On uniqueness of invariant measures for finite and infinite dimensional diffusions

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Abstract

We prove uniqueness of "invariant measures", i.e., solutions to the equation $L^*\mu = 0$ where $L = \Delta + B \cdot \nabla$ on $\mathbb{R}^n$ with B satisfying some mild integrability conditions and μ is a probability measure on $\mathbb{R}^n$. This solves an open problem posed by S.R.S. Varadhan in 1980. The same conditions are shown to imply that the closure of L on $L^1(\mu)$ generates a strongly continuous semigroup having μ as its unique invariant measure. The question whether an extension of L generates a strongly continuous semigroup on $L^1(\mu)$ and whether such an extension is unique is addressed separately and answered positively under even weaker local integrability conditions on B. The special case when B is a gradient of a function (i.e., the "symmetric case") is in particular studied and conditions are identified ensuring that $L^*\mu = 0$ implies that L is symmetric on $L^2(\mu)$ resp. $L^*\mu = 0$ has a unique solution. We also prove infinite dimensional analogues of the latter two results and a new elliptic regularity theorem for invariant measures in infinite dimensions.

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1 Introduction and main results

This paper is concerned with the problem which was raised by S.R.S. Varadhan in [31] of uniqueness of invariant measures for diffusions with singular drifts on a finite or infinite dimensional state space X. More precisely, if $X = \mathbb{R}^n$, these diffusions are weak solutions of stochastic differential equations of type

$$dX_t = dW_t + B(X_t)dt \tag{1.1}$$

where $(W_t)_{t\geq 0}$ denotes Brownian motion on $\mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ is merely Borel-measurable. We emphasize, however, that many of our results also hold in case a smooth strictly positive definite diffusion matrix $\sigma(X_t)$ replaces the identity in front of dW_t in (1.1).

Since we do not want to impose any a-priori conditions that ensure the existence of a solution to (1.1) resp. a corresponding semigroup describing its transition probabilities, we formulate our problem purely analytically, i.e., in terms of the underlying generator $\Delta + B \cdot \nabla$, where Δ denotes the Laplacian on $\mathbb{R}^n$. Let us do so precisely, first sticking to the finite dimensional case $X = \mathbb{R}^n$.

For a positive Radon measure μ on a topological space X let $L^p(\mu)$ ((:= $L^p(X;\mu)$) resp. $L^p_{loc}(\mu)$ ((:= $L^p_{loc}(X;\mu)$), $p \geq 1$, denote the corresponding spaces of p-integrable resp. locally p-integrable real-valued functions on X. As usual for $p \geq 1$, $r > 0$, $H^{p,r}(\mathbb{R}^n)$ denotes the Sobolev space of all functions on $\mathbb{R}^n$ with generalized derivatives up to order r in $L^p(dx)$, where dx denotes Lebesgue measure on $\mathbb{R}^n$. $H^{p,r}_{loc}(\mathbb{R}^n)$ denotes the corresponding local Sobolev space, i.e., $f \in H^{p,r}_{loc}(\mathbb{R}^n)$ if $\zeta f \in H^{p,r}(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(\mathbb{R}^n)$.

Set

$$Lf := \Delta f + \langle B, \nabla f\rangle \quad , \ f \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$. $C_0^\infty(\mathbb{R}^n)$ is the set of all real-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact support. For a (signed) Radon measure μ on $\mathbb{R}^n$ of finite total variation we write

$$L^*\mu = 0\ , \tag{1.3}$$

if the following two conditions are satisfied:

$$|B| \in L^1_{loc}(|\mu|) \tag{1.4}$$

and

$$\int Lf\, d\mu = 0 \text{ for all } f \in C_0^\infty(\mathbb{R}^n)\ . \tag{1.5}$$

Here $|\ | := \langle\ ,\ \rangle^{1/2}$ and $|\mu| := \mu^+ + \mu^-$, where μ^+, μ^- denote the positive and negative part of μ respectively. Note that condition (1.4) is necessary for (1.5) to make sense.

Remark 1.1 (i) Suppose that the closure of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^1(\mu)$ generates a strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$. Then (1.3) implies that μ is (T_t^μ)*–invariant*, i.e., for all $t > 0$

$$\int T_t^\mu f\ d\mu = \int f\ d\mu \quad \text{for all } f \in L^\infty(\mu)$$

(cf. Proposition 2.6 (ii) below).

(ii) Since $|\mu|(\mathbb{R}^n) < \infty$, it is an easy exercise to show that in (1.5) $C_0^\infty(\mathbb{R}^n)$ can be replaced by $C_b^2(\mathbb{R}^n)$ (i.e., the set of all twice differentiable functions on $\mathbb{R}^n$ with all partial derivatives up to order two bounded) provided $|B| \in L^1(|\mu|)$.

Let us denote the set of all probability measures on $\mathbb{R}^n$ satisfying the elliptic equation (1.3) by $\mathcal{M}_{ell}$. So, our aim is to give sufficient conditions on the vector field B that ensure that $\#\mathcal{M}_{ell} \le 1$. For the best possible results it is necessary to separate the issue of uniqueness from that of existence. Therefore, the question of existence, i.e., whether $\mathcal{M}_{ell} \neq \emptyset$ is treated in a separate paper (cf. [12] and for the infinite dimensional case also [11, Section 5]). Let us restate the most relevant result from [12] here:

Theorem (cf. [12, Theorem 1.6]) *Suppose that $A = (A^{ij})$ is a mapping on $\mathbb{R}^n$ with values in the nonnegative symmetric linear operators on $\mathbb{R}^n$ and let*

B be a Borel mapping on $\mathbb{R}^n$ such that, for every ball U_R of radius R there exist numbers $\alpha_R > n$ and $\beta_R > 0$ such that $\beta_R I \le A(x)$ for all $x \in U_R$ and $A^{ij}|_{U_R} \in H^{\alpha_R,1}(U_R)$, $\left|B|_{U_R}\right| \in L^{\alpha_R}(U_R)$. Assume, in addition, that there exists a function $V \in C^2(\mathbb{R}^n)$ such that

$$V(x) \to +\infty \quad \textit{and} \quad L_A V(x) \to -\infty \quad \textit{as } |x| \to \infty \tag{1.6}$$

where $L_A f := \mathrm{trace}\,(Af'') + (B, \nabla f)$. Then there exists a probability measure μ such that

$$|B| \in L^1_{loc}(\mu) \quad \textit{and} \quad L_A^* \mu = 0$$

in the sense of (1.3). Moreover, μ admits a continuous strictly positive density p w.r.t. Lebesgue measure such that $p|_{U_{R-1}} \in H^{\alpha_R,1}(U_{R-1})$ for every $R > 1$.

Conditions ensuring the existence of (the Lyapunov–type) functions V in the above theorem are well–known (cf. e.g. [12, Proof of Corollary 1.4]).

Of course, it can happen that $\mathcal{M}_{ell} = \emptyset$ (e.g. if $B \equiv 0$). But there are also other types of conditions ensuring $\mathcal{M}_{ell} \neq \emptyset$. For example, if $B = \frac{\nabla\rho}{\rho} + v$ where v is bounded and ρ is a strictly positive lower semicontinuous function such that $\mathcal{E}_\rho(f,g) := \int \langle \nabla f, \nabla g \rangle_{\mathbb{R}^n}\, \rho\, dx$, $f, g \in C_0^\infty(\mathbb{R}^n)$, satisfies the Logarithmic Sobolev inequality (cf. [18]), then $\mathcal{M}_{ell} \neq \emptyset$. For the underlying general results, which also hold in infinite dimensions, we refer to [13, Theorems 3.4 and 3.6] for details.

By [10, Theorem 1(ii)] we know that each $\mu \in \mathcal{M}_{ell}$ has a density, say ρ, w.r.t. Lebesgue measure dx on $\mathbb{R}^n$ and that $\rho \in H^{1,1}_{loc}(\mathbb{R}^n)$ provided $|B| \in L^p_{loc}(\mu)$ for some $p > 1$ (cf. Theorem 2.5 below for details).

Our main result concerning $\#\mathcal{M}_{ell} \le 1$ is the following:

Theorem 1.2 *Suppose $|B| \in L^p_{loc}(dx)$ for some $p > n \ge 2$. Then there exists at most one $\mu \in \mathcal{M}_{ell}$ such that its density ρ w.r.t. Lebesgue measure has the property that*

$$|B\rho - \nabla\rho| \in L^1(dx)\ . \tag{1.7}$$

The local integrability condition $|B| \in L^p_{loc}(dx)$, $p > n$, is crucial for our proof. However, in the "symmetric case" (i.e., the case where B is a gradient) this condition can be drastically weakened (see Theorem 1.7(ii)). Whether the local condition on $|B|$ is sharp for general B satisfying (1.6), we do not know at present. We have examples that $\#\mathcal{M}_{ell} > 1$ only if $|B| \notin L^1_{loc}(dx)$ (see [11, Example 6.1]).

Before we proceed, let us comment on condition (1.7). As follows from the next theorem, on which the proof of Theorem 1.2 relies in an essential way, condition (1.7) specifies a convex subset of measures μ in $\mathcal{M}_{ell}$ having the following property:

(G) The closure of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^1(\mu)$ generates a strongly continuous semigroup on $L^1(\mu)$.

In fact, the proof of Theorem 1.2, presented in this paper, proves the uniqueness of $\mu \in \mathcal{M}_{ell}$ in the largest convex subset consisting of all measures $\mu \in \mathcal{M}_{ell}$ satisfying (G). Unfortunately, so far we cannot prove that the set of all $\mu \in \mathcal{M}_{ell}$ with (G) is convex. In any case (G) and hence (1.7) are reasonable assumptions, in particular, if one is interested in invariant measures rather than merely in solutions of (1.3) (cf. Remark 1.1 (i)).

Theorem 1.3 *Suppose that $|B| \in L^p_{loc}(dx)$ for some $p > n \geq 2$, and let $\mu \in \mathcal{M}_{ell}$ with $\rho := \frac{d\mu}{dx}$ satisfying (1.7). Then:*

(i) There exists exactly one strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$ such that its generator $(L_\mu, D(L_\mu))$ extends L, i.e., we have that $C_0^\infty(\mathbb{R}^n) \subset D(L_\mu)$ and $L_\mu = L$ on $C_0^\infty(\mathbb{R}^n)$. Indeed, $(L_\mu, D(L_\mu))$ is the closure of $(L, C_0^\infty(\mathbb{R}^n)))$ on $L^1(\mu)$.

(ii) $(T_t^\mu)_{t>0}$ is Markovian *(i.e., for all $t > 0$, $T_t^\mu 1 = 1$ and $T_t^\mu f \geq 0$ provided $f \geq 0$).*

(iii) If $(G_\alpha^\mu)_{\alpha>0}$ denotes the corresponding resolvent, then $G_\alpha^\mu f$ has a (unique) continuous μ–version for all $f \in L^{np/(n+p)}(\mu)$ and all $\alpha > 0$. In particular, $(G_\alpha^\mu)_{\alpha>0}$ is strongly Feller *(i.e., $G_\alpha^\mu f$ has a continuous μ–version for all $f \in L^\infty(\mu)$, $\alpha > 0$).*

(iv) $T_t^\mu f$ has a (unique) continuous μ–version $\widetilde{T_t^\mu f}$ for every $f \in C_0^\infty(\mathbb{R}^n)$, $t > 0$, and μ is the only probability measure ν on $\mathbb{R}^n$ such that

$$\int \widetilde{T_t^\mu f}\, d\nu = \int f\, d\nu \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n) \text{ and all } t > 0. \tag{1.8}$$

Important ingredients of the proof of Theorem 1.3, which will be given in Section 3 below, are the new regularity results in [10] and a classical result by N.S. Trudinger in [29] on the Harnack inequality for operators of type (1.2). The latter result ensures that ρ (which is continuous by [10, Theorem 1(iii)]) is, in fact, strictly positive. Furthermore, the technical (but important) Corollary 2.3 is used.

As already mentioned, the proof of Theorem 1.2 uses that of Theorem 1.3. One reason for this is that without having a corresponding semigroup we cannot prove that the set of all solutions of (1.3) or, more precisely, their densities w.r.t. Lebesgue measure, has lattice structure. So, it is both interesting and important to study the question whether (an extension of) L generates a semigroup, as a problem of its own. Concerning this, (in addition to Theorem 1.3) there are the following results, where partly the local integrability condition on B in Theorem 1.3 is relaxed. However, we need a stronger global one (i.e., condition (1.9) instead of (1.7)). We first recall that a Markovian strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$ can be restricted to a strongly continuous semigroup on $L^p(\mu)$ for all $p > 1$ (cf. [23]).

Theorem 1.4 *(i) Suppose $\mu \in \mathcal{M}_{ell}$ such that $|B| \in L^2_{loc}(\mu)$ and such that its density ρ w.r.t. dx has the property that*

$$|B - \nabla\rho/\rho| \in L^2(\mu) \tag{1.9}$$

(where as usual $|\nabla\rho|/\rho := 0$ on $\{\rho = 0\}$). Then there exists a strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$ with generator $(L_\mu, D(L_\mu))$ extending L. Furthermore, $(T_t^\mu)_{t>0}$ is Markovian and μ is $(T_t^\mu)_{t>0}$–invariant.

(ii) Suppose $\mu \in \mathcal{M}_{ell}$ such that $|B| \in L^2(\mu)$. Then for $\rho := \frac{d\mu}{dx}$ condition (1.9) holds and the semigroup $(T_t^\mu)_{t>0}$ in (i) is the only strongly continuous semigroup on $L^1(\mu)$ whose generator $(L_\mu, D(L_\mu))$ extends L.

