4.1 *The image $\pi(V)$ is dense in $\mathrm{GL}_n(K)^{g-1}$.*

Proof. Recall that the variety $R_n(\Gamma_g)$ is defined by a single matrix equation. Thus, $\dim V \geq (g-1)n^2$ and we need only to prove that π is a morphism of finite degree or, equivalently, for any point $(X_1, \ldots, X_g) \in V$ which is in "generic" position we have $\ker d\pi_{(X_1,\ldots,X_g)} = 0$,, where $d\pi_{(X_1,\ldots,X_g)}$ is the differential of π.

Using the dual numbers one easily checks that

$$\ker d\pi_{(X_1,\ldots,X_g)} \subset \{(X, 0, \ldots, 0) \in M_n(K) \times \ldots \times M_n(K) \mid X_1 X + X X_1 = 0\}.$$

Pick any $X \in M_n(K)$ such that $X_1 X + X X_1 = 0$. We may assume in addition that X_1^2 is regular, since by Proposition 2.1, the set

$$V' = \big\{(V_1, \ldots, V_g) \in V \mid V_1^2 \text{ is regular}\big\}$$

is non-empty. Then the above equality gives $X_1^2 X = X X_1^2$. But all matrices commuting with the regular matrix X_1^2 can be written as polynomials of X_1^2; in particular X simultaneously commutes and anticommutes with X_1, whence $X = 0$, as required. □

As a by-product of the argument above we have

Corollary 4.1 $\dim V = (g-1)n^2$.

We now turn to construction of the variety W. Suppose that $V \subset W_1$ (the case $V \subset W_2$ is handled in exactly the same manner). For any matrix $A \in M_n(K)$ denote the coefficients (up to signs) of its characteristic polynomial by $\sigma_1(A), \ldots, \sigma_n(A)$, i.e. let

$$\det(tE_n - A) = t^n - \sigma_1(A)t^{n-1} + \ldots + (-1)^n \sigma_n(A).$$

In particular, for a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ the polynomials $\sigma_1(A), \ldots, \sigma_n(A)$ are the elementary symmetric polynomials of $a_1, \ldots, a_n$.

Let $t_1, \ldots, t_n$ be algebraically independent variables over K and $T = \mathrm{diag}(t_1, \ldots, t_n)$. Consider the following fields:

$$L_1 = K(\sigma_1(T^2), \ldots, \sigma_n(T^2)), \qquad L_2 = K(\sigma_1(T^2), \ldots, \sigma_{n-1}(T^2), \sigma_n(T)),$$
$$L_3 = K(\sigma_1(T), \ldots, \sigma_n(T)), \qquad L_4 = K(t_1, \ldots, t_n).$$

By construction, $L_1 \subset L_2 \subset L_3 \subset L_4$ and L_4 is clearly a splitting field of the polynomial

$$g(t) = (t^2)^n - \sigma_1(T^2)(t^2)^{n-1} + \ldots + (-1)^n \sigma_n(T^2),$$

defined over L_1. This means that $\mathrm{Gal}(L_4/L_1)$ is the semidirect product of its subgroup H generated by the automorphisms $\tau_1, \ldots, \tau_n$ such that

$$\tau_i(t_j) = \begin{cases} -t_j, & \text{if } i = j \\ t_j, & \text{if } i \neq j \end{cases}$$

and the permutation group S_n of the set $\{t_1, \ldots, t_n\}$. Put also

$$H_0 = \left\{ \tau_1^{\alpha_1} \ldots \tau_n^{\alpha_n} \mid \sum_{i=1}^{n} \alpha_i \text{ is even} \right\}.$$

Since elements of the group H_0 induce the identity on L_2 and $[L_2 : L_1] = 2$ we get that $\mathrm{Gal}(L_4/L_2) = H_0 \ltimes S_n$, where the sign $\ltimes$ denotes the semidirect product.

Lemma 4.1 $[L_2(\sigma_1(T)) : L_2] = 2^{n-1}$.

Proof. Obviously, we have

$$[L_4 : L_1] = 2^n n!, \qquad [L_2 : L_1] = 2, \qquad [L_4 : L_3] = n!.$$

It follows then that $[L_3 : L_2] = 2^{n-1}$ and so we need only to prove that $L_2(\sigma_1(T)) = L_3$.

Assume the contrary. Since $L_2(\sigma_1(T)) \subset L_3$ and $L_3 = L_4^{S_n}$ there exists a non-trivial element $\tau \in H_0$ inducing the identity on $L_2(\sigma_1(T))$; in particular,

$$\tau(\sigma_1(T)) = \tau(t_1 + \ldots + t_n) = t_1 + \cdots + t_n.$$

However $\tau \neq 1$ and hence τ changes the sign of at least one variable t_i — a contradiction, proving the lemma. □

Denote by $f(z)$ the irreducible polynomial of $\sigma_1(T)$ over L_2. By Lemma 4.1, its degree is 2^{n-1}. Therefore $f(z)$ must coincide with $\prod_{\tau\in H_0}(z-\tau(\sigma_1(T)))$, since the latter polynomial is invariant under all $\tau \in H_0 \ltimes S_n$ and has the same degree as $f(z)$. Note also that all coefficients of $f(z)$ are polynomials of $\sigma_1(T^2)$, $\ldots$, $\sigma_{n-1}(T^2)$, $\sigma_n(T)$, because obviously, one has

$$K[t_1, \ldots, t_n] \cap L_2 = K[\sigma_1(T^2), \ldots, \sigma_{n-1}(T^2), \sigma_n(T)]$$

Let

$$\left\{x_{ij}^{(l)} \mid 1 \leq i, j \leq n, 2 \leq l \leq g\right\}$$

be algebraically independent variables over $K(t_1, \ldots, t_n)$ and let $X_2 = (x_{ij}^{(2)}), \ldots, X_g = (x_{ij}^{(g)})$. Substituting

$$\sigma_1(X_g^{-2}, \ldots X_2^{-2}),\ \ldots\ \sigma_{n-1}(X_g^{-2} \ldots X_2^{-2}),\ \sigma_n(X_g^{-1} \ldots X_2^{-1})$$

for $\sigma_1(T^2), \ldots, \sigma_{n-1}(T^2), \sigma_n(T)$ respectively into the coefficients of $f(z)$ we get a polynomial which will be denoted by $\overline{f}$.

Theorem 4.1 *The variety*

$$W = \left\{(z, X_2, \ldots, X_g) \in \mathbb{A}^1 \times \mathrm{GL}_n(K)^{g-1} \mid \overline{f}(z, x_{11}^{(2)}, \ldots, x_{nn}^{(g)}) = 0\right\}$$

is irreducible of dimension $(g-1)n^2$.

Proof. Let $F = K(x_{11}^{(2)}, \ldots, x_{nn}^{(g)})$. The leading coefficient of $\overline{f}$ viewed as a polynomial of z with coefficients in F is 1. So it suffices to show that $\overline{f}$ is irreducible over F. Denote the composita $F \cdot L_2$, $F \cdot L_3$, $F \cdot L_4$ by F_2, F_3, F_4 respectively and let

$$\begin{aligned} A_2 &= F[\sigma_1(T^2), \ldots \sigma_{n-1}(T^2), \sigma_n(T)], \\ A_3 &= F[\sigma_1(T), \ldots \sigma_n(T)], \quad A_4 = F[t_1, \ldots, t_n]. \end{aligned}$$

Obviously, the integral closure of A_2 in F_3 and F_4 coincides with A_3 and A_4 respectively. Consider the maximal ideal $I_2 \subset A_2$ generated by the polynomials $\sigma_1(T^2) - \sigma_1(X_g^{-2} \ldots X_2^{-2})$, $\ldots$, $\sigma_{n-1}(T^2) - \sigma_{n-1}(X_g^{-2} \ldots X_2^{-2})$, $\sigma_n(T) - \sigma_n(X_g^{-1} \ldots X_2^{-1})$ and let $I_4 \subset A_4$ be any maximal ideal lying above I_2.

Proposition 4.2 *The field A_4/I_4 is a finite extension of $F = A_2/I_2$ of degree $2^{n-1}n!$.*

Assuming for the moment Proposition 4.2 we show how it implies Theorem 4.1. To this end note that by Proposition 4.2 the decomposition group G_{I_4} of the maximal ideal I_4 has order not less than $2^{n-1}n!$. However, $|\,\mathrm{Gal}(F_4/F_2)| = |\,\mathrm{Gal}(L_4/L_2)| = 2^{n-1}n!$ and hence G_{I_4} coincides with $\mathrm{Gal}(F_4/F_2)$, implying in particular, that I_4 is the only maximal ideal lying above I_2.

Furthermore, by the construction above, under the natural map $A_2 \to \overline{A}_2 = A_2/I_2$ the reduced polynomial of $f(z)$ coincides with $\overline{f}(z)$. It follows then that $\overline{f}(z)$ is a power of some irreducible polynomial in $\overline{A}_2[z]$, since clearly

$$f(z) \in K[\sigma_1(T^2), \dots \sigma_{n-1}(T^2), \sigma_n(T)][z] \subset A_2[z]$$

is irreducible in $A_2[z]$ and splits into linear factors in A_4 (see [9], Chapter 9, § 2).

On the other hand, the specialization $X_2 \mapsto T^{-1}$, $X_3 \mapsto E_n, \dots, X_g \mapsto E_n$ defines a well-defined ring homomorphism $K[x_{11}^{(2)}, \dots, x_{nn}^{(g)}] \to L_4$ and under this homomorphism the polynomial $\overline{f}$ maps back to f which has no multiple roots, whence $\overline{f}$ is irreducible, as required. $\Box$

The proof of Proposition 4.2 is based on the following two statements.

Lemma 4.2 *Let $H \le S_n$ and $|H| \ge (n-1)!$. Then H is a maximal subgroup of S_n.*

Proof. The cases $n = 2, 3, 4$ are trivial. Let $n \ge 5$ and $d = [S_n : H]$. Consider the set $X = \{g_i H\}$ of left cosets and let φ be the permutation representation $\varphi : S_n \to S_d$ into the permutation group of the set X.

If $H = A_n$ then there is nothing to prove. Now let $H \ne A_n$. Then the restriction $\varphi\,|_{A_n}$ is injective, since A_n, $n \ge 5$, is a simple group, and so $n!/2 = |\varphi(A_n)| \le d!$. It follows that $d = n$ and hence φ is an isomorphism. It remains to observe that the image $\varphi(H)$ coincides with the stabilizer of the coset $1 \cdot H = H$, whence $\varphi(H)$ is maximal. $\Box$

Lemma 4.3 *The polynomial*

$$\overline{g}_1(t) = t^n - \sigma_1(X_g^{-2} \dots X_2^{-2})t^{n-1} + \dots + (-1)^n \sigma_n(X_g^{-2} \dots X_2^{-2})$$

is irreducible over F and its splitting field E has degree $n!$ over F.

Proof. It suffices to prove the same statement for the polynomial

$$t^n - \sigma_1(X_g^2 \dots X_2^2)t^{n-1} + \dots + (-1)^n \sigma_n(X_g^2 \dots X_2^2).$$

which will be denoted for simplicity by the same letter $\overline{g}_1(t)$. We argue by induction on n. The case $n = 2$ is handled by straightforward computations. Let $n > 2$ and let

$$A = K_1[x_{n1}^{(2)}, \dots, x_{nn}^{(2)}, x_{1n}^{(2)}, \dots, x_{n-1,n}^{(2)}, \dots, x_{n1}^{(g)}, \dots, x_{nn}^{(g)}, \dots, x_{1n}^{(g)}, \dots, x_{n-1,n}^{(g)}],$$

where

$$K_1 = K(x^{(2)}_{11}, \dots, x^{(2)}_{1,n-1}, \dots, x^{(2)}_{n-1,1}, \dots, x^{(2)}_{n-1,n-1}, \dots, x^{(g)}_{11}, \dots, x^{(g)}_{1,n-1}, \dots, x^{(g)}_{n-1,1}, \dots, x^{(g)}_{n-1,n-1}).$$

Clearly, $A \subset F$. Denote the integral closure of A in E by B. Let $\mathfrak{p} \subset A$ be a maximal ideal generated by the elements $x^{(i)}_{n1}, \dots, x^{(i)}_{n,n-1}, x^{(i)}_{1n}, \dots, x^{(i)}_{n-1,n}, x^{(i)}_{nn} - 1$, where $2 \leq i \leq g$. Let also $\mathfrak{P} \subset B$ be any maximal ideal lying above $\mathfrak{p}$, $\Gamma = \mathrm{Gal}(E/K)$ and $\Gamma_{\mathfrak{P}} \leq \Gamma$ the decomposition group of $\mathfrak{P}$.

There is the natural surjective homomorphism $\Gamma_{\mathfrak{P}} \to \mathrm{Gal}(\overline{B}/\overline{A})$, where $\overline{B} = B/\mathfrak{P}$, $\overline{A} = A/\mathfrak{p}$. We claim that $|\operatorname{Gal}(\overline{B}/\overline{A})| \geq (n-1)!$. Indeed, since

$$\overline{g}_1(t) = \det(tE_n - X_g^2 \dots X_2^2)$$

and $\overline{g}_1(t)$ splits into linear factors in B, its reduced polynomial under the natural map $A \to \overline{A}$ also splits into linear factors in $\overline{B}$ and thus $\overline{B}$ contains its splitting field. However, under the map $A \to \overline{A}$ the polynomial $\overline{g}_1(t)$ is mapped to $\det(tE_n - \overline{X}_g^2 \dots \overline{X}_2^2)$, where

$$\overline{X}_2 = \begin{pmatrix} x^{(2)}_{11} & \dots & x^{(2)}_{1,n-1} & 0 \\ & \ddots & & \\ x^{(2)}_{n-1,1} & \dots & x^{(2)}_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \dots, \overline{X}_g = \begin{pmatrix} x^{(g)}_{11} & \dots & x^{(g)}_{1,n-1} & 0 \\ & \ddots & & \\ x^{(g)}_{n-1,1} & \dots & x^{(g)}_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

and hence by induction, the splitting field of the latter polynomial has degree $(n-1)!$ over $\overline{A}$.

Thus, we established that $|\Gamma_{\mathfrak{P}}| \geq |\operatorname{Gal}(\overline{B}/\overline{A})| \geq (n-1)!$. Then by Lemma 4.2, the subgroup $\Gamma_{\mathfrak{P}}$ is maximal in S_n, whence either $\Gamma = S_n$ or $\Gamma = \Gamma_{\mathfrak{P}} \neq S_n$. In the first case there is nothing to prove. Let $\Gamma = \Gamma_{\mathfrak{P}} \neq S_n$. Then there is only one maximal ideal in B lying above $\mathfrak{p}$. This means that the polynomial $\det(tE_n - \overline{X}_g^2 \dots \overline{X}_2^2)$, being the reduced polynomial of $\overline{g}_1(t)$ under the map $A \to \overline{A}$, is a power of some irreducible polynomial which is impossible since $t-1$ divides it. The proof is complete. $\square$

We are now in position to finish the proof of Proposition 4.2. The polynomial $g(t)$ splits into linear factors in A_4 and under the map $A_4 \to \overline{A}_4$ it is mapped to

$$\overline{g}(t) = (t^2)^n - \sigma_1(X_g^{-2} \dots X_2^{-2})(t^2)^{n-1} + \dots + (-1)^n \sigma_n(X_g^{-2} \dots X_2^{-2}),$$

implying that $\overline{g}(t)$ also splits into linear factors in $\overline{A}_4$. However, we get $[\overline{A}_4 : \overline{A}_2] \leq 2^{n-1} n!$ and so it suffices to prove that a splitting field of $\overline{g}(t)$ has degree $2^{n-1} n!$ over $\overline{A}_2 = F$.

A splitting field of $\overline{g}(t)$ can be obtained in two steps. First one can take a splitting field E of the polynomial $\overline{g}_1(t)$. Let $\overline{g}_1(t) = \prod_{i=1}^n (t - \alpha_i)$, $\alpha_i \in E$, be its factorization in E. Next, we should add $\sqrt{\alpha_1}, \dots, \sqrt{\alpha_n}$ to the field E, thus obtaining a splitting field of $\overline{g}(t)$.

By virtue of Lemma 4.3, we have $[E : F] = n!$ and so it remains to prove that $[E(\sqrt{\alpha_1}, \dots, \sqrt{\alpha_n}) : E] = 2^{n-1}$. In other words, we need to show that there exists only one relation between the roots $\alpha_1, \dots, \alpha_n$ of $\overline{g}_1(t)$ modulo $(E^*)^2$. One relation is trivial: $\prod_{i=1}^n \alpha_i = \det(X_g^{-1} \dots X_2^{-1})^2$. Furthermore, we saw above that under the map $B \to \overline{B} = B/\mathfrak{P}$ the residue of one of the roots, say α_1, is 1 and of the other ones are the roots $\overline{\alpha}_2, \dots, \overline{\alpha}_n$ of the polynomial $\det(tE_n - \overline{X}_g^{-2} \dots \overline{X}_2^{-2})$. Then by induction, we get $[\overline{B}(\sqrt{\overline{\alpha}_1}, \dots, \sqrt{\overline{\alpha}_n}) : \overline{B}] = 2^{n-2}$, implying

$$[E(\sqrt{\alpha_1}, \dots, \sqrt{\alpha_n}) : E] \geq [\overline{B}(\sqrt{\overline{\alpha}_1}, \dots, \sqrt{\overline{\alpha}_n}) : \overline{B}] = 2^{n-2}$$

However the module of the relations between $\alpha_1, \dots, \alpha_n$ is invariant under the group $S_n = \mathrm{Gal}(E/F)$. It follows then that there is at most one relation between $\alpha_1, \dots, \alpha_n$ modulo $(E^*)^2$, completing the proof of Proposition 4.2.

To finish the proof of the irreducibility of the varieties W_1, W_2 it remains to construct dominant morphisms

$$\varphi : W \to \mathrm{GL}_n(K)^{g-1}, \qquad \psi : V \to W$$

such that diagram (4.1) is commutative and $[K(W) : \varphi^*(K(\mathrm{GL}_n(K)^{g-1}))] = 2^{n-1}$.

Lemma 4.4 *The morphism*

$$\varphi : W \to \mathrm{GL}_n(K)^{g-1}, \qquad (z, X_2, \dots, X_g) \mapsto (X_2, \dots, X_g),$$

is dominant of degree 2^{n-1}.

The proof is trivial, since $\overline{f}$ viewed as a polynomial of z has degree 2^{n-1}. □

Lemma 4.5 *The morphism*

$$\psi : V \to W, \qquad (X_1, X_2, \dots, X_g) \mapsto (\mathrm{tr}(X_1), X_2, \dots, X_g),$$

where $\mathrm{tr}(X_1)$ *is the trace of* X_1*, is dominant.*

Proof. We need only to check that ψ is well-defined. If $(X_1, X_2, \dots, X_g) \in V$ and X_1 is semisimple, then the equality $\overline{f}(\mathrm{tr}(X_1), X_2, \dots, X_g) = 0$ follows immediately from the way of constructing f and $\overline{f}$. Furthermore, the set

$$\big\{(X_1, \dots, X_g) \in V \mid X_1^2 \text{ is semisimple regular}\big\}$$

is open and non-empty (see Proposition 2.1) and hence $\overline{f}(\mathrm{tr}(X_1), X_2, \dots, X_g)$ is zero for all $(X_1, \dots, X_g) \in V$. □

Theorem 4.2 *The varieties* W_1, W_2 *are (absolutely) irreducible* Q*-rational varieties of dimension* $(g-1)n^2$.