(iii) Suppose $\mu \in \mathcal{M}_{ell}$ such that $|B| \in L^p_{loc}(dx)$ for some $p > n \geq 2$ such that its density ρ w.r.t. dx satisfies

$$|B - \nabla\rho/\rho| \in L^q(\mu) \quad \text{for some } q \in [1, \infty]. \tag{1.10}$$

Let $r := 2 - \frac{2}{q+1}$ (where $\frac{1}{\infty} := 0$). Then the restriction of $(T_t^\mu)_{t>0}$ to $L^r(\mu)$ is the unique strongly continuous semigroup on $L^r(\mu)$ whose generator extends L. This generator is, in fact, the closure of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^r(\mu)$.

Part (i) of Theorem 1.4 is not due to us and has only been added for completness. It is a special case of a general result due to W. Stannat (cf. [27, Theorem 1.1] and also [28] where the moment condition in [27] is dropped and where proofs are purely analytic). The proof of (iii) is similar to that of the corresponding part of Theorem 1.3. The proof of Theorem 1.4 (ii) is a consequence of the following result.

Proposition 1.5 *Suppose $\mu_i \in \mathcal{M}_{ell}$, $i = 1, 2$, such that μ_2 is absolutely continuous w.r.t. μ_1 and $|B|$ is μ_i–square integrable for both $i = 1$ and $i = 2$. Let $(T_t^{\mu_i})_{t>0}$ denote strongly continuous semigroups on $L^1(\mu_i)$, whose generators $(L_{\mu_i}, D(L_{\mu_i}))$ extend L, $i = 1, 2$. (They exist according to Theorem 1.4 (i)). Then $T_t^{\mu_1} f = T_t^{\mu_2} f$ μ_2–a.e. for all $t > 0$ and all bounded Borel-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$.*

The above proposition can be proved by a modification of an argument of V.I. Liskevich and Yu. A. Semenov in [21] based on Duhamel's formula.

Remark 1.6 (i) We would like to point out that all strongly continuous semigroups $(T_t^{\mu})_{t>0}$ appearing above are automatically contractions on all $L^p(\mu)$, since they are sub–Markovian and since μ is T_t^{μ}–invariant.

(ii) In the situation of Theorem 1.4 (i) there exists a "nice" Markov process on $\mathbb{R}^n$ whose transition probabilities are given by $(T_t^{\mu})_{t>0}$ and which is a weak solution of (1.1). This follows again from [27, 28] to which we refer for details and the precise definitions.

We conclude our discussion in finite dimensions by considering the *symmetric case*, i.e., the case where B is a gradient. More precisely,

$$B \text{ is a (fixed) Borel–measurable } dx\text{–version of } \frac{\nabla \rho_0}{\rho_0} \tag{1.11}$$

for some $\rho_0 \in H_{loc}^{1,1}(\mathbb{R}^n)$, $\rho_0 \geq 0$, $\int \rho_0 dx = 1$, where as before $\frac{\nabla \rho_0}{\rho_0} := 0$ on $\{\rho_0 = 0\}$. Then obviously, $\mu_0 := \rho_0 \, dx \in \mathcal{M}_{ell}$. We recall that B in (1.11) is called the *logarithmic derivative* (or *logarithmic gradient*) of μ_0 and is usually denoted by β^{μ_0}. Obviously, $|B| \in L_{loc}^1(\mu_0)$.

In this case concerning the question whether $\#\mathcal{M}_{ell} \leq 1$, we have the following result with a much weaker local condition on B than in Theorem 1.2, but again we need the global condition (1.9).

Theorem 1.7 *Let B be as in* (1.11) *and suppose that $|B| \in L^2_{loc}(\mu_0)$.*
(i) Let $\mu \in \mathcal{M}_{ell}$ with $|B| \in L^2_{loc}(\mu)$ such that $\rho := \frac{d\mu}{dx}$ satisfies (1.9)*. Then*

$$\frac{\nabla \rho}{\rho} = B \quad \mu\text{–a.e.}$$

and $(L, C_0^\infty(\mathbb{R}^n))$ is symmetric on $L^2(\mu)$.
(ii) If $|B| \in L^1_{loc}(U; dx)$ for some connected open subset U of $\mathbb{R}^n$ such that $dx(\mathbb{R}^n \setminus U) = 0$, then μ_0 is the only measure $\mu \in \mathcal{M}_{ell}$ with $|B| \in L^2_{loc}(\mu)$ and $\rho := \frac{d\mu}{dx}$ satisfying (1.9)*.*

The proof of Theorem 1.7 and that of its infinite–dimensional analogue Theorem 4.5 (as well as Theorem 4.3) below depend in an essential way on the results on "Markov uniqueness" in [24], [25], [7].

Remark 1.8 If we call any $\mu \in \mathcal{M}_{ell}$ an *invariant measure* (having in mind the notion of invariant measure for a solution to (1.1) resp. Remark 1.1 (i), then Theorem 1.7 (i) gives conditions implying that invariant measures are symmetrizing.

Concerning the existence of corresponding semigroups in the present symmetric case we note that for $B = \frac{\nabla \rho_0}{\rho_0}$ as in (1.11) and for $\rho := \rho_0$, $\mu := \rho_0\, dx$ condition (1.9) trivially holds, so Theorem 1.4 (i) applies provided $|B| \in L^2_{loc}(\mu)$. However, in the present symmetric case the latter restriction is unnecessary.

Proposition 1.9 *Let $B = \frac{\nabla \rho_0}{\rho_0}$ be as in* (1.11) *and $\mu_0 := \rho_0\, dx$. Then there exists a strongly continuous Markovian semigroup $(T_t^{\mu_0})_{t>0}$ on $L^1(\mu_0)$ whose generator $(L_{\mu_0}, D(L_{\mu_0}))$ extends L and which has μ_0 as a $(T_t^{\mu_0})_{t>0}$–invariant measure. Furthermore, the restriction of $(T_t^{\mu_0})_{t>0}$ to $L^2(\mu_0)$ consists of symmetric operators.*

Remark 1.10 (i) The *uniqueness* of $(T_t^{\mu_0})_{t>0}$ in Proposition 1.9 is ensured by Theorem 1.4 (ii) provided $|B| \in L^2(\mu_0)$. By [25, Theorem 3.1], however, we can relax the latter condition to $|B| \in L^2_{loc}(\mu_0)$ and still have uniqueness of $(T_t^{\mu_0})_{t>0}$ among all strongly continuous *symmetric* Markovian semigroups on $L^2(\mu_0)$ whose generators extend L. We also note that in the present symmetric case we can take $r = 2$ in Theorem 1.4(iii). This was, however, already known from [10, Corollary 8]).

(ii) In the situation of Proposition 1.9 it follows by its proof and the general theory in [16], [22] that there exists a diffusion process on $\mathbb{R}^n$ whose transition probabilities are given by $(T_t^{\mu_0})_{t>0}$.

Detailed proofs of all the above results are contained in Section 3 below, after we prove resp. recall some auxiliary results in Section 2 as preparation.

In Section 4 we turn to the infinite dimensional case. We prove an infinite dimensional version of Theorem 1.7, in particular, gaining a sufficient condition ensuring that invariant measures are symmetrizing (see Theorem 4.3 below). Furthermore, we extend one of our main results in [11] (namely, Theorem 3.5), i.e., an elliptic regularity result in infinite dimensions to the case where the reference measure is no longer Gaussian, but still satisfies a logarithmic Sobolov inequality and some approximation property (see Theorem 4.3 below).

Finally, in the Appendix for the convenience of the reader we provide some simple proofs for characterizations of extremal invariant resp. symmetrizing measures related to the results of this paper and also used in the final Remark 4.6.

2 Auxiliary results

In this section we prove resp. recall some results to be used later.

2.1 Some generalities

In this subsection let $(X, \mathcal{B}, \mu)$ be a probability space and let $L^p(\mu) := L^p(X, \mu)$, $p \geq 1$, denote the corresponding real L^p–spaces. Recall that a bounded linear operator T on $L^p(\mu)$ is called *positivity preserving* if $Tf \geq 0$ provided $f \geq 0$, and *sub–Markovian* if $0 \leq Tf \leq 1$ provided $0 \leq f \leq 1$.

Lemma 2.1 *Let μ be a probability measure on $(X, \mathcal{B})$ and S a positivity preserving bounded linear operator on $L^2(\mu)$. Let $g \in L^\infty(\mu)$, $g \geq 0$, be such that for the dual S^* of S on $L^2(\mu)$, $S^*1 \leq g$. Then S extends to a bounded linear operator on $L^1(\mu)$ again denoted by S. Let*

$$V := \{f \in L^1(\mu) \mid Sf = g \cdot f\}.$$

Then V is a vector lattice.

Proof. For all $f \in L^2(\mu)$

$$\int |Sf|\, d\mu \le \int S|f|\, d\mu = \int S^*1\, |f|\, d\mu \le \|g\|_\infty \int |f|\, d\mu.$$

Hence, S extends to $L^1(\mu)$. Obviously, V is a linear space. So, it suffices to show that $f \in V \Longrightarrow f^+ \in V$. But if $f \in V$, then $Sf^+ \ge Sf = gf$ and $Sf^+ \ge 0$, hence, $Sf^+ \ge (gf)^+ = gf^+$. Now the claim follows, since

$$0 \le \int (Sf^+ - gf^+)\, d\mu = \int S^*1\, f^+\, d\mu - \int gf^+\, d\mu \le 0.$$

$\square$

Proposition 2.2 *Let T be a sub–Markovian, bounded linear operator on $L^1(\mu)$ such that μ is T–*invariant*, i.e.,*

$$\int Tf\, d\mu = \int f\, d\mu \quad \textit{for all } f \in L^\infty(\mu).$$

Suppose $\rho_1,\ \rho_2 \in L^1(\mu)$ such that $\rho_1 \cdot \mu$ and $\rho_2 \cdot \mu$ are T–invariant. then $(\rho_1 \wedge \rho_2) \cdot \mu$ is T–invariant.

Proof. Define for $\rho \in L^1(\mu)$ and $n \in \mathbb{N}$

$$\rho^{(n)} := (\rho \wedge n) \vee (-n)\ .$$

Clearly, the restriction of T to $L^2(\mu)$ (again denoted by T) is bounded and its dual T^* on $L^2(\mu)$ is again sub–Markovian (cf. [22, Chap. II, Proposition 4.1]), hence extends to a bounded linear operator on $L^1(\mu)$ again denoted by T^*. Furthermore, if $\rho \in L^1(\mu)$, then $\rho \cdot \mu$ is T–invariant if and only if $T^*\rho = \rho$, since for all $f \in L^\infty(\mu)$

$$\begin{aligned} \int Tf\ \rho\, d\mu &= \lim_{n\to\infty} \int Tf\ \rho^{(n)}\, d\mu \\ &= \lim_{n\to\infty} \int f\, T^*\rho^{(n)} d\mu \\ &= \int f\, T^*\rho\ d\mu\ . \end{aligned}$$

Therefore, Lemma 2.1 applies and the assertion follows. $\square$

Corollary 2.3 *Let T_t, $t > 0$, be sub–Markovian bounded linear operators on $L^1(\mu)$ such that μ is (T_t)–invariant (i.e., T_t–invariant for all $t > 0$). Define $\mathcal{M}_\mu$ to be the set of all probability measures on $(X, \mathcal{B})$ which are absolutely continuous w.r.t. μ and (T_t)–invariant. Let*

$$\mathcal{M}_{\mu,b} := \{\nu \in \mathcal{M}_\mu \mid \frac{d\nu}{d\mu} \textit{ is bounded}\},$$
$$\mathcal{M}_{\mu,>0} := \{\nu \in \mathcal{M}_\mu \mid \frac{d\nu}{d\mu} \textit{ is strictly positive}\}.$$

Then the following assertions are equivalent:

(i) $\mathcal{M}_\mu = \{\mu\}$.

(ii) $\mathcal{M}_\mu = \mathcal{M}_{\mu,>0}$.

(iii) $\mathcal{M}_{\mu,>0} = \{\mu\}$.

(iv) $\mathcal{M}_{\mu,b} \subset \mathcal{M}_{\mu,>0}$.

Proof. Obviously, (i) implies (ii) and (iv).
(ii)$\Rightarrow$ (iii): Let $g \in L^1(\mu)$, $g > 0$, such that $g \cdot \mu \in \mathcal{M}_{\mu,>0}$. Then $(1-g)^+\mu$, $(1-g)^-\mu$ are both (T_t)–invariant by Proposition 2.2. Hence by assumption (ii), $(1-g)^+$ or $(1-g)^-$ is identically zero. In either case it follows that $g \equiv 1$ since $g \cdot \mu$ is a probability measure, hence $\mathcal{M}_{\mu,>0} = \{\mu\}$.

(iii)$\Rightarrow$(i): Let $\nu = g \cdot \mu \in \mathcal{M}_\mu$ and define

$$\nu_1 := \frac{1}{2}\mu + \frac{1}{2}\nu \ (\in \mathcal{M}_\mu, \text{ since } \mathcal{M}_\mu \text{ is convex}).$$

But then $\nu_1 = (\frac{1}{2} + \frac{1}{2}g) \cdot \mu$, hence $\nu_1 \in \mathcal{M}_{\mu,>0}$. Therefore, assumption (iii) implies that $\nu_1 = \mu$. Consequently, $g \equiv 1$, i.e., $\nu = \mu$.

It remains to prove (iv)$\Rightarrow$(ii): Let $\nu = g \cdot \mu \in \mathcal{M}_\mu$. Then by Proposition 2.2 for all $n \in \mathbb{N}$ (n so large that $\int g \wedge n \, d\mu \neq 0$)

$$\left(\int (g \wedge n) d\mu\right)^{-1} \cdot (g \wedge n) \cdot \mu \in \mathcal{M}_{\mu,b}.$$

Hence assumption (iv) implies that $g \wedge n > 0$, hence $g > 0$, i.e., $\nu \in \mathcal{M}_{\mu,>0}$.
$\square$

Remark 2.4 (i) In the appendix of this paper we shall prove that in Corollary 2.3 assertions (i) – (iv) are in addition equivalent to:

(v) If $f \in L^1(\mu)$ such that $T_t f = f$ for all $t > 0$, then $f = 0$ const.

(ii) We note that since above each T_t is sub–Markovian and μ is (T_t)–invariant, it follows automatically that $T_t 1 = 1$ for each $t > 0$.

(iii) Since $(T_t)_{t>0}$ in Corollary 2.3 is a semigroup of operators on $L^1(\mu)$, the notion of (T_t)–invariance only makes sense for measures absolutely continuous w.r.t. μ, unless $T_t f$, $t > 0$, has a unique continuous μ–version for sufficiently many f as in Theorem 1.3 (iv). Therefore, in general one has to consider the set $\mathcal{M}_\mu$ defined above.

2.2 Preliminary results for $X = I\!R^n$

In this subsection let $X := \mathbb{R}^n$ and $\mathcal{B}$ denotes its Borel σ–algebra. We start with summarizing a few known results to be used below in the following theorem.

Theorem 2.5 *Let $n \geq 2$ and let μ be a (signed) Radon measure on $\mathbb{R}^n$. Let $B : \mathbb{R}^n \to \mathbb{R}^n$, $c : \mathbb{R}^n \to \mathbb{R}$ be maps such that $|B|$, $c \in L^1_{loc}(|\mu|)$. Assume that*

$$\int Lf \; d\mu = 0 \text{ for all } f \in C_0^\infty(\mathbb{R}^n), \tag{2.1}$$

where

$$Lf := \Delta f + \langle B, \nabla f \rangle + cf \; . \tag{2.2}$$

Then:

(i) μ is absolutely continuous w.r.t. dx and

$$\rho := \frac{d\mu}{dx} \in L^p_{loc}(dx) \quad \text{for all } p \in [1, \frac{n}{n-1}).$$

(ii) If $|B|$, $\sqrt{|c|} \in L^\gamma_{loc}(|\mu|)$ for some $\gamma \in (1, n]$ then $\rho \in H^{p,1}_{loc}(\mathbb{R}^n)$ for all $p \in [1, \frac{n}{n-\gamma+1})$.

(iii) If $|B| \in L^\gamma_{loc}(dx)$, $c \in L^{\gamma n/(n+\gamma)}_{loc}(dx)$ for some $\gamma \in (n, \infty)$, then $\rho \in H^{\gamma,1}_{loc}(\mathbb{R}^n)$ $(\subset C^{1-n/\gamma}(\mathbb{R}^n))$ and, in particular, ρ has a continuous dx–version $\tilde{\rho}$. Furthermore, $\tilde{\rho}$ is strictly positive.

(iv) If $|B| \in L^2(\mu)$ and $c \equiv 0$, then $\rho \in H^{1,1}(\mathbb{R}^n)$ and $|\nabla\rho|/\rho \in L^2(\mu)$.

Assertions (i), (ii) are special cases of [10, Theorem 1]. The same is true for (iii) except for the last part which is a direct consequence of the Harnack inequality for L proved by N.S. Trudinger in [29]. (iv) is just Theorem 3.1 (i), (ii) in [11].

For the rest of this subsection fix $\mu \in \mathcal{M}_{ell}$, where $\mathcal{M}_{ell}$ is as defined in the introduction. Suppose there exists a strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$, whose generator $(L_\mu, D(L_\mu))$ extends L where L with domain $C_0^\infty(\mathbb{R}^n)$ is as in (1.2).

Proposition 2.6 *Let $\rho \in L^\infty(\mu)$ such that $\nu := \rho\mu$ is a probability measure.*

(i) If ν is (T_t^μ)–invariant, then $\nu \in \mathcal{M}_{ell}$.

(ii) Suppose $C_0^\infty(\mathbb{R}^n)$ is dense in $D(L_\mu)$ w.r.t. the graph norm given by L_μ. If $\nu \in \mathcal{M}_{ell}$, then ν is (T_t^μ) – invariant.

Proof. (i): Since ρ is bounded, $(T_t^\mu)_{t>0}$ is a strongly continuous semigroup on $L^1(\nu)$, $|B| \in L^1_{loc}(\nu)$, and for all $f \in C_0^\infty(\mathbb{R}^n)$

$$\int Lf\, d\nu = \frac{d}{dt}\left(\int T_t^\mu f\ d\nu\right)\Big|_{t=0} = \frac{d}{dt}\left(\int f\, d\nu\right)\Big|_{t=0} = 0\ .$$

(ii): Let $f \in C_0^\infty(\mathbb{R}^n)$. Then, since ρ is bounded,

$$\int (T_t^\mu f - f)\, d\nu = \int_0^t \int L_\mu T_s^\mu f\ d\nu ds \quad \text{for all } t > 0. \tag{2.3}$$

But for $s > 0$ by assumption there exist $f_k \in C_0^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$, such that $Lf_k \to L_\mu T_s^\mu f$ in $L^1(\mu)$ hence in $L^1(\nu)$. Therefore, since $\nu \in \mathcal{M}_{ell}$, it follows that

$$\int L_\mu T_s^\mu f\ d\nu = 0 \quad \text{for all } s > 0,$$

and, consequently, ν is (T_t^μ)–invariant by (2.3). $\square$

3 Proofs of the finite dimensional results

In this section let $X := \mathbb{R}^n$ and $\mathcal{B}$ denotes its Borel σ–algebra. We adopt all notations introduced in the introduction. We first prove Theorem 1.3. We need the following lemma.

Lemma 3.1 *Let $\mu \in \mathcal{M}_{ell}$ such that $|B| \in L^{\gamma}_{loc}(\mu)$ for some $\gamma > 1$. Then for every right–continuous (or left–continuous) non–decreasing function $\zeta : \mathbb{R} \to \mathbb{R}$*

$$\int Lf\ \zeta(f)\ d\mu \leq 0 \quad \textit{for all } f \in C_0^\infty(\mathbb{R}^n). \tag{3.1}$$

In particular, $(L, C_0^\infty(\mathbb{R}^n))$ is closable on $L^p(\mu)$ for all $p \in [1, \gamma)$ and (3.1) extends to all f in the closure $D_p(L, \mu)$ of $C_0^\infty(\mathbb{R}^n)$ w.r.t. the graph norm of L on $L^p(\mu)$ provided $\zeta(f) \in L^q(\mu)$, where $q \in (1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, for all $p \geq 1$, $\lambda > 0$

$$\|(\lambda - L)f\|_{L^p(\mu)} \geq \lambda \|f\|_{L^p(\mu)} \ \textit{for all } f \in D_p(L, \mu) \tag{3.2}$$

(where we denote the closure of L again by L).

Proof. By Theorem 2.5 (ii) we know that $\rho := \frac{d\mu}{dx} \in H^{p',1}_{loc}(\mathbb{R}^n)$ with some $p > 1$. Hence, letting $v := B - \nabla\rho/\rho$ we have $|v| \in L^1_{loc}(\mu)$. Therefore, the following integration by parts is justified for every $f \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned} 0 = \int Lf\, d\mu &= -\int \langle \nabla f, \nabla \rho \rangle\, dx + \int \langle \nabla f, B \rangle \rho\, dx \\ &= \int \langle \nabla f, v \rangle \rho\, dx = \int \langle \nabla f, v \rangle\, d\mu. \end{aligned} \tag{3.3}$$

Fix $f \in C_0^\infty(\mathbb{R}^n)$ and $R > 0$ such that $f(\mathbb{R}^n) \subset [-R, R]$. Assume first that ζ is smooth. We can find a smooth compactly supported function ξ such that ξ agrees with ζ on $[-R, R]$ and $\int \xi(s)\, ds = 0$. Then $\eta(t) := \int_{-\infty}^t \xi(s)\, ds \in C_0^\infty(\mathbb{R}^n)$. Note that $\zeta(f) = \xi(f) = \eta'(f)$. Integrating by parts one has

$$\begin{aligned} \int Lf\, \zeta(f)\, d\mu &= -\int \zeta'(f)\, \langle \nabla f, \nabla f \rangle\, d\mu + \int \langle \nabla f, v \rangle\, \zeta(f)\, d\mu \\ &= -\int \zeta'(f)\, \langle \nabla f, \nabla f \rangle\, d\mu + \int \langle \nabla f, v \rangle\, \eta'(f)\, d\mu \\ &= -\int \zeta'(f)\, \langle \nabla f, \nabla f \rangle\, d\mu \leq 0, \end{aligned} \tag{3.4}$$

since $\zeta' \geq 0$ and $\int \langle \nabla f, v \rangle \, \eta'(f) \, d\mu = \int \langle \nabla(\eta \circ f), v \rangle \, d\mu = 0$ by (3.3) (which trivially extends to all $f \in C_b^\infty(\mathbb{R}^n)$).

In the case of a right-continuous nondecreasing ζ we can find a sequence of smooth nondecreasing functions $(\zeta_k)_{k \in \mathbb{N}}$ such that $\zeta_k(t) \underset{k \to \infty}{\to} \zeta(t)$ for every $t \in [-R, R]$ (e.g., $\zeta_k(t) := \int_0^1 \zeta(t + s/k) \sigma(s) \, ds$, where σ is a smooth probability density with support in $[0,1]$). Then the sequence $(\zeta_k(f))_{k \in \mathbb{N}}$ is uniformly bounded and converges to $\zeta(f)$ pointwise, which gives (3.1). For left-continuous ζ the proof is analogous.

The rest of the assertion is now obtained by standard arguments: let $p \geq 1$ and define the function ζ by $\zeta(t) := -|t|^{p-1}$ if $t < 0$, and $\zeta(t) = t^{p-1}$ if $t \geq 0$. If $\|f\|_{L^p(\mu)} = 1$, the norm of $\zeta(f)$ in the dual L^q-space is also 1. Hence by Hölder's inequality and (3.4)

$$\|(\lambda - L)f\|_{L^p(\mu)} \geq \int (\lambda - L)f \, \zeta(f) \, d\mu \geq \lambda \int f \zeta(f) \, d\mu = \lambda \int |f|^p \, d\mu = \lambda.$$

In particular, $(-L, C_0^\infty(\mathbb{R}^n))$ is accretive, hence closable on $L^p(\mu)$ (cf. e.g. [23, Sect. X.8]). Therefore, (3.2) follows and (3.1) for $f \in D_p(L, \mu)$ as well, provided $\zeta \in C_b(\mathbb{R}^n)$. By the same approximation argument as above we can, however, drop the continuity assumption on ζ and finally, considering $(\zeta \wedge k) \vee (-k)$, $k \in \mathbb{N}$, the boundedness assumption provided $\zeta(f) \in L^q(\mu)$. □

Proof of Theorem 1.3. (i): By Theorem 2.5 (iii) we know that $\rho := \frac{d\mu}{dx} \in H_{loc}^{p,1}(\mathbb{R}^n) \subset C_{loc}^{1-n/p}$ and that

$$\operatorname*{ess\,inf}_K \rho > 0 \text{ for all compact } K \subset \mathbb{R}^n. \tag{3.5}$$

Set $\beta^\mu := \frac{\nabla \rho}{\rho}$ and $v := B - \beta^\mu$. Then $v \in L_{loc}^p(\mu) \cap L^1(\mu)$ by (1.7). We shall prove that the closure $(L_\mu, D(L_\mu))$ of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^1(\mu)$ (which exists by Lemma 3.1) generates a strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^1(\mu)$. Since such generators cannot have proper extensions that generate strongly continuous semigroup on $L^1(\mu)$ (cf. e.g. [23, p. 241]), the first part of the assertion will follow then. Thus, because of (3.2) and [23, Theorem X.48] we only have to show the following:

Claim 1. $(L - 1)(C_0^\infty(\mathbb{R}^n))$ *is dense in* $L^1(\mu)$.