Proof. We have already established that W_1, W_2 are irreducible and thus, it remains to prove the Q-rationality. To this end, first we simplify (in some sense) the matrix equation defining W_1, W_2. Namely, the morphism

$$\nu : \mathrm{GL}_n(K)^g \to \mathrm{GL}_n(K)^g, \qquad (X_1, X_2, \ldots, X_g) \mapsto (V_1, V_2, \ldots, V_g),$$

where

$$\begin{aligned} V_1 &= X_1X_3^{-1}X_2^{-1}X_1^{-1}, \quad V_2 = X_1X_2X_3X_1X_2(X_1X_2X_3)^{-1}, \\ V_3 &= X_1X_2X_3, \quad V_4 = X_4, \ \ldots, \ V_g = X_g \end{aligned}$$

is clearly a biregular isomorphism, since ν^{-1} is defined by the formulas

$$\nu^{-1} : \mathrm{GL}_n(K)^g \to \mathrm{GL}_n(K)^g, \qquad (V_1, V_2, \ldots, V_g) \mapsto (X_1, X_2, \ldots, X_g),$$

where

$$\begin{aligned} X_1 &= V_1V_3, \quad X_2 = V_3^{-1}V_1^{-1}V_3^{-1}V_2V_3, \\ X_3 &= V_3^{-1}V_2^{-1}V_3^2, \qquad X_4 = V_4, \ldots, X_g = V_g. \end{aligned}$$

Then we get $W_1 \simeq \nu(W_1)$, $W_2 \simeq \nu(W_2)$ and the images $\nu(W_1)$, $\nu(W_2)$ coincide with

$$\begin{aligned} U_1 &= \left\{(V_1, \ldots, V_g) \in \mathrm{GL}_n(K)^g \mid [V_1, V_2]V_3^2 \ldots V_g^2 = 1, \det(V_3 \ldots V_g) = 1\right\}, \\ U_2 &= \left\{(V_1, \ldots, V_g) \in \mathrm{GL}_n(K)^g \mid [V_1, V_2]V_3^2 \ldots V_g^2 = 1, \det(V_3 \ldots V_g) = -1\right\} \end{aligned}$$

respectively, where $[V_1, V_2] = V_1V_2V_1^{-1}V_2^{-1}$.

Fix any generic point $\omega_0 = (\widetilde{V}_3, \ldots, \widetilde{V}_g)$ of the variety

$$M_i = \left\{(V_3, \ldots, V_g) \in \mathrm{GL}_n(K)^{g-2} \mid \det(V_3 \ldots V_g) = (-1)^{i-1}\right\}$$

which is clearly Q-rational and let $K_i = Q(\omega_0)$ be the field generated over Q by all coefficients of the matrices $\widetilde{V}_3, \ldots, \widetilde{V}_g$.

Proposition 4.3 *There exists a Q-defined Zariski open set $B \subset \mathrm{SL}_n(K)$ such that for any extension L/Q and any point $b \in B(L)$ the commutator variety*

$$W_b = \{(U, V) \in \mathrm{GL}_n(K) \times \mathrm{GL}_n(K) \mid [U, V] = b\}$$

is an (absolutely) irreducible L-rational variety of dimension $n^2 + 1$.

Proof: see [14], Proposition 5. □

Put $h = \widetilde{V}_3^2 \ldots \widetilde{V}_g^2$. Since ω_0 is a generic point, h is in "generic" position, whence h^{-1} belongs to the Q-defined Zariski open set $B \subset \mathrm{SL}_n$ specified in Proposition 4.3. Therefore, the field of Q-rational functions $Q(U_i)$ on U_i can be identified with the field of K_i-rational functions $K_i(W_{h^{-1}})$ on the variety $W_{h^{-1}}$. However by Proposition 4.3, $W_{h^{-1}}$ is rational over K_i. This means that $K_i(W_{h^{-1}})$ is a purely transcendental extension of K_i and since K_i is also purely transcendental extension of $\mathbb{Q}$ so is $\mathbb{Q}(U_i)$, yielding the required result. □

Finally, we should handle the case $(n,g) = (2,3)$. Obviously, $R_2(\Gamma_3)$ can be rewritten in the form

$$R_2(\Gamma_3) = \left\{(X,Y,Z) \in \mathrm{GL}_2(K)^3 \mid X^2 = Y^2 Z^2\right\}.$$

As in the previous case, put

$$\begin{aligned} W_1 &= \{(X,Y,Z) \in R_2(\Gamma_3) \mid \det X = \det(YZ)\}, \\ W_2 &= \{(X,Y,Z) \in R_2(\Gamma_3) \mid \det X = -\det(YZ)\}, \end{aligned}$$

One can easily check that all the arguments above can be carried over to our situation yielding that there exists a unique irreducible component S_i of the variety W_i $(i = 1,2)$ whose image under the natural projection

$$W_i \to \mathrm{GL}_2(K) \times \mathrm{GL}_2(K), \qquad (X,Y,Z) \mapsto (Y,Z)$$

is dense in $\mathrm{GL}_2(K) \times \mathrm{GL}_2(K)$. Clearly, S_i is the closure of the set

$$\left\{(X,Y,Z) \in W_i \mid X^2 \text{ is regular}\right\}$$

and hence the union V of all other irreducible components of $R_2(\Gamma_3)$ coincides with the closure of

$$\widetilde{V} = \left\{(X,Y,Z) \in \mathrm{GL}_2(K)^3 \mid X^2 = Y^2 Z^2; \quad X^2, Y^2, Z^2 \text{ are non-regular}\right\}$$

We also know (see Proposition 2.3) that any irreducible component of W_i contains a point (X,Y,Z) with regular components X,Y,Z. However, the condition X is regular, but X^2 is non-regular implies that X is conjugate to $\mathrm{diag}(a,-a)$ for some $a \in K$; in particular, X^2 is a scalar matrix. This means that V coincides with the closure of the set

$$\begin{aligned} V_1 = \big\{(X \,\mathrm{diag}(ab,-ab)X^{-1}, Y \,\mathrm{diag}(a,-a)Y^{-1}, Z \,\mathrm{diag}(b,-b)Z^{-1}) \mid \\ a,b \in K^*, X,Y,Z \in \mathrm{GL}_2(K)\big\}. \end{aligned}$$

It follows that V_1 (and hence V) is irreducible and lies in W_2 and so $R_2(\Gamma_3)$ is the union of the irreducible components $S_1 = W_1$, S_2, V of dimension 8.

Proposition 4.4 *S_1, S_2, V are irreducible $\mathbb{Q}$-rational varieties.*

Proof. The $\mathbb{Q}$-rationality of W_1 was established in Theorem 4.2 and for V the statement is trivial. In the last case, i.e. for S_2, the proof is purely computational. Namely, we know that S_2 is birationally isomorphic to

$$W = \left\{(z,Y,Z) \in \mathbb{A}^1 \times \mathrm{GL}_2(K) \times \mathrm{GL}_2(K) \mid \overline{f}(z, \sigma_1(Y^2Z^2), -\sigma_2(YZ)) = 0\right\}.$$

It is easy to check that for $n = 2$, the polynomial $f(z)$ is of the form

$$f(z) = z^2 - \sigma_1(T^2) - 2\sigma_2(T)$$

and hence

$$\overline{f}(z, \sigma_1(T^2Z^2), -\sigma_2(YZ)) = z^2 - \mathrm{tr}(Y^2Z^2) + 2\det(YZ).$$

Let $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$. Then we get

$$\mathrm{tr}(Y^2Z^2) = (y_{11}^2 + y_{12}y_{21})(z_{11}^2 + z_{12}z_{21}) + (y_{22}^2 + y_{12}y_{21})(z_{221}^2 + z_{12}z_{21}) + \\ + y_{12}z_{21}(y_{11} + y_{22})(z_{11} + z_{22}) + y_{21}z_{12}(y_{11} + y_{22})(z_{11} + z_{22}).$$

It follows that the variable y_{12} occurs in the equation

$$z^2 - \mathrm{tr}(Y^2Z^2) + 2\det(YZ) = 0$$

in the first degree and so it can be expressed as a rational function of z, y_{11}, y_{22}, y_{21}, z_{11}, z_{11}, z_{12}, z_{21}, z_{22}, implying that $\mathbb{Q}(W)$ is a purely transcendental extension of $\mathbb{Q}$. The proposition is proved. □

Corollary 4.2 *For all $n \geq 2$, $g \geq 3$ the variety $R_n(\Gamma_g)_{\mathbb{C}}$ has two connected components with respect to the complex topology.*

Proof. If $(n, g) \neq (2, 3)$, then by Theorem 4.2, $(W_1)_{\mathbb{C}}$ and $(W_2)_{\mathbb{C}}$ are connected and the assertion follows. If $(n, g) = (2, 3)$, then in the notation above we have obviously $V \cap S_2 \neq \emptyset$, implying $(W_2)_{\mathbb{C}} = V_{\mathbb{C}} \cup (S_2)_{\mathbb{C}}$ is also connected. □

5 Representations in $\mathrm{SL}_\mathbf{n}(\mathbf{K})$.

The goal of this section is to prove the (partial) analog of Theorem A for the representation variety $R(\Gamma_g, \mathrm{SL}_n(K))$.

Theorem 5.1 *The number of irreducible components of $R(\Gamma_g, \mathrm{SL}_n(K))$ equals*

1) $[n/2] + 1$, if $g = 1$;
2) $(n^2 + 6n + 8)/8$, if $g = 2$ and n is even;
3) $(n^2 + 4n + 3)/8$, if $g = 2$ and n is odd;
4) 1, if $g \geq 3$.

Moreover, for all n and g all components of $R(\Gamma_g, \mathrm{SL}_n(K))$ are $\mathbb{Q}$-unirational varieties.

We begin with the last case $g \geq 3$. Let W_1 be the component of $R_n(\Gamma_g)$ specified in § 4. Since $R(\Gamma_g, \mathrm{SL}_n(K)) \subset W_1$ is defined in W_1 by the equations $\det X_i = 1$ it is natural to try to reduce our claim to the irreducibility of W_1.

Assume that $R(\Gamma_g, \mathrm{SL}_n(K))$ is reducible and let $R(\Gamma_g, \mathrm{SL}_n(K)) = R_1 \cup \ldots \cup R_t$ be the decomposition into irreducible components. Consider the morphism

$$\lambda : R(\Gamma_g, \mathrm{SL}_n(K)) \times \mathbb{A}^{g-1} \to W_1 \subset R_n(\Gamma_g),$$

$$(X_1, \ldots, X_g, a_1, \ldots, a_{g-1}) \mapsto (X_1a_1, \ldots, X_{g-1}a_{g-1}, X_g(a_1 \ldots a_{g-1})^{-1}).$$

Lemma 5.1 *For any point*

$$X = (X_1, \dots, X_g, a_1, \dots, a_{g-1}) \in R(\Gamma_g, \mathrm{SL}_n(K)) \times \mathbb{A}^{g-1}$$

the fibre $\lambda^{-1}(\lambda(X))$ *is finite.*

Proof. Indeed, let

$$\widetilde{X} = (\widetilde{X}_1, \dots, \widetilde{X}_g, \widetilde{a}_1, \dots, \widetilde{a}_{g-1}) \in \lambda^{-1}(\lambda(X)).$$

Then $X_1 a_1 = \widetilde{X}_1 \widetilde{a}_1, \dots, X_{g-1} a_{g-1} = \widetilde{X}_{g-1} \widetilde{a}_{g-1}$. Consideration of the determinants shows that $\widetilde{a}_1^n = a_1^n, \dots, \widetilde{a}_{g-1}^n = a_{g-1}^n$ and hence $\widetilde{a}_i = a_i \zeta_i$, $\widetilde{X}_i = X_i \widetilde{\zeta}_i^{-1}$, where ζ_i is n-th root of unity. $\square$

The subvariety $R(\Gamma_g, \mathrm{SL}_n(K)) \subset W_1$ is defined by $(g-1)$ equations and so the dimension of any component R_i is at least

$$\dim W_1 - (g-1) = n^2(g-1) - (g-1) = (n^2-1)(g-1).$$

This fact combined with Lemma 5.1 implies that

$$\dim \lambda(R_i \times \mathbb{A}^{g-1}) = \dim(R_i \times \mathbb{A}^{g-1}) \geq n^2(g-1),$$

i.e. $\lambda(R_i \times \mathbb{A}^{g-1})$ is dense in W_1 for all $i = 1, \dots, t$.

Let $R_i' = R_i \setminus \{\cup_{j \neq i}(R_i \cap R_j)\}$. Clearly, R_i' is Zariski open in R_i and hence the intersection $Z = \bigcap_{i=1}^t \lambda(R_i' \times \mathbb{A}^{g-1})$ is non-empty. It follows then that Z contains some Zariski open subset $Z' \subset W_1$ and for any $w \in Z'$ the fibre $\lambda^{-1}(w)$ is finite and contains points belonging to pairwise disjoint sets $R_1' \times \mathbb{A}^{g-1}, \dots, R_t' \times \mathbb{A}^{g-1}$. However if

$$(X_1, \dots, X_g, a_1, \dots, a_{g-1}) \in \lambda^{-1}(w)$$

then we showed above that any other point lying in the fibre $\lambda^{-1}(w)$ is of the form

$$(X_1\varepsilon_1, \dots, X_{g-1}\varepsilon_{g-1}, X_g(\varepsilon_1 \dots \varepsilon_{g-1})^{-1}, a_1\varepsilon_1^{-1}, \dots, a_{g-1}\varepsilon_{g-1}^{-1}),$$

where $\varepsilon_1, \dots, \varepsilon_{g-1}$ are n-th roots of unity. So the proof is completed by the following

Proposition 5.1 *If* $(X_1, \dots, X_g) \in R_i'$ *is in "generic" position then for all n-th roots of unity* $\varepsilon_1, \dots, \varepsilon_{g-1}$ *the element*

$$(X_1\varepsilon_1, \dots, X_{g-1}\varepsilon_{g-1}, X_g(\varepsilon_1 \dots \varepsilon_{g-1})^{-1})$$

also belongs to R_i.

Proof. We are going to find an irreducible variety $V \subset R(\Gamma_g, \mathrm{SL}_n(K))$ containing both $(X_1, \dots, X_g)$ and $(X_1\varepsilon_1, \dots, X_{g-1}\varepsilon_{g-1})$. Since $(X_1, \dots, X_g)$ does not belong to any other component R_j, $j \neq i$, this will imply that $V \subset R_i$, yielding the required result.

To construct such a variety we will use the substitutions specified in the proof of Proposition 2.3. Let

$$\widetilde{W}_1 = \{(X_1, \dots, X_g) \in W_1 \mid X_1X_2, \dots, X_{g-1}X_g \text{ are regular semisimple elements}\}$$

Since $\widetilde{W}_1$ is a non-empty Zariski open subset in W_1 and $\lambda(R_i \times \mathbb{A}^{g-1})$ is dense in W_1, the intersection $\widetilde{R}_i = \widetilde{W}_1 \cap R_i$ is again non-empty and open in R_i for all $i = 1, \dots, t$. Now fix any irreducible component R_i of $R(\Gamma_g, \mathrm{SL}_n(K))$ and let U_1, $U_1^{(1)}$ be the following sets:

$$U_1 = \left\{((X_1, \dots, X_g), A) \in \widetilde{R}_i \times \mathrm{GL}_n(K) \mid A(X_1X_2) = (X_1X_2)A\right\},$$

$$U_1^{(1)} = \{((X_1, \dots, X_g), A) \in U_1 \mid \det A = 1\}.$$

Let also

$$\varphi : U_1^{(1)} \to R(\Gamma_g, \mathrm{SL}_n(K)), \qquad ((X_1, \dots, X_g), A) \mapsto (X_1A, A^{-1}X_2, X_3, \dots, X_g).$$

Lemma 5.2 *$U_1^{(1)}$ is irreducible.*

Proof. Let $U_1^{(1)} = S_1 \cup \ldots \cup S_l$ be the decomposition into irreducible components and π be the projection

$$\pi : U_1^{(1)} \to \widetilde{R}_i, \qquad ((X_1, \dots, X_g), A) \mapsto (X_1, \dots, X_g).$$

We claim that the image $\pi(S_j)$ is dense in $\widetilde{R}_i$ for all $j = 1, \dots, l$. Indeed, since X_1X_2 is regular, Lemma 3.1 implies that U_1 is an irreducible variety of dimension $\dim R_i + n$ and so the dimension of any irreducible component of the variety $U_1^{(1)}$ equals $\dim U_1 - 1 = \dim R_i + n - 1$. Furthermore, the dimension of the fibre $\pi_j^{-1}(\alpha)$, where $\pi_j = \pi|_{S_j}$ and $\alpha \in \widetilde{R}_i$, is at most $n - 1$. Therefore, $\dim \pi(S_j) \geq \dim R_i$, i.e. $\pi(S_j)$ is dense in R_i.

Consider the sets $\widetilde{S}_j = S_j \setminus \left\{\bigcup_{k \neq j} S_k\right\}$ and their images $\pi_j(\widetilde{S}_j)$. We already know that $\bigcap_{j=1}^{l} \pi_j(\widetilde{S}_j) \neq \varnothing$. So if

$$\alpha = (X_1, \dots, X_g) \in \bigcap_{j=1}^{l} \pi_j(\widetilde{S}_j),$$

then $\pi^{-1}(\alpha)$ is irreducible in view of the fact that X_1X_2 is regular. On the other hand by our construction, $\pi^{-1}(\alpha)$ contains points from disjoint sets $\widetilde{S}_1, \dots, \widetilde{S}_l$, which contradicts the irreducibility of $\pi^{-1}(\alpha)$. □

The set $V_1 = \varphi_1(U_1^{(1)})$, being the image of $U_1^{(1)}$, is irreducible. Then arguing as above one can show that the sets

$$U_2 = \left\{((X_1, \dots, X_g), A) \in \widetilde{V}_1 \times \mathrm{GL}_n(K) \mid A(X_2X_3) = (X_2X_3)A\right\},$$

where $\widetilde{V}_1 = V_1 \cap \widetilde{W}_1$, and

$$U_2^{(1)} = \{((X_1, \dots, X_g), A) \in U_2 \mid \det A = 1\}$$

are irreducible, implying $V_2 = \varphi_2(U_2^{(1)})$ is so, where

$$\varphi_2 : U_2^{(1)} \to R(\Gamma_g, \mathrm{SL}_n(K)), \qquad ((X_1, \dots, X_g), A) \mapsto (X_1, X_2 A, A^{-1} X_3, X_4 \dots, X_g),$$

and so on. Finally, we may put $V = V_{g-1}$. Obviously, for any $(X_1, \dots, X_g) \in \widetilde{R}_i$ and any set $(\varepsilon_1, \dots, \varepsilon_{g-1})$ of n-th roots of unity we have

$$\begin{aligned}
(X_1\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_1^{-1} X_2, X_3, \dots, X_g) &= (X_1\varepsilon_1, X_2\varepsilon_1^{-1}, X_3, \dots, X_g) \in \widetilde{V}_1, \\
(X_1\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_1^{-1} X_2 \widetilde{\varepsilon}_2, \widetilde{\varepsilon}_2^{-1} X_3, X_4, \dots, X_g) &= (X_1\varepsilon_1, X_2\varepsilon_2, X_3(\varepsilon_1\varepsilon_2)^{-1}, X_4, \dots, X_g) \in \widetilde{V}_2, \\
&\vdots \\
(X_1\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_1^{-1} X_2 \widetilde{\varepsilon}_2, \dots, \widetilde{\varepsilon}_{g-2}^{-1} X_{g-1} \widetilde{\varepsilon}_{g-1}, \widetilde{\varepsilon}_{g-1}^{-1} X_g) &= (X_1\varepsilon_1, \dots, X_{g-1}\varepsilon_{g-1}, X_g(\varepsilon_1 \dots \varepsilon_{g-1})^{-1}) \in V_{g-1}
\end{aligned}$$

where $\widetilde{\varepsilon}_1 = \varepsilon_1$, $\widetilde{\varepsilon}_2 = \varepsilon_1\varepsilon_2$, $\dots$, $\widetilde{\varepsilon}_{g-1} = \varepsilon_1 \dots \varepsilon_{g-1}$, completing the proof of Proposition 5.1 as well as Theorem 5.1 the case $g \geq 3$.