So, let $g \in L^\infty(\mu)$ such that

$$\int g(L-1)f \; d\mu = 0 \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n). \tag{3.6}$$

Applying Theorem 2.5 (iii) to the measure $\mu_1 := g \cdot \mu = g \cdot \rho \, dx$, we conclude that $g \cdot \rho \in H^{p,1}(\mathbb{R}^n) \subset C_{loc}^{1-n/p}(\mathbb{R}^n)$, hence by (3.5)

$$g \in H_{loc}^{p,1}(\mathbb{R}^n) \ (\subset C_{loc}^{1-n/p}(\mathbb{R}^n)).$$

Therefore, since also

$$g\beta^\mu, \ g|B| \in L_{loc}^p(dx),$$

we can integrate by parts in equality (3.6) which yields for every $f \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} 0 &= -\int \langle \nabla f, \nabla g \rangle \, d\mu - \int \langle \nabla f, \beta^\mu \rangle g \, d\mu + \int \langle \nabla f, B \rangle g \, d\mu - \int f g \, d\mu \\ &= -\int \langle \nabla f, \nabla g \rangle \, d\mu - \int f g \, d\mu + \int \langle \nabla f, v \rangle g \, d\mu. \end{aligned} \tag{3.7}$$

Now let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \in H_{loc}^{2,1}(\mathbb{R}^n)$. Then by the product rule

$$\langle \nabla \varphi, \nabla(\psi g) \rangle = \langle \nabla(\psi\varphi), \nabla g \rangle - \varphi \langle \nabla \psi, \nabla g \rangle + g \langle \nabla \varphi, \nabla \psi \rangle. \tag{3.8}$$

Since equality (3.7) extends to all $f \in H^{2,1}(\mathbb{R}^n)$ with compact support, we can apply (3.7) to $f := \psi\varphi$ and use (3.8) to obtain

$$\begin{aligned} &\int \langle \nabla \varphi, \nabla(\psi g) \rangle \, d\mu + \int \varphi \psi g \, d\mu \\ &\underset{(3.8)}{=} \int \psi \langle \nabla \varphi, \nabla g \rangle \, d\mu + \int g \langle \nabla \varphi, \nabla \psi \rangle \, d\mu + \int \varphi \psi g \, d\mu \\ &\underset{(3.8)}{=} \int \langle \nabla(\psi\varphi), \nabla g \rangle \, d\mu - \int \varphi \langle \nabla \psi, \nabla g \rangle \, d\mu + \int g \langle \nabla \varphi, \nabla \psi \rangle \, d\mu + \int \varphi \psi g \, d\mu \\ &\underset{(3.7)}{=} -\int \varphi \langle \nabla \psi, \nabla g \rangle + \int g \langle \nabla \varphi, \nabla \psi \rangle \, d\mu + \int \langle \nabla(\psi\varphi), v \rangle g \, d\mu. \end{aligned}$$

Taking $\varphi = \psi g$, one gets

$$\begin{aligned} &\int \langle \nabla(\psi g), \nabla(\psi g) \rangle \, d\mu + \int (\psi g)^2 \, d\mu \\ &= -\int \psi g \langle \nabla \psi, \nabla g \rangle \, d\mu + \int g \langle \nabla(\psi g), \nabla \psi \rangle \, d\mu + \int \langle \nabla(\psi^2 g), v \rangle g \, d\mu \\ &= \int g^2 \langle \nabla \psi, \nabla \psi \rangle \, d\mu + \int \langle \nabla(\psi^2 g), v \rangle g \, d\mu. \end{aligned} \tag{3.9}$$

For $k \in \mathbb{N}$ let $\psi_k \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \psi_k \le 1$, $\psi_k(x) = 1$ if $|x| \le k$, $\psi_k(x) = 0$ if $|x| \ge k+1$, and $\sup_k \|\nabla\psi_k\| < \infty$.

Since g is in particular in $L^2(\mu)$, we have $\int g^2 \langle \nabla\psi_k, \nabla\psi_k\rangle \, d\mu \to 0$. Taking into account that $\psi_k g^2 \in H^{p,1}(\mathbb{R}^n)$ and has compact support we can find $f_l \in C_0^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$, such that $|\nabla(\psi_k f_l) - \nabla(\psi_k g)^2| \underset{l\to\infty}{\longrightarrow} 0$ in $L^p(\mu)$, hence $\langle \nabla(\psi_k f_l), v\rangle \underset{l\to\infty}{\longrightarrow} \langle \nabla(\psi_k g)^2, v\rangle$ in $L^1(\mu)$, because $v \in L^p_{loc}(\mu)$ and $p > n \ge 2$. Since $\mu \in \mathcal{M}_{ell}$, it follows that

$$
\begin{aligned}
& \int \psi_k g \langle \nabla(\psi_k g), v\rangle \, d\mu = \frac{1}{2}\int \langle \nabla(\psi_k g)^2, v\rangle \, d\mu \\
= \;& \frac{1}{2}\lim_{l\to\infty}\int \langle \nabla(\psi_k f_l), v\rangle \, d\mu = \frac{1}{2}\lim_{l\to\infty}\int L(\psi_k f_l)\, d\mu = 0.
\end{aligned}
$$

Hence we have for the second term on the right in (3.9) (with ψ_k replacing ψ) that

$$
\int \langle \nabla(\psi_k^2 g), v\rangle g \, d\mu = \int \psi_k g^2 \langle \nabla\psi_k, v\rangle \, d\mu,
$$

which because of (1.7) as $k \to \infty$ tends to zero by Lebesgue's dominated convergence theorem. Therefore, (3.9) implies that $\displaystyle\int \psi_k^2 \, g^2 \, d\mu \underset{k\to\infty}{\longrightarrow} 0$.

Taking into account that $\psi_k \to 1$, we conclude that $g = 0$ and the claim is proven.

(ii): We first show that $(T_t^\mu)_{t>0}$ is positivity preserving. It suffices to show that $f \ge 0$ if $f \in D(L_\mu)$ and $(\lambda - L_\mu)f \ge 0$, where $\lambda > 0$. Set $g := (\lambda - L_\mu)f$. Letting $\zeta = 1_{(0,\infty)}$ in Lemma 3.1, we get

$$
\begin{aligned}
0 \;\ge\; & \int L_\mu(-f)\;\zeta(-f)\, d\mu = \int (\lambda - L_\mu)f\;\zeta(-f)\, d\mu - \lambda\int f\;\zeta(-f)\, d\mu \\
=\; & \int g\;\zeta(-f)\, d\mu - \lambda\int f\;\zeta(-f)\, d\mu \ge \lambda \int |f|\; 1_{\{f<0\}}\, d\mu \ge 0\;.
\end{aligned}
$$

Hence, $|f|\, 1_{\{f<0\}} = 0$ μ–a.e., i.e., $f \ge 0$ μ–a.e.

Let us now show that $1 \in D(L_\mu)$ and that $L_\mu 1 = 0$. By Claim 1 above and (3.2) it follows that there exists $h \in D(L_\mu)$ with $(1 - L_\mu)h = 1$. As shown above, this yields $h \ge 0$. Applying Lemma 3.1 to the function $\zeta = 1_{(1,\infty)}$, we get

$$
\int h\;\zeta(h)\, d\mu \le \int h\;\zeta(h)\, d\mu - \int L_\mu h\;\zeta(h)\, d\mu = \int \zeta(h)\, d\mu,
$$

whence $h \leq 1$ μ–a.e. Since $\int h\, d\mu = 1$ due to $\int L_\mu h\, d\mu = 0$ (because $\mu \in \mathcal{M}_{ell}$), we conclude that $h = 1$. Consequently, $L_\mu 1 = 0$. Therefore, $(T_t^\mu)_{t\geq 0}$ is a Markovian semigroup.

(iii): We first recall that for $\alpha > 0$, $f \in L^2(\mu)$

$$G_\alpha^\mu f = (\alpha - L_\mu)^{-1} f = \int_0^\infty e^{-\alpha t} T_t^\mu f \; dt. \tag{3.10}$$

Define

$$\hat{L} f := \Delta f + 2\rho^{-1}\langle \nabla \rho, \nabla f\rangle - \langle B, \nabla f\rangle, \;\; f \in C_0^\infty(\mathbb{R}^n). \tag{3.11}$$

Claim 2. For all $g \in D(L_\mu)$

$$\int f \; L_\mu g \; d\mu = \int \hat{L} f \; g \; d\mu \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n).$$

To prove the claim we first note that (i), (ii) above also apply to $\hat{L}$ instead of L. Hence the closure of $(\hat{L}, C_0^\infty(\mathbb{R}^n))$ on $L^1(\mu)$ generates a strongly continuous semigroup $(\hat{T}_t^\mu)_{t>0}$ with all the respective properties. Let $(\hat{G}_\alpha^\mu)_{\alpha>0}$ denote the corresponding resolvent. Then for all $f, g \in C_0^\infty(\mathbb{R}^n)$, $\alpha > 0$, by the definition of $\hat{L}$

$$\begin{aligned} & \int G_\alpha^\mu((\alpha - L)f) \; (\alpha - \hat{L})g \; d\mu \\ = & \int f \; (\alpha - \hat{L})g \; d\mu \\ = & \int (\alpha - L)f \; g \; d\mu \\ = & \int (\alpha - L)f \; \hat{G}_\alpha^\mu((\alpha - \hat{L})g) \; d\mu \; . \end{aligned}$$

Since both $(\alpha - L)(C_0^\infty(\mathbb{R}^n))$ and $(\alpha - \hat{L})(C_0^\infty(\mathbb{R}^n))$ are dense in $L^1(\mu)$, it follows that for all $\alpha > 0$

$$\int G_\alpha^\mu f \; g \; d\mu = \int f \; \hat{G}_\alpha^\mu g \; d\mu \tag{3.12}$$

for all $f, g \in L^1(\mu)$ such that f or g is bounded.

Let now $g \in D(L_\mu)$. Then there exists $\psi \in L^1(\mu)$ such that $g = G_1^\mu \psi$. Hence by (3.12) for all $f \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int f\ (1-L_\mu)g\ d\mu &= \int f\ \psi\ d\mu \\ &= \int \hat{G}_1^\mu((1-\hat{L})f)\ \psi\ d\mu \\ &\underset{(3.12)}{=} \int (1-\hat{L})f\ g\ d\mu.\end{aligned}$$

This proves Claim 2.

Let $f \in L^{np/(n+p)}(\mu)$. Then $G_\alpha^\mu f \in L^{np/(n+p)}(\mu)$, because each T_t^μ hence each αG_α^μ is a contraction on $L^{np/(n+p)}(\mu)$ by (3.10). Since

$$L_\mu G_\alpha^\mu f = -f + \alpha G_\alpha^\mu f\ ,$$

it follows by Claim 2, (3.5), and Theorem 2.5 (iii) that $G_\alpha^\mu f$ has a continuous version, and (iii) is proven.

(iv): Let $t > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$, then $Lf \in L^p(\mu) \subset L^{np/(n+p)}(\mu)$. Hence $T_t^\mu((1-L)f) \in L^{np/(n+p)}(\mu)$. Furthermore,

$$T_t^\mu f = T_t^\mu(G_1^\mu((1-L)f)) = G_1^\mu(T_t^\mu((1-L)f)).$$

But the latter has a continuous μ–version by (iii).

To prove the second part of assertion (iv) we first note that by Proposition 2.6 (ii) μ is (T_t^μ)–invariant. To prove the converse we need some preparations: Let σ be a smooth probability density on $\mathbb{R}^n$ with compact support and define

$$\delta_k(x) := k^n \sigma(kx),\ \ k \in \mathbb{N},\ \ x \in \mathbb{R}^n.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$ and let $\widetilde{T_t^\mu f}$ denote the continuous μ–version of $T_t^\mu f$ for $t > 0$.

Claim 3. $(t,x) \mapsto \widetilde{T_t^\mu f}(x)$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)$–measurable.

To prove this claim we first note that clearly

$$(t,x) \mapsto \int T_t^\mu f(y)\ \delta_k(x-y)\ \rho(y)\ dy = \Big((\rho \cdot \widetilde{T_t^\mu f}) * \delta_k\Big)(x)$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$ for all $k \in \mathbb{N}$. But for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

$$\widetilde{T_t^\mu f}(x) = \lim_{k\to\infty} \Big((\rho \cdot \widetilde{T_t^\mu f}) * \delta_k\Big)(x)$$

(since ρ is continuous), and Claim 3 is proven.

Now we can complete the proof of (iv). It remains to show that if ν is a probability measure on $\mathbb{R}^n$ satisfying (1.8), then $\nu = \mu$.

Claim 4. Let ν satisfy (1.8). Then ν is absolutely continuous w.r.t. μ.

To prove this claim let $f \in L^\infty(\mu)$. Let $\widetilde{G_1^\mu f}$ denote the continuous μ–version of $G_1^\mu f$ (which exists according to (iii) and is unique since supp $\mu = \mathbb{R}^n$). If $f \in C_0^\infty(\mathbb{R}^n)$, due to Claim 3 we can apply Fubini's Theorem to obtain that

$$\int \widetilde{G_1^\mu f}\ d\nu = \int_0^\infty e^{-t} \int \widetilde{T_t^\mu f}(x)\, \nu(dx)\, dt = \int f\, d\nu \tag{3.13}$$

where we used that

$$x \mapsto \int_0^\infty e^{-t} \widetilde{T_t^\mu f}(x)\ dt$$

is also continuous, and thus by (3.10) and since supp $\mu = \mathbb{R}^n$,

$$\widetilde{G_1^\mu f}(x) = \int_0^\infty e^{-t} \widetilde{T_t^\mu f}(x)\ dt \quad \text{for } all\ x \in \mathbb{R}^n .$$

By a monotone class argument (3.13) implies that

$$\int \widetilde{G_1^\mu f}\ d\nu = \int f\ d\nu \tag{3.14}$$

for all bounded Borel–measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. So, if $N \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(N) = 0$, then

$$\int \widetilde{G_1^\mu 1_N}\ d\mu = \mu(N) = 0,$$

hence $\widetilde{G_1^\mu 1_N}(x) = 0$ for *all* $x \in \mathbb{R}^n$, hence by (3.14) it follows that $\nu(N) = 0$, and Claim 4 is proven.

The proof is, therefore, completed by

Claim 5. Let $\mathcal{M}_\mu$ denote the set of all probability measures on $\mathbb{R}^n$ which are (T_t^μ)–invariant and absolutely continuous w.r.t. μ. Then

$$\mathcal{M}_\mu \subset \{\mu\} .$$

Since $(T_t^\mu)_{t>0}$ is sub–Markovian, by Corollary 2.3 it suffices to prove that $\mathcal{M}_{\mu,b} \subset \mathcal{M}_{\mu,>0}$. So, let $\nu := g \cdot \mu \in \mathcal{M}_{\mu,b}$. Then by Proposition 2.6 (i), $\nu \in \mathcal{M}_{ell}$. Hence (because $\mu = \rho dx$) it follows by Theorem 2.5(iii) that $g \cdot \rho > 0$. Therefore, $g > 0$ μ–a.e. by (3.5) and the proof is complete. $\square$.

Proof of Theorem 1.2. For $i = 1, 2$ let $\mu_i \in \mathcal{M}_{ell}$ such that $\rho_i := \frac{d\mu_i}{dx}$ satisfies (1.7). Define

$$\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = \frac{1}{2}(\rho_1 + \rho_2)dx \ .$$

Then $\mu \in \mathcal{M}_{ell}$ and $\rho := \frac{1}{2}1(\rho_1 + \rho_2)$ satisfies (1.7). Hence Theorem 1.3 applies to μ. Let $(T_t^\mu)_{t>0}$ be the corresponding semigroup. Since $\frac{d\mu_i}{d\mu} = 2\frac{\rho_i}{\rho_1+\rho_2}$, $i = 1, 2$, are bounded functions, we can apply Proposition 2.6 (ii) with $\mu_i = \nu$ to conclude that $\mu_i \in \mathcal{M}_\mu$. Hence by Theorem 1.3

$$\mu_i = \mu \ , \ i = 1, 2,$$

which proves the assertion. $\square$

Proof of Proposition 1.5. By Theorem 2.5 (iv) we know that $\rho_i := \frac{d\mu_i}{dx} \in H^{1,1}(\mathbb{R}^n)$ and that $\beta^{\mu_i} := \frac{\nabla\rho_i}{\rho_i} \in L^2(\mu_i)$.

Case 1. $\mu_1 = \mu_2 =: \mu$.

Let $(T_t^\mu)_{t>0}$ be *any* strongly continuous semigroup on $L^1(\mu)$ such that its generator $(L_\mu, D(L_\mu))$ on $L^1(\mu)$ extends $(L, C_0^\infty(\mathbb{R}^n))$. Then since $\mu \in \mathcal{M}_{ell}$ and by Remark 1.1 (ii) we obtain that

$$\int \langle B - \beta^\mu, \nabla f\rangle \ d\mu = 0 \ \text{ for all } f \in C_b^\infty(\mathbb{R}^n). \tag{3.15}$$

There exist $B_j \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $j \in \mathbb{N}$, such that

$$|B - B_j| \underset{j\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(\mu). \tag{3.16}$$

Let $(T_t^{(j)})_{t>0}$ be the strongly continuous semigroup on $C_\infty(\mathbb{R}^n)$ (i.e., the completion of $C_0(\mathbb{R}^n)$ w.r.t. supremum norm) generated by $L_j := \Delta + \langle B_j, \nabla\cdot\rangle$, $j \in \mathbb{N}$. It follows by arguments similar to those given in [21] in the symmetric case that for every $f \in C_0^\infty(\mathbb{R}^n)$ and $t > 0$ one has

$$T_t^{(j)} f \to T_t^\mu f \quad \text{in } L^1(\mu) \text{ as } j \to \infty.$$

Indeed, letting $u_j(t) := T_t^{(j)} f$ for $f \in C_0^\infty(\mathbb{R}^n)$, $t > 0$, $j \in \mathbb{N}$, it is easily seen that $u_j(t)$ is in the Schwartz class of rapidly decreasing smooth functions, hence in $D(L_\mu)$ (which follows by multiplying by bump functions and the square-integrability of $|B|$). Therefore, we get by the Duhamel formula

$$T_t^\mu f - T_t^{(j)} f = \int_0^t T_{t-s}^\mu (L - L_j) T_s^{(j)} f \, ds = \int_0^t T_{t-s}^\mu \langle B - B_j, \nabla u_j(s) \rangle \, ds.$$

Because T_t^μ, $t > 0$, are contractions, this yields the estimate

$$\|T_t^\mu f - T_t^{(j)} f\|_{L^1(\mu)} \leq \|B - B_j\|_{L^2(\mu)} \int_0^t \|\nabla u_j(s)\|_{L^2(\mu)} \, ds. \tag{3.17}$$

Now it suffices to note that the integral on the right hand side of (3.17) can be estimated as follows:

$$\int_0^t \|\nabla u_j(s)\|_{L^2(\mu)} \, ds \leq \sqrt{t} \|f\|_\infty \sqrt{1 + t\|B - B_j\|_{L^2(\mu)}^2}. \tag{3.18}$$

Indeed, using that $\frac{d}{dt} u_j(t) = L_j u_j(t)$ we get

$$\begin{aligned}
\frac{d}{dt}(u_j(t), u_j(t))_{L^2(\mu)} &= 2(L_j u_j(t), u_j(t))_{L^2(\mu)} \\
&= -2 \int \langle \nabla u_j(t), \nabla u_j(t) \rangle \, d\mu \\
&\quad -2 \int \langle \nabla u_j(t), \beta^\mu \rangle u_j(t) \, d\mu \\
&\quad +2 \int \langle \nabla u_j(t), B_j \rangle u_j(t) \, d\mu \\
&= -2 \int \langle \nabla u_j(t), \nabla u_j(t) \rangle \, d\mu \\
&\quad +2 \int \langle \nabla u_j(t), B_j - B \rangle u_j(t) \, d\mu,
\end{aligned}$$

since

$$\int \langle \nabla u_j(t), \beta^\mu - B \rangle u_j(t) \, d\mu = \frac{1}{2} \int \langle \nabla (u_j(t)^2), \beta^\mu - B \rangle \, d\mu = 0$$

because of (3.15).

Hence

$$\begin{aligned}2\|\nabla u_j(t)\|^2_{L^2(\mu)} &\le -\frac{d}{dt}(u_j(t),u_j(t))_{L^2(\mu)}\\ &\quad +\|u_j(t)\|^2_\infty\|B_j-B\|^2_{L^2(\mu)}+\|\nabla u_j(t)\|^2_{L^2(\mu)},\end{aligned}$$

which gives

$$\|\nabla u_j(t)\|^2_{L^2(\mu)} \le -\frac{d}{dt}(u_j(t),u_j(t))_{L^2(\mu)}+\|f\|^2_\infty\|B_j-B\|^2_{L^2(\mu)}.$$

Integrating this estimate over t we get

$$\begin{aligned}\int_0^t \|\nabla u_j(s)\|^2_{L^2(\mu)}\,ds &\le \|f\|_{L^2(\mu)}+t\|f\|^2_\infty\|B_j-B\|^2_{L^2(\mu)}\\ &\le \|f\|^2_\infty(1+t\|B_j-B\|^2_{L^2(\mu)}),\end{aligned}$$

whence (3.18) is proven.

(3.16)–(3.18) imply that all such semigroups $(T^\mu_t)_{t>0}$ as above must coincide which proves Case 1.

Case 2. μ_1,μ_2 as in the assertion.

Define $\mu := \frac{1}{2}\mu_1+\frac{1}{2}\mu_2$ and let $(T^\mu_t)_{t>0}$ be as in Case 1. Fix $i\in\{1,2\}$. Clearly, $\rho_i := \frac{d\mu_i}{d\mu}$ is bounded, hence $(T^\mu_t)_{t>0}$ is a strongly continuous semigroup on $L^1(\mu_i)$ and its generator on $L^1(\mu_i)$ also extends $(L, C_0^\infty(\mathbb{R}^n))$. Consequently, by Case 1

$$T^\mu_t f = T^{\mu_i}_t f \quad \mu_i - a.e.$$

for all $t>0$ and all bounded Borel–measurable $f : X\to\mathbb{R}$ and the proposition is proven. □

Proof of Theorem 1.4. (i): As already mentioned in the introduction this is a special case of a result in [27, 28].
(ii): The fact that (1.9) holds follows from Theorem 2.5 (iv). The rest of the assertion is thus a consequence of Proposition 1.5.
(iii): Exactly as in the proof of Theorem 1.3 it suffices to prove the following

Claim. $(L-1)(C_0^\infty(\mathbb{R}^n))$ is dense in $L^r(\mu)$.

So, let $g \in L^{r'}(\mu)$ with $r' := \frac{r}{r-1}$, such that

$$\int g\ (L-1)f\ d\mu = 0 \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n).$$

We note that also in this case equality (3.9) holds. Since $r \leq 2$, we have $g \in L^2(\mu)$, and also

$$\int \psi_k g^2 \langle \nabla \psi_k, v \rangle\, d\mu \longrightarrow 0 \text{ as } k \to \infty,$$

since $v \in L^q(\mu)$ by (1.10) and $g^2 \in L^{r/2(r-1)}(\mu) = L^{q/(q-1)}(\mu)$. So, the same arguments as in the proof of Theorem 1.3 prove the claim.

$\square$

For the proof of Theorem 1.7 we need the following

Lemma 3.2 *The set $\mathcal{M}$ of all $\mu \in \mathcal{M}_{ell}$ such that $|B| \in L^2_{loc}(\mu)$ and $\rho := \frac{d\mu}{dx}$ satisfies* (1.9) *is convex.*

Proof. Let $\mu_i = \rho_i\, dx \in \mathcal{M}$, $i = 1, 2$. Let $\alpha \in (0,1)$ and $\mu := \alpha\mu_1 + (1-\alpha)\mu_2$. Clearly, $\mu \in \mathcal{M}_{ell}$ with $|B| \in L^2_{loc}(\mu)$. Furthermore, $\rho := \frac{d\mu}{dx} = \alpha\rho_1 + (1-\alpha)\rho_2$ satisfies (1.9), since

$$\begin{aligned}
& |B - \frac{\nabla\rho}{\rho}|^2 \rho \\
= \ & \frac{|\alpha\rho_1(B - \frac{\nabla\rho_1}{\rho_1}) + (1-\alpha)\rho_2(B - \frac{\nabla\rho_2}{\rho_2})|^2}{\alpha\rho_1 + (1-\alpha)\rho_2} \\
= \ & \left[\sqrt{\alpha}\left(\frac{\alpha\rho_1}{\alpha\rho_1 + (1-\alpha)\rho_2}\right)^{1/2} f_1 + \sqrt{1-\alpha}\left(\frac{(1-\alpha)\rho_2}{\alpha\rho_1 + (1-\alpha)\rho_2}\right)^{1/2} f_2\right]^2
\end{aligned}$$

where $f_i := \frac{\rho_i B - \nabla\rho_i}{\sqrt{\rho_i}} \in L^2(dx)$, $i = 1, 2$ by assumption. Hence $|B - \nabla\rho/\rho|^2\rho \in L^1(dx)$. $\square$

Proof of Theorem 1.7. (i): Let μ, ρ be as in the assertion. By Theorem 2.5(ii) we know that $\rho \in H^{1,1}_{loc}(\mathbb{R}^n)$. Since $\mu \in \mathcal{M}_{ell}$ with $B = \nabla\rho_0/\rho_0$, (cf (1.11)) it follows that

$$\int \langle \nabla f, \left(\frac{\nabla\rho_0}{\rho_0} - \frac{\nabla\rho}{\rho}\right) \rangle\, d\mu = 0 \text{ for all } f \in C_0^\infty(\mathbb{R}^n). \tag{3.19}$$

In order to prove that this implies that $\frac{\nabla\rho_0}{\rho_0} - \frac{\nabla\rho}{\rho} = 0$, by Lemma 3.2 we may assume that $\rho \geq \frac{1}{2}\rho_0$ (otherwise consider $\frac{1}{2}\mu + \frac{1}{2}\mu_0$ instead of μ). In particular, $\rho_0/\rho \in L^\infty(\mu)$, hence for every $k \in \mathbb{N}$

$$\log\left(\frac{\rho_0}{\rho} + \frac{1}{k}\right) \in L^\infty(\mu) \ .$$

Furthermore, for all $f \in C_0^\infty(\mathbb{R}^n)$ and $\partial_i := \frac{\partial}{\partial x_i}$ $1 \leq i \leq n$,

$$\int \partial_i f \ \log\left(\frac{\rho_0}{\rho} + \frac{1}{k}\right) d\mu = \lim_{\varepsilon \searrow 0} \int \partial_i f \ \log\left(\frac{k\rho_0 + \rho + \varepsilon}{k\rho + \varepsilon}\right) d\mu$$

by Lebesgue's dominated convergence theorem, since for

$$0 < \varepsilon \leq 1$$

$$\frac{1}{k} \leq \frac{\rho + \varepsilon}{k\rho + \varepsilon} \leq \frac{k\rho_0 + \rho + \varepsilon}{k\rho + \varepsilon} \leq \frac{(2k+1)\rho + \varepsilon}{k\rho + \varepsilon} \leq \frac{2k+1}{k} \quad \mu - a.e.$$

(because for $a, b > 0$ we have that $\frac{d}{d\varepsilon}\frac{a+\varepsilon}{b+\varepsilon} \geq 0 \Leftrightarrow b \geq a$).
But

$$\begin{aligned} & \int \partial_i f(\log(k\rho_0 + \rho + \varepsilon) - \log(k\rho + \varepsilon))d\mu \\ = & -\int f\left(\frac{\partial_i(k\rho_0 + \rho)}{k\rho_0 + \rho + \varepsilon} - \frac{k\partial_i\rho}{k\rho + \varepsilon}\right) d\mu \\ & \quad -\int f \ \log\left(\frac{k\rho_0 + \rho + \varepsilon}{k\rho + \varepsilon}\right) \frac{\partial_i\rho}{\rho} \ d\mu \ . \end{aligned} \tag{3.20}$$

Therefore, as above (and since $|\nabla\rho|/\rho$, $|\nabla\rho_0|/\rho_0 \in L^2_{loc}(\mu)$ by assumption)

$$\begin{aligned} & -\int \partial_i f \ \log\left(\frac{\rho_0}{\rho} + \frac{1}{k}\right) d\mu \\ = & \int f \ \frac{k\rho_0}{k\rho_0 + \rho}\left(\frac{\partial_i\rho_0}{\rho_0} - \frac{\partial_i\rho}{\rho}\right) d\mu \\ & \quad + \int f \ \log\left(\frac{\rho_0}{\rho} + \frac{1}{k}\right) \ \frac{\partial_i\rho}{\rho} \ d\mu \ . \end{aligned} \tag{3.21}$$