We now turn to the case $g = 2$. In contrast to the previous case the representation variety $R(\Gamma_2, \mathrm{SL}_n(K))$ is reducible. Nevertheless, its irreducible components can be described exactly in the same manner. Namely, repeating verbatim the arguments from Lemmas 5.1, 5.2 and Proposition 5.1 one can easily check that if the intersection $V_{s,t}^{(1)} = V_{s,t} \cap R(\Gamma_2, \mathrm{SL}_n(K))$ is non-empty, then it is irreducible (here $V_{s,t}$ are the irreducible components of $R_n(\Gamma_2)$ specified in § 3). It is also easy to see that $V_{s,t}^{(1)} \neq \varnothing$ iff t is even.

Proposition 5.2 *The number of irreducible components of $R(\Gamma_2, \mathrm{SL}_n(K))$ equals*

$$\begin{cases} (n^2 + 6n + 8)/8, & \text{if } n \text{ is even;} \\ (n^2 + 4n + 3)/8, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, all irreducible components are $\mathbb{Q}$-unirational varieties.

Proof. If $(s,t) \neq (s_1, t_1)$ and t, t_1 are even then clearly we have $V_{s,t}^{(1)} \not\subseteq V_{s_1,t_1}^{(1)}$, since otherwise the image of

$$\lambda_{s,t} : V_{s,t}^{(1)} \times \mathbb{A}^1 \to V_{s,t}, \qquad (X, Y, a) \mapsto (aX, aY)$$

being dense in $V_{s,t}$ is contained in the analogous image $\lambda_{s_1,t_1}(V_{s_1,t_1}^{(1)} \times \mathbb{A}^1)$, implying that $V_{s,t} \subseteq V_{s_1,t_1}$ which is impossible. Thus, the number of irreducible components of $R(\Gamma_2, \mathrm{SL}_n(K))$ equals the number of pairs (s,t) such that $0 \leq s \leq [n/2]$, $0 \leq t \leq n - 2s$ and t is even.

If $n = 2l$ is even then for a given s we have $(l - s) + 1$ possibilities for t. This means that the number of the above pairs (s,t) is equal to

$$\sum_{s=0}^{l} ((l - s) + 1) = 1 + 2 + \ldots + (l - 1) = (l + 1)(l + 2)/2 = (n^2 + 6n + 8)/8.$$

Analogously, if $n = 2l+1$, then the number of the above pairs (s,t) is equal to

$$\sum_{s=0}^{l}((l-s)+1) = 1+2+\ldots+(l+1) = (l+1)(l+2)/2 = (n^2+4n+3)/8.$$

To prove the Q-rationality of $V^{(1)}_{s,t}$, consider the restriction $\varphi_{s,t}|_V$, where

$$V \subset \mathbb{A}^s \times \mathbb{A}^{n-2s} \times \mathrm{GL}_2^s \times \mathrm{GL}_n(K)$$

consists of all points $\alpha = (a_1, \ldots, a_s, b_1, \ldots, b_{n-2s}, Z_1, \ldots, Z_s, A)$ such that

$$(-1)^s \prod_{i=1}^{s} \alpha_i^2 \prod_{i=1}^{n-2s} b_i = 1.$$

By dimension arguments $\operatorname{Im}\varphi_{s,t}(V)$ is dense in $V^{(1)}_{s,t}$. It remains to observe that if $2s < n$ then the variety V is irreducible and Q-rational and in case $n = 2s$ it consists of two irreducible Q-rational components. □

Let us proceed with the proof of Q-unirationality of $R(\Gamma_g, \mathrm{SL}_n(K))$ in the case $g \geq 3$. Our proof will rest on the notion of two matrices to be chain equivalent which was introduced and used by Rapinchuk and by the authors in [14], where we studied the representation varieties of the fundamental groups of compact orientable surfaces. In fact, in the non-orientable case the argument is completely analogous to one in [14]. So we give only a sketch of the proof of the Q-unirationality, omitting all technical details which can be found in [14].

To begin with, recall that $R(\Gamma_g, \mathrm{SL}_n(K))$ is biregularly isomorphic to the variety

$$S = \left\{(V_1, \ldots, V_g) \in \mathrm{SL}_n(K)^g \mid [V_1, V_2]V_3^2 \ldots V_g^2 = 1\right\}.$$

It was proved by Thompson [18] that almost any $Z \in \mathrm{SL}_n(K)$ (in fact, any Z which does not belong to the center) is a commutator of two matrices $X, Y \in \mathrm{SL}_n(K)$. This means that the natural projection onto the last $(g-2)$ components

$$\pi : S \to \mathrm{SL}_n(K)^{g-2}, \qquad (V_1, V_2, \ldots, V_g) \mapsto (V_3, \ldots, V_g),$$

is a dominant morphism and hence it suffices to establish L-unirationality of the fibre $\pi^{-1}(\alpha)$, where α is a generic point of $\mathrm{SL}_n(K)^{g-2}$ and $L = K(\alpha)$.

Proposition 5.3 *There exists a Q-defined Zariski open subset $U \subset \mathrm{SL}_n(K)$ such that for any extension L/Q and any point $Z \in U(L)$ the variety*

$$W'_Z = \{(X, Y) \in \mathrm{SL}_n(K) \times \mathrm{SL}_n(K) \mid [X, Y] = Z\}$$

is (absolutely) irreducible and unirational over L.

Proof. Let us fix any point $(X,Y)\in W'_Z$ and show that any other point of W'_Z can be obtained from (X,Y) by means of a natural procedure which can be interpreted as a motion along some unirational subvariety. For technical reasons, it is more convenient to define and analyze this procedure first for elements in $\mathrm{GL}_n(K)$.

Definition 5.1 *Two pairs of matrices $a_i=(X_i,Y_i)\in \mathrm{GL}_n(K)\times \mathrm{GL}_n(K)$ $(i=1,2)$ are equivalent if $[X_1,Y_1]=[X_2,Y_2]$ and either $X_1=X_2$ or $Y_1=Y_2$.*

This means that a_2 is obtainable from a_1 by a single standard transformation by which we mean multiplication of either of the components by an element from the centralizer of the other. Now, let $R\subset \mathrm{GL}_n(K)$ be the set of regular semisimple elements.

Definition 5.2 *Elements $a,b\in \mathrm{GL}_n(K)\times \mathrm{GL}_n(K)$ (resp. $a,b\in R\times R$) are said to be chain equivalent (resp. strictly chain equivalent) if there are elements $c_1,\dots,c_k$ in $\mathrm{GL}_n(K)\times \mathrm{GL}_n(K)$ (resp. in $R\times R$) such that $c_1=a$, $c_k=b$ and for any $i=1,\dots,k-1$ the elements c_i and c_{i+1} are equivalent.*

Thus, chain equivalent pairs are those obtainable from one another by a sequence of standard transformations. Clearly, the relation of being chain equivalent (resp. strictly chain equivalent) is an equivalence relation.

It follows from our definitions that any two chain equivalent pairs $a_i=(X_i,Y_i)$ $(i=1,2)$ have the same commutator: $[X_1,Y_1]=[X_2,Y_2]$. Let us put the question the other way around: does the fact that $[X_1,Y_1]=[X_2,Y_2]$ imply that a_1 and a_2 are chain equivalent? We are going to show that this is, indeed, true "generically", i.e. on a Zariski open set, and this fact will play a crucial role in the proof of Proposition 5.3.

Theorem 5.2 *There exists a Q-defined Zariski open set $V\subset R^4$ with the following property: any $a_i=(X_i,Y_i)\in R^2$ $(i=1,2)$ such that $[X_1,Y_1]=[X_2,Y_2]$ and $(X_1,Y_1,X_2,Y_2)\in V$, are strictly chain equivalent.*

Proof. Fix an integer $t\geq 0$ and consider a subvariety

$$X_t\subset R^2\times \mathrm{GL}_n(K)^{2t}$$

consisting of points $(X,Y,A_1,\dots,A_t,B_1,\dots,B_t)$ subject to the following conditions:

$$[X,B_1]=1,\quad [XA_1\dots A_i,B_{i+1}]=1\qquad \text{for } i=1,\dots,t-1$$
$$[A_i,YB_1\dots B_i]=1,\quad XA_1,\dots,A_i,\ YB_1\dots B_i\in R\qquad \text{for } i=1,\dots,t.$$

Furthermore, define a morphism $\varphi_t: X_t\to R^4$ by the formula

$$\varphi_t(X,Y,A_1,\dots,A_t,B_1,\dots,B_t)=(X,Y,XA_1\dots A_t,YB_1\dots B_t).$$

It follows from this definition that if $(X_1,Y_1,X_2,Y_2)\in \operatorname{Im}\varphi_t$ then (X_1,Y_1) can be linked with (X_2,Y_2) by the following chain of elements:

$$\dots,(XA_1\dots A_i,YB_1\dots B_i),\ (XA_1\dots A_i,YB_1\dots B_{i+1}),(XA_1\dots A_{i+1},YB_1\dots B_{i+1}),\dots$$

Moreover, all pairs above belong to $R\times R$ and each next pair is obtained from the previous one by a standard transformation, implying that (X_1, Y_1) and (X_2, Y_2) are strictly chain equivalent; in particular $[X_1, Y_1] = [X_2, Y_2]$. Put

$$Y = \left\{(X_1, Y_1, X_2, Y_2) \in R^4 \mid [X_1, Y_1] = [X_2, Y_2]\right\}.$$

Then $\varphi_t(X_t) \subset Y$, and to prove Theorem 5.1 it suffices to show that $\varphi_t(X_t)$ is dense in Y for some t.

Lemma 5.3 *X_t and Y are irreducible varieties of dimension $2(n^2 + t)$ and $3n^2 + 1$, respectively.*

Proof. See [14], Lemma 9.