Here we used the fact that on $\{\rho_0 = 0\}$ we have that $\partial_i\rho_0 = 0$ dx–a.e., hence both integrands of the first integrals on the right hand sides of (3.20), (3.21) respectively are dx–a.e. equal to zero on $\{\rho_0 = 0\}$ as well. Since $|\nabla\rho_0/\rho_0 - \nabla\rho/\rho| \in L^2(\mu)$ by assumption, it follows by [7, Lemma 2.2] and [25, Theorems 2.3 and 3.1] that there exist $f_l \in C_0^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$, such that both $f_l \to \log(\frac{\rho_0}{\rho} + \frac{1}{k})$ and

$$\Big|\nabla f_l - \frac{k\rho_0}{k\rho_0 + \rho}\left(\frac{\nabla\rho_0}{\rho_0} - \frac{\nabla\rho}{\rho}\right)\Big| \longrightarrow 0 \text{ in } L^2(\mu)$$

as $l \to \infty$. But, clearly, also

$$\Big|\left(\frac{k\rho_0}{k\rho_0 + \rho} - 1\right)\left(\frac{\nabla\rho_0}{\rho_0} - \frac{\nabla\rho}{\rho}\right)\Big| \longrightarrow 0 \text{ in } L^2(\mu)$$

as $k \to \infty$. Hence it follows by (3.19) that

$$\frac{\nabla\rho}{\rho} = \frac{\nabla\rho_0}{\rho_0} \quad \mu - a.e.,$$

and, therefore, for all $f, g \in C_0^\infty(\mathbb{R}^n)$

$$-\int g\, L\, f\ d\mu = \int \langle \nabla g, \nabla f\rangle\ d\mu,$$

i.e., $(L, C_0^\infty(\mathbb{R}^n))$ is symmetric on $L^2(\mu)$. Thus (i) is proven.

(ii): We first note that $\rho_0 = \frac{d\mu_0}{dx}$, of course, satisfies (1.9). The converse of the assertion is an immediate consequence of (i) and [11, Theorem 6.2]. The latter, indeed, immediately extends to the situation considered here (as follows easily from its proof). □

Remark 3.3 (i) We want to emphasize that in Theorem 1.7 (i) it is essential that in (1.11) B is a dx–version and not merely a μ_0–version of $\nabla\rho_0/\rho_0$ (in particular, $B = 0$ on $\{\rho_0 = 0\}$ dx–a.e.), as the following example shows: let $n = 2$ and $\varphi \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, radially symmetric, such that $\int \varphi^2 dx = 1$ and

$$d(x) := \begin{cases} (-x_2, x_1) & \text{if } x = (x_1, x_2) \in \operatorname{supp}\varphi \\ 0 & \text{else .} \end{cases}$$

Let $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{supp}\psi \cap \operatorname{supp}\varphi = \emptyset$, $\rho_0 := \psi^2$, and

$$B := 2\frac{\nabla\psi}{\psi} + 2\frac{\nabla\varphi}{\varphi} + d\ .$$

Then $|\nabla\rho_0|/\rho_0 = 2|\nabla\psi|/\psi \in L^2(\rho_0\,dx)$ and

$$B = \frac{\nabla\rho_0}{\rho_0} \qquad \rho_0 dx - a.e. \tag{3.22}$$

(which is *not* the same as (1.11)!). Therefore,

$$-\int g\ Lf\ \rho_0 dx = \int \langle\nabla g, \nabla f\rangle\, \rho_0\, dx \text{ for all } f, g \in C_0^\infty(\mathbb{R}^2),$$

i.e., $(L, C_0^\infty(\mathbb{R}^2)$ is symmetric on $L^2(\mu)$ (where L is as in (1.2)). But if $\mu := \varphi^2 dx$, then $|\nabla\varphi^2|/\varphi^2 = 2|\nabla\varphi|/\varphi \in L^2_{loc}(\mu)$ and

$$B - 2\frac{\nabla\varphi}{\varphi} = d \quad \mu - a.e..$$

Furthermore, $|d| \in L^\infty(\mu) \subset L^2(\mu)$, and also $\mu \in \mathcal{M}_{ell}$, because (if $\varphi^2 = h(|x|)$, then $\operatorname{div}(\varphi^2 d) = \varphi \operatorname{div} d + \frac{h'(|x|)}{|x|}\langle d, x\rangle = 0$, hence) for all $f \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} \int Lf\, d\mu &= \int \langle B - \frac{\nabla\varphi^2}{\varphi^2}, \nabla f\rangle d\mu \\ &= \int \langle d, \nabla f\rangle d\mu \\ &= \int f \text{ div } (d\varphi^2) d\mu = 0 \end{aligned}$$

(i.e., μ is *invariant*, see the introduction).

But for all $f, g \in C_0^\infty(\mathbb{R}^2)$

$$-\int g\ Lf\ d\mu = \int \langle\nabla g, \nabla f\rangle d\mu - \int g\langle d, \nabla f\rangle d\mu\ ,$$

i.e., $(L, C_0^\infty(\mathbb{R}^2))$ is not symmetric on $L^2(\mu)$.

(ii) The additional condition in Theorem 1.7 (ii) cannot be dropped. There are easy counterexamples even if $n = 1$. We refer to [11, Example 6.1].

(iii) Concerning the relation of the above results with those in [15], we emphasize that the class of invariant measures considered there is defined in terms of an associated diffusion and much more restrictive than the one in this paper resp. [11]. Otherwise, the one–dimensional examples mentioned in (ii) would serve as counterexamples to some of the statements in [15, Section 6]. In particular, one of the final claims in [15, Sect. 6] that the results there are more complete than those in [11], is incorrect for the same reason, namely that the two notions of invariant measures are incompatible in general.

Proof of Proposition 1.9. Consider the quadratic form

$$\mathcal{E}(f,g) := \int \langle \nabla f, \nabla g\rangle d\mu_0;\ f,g \in C_0^\infty(\mathbb{R}^n)$$

on $L^2(\mu_0)$. Since $\rho_0 \in H^{1,1}_{loc}(\mathbb{R}^n)$, it follows that on dx–a.e. line parallel to the axes of $\mathbb{R}^n$, ρ_0 is absolutely continuous. Hence by [6, Theorem 5.3], $(\mathcal{E}, C_0^\infty(\mathbb{R}^n))$ is closable on $L^2(\mu_0)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form (see e.g. [6, Theorem 3.8]). Hence the existence of the desired $(T_t^{\mu_0})_{t>0}$ as a sub–Markovian strongly continuous symmetric semigroup on $L^2(\mu_0)$ follows by standard theory (see e.g. [22, Chap. I] or [16]). In particular, $(T_t^{\mu_0})_{t>0}$ extends to $L^1(\mu_0)$. Since $1 \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$, $(T_t^{\mu_0})_{t>0}$ is in fact Markovian, hence μ_0 is $(T_t^{\mu_0})_{t>0}$–invariant. □

4 The infinite dimensional case

In this section we shall study problems analogous to those in Sections 2 and 3 in infinite dimensions. Though there is no direct analogue of Lebesgue measure in infinite dimensions, a few results can be achieved. This is due to the fact that there exists an exact analogue of a logarithmic derivative resp. gradient, though it cannot be written as $\nabla\rho/\rho$. Before we recall the definition, let us describe our framework.

Let X be a locally convex space over $\mathbb{R}$, X^* its topological dual, $H \subset X$ a continuously embedded separable Hilbert space. Then the dual map defines an injection $j_H : X^* \to H^* \cong H$ by the Riesz representation theorem: $(j_H(k), h)_H = {}_{X^*}\langle k, h\rangle_X$ for all $k \in X^*$, $h \in H$. If there is no risk of ambiguity, we denote j_H by j. An instructive example is $X = \mathbb{R}^\infty$ (the space of all sequences with the coordinate-wise convergence), $H = l^2$. Then $(\mathbb{R}^\infty)^*$ is identified with $\mathbb{R}_0^\infty$ (all finite sequences). The class of all functions on X of the form $f = \varphi(l_1, \ldots, l_n)$, $\varphi \in C_b^\infty(\mathbb{R}^n)$, $l_i \in X^*$ is denoted by $\mathcal{F}C_b^\infty$. Such functions are called smooth bounded cylindrical.

Let μ be a (signed) Radon measure of finite total variation on X.

Definition 4.1 *A Borel (or just μ-measurable) map $\beta : X \to X$ is said to be a vector logarithmic derivative of μ along H if for all $f \in \mathcal{F}C_b^\infty$ and all $k \in X^*$ the following integration by parts formula holds:*

$$\int_X \partial_{j(k)} f(x)\, \mu(dx) = -\int_X f(x)\, {}_{X^*}\langle k, \beta(x)\rangle_X\, \mu(dx), \tag{4.1}$$

where it is assumed that ${}_{X^*}\langle k, \beta\rangle_X \in L^2(|\mu|)$ for all $k \in X^*$ (and $\partial_h f$ denotes the usual partial derivative along h).

It is clear that β is uniquely determined by μ if it exists. By β_H^μ we denote its dependence on μ and H. Obviously, if $X = H = \mathbb{R}^n$ (with the same inner product), then the existence of β is equivalent to the representation $\mu = \rho dx$, $\rho \in H^{1,1}(\mathbb{R}^n)$, and in this case $\beta = \nabla\rho/\rho$. In infinite dimensions Gibbs measures provide interesting examples of measures possessing vector logarithmic derivatives (see [2]).

Vector logarithmic derivatives were introduced in [1]. Earlier S.V. Fomin defined partial logarithmic derivatives in his theory of differentiable measures (see [9], [14] for surveys and references). Clearly, the existence of a vector logarithmic derivative β_H^μ implies the partial differentiability in Fomin's sense along all vectors $j_H(k)$, $k \in X^*$ (and the equality $d_{j(k)}\mu = {}_{X^*}\langle k, \beta_H^\mu\rangle_X \cdot \mu$).

In infinite dimensions equation (1.3) can be given sense similarly to the finite dimensional case: L is now defined by

$$Lf := \Delta_H f + {}_{X^*}\langle f', B\rangle_X, \quad f \in \mathcal{F}C_b^\infty, \tag{4.2}$$

where $B : X \to X$ is Borel–measurable and

$$\Delta_H f := \mathrm{Trace}_H\, f''(x) = \sum_{i,j=1}^n \frac{\partial^2\varphi}{\partial y_i \partial y_j}(l_1(x), \ldots, l_n(x))(j_H(l_i), j_H(l_j))_H \tag{4.3}$$

if $f = \varphi(l_1, \ldots, l_n)$, $n \in \mathbb{N}$, $l_i \in X^*$, $\varphi \in C_b^\infty(\mathbb{R}^n)$. Here f', f'' denote first resp. second Fréchet dreivatives of f. We write

$$L^*\mu = 0 \tag{4.4}$$

if the following two conditions are satisfied:

$${}_{X^*}\langle k, B\rangle_X \in L^2(|\mu|) \tag{4.5}$$

(which is stronger than (1.4) if $X = H = \mathbb{R}^n$) and

$$\int Lf\; d\mu = 0 \ \text{ for all } f \in \mathcal{F}C_b^\infty. \tag{4.6}$$

For sufficient conditions implying the existence of vector logarithmic derivatives see [11]. Much less is known about infinite dimensional analogues of the results presented in the previous section. In particular, the crucial uniqueness problem has been given only rather partial solutions. For example, it has been shown in [11, Sect. 6a)] that even if β is a continuous linear operator on a Hilbert space X, there is no uniqueness of a probability measure with logarithmic derivative β. According to [11], there exist two different centered Gaussian measures μ and ν with common β which is a continuous linear operator (in particular, $(\mu+\nu)/2$ is a non-Gaussian measure with the same linear logarithmic derivative). Nevertheless, we have the following result:

Proposition 4.2 *Let μ and ν be two Radon probability measures on X with vector logarithmic derivatives β_H^μ, β_H^ν resp. Assume that there exists a set $K \subset X^*$ distinguishing Fourier transforms of Radon measures which is a union of finite dimensional subspaces V with the following property: for any $l \in V$ the functions ${}_{X^*}\langle l, \beta_H^\mu\rangle_X$ and ${}_{X^*}\langle l, \beta_H^\nu\rangle_X$ admit equal versions of the form $f_l(l_1,...,l_n)$, where $l_i \in V$, $n = dimV$ and f_l is locally dx–integrable on $\mathbb{R}^n$. Then $\mu=\nu$.*

Proof. Let $l_1,...,l_n$ be a basis in V. Set $Px = (l_1(x),...,l_n(x))$. Then the function f_l can be taken for the logarithmic derivative along $P(j_H(V))$ for both measures $\mu\circ P^{-1}$ and $\nu \circ P^{-1}$. Applying Theorem 6.2 from [11], we get $\mu\circ P^{-1} = \nu\circ P^{-1}$. By our assumption on K this implies that $\mu=\nu$. □

Before we state our next theorem which is an extension of [11, Theorem 3.5], we need some preparations.

For $x \in X$, $f \in \mathcal{F}C_b^\infty(\mathbb{R}^n)$ let $\nabla_H f(x)$ be the unique element in H such that

$$(\nabla_H f(x), h)_H = \partial_h f(x) = {}_{X^*}\langle f'(x), h\rangle_X \text{ for all } h\in H. \tag{4.7}$$

Clearly, if $f = \varphi(l_1,\dots,l_n)$, $n\in\mathbb{N}$, $l_i \in X^*$, $\varphi\in C_b^\infty(\mathbb{R}^n)$, then

$$\nabla_H f(x) = \sum_{i=1}^n \partial_i \varphi(l_1(x),\dots,l_n(x))\; j_H(l_i),\; x\in X.$$

For a Radon probability measure ν on X define

$$\mathcal{E}_\nu(f,g) = \int (\nabla_H f, \nabla_H g)\; d\nu;\;\; f,g\in \mathcal{F}C_b^\infty(\mathbb{R}^n). \tag{4.8}$$

$(\mathcal{E}_\nu, \mathcal{F}C_b^\infty)$ is said to satisfy the Log–Sobolev inequality (cf. [18]) if there exists a constant $c>0$ such that

$$\mathcal{E}_\nu(f,f) \geq c\int f^2\; \log\left(\frac{f}{\|f\|_{L^2(\nu)}}\right)^2\; d\nu \;\text{ for all } f\in\mathcal{F}C_b^\infty. \tag{4.9}$$

If for all $f, g \in \mathcal{F}C_b^\infty$, $f = g$ ν–a.e. implies that $\nabla_H f = \nabla_H g$ ν–a.e.and if $\widetilde{\mathcal{F}C_b^\infty}^\nu$ denotes the ν–classes givben by $\mathcal{F}C_b^\infty$, then $(\mathcal{E}_\nu, \widetilde{\mathcal{F}C_b^\infty}^\nu)$ is a symmetric positive definite bilinear form on $L^2(\nu)$. If it is closable on $L^2(\nu)$, its closure $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is a Dirichlet form (cf. [6, Sect. 3]). Then ∇_H naturally extends to all functions in $D(\mathcal{E}_\nu)$. We denote this extension again by ∇_H. Clearly, also (4.9) extends to all $f \in D(\mathcal{E}_\nu)$.

It is easy to check and well–known that if ν has a logarithmic derivative β_H^ν, then we are in the above case, hence $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is defined as above (cf. e.g. [5]).

We recall that ν is called quasi–invariant along a linear subspace V of X if all image measures of ν under translations by elements in V are absolutely continuous w.r.t. ν

Theorem 4.3 *Let ν be a probability measure on a Souslin locally convex space X which satisfies the Log–Sobolev inequality* (4.9)*, is quasi-invariant along $j_H(X^*)$, and has a logarithmic derivative $\beta := \beta_H^\nu$ associated with H such that ${}_{X^*}\langle k, \beta\rangle_X \in L^2(\nu)$ for all $k \in X^*$. Furthermore, suppose that there exist $l_n \in X^*$, $n \in \mathbb{N}$, separating the points of X such that $e_n := j_H(l_n)$, $n \in \mathbb{N}$, form an orthonormal basis of H and*

$$\sup_{n\in\mathbb{N}} \ \nu\text{-}\operatorname*{esssup}_{x\in X} \sum_{j=1}^{n} \left({}_{X^*}\langle l_j, \beta(x)\rangle_X - E_\nu[\, {}_{X^*}\langle l_j, \beta\rangle_X \mid \sigma_n](x)\right)^2 < \infty \ . \qquad (4.10)$$

Assume that μ is a probability measure satisfying equation (4.4)*, with $B := \beta + v$, $v : X \to H$, $|v|_H \in L^2(\mu)$ (recall B is, in particular, assumed to satisfy* (4.5)*). Then:*

(i) μ is absolutely continuous with respect to ν and its Radon-Nikodym derivative ρ admits the representation $\rho = \varphi^2$, where $\varphi \in D(\mathcal{E}_\nu)$.

(ii) β_H^μ exists and $\beta_H^\mu = \beta + u$, where $u : X \to H$ is the orthogonal projection of v onto the closure of the set $\{\nabla_H f \mid f \in \mathcal{F}C_b^\infty\}$ in the Hilbert space $L^2(\mu, H)$ $(:= L^2(X \to H, \mu))$. Furthermore, $u = 2\nabla_H \varphi/\varphi$.

(iii) μ is differentiable along all vectors $h \in j_H(X^*)$ (in the sense of Fomin).

Proof. (i): The method of the proof of Theorem 3.5 from [11] applies here, however, with some non–trivial modifications. We give a complete and partially improved proof here.

We may assume that $u = v$, since μ satisfies (4.4) with $B_1 := \beta + u$. This follows from the equality

$$\begin{aligned}&\int {}_{X^*}\langle f'(x), u(x) - v(x)\rangle_X \, \mu(dx)\\ &= \int (\nabla_H f(x), u(x) - v(x))_H \; \mu(dx) = 0 \;\forall f \in \mathcal{F}C_b^\infty.\end{aligned}$$

For any μ-square-integrable map $F: X \to H$ the sequence $F_n = E_\mu[F|\sigma_n]$ of the conditional expectations of F with respect to the σ-fields σ_n, generated by $\ell_1, ..., \ell_n$, converges to F in $L^2(\mu, H)$ (see e.g. [30, Chapter II, Theorem 4.1]). Let $H_n :=$ span $(e_1, ..., e_n)$ and $P_n x := \ell_1(x)e_1 + ... + \ell_n(x)e_n$, $x \in X$. The space H_n, $x \in X$ is regarded with the inner product from H. Note that $\ell_i(e_j) = (e_i, e_j)_H$ for any i, j. Hence $P_n|_H$ is the orthogonal projection on H_n with respect to the inner product of H and $\|P_n h - h\|_H \to 0$ as $n \to \infty$ $\forall h \in H$. Therefore, if F and F_n are as above then $(P_n F_n)_{n\in\mathbb{N}}$ converges to F in $L^2(\mu, H)$. Indeed,

$$\begin{aligned}&\int |F(x) - P_n F_n(x)|_H^2 \, \mu(dx)\\ &\le \; 2\int |F(x) - P_n F(x)|_H^2 \, \mu(dx) + 2\int |P_n F(x) - P_n F_n(x)|_H^2 \, \mu(dx)\\ &\le \; 2\int |F(x) - P_n F(x)|_H^2 \, \mu(dx) + 2\int |F(x) - F_n(x)|_H^2 \, \mu(dx) \underset{n\to\infty}{\longrightarrow} 0,\end{aligned} \tag{4.11}$$

since the first term in the right-hand side tends to 0 by Lebesgue's dominated convergence theorem.

Fix $n \in \mathbb{N}$ and set $\nu_n := \nu \circ P_n^{-1}$, $\mu_n := \mu \circ P_n^{-1}$. By [7, Proposition 4.1 resp. Lemma 4.2] $q_n := \frac{d\nu_n}{dx}$ (where dx denotes Lebesgue measure on H_n) exists and $\sqrt{q_n} \in H^{2,1}_{loc}(H_n)$. Hence ν_n has logarithmic derivative $\beta_{H_n}^{\nu_n} = \frac{\nabla_H q_n}{q_n} \in L^2(\nu_n, H_n)$. Since ν is quasi–invariant along $j_H(X^*)$, we have that $q_n > 0$ dx–a.e. on H_n. Furthermore, it is easy to see that

$$E_\nu[P_n\beta \mid \sigma_n] = \beta_{H_n}^{\nu_n} \circ P_n \quad \nu\text{–a.e.} \tag{4.12}$$

Here $\beta_{H_n}^{\nu_n}$ also denotes a Borel–measurable ν_n–version of $\beta_{H_n}^{\nu_n}$ which we fix from now on. By (4.10) we can hence choose a Borel–measurable ν–version of β such that

$$C := \sup_{x\in X}\sup_{n} \|P_n\beta(x) - \beta_{H_n}^{\nu_n}\circ P_n(x)\|_H < \infty\ . \tag{4.13}$$

Below β will always denote this particular Borel ν–version.

Set

$$v_n := E_\mu[P_n v|\sigma_n](= P_n E_\mu[v|\sigma_n])$$

and

$$b_n := E_\mu[P_n B|\sigma_n], \quad \beta_n := E_\mu[P_n\beta|\sigma_n]\ .$$

By (4.11) we have that

$$\|v - v_n\|_H \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2(\mu). \tag{4.14}$$

There exist $\widetilde{b}_n$, $\widetilde{\beta}_n$, $\widetilde{v}_n : H_n \to H_n$ Borel–measurable such that $b_n = \widetilde{b}_n\circ P_n$, $\beta_n = \widetilde{b}_n \circ P_n$, and $v_n = \widetilde{v}_n\circ P_n$ μ–a.e. Furthermore,

$$\widetilde{b}_n = \widetilde{\beta}_n + \widetilde{v}_n \qquad \mu_n\text{–a.e} \tag{4.15}$$

Clearly,

$$\widetilde{\beta}_n\circ P_n - \beta_{H_n}^{\nu_n}\circ P_n = E_\mu[P_n\beta - \beta_{H_n}^{\nu_n}\circ P_n \mid \sigma_n] \quad \mu\text{–a.e.}$$

Hence (4.13) implies that

$$\|\widetilde{\beta}_n - \beta_{H_n}^{\nu_n}\|_H \le C \quad \mu_n\text{–a.e.} \tag{4.16}$$

As shown in the proof of Proposition 3.3 in [11] (and as can be easily checked directly), the measure μ_n on H_n satisfies the equation

$$L_n^*\,\mu_n = 0\ , \tag{4.17}$$

where

$$L_n\,u := \sum_{i=1}^{n} \partial_{e_i}^2 u + (\nabla_H u, \widetilde{b}_n)_H \quad \forall u \in C_b^\infty(H_n).$$

Therefore, according to Theorem 2.5 (iv), $\rho_n := \frac{d\mu_n}{dx}$ exists and $\beta_{H_n}^{\mu_n} := \frac{\nabla_H\rho_n}{\rho_n} \in L^2(\mu_n, H_n)$. Since $q_n > 0$ dx–a.e., $f_n := \frac{d\mu_n}{d\nu_n} = \frac{\rho_n}{q_n}$ exists resp. is well–defined.

Claim 1. $\sqrt{f_n} \in D(\mathcal{E}_{\nu_n})$ and μ_n–a.e.

$$\beta_{H_n}^{\mu_n} - \beta_{H_n}^{\nu_n} = 2\frac{\nabla_H \sqrt{f_n}}{\sqrt{f_n}} \in L^2(\mu_n).$$

To prove this claim let $\chi := \sqrt{\rho_n}$, $\psi := \sqrt{q_n}$. Then $\chi, \psi \in H^{2,1}(\mathbb{R}^n)$ and for all $g \in C_b^\infty(H_n)$, $1 \leq i \leq n$,

$$\begin{aligned}
& \int \partial_i g \ \sqrt{f_n} \ d\nu_n \\
= \ & \lim_{k\to\infty} \int \partial_i g \ \frac{\chi}{\psi + \frac{1}{k}} \ \psi^2 \ dx \\
= \ & -\lim_{k\to\infty} \int g \left(\frac{\partial_i \chi}{\psi + \frac{1}{k}} - \frac{\chi \partial_i \psi}{(\psi + \frac{1}{k})^2} \right) \ \psi^2 \ dx \\
& \quad - \lim_{k\to\infty} \int g \ \frac{\chi}{\psi + \frac{1}{k}} \ \beta_{H_n}^{\nu_n} \ \psi^2 \ dx \\
= \ & -\int g \ \left(\frac{\partial_i \chi}{\psi} - \frac{\chi \partial_i \psi}{\psi^2} \right) \ d\nu_n - \int g \ \sqrt{f_n} \ \beta_{H_n}^{\nu_n} \ \psi^2 \ dx \ .
\end{aligned}$$

Clearly, $\partial_i \chi / \psi \in L^2(\nu_n)$. Since $\widetilde{\beta}_n \in L^2(\mu_n)$ (because $\beta \in L^2(\mu)$ by assumption), it follows by (4.16) that $2|\partial_i \psi / \psi| \leq |\beta_{H_n}^{\nu_n}|_H \in L^2(\mu_n) = L^2(\chi^2 dx)$. Hence also $(\chi \partial_i \psi)/\psi^2 \in L^2(\nu_n)$. Consequently, by [7, Lemma 2.2] and [25, Theorems 2.3 and 3.1] $\sqrt{f_n} \in D(\mathcal{E}_{\nu_n})$ and

$$\nabla_H \sqrt{f_n} = \left(\frac{\partial_i \chi}{\chi} - \frac{\partial_i \psi}{\psi} \right) \sqrt{f_n} = \frac{1}{2} (\beta_{H_n}^{\mu_n} - \beta_{H_n}^{\nu_n}) \sqrt{f_n} \ .$$

Thus, Claim 1 is proven.