By the Dimension Theorem, $\varphi_t(X_t)$ is dense in Y iff for some $d \in Y$ the following holds

$$\dim \varphi_t^{-1}(d) \leq \dim X_t - \dim Y = 2nt - n^2 - 1.$$

For fixed $(U, V) \in R^2$ let

$$Z_t(U, V) = X_t \cap ((U, V) \times \mathrm{GL}_n(K)^{2t}).$$

Lemma 5.4 *1) $Z_t(U, V)$ is an irreducible variety of dimension $2nt$.*

2) There exists a pair $(U_0, V_0) \in R^2$ such that for some t we have $\dim \overline{\varphi_t(Z_t(U_0, V_0))} \geq n^2 + 1$.

Using Lemma 5.4 it is easy to complete the proof of Theorem 5.1. Indeed, for any $d \in \varphi_t(Z_t(U_0, V_0))$ we have $\varphi_t^{-1}(d) \subset Z_t(U_0, V_0)$. So using the dimension values from Lemma 5.4 we conclude that there must be a point $d \in \varphi_t(Z_t(U_0, V_0))$ such that

$$\dim \varphi_t^{-1}(d) = \dim Z_t(U_0, V_0) - \dim \overline{\varphi_t(Z_t(U_0, V_0))} \leq 2tn - n^2 - 1$$

as required. □

Proof of Lemma 5.4. The first statement is proved in exactly the same fashion as Lemma 5.3. To prove 2) let us take

$$U_0 = \mathrm{diag}(1, \rho, \ldots, \rho^{n-1}), \qquad V_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

where ρ is a primitive n-th root of unity. It is easy to check that $(U_0, V_0) \in R$ and the commutator $[U_0, V_0]$ equals $\xi = \mathrm{diag}(\rho, \ldots, \rho)$. Composing φ_t with the projection $Y \to \mathrm{GL}_n(K) \times \mathrm{GL}_n(K)$ onto the second pair of components, we obtain a morphism

$$\rho_t : Z_t(U, V) \to W_Z = \left\{(X, Y) \in \mathrm{GL}_n(K)^2 \mid [X, Y] = Z\right\},$$

where $Z = [U, V]$, and we need to prove that for some t

$$\dim \overline{\rho_t(Z_t(U_0, V_0))} \geq n^2 + 1.$$

It easily follows from our definitions that

$$\ldots \subset \rho_t(Z_t(U_0, V_0)) \subset \rho_{t+1}(Z_{t+1}(U_0, V_0)) \subset \ldots$$

Pick t such that the dimension of $L = \overline{\rho_t(Z_t(U_0, V_0))}$ is maximal possible. Then from the above we get $\rho_s(Z_s(U_0, V_0)) \subset L$ for any s. However the union $\bigcup_{s \geq 0} \rho_s(Z_s(U_0, V_0))$ coincides with the equivalence class $E = [(U_0, V_0)]$ with respect to the strict chain equivalence. So, all that remains to be proven is that $\dim \overline{E} \geq n^2 + 1$.

Since ξ belongs to the center of $\mathrm{GL}_n(K)$, this group acts on W_ξ by conjugation. Furthermore, for any point $(U, V) \in W_\xi$ and any $g \in \mathrm{GL}_n(K)$ we obviously have $g[(U, V)] = [(gUg^{-1}, gVg^{-1})]$. This implies, in particular, that E is invariant under $Z_{\mathrm{GL}_n(K)}(U_0)$, $Z_{\mathrm{GL}_n(K)}(V_0)$, and therefore also under the subgroup H generated by them.

Lemma 5.5 $H = \mathrm{GL}_n(K)$.

Proof: see [14], Lemma 11.

It follows from Lemma 5.5 that the image of the morphism

$$\delta : G_m \times G_m \times \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)^2, \qquad (\alpha, \beta, h) \mapsto (h(\alpha U_0)h^{-1}, h(\beta V_0)h^{-1})$$

is contained in E and the following lemma completes the proof of Theorem 5.1.

Lemma 5.6 *For any* $l = (\alpha, \beta, h)$ *we have* $\dim \delta^{-1}(\delta(l)) = 1$.

Proof: see [14].

We conclude this section with the proof of Proposition 5.3. Put $R' = R \cap \mathrm{SL}_n(K)$, and for $(U, V) \in R' \times R'$ let

$$Z'_t(U, V) = Z_t(U, V) \cap ((U, V) \times \mathrm{SL}_n(K)^{2t}).$$

Clearly, the restriction of ρ_t to $Z'_t(U, V)$ defines a morphism

$$\rho'_t : Z'_t(U, V) \to W'_Z = W_Z \cap \mathrm{SL}_n(K)^2$$

and ρ'_t is dominant if ρ_t is (and W'_Z is irreducible).

Lemma 5.7 *1) For any $t \geq 0$, any extension K/Q and any $(U, V) \in R'(L)^2$ the variety $Z'_t(U, V)$ is irreducible and K-unirational.*

2) There exist an integer $t > 0$ and a Q-defined Zariski open set $C' \subset R' \times R'$ such that for $(U, V) \in C'$ the morphism $\rho'_t : Z'_t(U, V) \to W'_Z$ is dominant.

Proof: see [14], Lemma 12.

Lemma 5.7 implies that for any $(U, V) \in C'_L$ the corresponding commutator variety W'_Z, where $Z = [U, V]$, is L-unirational. So, to complete the proof of proposition 5.3 we need to show that there exists a Q-defined Zariski open set $P \subset \mathrm{SL}_n(K)$ such that for any $Z \in P(L)$ the variety W'_Z contains a point $(U, V) \in C'_L$. This follows from the statement below which is interesting in its own right.

Proposition 5.4 *Let $D \subset \mathrm{SL}_n(K) \times \mathrm{SL}_n(K)$ be a Q-defined Zariski open set. Then there exists a Q-defined rational map $\theta : \mathrm{SL}_n(K) \to \mathrm{SL}_n(K) \times \mathrm{SL}_n(K)$ such that for any Z from the domain of θ we have $\theta(Z) \in W'_Z$ and $\mathrm{Im}\,\theta \cap D \neq \varnothing$.*

Proof: see [14], Proposition 10.

6 Character varieties.

Theorem 6.1 *The number of irreducible components of the character variety $X_n(\Gamma_g)$ equals:*

a) $n + 1$, if $g = 1$;

b) $(n^2 + 4n + 4)/4$, if $g = 2$ and n is even;

c) $(n^2 + 4n + 3)/4$, if $g = 2$ and n is odd;

d) 3, if $g = 3$ and $n = 2$;

e) 2, if $g \geq 3$ and $(n, g) \neq (2, 3)$.

Proof. Consider the canonical projection

$$\mu : R_n(\Gamma_g) \to X_n(\Gamma_g).$$

If $R_n(\Gamma_g) = V_1 \cup \ldots \cup V_d$ is the decomposition into the irreducible components, then clearly, $X_n(\Gamma_g) = \bigcup_{i=1}^{d} \mu(V_i)$. Comparing the number of irreducible components of $R_n(\Gamma_g)$ with the number indicated in the statement above we see that they are equal. So it suffices to prove that $\overline{\mu(V_i)}$ is irreducible component of $X_n(\Gamma_g)$. Below we will show that $\overline{\mu(V_i)}$ and $\overline{\mu(V_j)}$ have the same dimension. Therefore, we need only to prove that $\overline{\mu(V_i)} \neq \overline{\mu(V_j)}$ for all $i \neq j$

Put $V'_i = \{\rho \in V_i \mid \rho \text{ is fully reducible representation}\}$.

Proposition 6.1 *V'_i is non-empty Zariski open set for all $i = 1, \ldots, d$.*

The proof follows from the description of the irreducible components of $R_n(\Gamma_g)$ (see Theorems 3.1, 3.2, 4.2, Propositions 4.1, 4.4). □

We also need the following well-known

Proposition 6.2 *If $\rho, \rho' \in R_n(\Gamma_g)$ are fully reducible representations, then $\rho' \in \mu^{-1}(\mu(\rho))$ iff ρ' belongs to the $\mathrm{GL}_n(K)$-orbit $\mathcal{O}(\rho)$, i.e. iff ρ' and ρ are equivalent.*

Proof: see [10].

Assume for the moment that $\dim \overline{\mu(V_i)} = \dim \overline{\mu(V_j)}$ for all i, j. We show how this implies Theorem 6.1. Indeed, assume $\overline{\mu(V_i)} = \overline{\mu(V_j)}$ for some $i \neq j$. Then there exist fully reducible representations $\rho_i \in V_i \setminus V_j$, $\rho_j \in V_j \setminus V_i$ such that $\mu(\rho_i) = \mu(\rho_j)$. By Proposition 6.2, ρ_j belongs to the $\mathrm{GL}_n(K)$-orbit $\mathcal{O}(\rho_i)$. However all irreducible components $V_1, \dots, V_d$ are invariant under conjugation (see Lemma 6.1), whence the sets $V_i \setminus V_j$, $V_j \setminus V_i$ are so. This means, in particular, that $\mathcal{O}(\rho_i) \cap \mathcal{O}(\rho_j) = \varnothing$ — a contradiction, proving Theorem 6.1. □

Proposition 6.3 *All irreducible components of the character variety $X_n(\Gamma_g)$ are $\mathbb{Q}$-unirational varieties of dimension*

a) 0, if $g = 1$;

b) n, if $g = 2$;

c) $n^2(g-2)+1$, if $g \geq 3$.

Proof. The $\mathbb{Q}$-unirationality follows immediately from the $\mathbb{Q}$-rationality irreducible components of $R_n(\Gamma_g)$. Furthermore, by the Dimension Theorem, we get $\dim \overline{\mu(V_i)} = \dim V_i - \dim \mathcal{O}(\rho)$, where $\rho \in V_i$ is an element in "generic" position, and $\dim \mathcal{O}(\rho) = n^2 - \dim \mathrm{St}(\rho)$, where $\mathrm{St}(\rho)$ denotes the stabilizer of ρ.

If $g = 1$, then by Theorem 3.1, we get $V_i = \mathcal{O}(\rho)$, implying that $\dim \overline{\mu(V_i)} = 0$. Now let $g = 2$. Consider a representation $\rho = (X, Y) \in V_{s,t}$, where

$$\begin{aligned} X &= \mathrm{diag}(a_1, -a_1, \dots, a_s, -a_s, c_1, \dots, c_t, d_1, \dots, d_{n-2s-t}), \\ Y &= \mathrm{diag}\left(\begin{pmatrix} b_1 & 1 \\ a_1^2 - b_1^2 & -b_1 \end{pmatrix}, \dots, \begin{pmatrix} b_s & 1 \\ a_s^2 - b_s^2 & -b_s \end{pmatrix}, -c_1, \dots, -c_t, d_1, \dots, d_{n-2s-t}\right) \end{aligned}$$

and $V_{s,t}$ is the component of $R_n(\Gamma_2)$ specified in Theorem 3.2. If X is regular, then

$$\mathrm{St}(\rho) = \{\mathrm{diag}(u_1, u_1, \dots, u_s, u_s, u_{s+1}, \dots, u_{n-s}) \mid u_i \in K^*\},$$

whence $\dim \mathcal{O}(\rho) = n^2 - n - s$. Since $\dim V_{s,t} = n^2 + s$, finally we get $\dim \overline{\mu(V_{s,t})} = n$, as required.

Finally, let $g \geq 3$. We know (Theorem 4.2) that if $(n, g) \neq (2, 3)$ then $R_n(\Gamma_g)$ consists of two irreducible components W_1, W_2. Otherwise, it has three irreducible components S_1, S_2, V (see Proposition 4.4). One can easily check that in all these cases every component contains irreducible representations. This means that $\dim \mathcal{O}(\rho) = n^2 - 1$ and hence

$$\dim \overline{\mu(V_i)} = n^2(g-1) - n^2 + 1 = n^2(g-2) + 1.$$

The proof of the proposition is complete. □

For small values of n and g one can strengthen the statement of Proposition 6.3.

Proposition 6.4 *All irreducible components of the character variety $X_n(\Gamma_g)$ are $\mathbb{Q}$-rational varieties in the following cases:*

1) $g=2$ and n arbitrary;
2) $g\geq 4$ and $n\leq 3$;
3) $g=3$ and $n=2$.

Proof. 1) For $g=2$ one can argue as in Theorem 3.3. Namely, let $U\subset V_{s,t}$ be the $\mathbb{Q}$-rational subvariety, consisting of all pairs (X,Y) of the form

$$
\begin{aligned}
X &= \operatorname{diag}(a_1,-a_1,\ldots,a_s,-a_s,c_1,\ldots,c_t,d_1,\ldots,d_{n-2s-t}),\\
Y &= \operatorname{diag}\left(\begin{pmatrix} b_1 & 1\\ a_1^2-b_1^2 & -b_1\end{pmatrix},\ldots,\begin{pmatrix} b_s & 1\\ a_s^2-b_s^2 & -b_s\end{pmatrix},-c_1,\ldots,-c_t,d_1,\ldots,d_{n-2s-t}\right).
\end{aligned}
$$

Define the action of the finite group

$$G=S_s\times S_t\times S_{n-2s-t}\times\prod_{i=1}^{s}H_i,$$

where $H_i=\langle\sigma_i\mid\sigma_i^2=1\rangle$, on the variety U as follows. Elements of S_s permute elements of the sets $\{a_1,\ldots,a_s\}$, $\{b_1,\ldots,b_s\}$; elements of S_t and S_{n-2s-t} permute elements of the sets $\{c_1,\ldots,c_t\}$ and $\{d_1,\ldots,d_{n-2s-t}\}$ respectively; the generator σ_i of H_i maps variables a_i, b_i into $-a_i$, $-b_i$ respectively and acts as identity on others.

It is easy to make sure that the images $\mu(u_1)$, $\mu(u_2)$ of any two elements u_1, $u_2\in U$, which are in "generic" position, coincide iff u_2 belongs to the GL_n-orbit of u_1. This means that the field $\mathbb{Q}\left(\overline{\mu(V_{s,t})}\right)$ of $\mathbb{Q}$-rational functions on $\overline{\mu(V_{s,t})}$ can be identified with the invariant subfield

$$\mathbb{Q}(a_1,\ldots,a_s,b_1,\ldots,b_s,c_1,\ldots,c_t,d_1,\ldots,d_{n-2s-t})^G.$$

However, the latter field is the compositum of

$$
\begin{aligned}
K_1 &= \mathbb{Q}(a_1,\ldots,a_s,b_1,\ldots,b_s)^{S_s\times\prod_{i=1}^{s}H_i},\\
K_2 &= \mathbb{Q}(c_1,\ldots,c_t)^{S_t},\\
K_3 &= \mathbb{Q}(d_1,\ldots,d_{n-2s-t})^{S_{n-2s-t}}
\end{aligned}
$$

and as in § 3 we can conclude that K_1, K_2, K_3 are purely transcendental extensions of $\mathbb{Q}$, implying $\overline{\mu(V_{s,t})}$ is also.

2) Let $g\geq 4$, $n\leq 3$. The group Γ_g can be represented (see § 4) in the form

$$\Gamma_g=\langle x_1,\ldots,x_g\mid[x_1,x_2]=x_3^2\ldots x_g^2\rangle.$$

Recall also (see Theorem 4.2) that the irreducible components of $R_n(\Gamma_g)$ are of the form

$$W_i=\left\{(X_1,\ldots,X_g)\in\mathrm{GL}_n(K)^g\mid[X_1,X_2]=X_3^2\ldots X_g^2,\ \det(X_3\ldots X_g)=(-1)^{i-1}\right\}$$

where $i = 1, 2$. Consider the following commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\mu} & \mu(W_1) \subset X_n(\Gamma_g) \\ {}^{\varphi}\downarrow & & \downarrow^{\delta} \\ \mathrm{GL}_n(K)^{g-3} \times \mathrm{SL}_n(K) & \xrightarrow{\tau} & Y \end{array} , \tag{6.1}$$

where Y is the categorical quotient of $\mathrm{GL}_n(K)^{g-3} \times \mathrm{SL}_n(K)$ modulo the action of $\mathrm{GL}_n(K)$ by conjugation, $\varphi(X_1, \ldots, X_g) = (X_3, \ldots, X_{g-1}, X_3 \ldots X_g)$, δ is the morphism corresponding to φ and τ, μ are the canonical projections.

Let ξ be a generic point of Y over Q, so that $L = Q(\xi)$ coincides with $Q(Y)$. Let $Z = \delta^{-1}(\xi)$ be the generic fibre of δ. Then the field $Q(\mu(W_1))$ can be identified with $L(Z)$ and our goal is to show that Z and Y are rational varieties over L and Q respectively, if $n \leq 3$.

Let $\eta \in \tau^{-1}(\xi)$ and $M = \varphi^{-1}(\eta)$ (note that μ and M are defined over some extension of L). We claim that μ induces a bijection (and consequently, a $\overline{L}$-defined birational isomorphism) between M and Z (note, that in the case $g = 3$ the morphism μ induces a surjection, but not an injection between M and Z). Indeed, the components of $\eta \in \mathrm{GL}_n(\overline{L})^{g-3} \times \mathrm{SL}_n(\overline{L})$ generate the matrix algebra $M_n(\overline{L})$ over $\overline{L}$. So for any $m \in M$ the corresponding representation of Γ_g is irreducible and from $\mu(m_1) = \mu(m_2)$ for $m_1, m_2 \in M$ we may now conclude (see Proposition 6.2) that m_1 and m_2 determine equivalent representations of Γ_g, i.e. $m_2 = (\mathrm{Int}\, g)m_1$ for some $g \in \mathrm{GL}_n(\overline{L})$. Applying φ, we get $\eta = (\mathrm{Int}\, g)\eta$ and hence g is, in fact, a scalar matrix, i.e. $m_2 = m_1$, as required.

This fact implies, that Z is birationally $\overline{L}$-isomorphic to the commutator variety

$$W_b = \left\{(X, Y) \in \mathrm{GL}_n(\overline{L}) \mid [X, Y] = b\right\},$$

where $b = \eta_1^2 \ldots \eta_{g-3}^2 (\eta_{g-3}^{-1} \ldots \eta_1^{-1} \eta_{g-2})^2$ (here η_i are the components of η), and so Z is rational at least over $\overline{L}$ (see [14]).

However what we really want is the rationality over L, and then to overcome the fact that μ is not L-defined we have to replace GL_n by its suitable Galois twist G over L. Since η is defined over L, for any $\theta \in \mathrm{Gal}(\overline{L}/L)$ we have $\tau(\theta(\eta)) = \tau(\eta)$, hence $\theta(\eta) = \mathrm{Int}(a_\theta)(\eta)$ for some uniquely defined $a_\theta \in \mathrm{PSL}_n$, since μ is irreducible. The family $a = \{a_\theta\}$ determines a cocycle in $Z^1(L, \mathrm{PSL}_n)$ and we can introduce the corresponding twisted group $H = {}^a\mathrm{GL}_n$. Consider the diagram

$$\begin{array}{ccc} {}^aW_1 & \xrightarrow{\widetilde{\mu}} & {}^a\mu(W_1) = \mu(W_1) \\ {}^{\widetilde{\varphi}}\downarrow & & \downarrow^{\widetilde{\delta}=\delta} \\ H^{g-3} \times {}^a\mathrm{SL}_n(K) & \xrightarrow{\widetilde{\tau}} & {}^aY = Y \end{array} , \tag{6.2}$$

which is obtained by twisting (6.1) (note that twisting by inner automorphisms we do not change the corresponding character varieties).

Then η corresponds to a L-defined point (to be denoted by the same letter) in $\widetilde{\tau}^{-1}(\xi)$, and the same argument as above shows that the fibre $\delta^{-1}(\xi) = \widetilde{\delta}^{-1}(\xi)$ is birationally L-isomorphic to the fibre $\widetilde{\varphi}^{-1}(\eta)$. On the other hand, obviously, $\widetilde{\varphi}^{-1}(\eta)$ equals the following commutator variety in H in the generic position:

$$\widetilde{W}_b = \{(X, Y) \in H \times H \mid [X, Y] = b\}$$

with the same b as above. It is well-known that $H = {}^a\mathrm{GL}_n$ equals $\mathrm{GL}_1(D)$ for some simple central algebra D over L of dimension n^2, so it remains to be proven that for $n \leq 3$ there exists a Zariski L-open set $U \subset {}^a\mathrm{SL}_n = \mathrm{SL}_1(D)$ with the following property: for any extension E/L and any $z \in U(E)$ the commutator variety $\widetilde{W}_z$ is rational over E.

To this end, let us introduce the functions $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ on D which are similar to the functions $\sigma_1, \ldots, \sigma_n$ used in § 4, and for $z \in \mathrm{SL}_1(D)$ define the corresponding variety

$$\widetilde{T}_z = \{y \in H \mid \widetilde{\sigma}_i(y) = \widetilde{\sigma}_i(zy), \quad i = 1, \ldots, n-1\}.$$

It is well-known that $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ are regular functions on D, satisfying the following

(i) $\widetilde{\sigma}_i$ is given by a homogeneous L-defined polynomial of degree i;

(ii) if C is an algebraically closed field containing L and $\chi : D \otimes_L C \to \mathrm{M}_n(C)$ is an isomorphism of C-algebras (which always exists, see [12]) then $\widetilde{\sigma}_i(d) = \sigma_i(\chi(d))$ for any $d \in D$.

We claim that if $n \leq 3$ then for "generic" $z \in \mathrm{SL}_1(D)$ the variety $\widetilde{T}_z$ is irreducible and rational over the field of definition. Indeed, if $n = 2$ then $\widetilde{T}_z$ is defined by a single linear equation, and our claim follows immediately. For $n = 3$ we have two conditions, one being linear and another quadratic in the coefficients of y. Solving the linear equation for one of the variables and plugging this expression into the quadratic equation, we see that $\widetilde{T}_z$ is actually a quadric. Therefore, to establish its rationality it suffices to find on it a rational L-point.

Let $F \subset D$ be a maximal subfield in D and C be an algebraically closed field containing F. Since F splits D, an isomorphism χ in (ii) can be found over F. Then $\widetilde{T}_z$ becomes F-isomorphic to

$$T_{\chi(z)} = \{y \in \mathrm{GL}_n(C) \mid \sigma_i(y) = \sigma_i(\chi(z)y), \quad i = 1, \ldots, n-1\}$$

and since by virtue of Thompson's theorem (see [18]) $W_{\chi(z)}(F) \neq \emptyset$ we may conclude that $T_{\chi(z)}(F) \neq \emptyset$. This means that the quadric equation defining $\widetilde{T}_z(F)$ has a rational point over an extension F of L of degree 3. But then by Springer's theorem (see [8]) it must have a point over L, proving our claim.

Now it is easy to complete the proof of the rationality of the commutator variety $\widetilde{W}_z$. Namely, for fixed $z \in \mathrm{SL}_1(D)$ consider a natural map

$$\nu : \widetilde{W}_z \to \widetilde{T}_z, \qquad (x, y) \mapsto y.$$

Then the fibre $\nu^{-1}(y)$ equals the variety of x's satisfying $xy = (zy)x$ which amounts to a system of linear equations for the coefficients of x with respect to a basis of D over L; in particular, if $\nu^{-1}(y) \neq \emptyset$ then $\nu^{-1}(y)$ is clearly rational over $L(z, y)$. Applying this to the fibre of ν over the generic point of $\widetilde{T}_z$ and taking into account the irreducibility of $\widetilde{W}_z$ (see [14]) we obtain the rationality of $\widetilde{W}_z$.

We now turn to the variety Y. It follows from [13, 6] that the categorical quotient $\mathrm{GL}_n(K)^{g-2}/\mathrm{GL}_n(K)$ is a $\mathbb{Q}$-rational variety if $n \leq 4$. Moreover, there exists a transcendental basis $u_1, \dots, u_t$ of the field $\mathbb{Q}(\mathrm{GL}_n(K)^{g-2}/\mathrm{GL}_n(K))$ over $\mathbb{Q}$ such that $u_1 = \det X_{g-2}$. Then since the variety $\mathrm{GL}_n(K)^{g-3} \times \mathrm{SL}_n(K)$ is defined in $\mathrm{GL}_n(K)^{g-3} \times \mathrm{GL}_n(K)$ by a single equation $\det(X_{g-2}) = 1$ we obtain that Y is also rational over $\mathbb{Q}$.

The argument for W_2 is entirely analogous and so we omit the proof.

3) In the notation of Proposition 4.4 we have three irreducible components S_1, S_2, V of the variety $R_2(\Gamma_3)$.

Recall that V coincides with the closure of

$$\left\{ X \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} X^{-1}, Y \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} Y^{-1},\ Z \begin{pmatrix} ab & 0 \\ 0 & -ab \end{pmatrix} Z^{-1} \right| \\ a, b \in K^*,\ X, Y, Z \in \mathrm{GL}_2(K) \Big\} .$$

Put

$$U = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & 1 \\ c & -b \end{pmatrix}, \begin{pmatrix} d & e \\ y & -d \end{pmatrix} \right\} \subset V,$$

where $y = (a^2(b^2 + c) - d^2)/e$, and let $\mu : U \to V/\mathrm{GL}_2(K)$ be the natural projection. Straightforward calculations show that the images of two points

$$\begin{aligned} u_1 &= \left(\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & 1 \\ c & -b \end{pmatrix}, \begin{pmatrix} d & e \\ y & -d \end{pmatrix} \right), \\ u_2 &= \left(\begin{pmatrix} \widetilde{a} & 0 \\ 0 & -\widetilde{a} \end{pmatrix}, \begin{pmatrix} \widetilde{b} & 1 \\ \widetilde{c} & -\widetilde{b} \end{pmatrix}, \begin{pmatrix} \widetilde{d} & \widetilde{e} \\ \widetilde{y} & -\widetilde{d} \end{pmatrix} \right) \end{aligned}$$

which are in a "generic" position, coincide iff

$$\widetilde{a} = -a, \quad \widetilde{b} = -b, \quad \widetilde{d} = -d, \quad \widetilde{c} = c, \quad \widetilde{e} = \frac{a^2(b^2 + c) - d^2}{ce}.$$

Since in addition we have $\dim U = \dim V/\mathrm{GL}_2(K)$ it follows that $\mathbb{Q}(V/\mathrm{GL}_2(K))$ can be identified with the invariant subfield of $\mathbb{Q}(a, b, c, d, e)$ with respect to the automorphism φ of order 2, mapping a, b, c, d, e into $-a$, $-b$, c, $-d$, $(a^2(b^2 + c) - d^2)/(ce)$ respectively (here a, b, c, d, e are algebraically independent variables over $\mathbb{Q}$).

Consider the rational functions

$$\begin{aligned} \overline{a} &= a(e - \frac{a^2(b^2 + c) - d^2}{ce}), & \overline{b} &= b(e - \frac{a^2(b^2 + c) - d^2}{ce}), & \overline{c} &= c, \\ \overline{d} &= d(e - \frac{a^2(b^2 + c) - d^2}{ce}), & \overline{e} &= e + \frac{a^2(b^2 + c) - d^2}{ce}. \end{aligned}$$

Obviously, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$ lies in $\mathbb{Q}(a,b,c,d,e)^{<\varphi>}$ and generate a subfield of $\mathbb{Q}(a,b,c,d,e)$ of degree 2. This means that $\mathbb{Q}(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e}) = \mathbb{Q}(a,b,c,d,e)^{<\varphi>}$, implying the rationality of V.

Finally, consider the categorical quotient $S_1/\mathrm{GL}_2(K)$. In § 4 we proved that

$$\psi : S_1 \to \mathbb{A}^1 \times \mathrm{GL}_2(K)^2, \qquad (X,Y,Z) \mapsto (\operatorname{tr} X, Y, Z)$$

is birational isomorphism between S_1 and the subvariety $W \subset \mathbb{A}^1 \times \mathrm{GL}_2(K)^2$ defined by a single equation

$$t^2 - \operatorname{tr}(Y^2Z^2) - 2\det(YZ) = 0. \tag{6.3}$$

Define the action of $\mathrm{GL}_2(K)$ on W as follows: $A \cdot (t,Y,Z) = (t, AYA^{-1}, AZA^{-1})$, where $A \in \mathrm{GL}_2(K)$ and $(t,Y,Z) \in W$. Then it is easy to see that ψ induces a birational isomorphism between $S_1/\mathrm{GL}_2(K)$ and $W/\mathrm{GL}_2(K)$. Clearly, $W/\mathrm{GL}_2(K)$ can be considered as a subvariety in $\mathbb{A}^1 \times (\mathrm{GL}_2(K)^2/\mathrm{GL}_2(K))$ defined by (6.3). Since

$$\operatorname{tr}(Y^2Z^2) = \operatorname{tr} Y \operatorname{tr} Z \operatorname{tr}(YZ) - (\operatorname{tr} Y)^2 \det Z - (\operatorname{tr} Z)^2 \det Y + 2\det(YZ) \tag{6.4}$$

(6.3) can be rewritten in the form

$$t^2 - \operatorname{tr} Y \operatorname{tr} Z \operatorname{tr}(YZ) + (\operatorname{tr} Y)^2 \det Z + (\operatorname{tr} Z)^2 \det Y + 2\det Y \det Z = 0 \tag{6.5}$$

Since in addition

$$\mathbb{Q}(\mathrm{GL}_2(K)^2/\mathrm{GL}_2(K)) = \mathbb{Q}(\operatorname{tr} Y, \det Y, \operatorname{tr} Z, \det Z, \operatorname{tr}(YZ)).$$

(see [6]) and equation (6.4) is linear with respect to $\operatorname{tr}(YZ)$ we finally obtain that $\mathbb{Q}(W/G) = \mathbb{Q}(t, \operatorname{tr} Y, \det Y, \operatorname{tr} Z, \det Z)$, i.e. $\mathbb{Q}(W/G)$ is a purely transcendental extension of $\mathbb{Q}$, as required.

The case of S_2 can be handled analogously. Proposition 6.4 is proved. □

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