Claim 2. Define $\varphi_n := \sqrt{f_n} \circ P_n$, $n \in \mathbb{N}$. Then

$$\sup_n \int \left\| \frac{\nabla_H \varphi_n}{\varphi_n} \right\|_H^2 \ d\mu < \infty.$$

To prove this claim we first note that by (4.15) and Claim 1

$$2\frac{\nabla_H \sqrt{f_n}}{\sqrt{f_n}} + \left(\widetilde{b}_n - \beta_{H_n}^{\mu_n} \right) = \widetilde{\beta}_n - \beta_{H_n}^{\nu_n} + \widetilde{v}_n \quad \mu_n - a.e.$$

Since $\mu_n = \mu \circ P_n^{-1}$, Claim 2 follows from (4.14) and (4.16), if we can show that for all $n \in \mathbb{N}$

$$\int \left(\frac{\nabla_H \sqrt{f_n}}{\sqrt{f_n}}, \widetilde{b}_n - \beta_{H_n}^{\mu_n} \right)_H d\mu_n = 0. \tag{4.18}$$

So, fix $n \in \mathbb{N}$. We know by (4.17) that

$$\int (\nabla_H g, \widetilde{b}_n - \beta_{H_n}^{\mu_n})_H \; d\mu_n = 0 \quad \text{for all } g \in C_0^\infty(H_n).$$

So, we have to show that $\nabla_H \sqrt{f_n}/\sqrt{f_n}$ can be approximated by a sequence $(\nabla_H g_k)_{k \in \mathbb{N}}$ in $L^2(\mu_n, H_n)$ for some $g_k \in C_0^\infty(H_n)$. But clearly,

$$\nabla_H \log(\sqrt{f_n} + \frac{1}{k}) \underset{k \to \infty}{\longrightarrow} \nabla_H \sqrt{f_n}/\sqrt{f_n} \text{ in } L^2(\mu_n, H_n).$$

On the other hand, for each $k \in \mathbb{N}$, $\log(\sqrt{f_n} + \frac{1}{k}) \in L^2(\mu_n)$. Hence again by [25, Theorems 2.3 and 3.1] there exist g_{kl}, $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$g_{kl} \underset{l \to \infty}{\longrightarrow} \log(\sqrt{f_n} + \frac{1}{k}) \text{ and } \|\nabla_H g_{kl} - \nabla_H \log(\sqrt{f_n} + \frac{1}{k})\|_H \underset{l \to \infty}{\longrightarrow} 0$$

in $L^2(\mu_n)$, and this altogether proves (4.18), hence Claim 2.

We shall now prove that $\{f_n \circ P_n : n \in \mathbb{N}\}$ is uniformly ν–integrable using Claim 2 and the assumption that the Log–Sobolev inequality (4.9) holds for ν, i.e., for all $f \in D(\mathcal{E}_\nu)$

$$\int_X f(x)^2 \log |f(x)| \, \nu(dx) \leq (2c)^{-1} \int_X \|\nabla_H f(x)\|_H^2 \, \nu(dx) + \|f\|_{L^2(\nu)}^2 \log \|f\|_{L^2(\nu)}. \tag{4.19}$$

Since $\sqrt{f_n} \in D(\mathcal{E}_{\nu_n})$, we have that

$$\varphi_n := \sqrt{f_n} \circ P_n \in D(\mathcal{E}_\nu)$$

(cf. the "Claim" in the proof of Theorem 1.4 in [7]), and we can apply the logarithmic Sobolev inequality to φ_n. Moreover,

$$\|\varphi_n\|_{L^2(\nu)}^2 = \int_X f_n(P_n x) \, \nu(dx) = \int_{H_n} f_n(x) \, \nu_n(dx) = \mu_n(H_n) = 1. \tag{4.20}$$

and by Claim 2 above

$$\sup_n \int \|\nabla_H \varphi_n\|_H^2 \, d\nu < \infty \ . \tag{4.21}$$

(4.19) – (4.21) imply that

$$\sup_n \int_X \varphi_n(x)^2 \log|\varphi_n(x)| \nu(dx) < \infty,$$

which in turn implies the uniform ν–integrability of $f_n \circ P_n = \varphi_n^2$, $n \in \mathbb{N}$. Furthermore, $(f_n \circ P_n)_{n\in\mathbb{N}}$ is a $(\sigma_n)_{n\in\mathbb{N}}$-martingale (under ν), thus it converges to some function $\rho \in L^1(\nu)$. Let $\lambda := \rho \cdot \nu$. Then for $\lambda_n := \varphi_n^2 \nu$, $n \in \mathbb{N}$, and for any $\ell = c_1\ell_1 + \ldots + c_m\ell_m$ one has that

$$\int_X \exp(i\ell(x)) \, \lambda_n(dx) \underset{n\to\infty}{\longrightarrow} \int_X \exp(i\ell(x)) \, \lambda(dx).$$

On the other hand, for $n \geq m$ we have that

$$\begin{aligned}
&\int_X \exp(i\ell(x))\lambda_n(dx) = \int_X \exp(i\ell(P_n x)) f_n(P_n x) \, \nu \, (dx) \\
= &\int_{H_n} \exp(i\ell(x)) f_n(x) \nu_n(dx) = \int_{H_n} \exp(i\ell(x)) \mu_n(dx) \\
= &\int_X \exp(i\ell(P_n x)) \mu(dx) = \int_X \exp(i\ell(x)) \mu(dx).
\end{aligned}$$

Therefore, $\mu = \lambda$. Furthermore, since $\varphi_n \to \sqrt{\rho} =: \varphi$ in ν-measure as $n \to \infty$, it follows from the uniform ν–integrability of $\{\varphi_n^2 : n \in \mathbb{N}\}$ that

$$\varphi_n \underset{n\to\infty}{\longrightarrow} \sqrt{\rho} \quad \text{in } L^2(\nu). \tag{4.22}$$

Because of (4.21) by a standard argument (cf. e.g. [22, Chap. I, Lemma 2.12]) we hence obtain that $\varphi \in D(\mathcal{E}_\nu)$ and even

$$\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi \text{ weakly in } (D(\mathcal{E}_\nu), \, \mathcal{E}_\nu + (\ , \ )_{L^2(\nu)}) \tag{4.23}$$

and assertion (i) is proven.

(ii) and (iii): Let $k \in X^*$ and $h := j_H(k)$. Since $\beta_h^\nu := (h, \beta_H^\nu)_H = {}_{X^*}\langle k, \beta\rangle_X$, we get by assumption that $\beta_h^\nu \in L^2(\mu)$, i.e., $\varphi\beta_h^\nu \in L^2(\nu)$. Hence it follows by [24, Proposition 2.1] that β_H^μ exists with

$$\beta_H^\mu = \beta(x) + 2\varphi^{-1}\,\nabla_H\varphi. \tag{4.24}$$

which, in particular, proves (iii). It remains to show that $2\varphi^{-1}\nabla_H\varphi = u$. Since $\varphi \in D(\mathcal{E}_\nu)$, exactly the same argument as the one at the end of the proof for Claim 2 above shows that $2\nabla_H\varphi/\varphi$ is in the closure of $\{\nabla_H f \mid f \in \mathcal{F}C_b^\infty\}$ in the Hilbert space $L^2(\mu, H)$. More precisely, instead of [25, Theorems 2.3 and 3.1] one just has to apply [7, Corollary 1.6 and Theorem 2.3]. (Note that condition (1.7) in [7, Corollary 1.6] is obviously fulfilled by (4.10)). Furthermore, since μ satisfies (4.4) and $B = \beta + v$, it follows for all $f \in \mathcal{F}C_b^\infty$ that

$$\begin{aligned} 0 &= \int {}_{X^*}\langle f', B - \beta_H^\mu\rangle_X \; d\mu \\ &= \int (\nabla_H f, v - 2\nabla_H\varphi/\varphi)_H \; d\mu \end{aligned}$$

by (4.24). Hence $u = 2\nabla_H\varphi/\varphi$ and the proof is complete. □

Remark 4.4 (i) Condition (4.10) on ν is a stronger version of the so–called "Høegh–Krohn condition", i.e., the above mentioned condition (1.7) in [7]. It is known to be satisfied for a large class of measures ν which are absolutely continuous w.r.t. certain Gaussian measures with densities having bounded gradient, but with H not necessarily being equal to the reproducing kernel Hilbert space of the Gaussian measure. For details we refer to [25, Sect. 4] and [7, Corollary 1.6 including its proof]. Also additional conditions for the validity of a Log–Sobolev inequality in these cases are well–known (cf. e.g. [23, Theorem X.61] and [3]).

(ii) Another typical situation where the previous theorem applies is the case where the left–hand side of (4.10) is equal to zero. Note that this is the case in the Gaussian cases mentioned in (i) More generally, this condition is satisfied for product-measures $\nu = \prod_n \rho_n dx$ on $\mathbb{R}^\infty$ if ρ_n are equal probability densities from $H^{1,1}(\mathbb{R}^1)$ and $H = l^2$. Clearly, the same is true for linear images of such product-measures. The Log–Sobolev inequality holds in this case provided it holds for the one-dimensional density ρ_n (see [18]). Note also that our conditions are fulfilled for certain Gibbs measures (namely, those corresponding to the "almost free" models where the interaction of any finite number of particles with all other particles is controlled in H norm).

(iii) If $|v_H|$ is bounded, it follows by [13] (see also [26] for ν Gaussian) that μ as in Theorem 4.3 exists.

(iv) By [27, 28] it follows that in the situation of Theorem 4.3 there exists a Markovian strongly continuous semigroup $(T_t^\mu)_{t>0}$ on $L^2(\mu)$ whose generator extends $(L, \mathcal{F}C_b^\infty)$. Furthermore, according to the same reference there exists an associated diffusion process.

Theorem 4.5 *Let ν, β be as in Theorem 4.3 and $\varphi_0 \in D(\mathcal{E}_\nu)$ with $\varphi_0 \geq 0$ and $\int \varphi_0^2 \, d\nu = 1$.*

(i) Suppose μ is a probability measure on X satisfying (4.4) with $B := \beta + 2\nabla_H \varphi_0/\varphi_0$ and $|\nabla_H \varphi_0|_H/\varphi_0 \in L^2(\mu)$ (where $\nabla_H \varphi_0 := 0$ on $\{\varphi_0 = 0\}$). Then $\mu = \varphi^2 \cdot \nu$ for some $\varphi \in D(\mathcal{E}_\nu)$ and

$$\frac{\nabla_H \varphi}{\varphi} = \frac{\nabla_H \varphi_0}{\varphi_0} \quad \mu\text{-a.e.}$$

In particular, $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$.

(ii) If $|\nabla_H \varphi_0|_H/\varphi_0 \in L^2(\nu)$, then $\mu_0 := \varphi_0 \cdot \nu$ is the only probability measure μ on X such that μ satisfies (4.4) with $B := \beta + 2\nabla_H \varphi_0/\varphi_0$ and $|\nabla_H \varphi_0|_H/\varphi_0 \in L^2(\mu)$.

Proof. (i): By Theorem 4.3 there exists $\varphi \in D(\mathcal{E}_\nu)$ such that $\mu = \varphi^2 \cdot \nu$. (4.4) implies that

$$\int \left(\nabla_H f, \frac{\nabla_H \varphi_0}{\varphi_0} - \frac{\nabla_H \varphi}{\varphi} \right)_H d\mu = 0 \quad \text{for all } f \in \mathcal{F}C_b^\infty.$$

Hence it satisfies to show that $\nabla_H \varphi_0/\varphi_0$, $\nabla_H \varphi/\varphi$ are in the closure of $\{\nabla_H f \mid f \in \mathcal{F}C_b^\infty\}$ in $L^2(\mu, H)$. To this end for $k \in \mathbb{N}$ let $g_k : \mathbb{R} \to \mathbb{R}$, $g_k(t) := (t \wedge k) \vee (-k)$, $t \in \mathbb{R}$. Then for $\psi \in \{\varphi, \varphi_0\}$

$$\begin{aligned} \nabla_H [g_k(\log(\psi + \frac{1}{k}))] \quad &= 1_{\{e^{-k/2} - \frac{1}{k} < \psi < e^{+k/2} - \frac{1}{k}\}} \frac{\nabla_H \psi}{\psi + \frac{1}{k}} \\ &\underset{k \to \infty}{\longrightarrow} \frac{\nabla_H \psi}{\psi} \quad \text{in } L^2(\mu, H). \end{aligned}$$

But since $g_k(\log(\psi + \frac{1}{k})) \in L^2(\mu)$, it follows by (4.10) that [7, Corollary 1.6 and Theorem 2.3] apply to complete the proof of (i).

(ii): Clearly, μ_0 satisfies (4.4) with $B := \beta + 2\nabla_H\varphi_0/\varphi_0$, and, of course, $|\nabla_H\varphi_0|_H/\varphi_0 \in L^2(\mu_0)$. If μ is another such measure then by (i) $\mu = \varphi^2 \cdot \nu$, $\varphi \in D(\mathcal{E}_\nu)$ and $\nabla_H\varphi/\varphi = \nabla_H\varphi_0/\varphi_0$ μ–a.e. But since now $|\nabla_H\varphi_0|_H/\varphi_0 \in L^2(\nu)$, it follows by [8, Lemma 3.2] that $\log\varphi_0 \in D(\mathcal{E}_\nu)$, $\nabla_H \log\varphi_0 = \nabla_H\varphi_0/\varphi_0$, and that $\varphi_0 > 0$ ν–a.e. Replacing μ by $\frac{1}{2}(\mu + \mu_0)$ resp. φ^2 by $\frac{1}{2}(\varphi^2 + \varphi_0^2)$ we may assume that $\varphi \geq \frac{1}{\sqrt{2}}\varphi_0 > 0$ ν–a.e. Hence

$$\nabla_H\varphi/\varphi = \nabla_H\varphi_0/\varphi_0 \quad \nu\text{–.a.e} \tag{4.25}$$

thus

$$\frac{|\nabla_H\varphi|_H}{\varphi} \in L^2(\nu),$$

and as above it follows that $\log\varphi \in D(\mathcal{E}_\nu)$ and $\nabla_H \log\varphi = \nabla_H\varphi/\varphi$ ν–a.e. Consequently by (4.25)

$$\nabla_H(\log\varphi - \log\varphi_0) = 0 \quad \nu\text{–a.e.} \tag{4.26}$$

Since the Log–Sobolev inequality (4.9) holds for ν by assumption, it follows by the well–known Rothaus–Simon mass gap theorem (see. e.g. [18]) that ν satisfies Poincaré's inequality with $p = 2$. Hence (4.26) implies that

$$\log\varphi = \log\varphi_0 \quad \nu\text{–a.e.}$$

Since both $\mu_0 = \varphi_0^2 \cdot \nu$ and $\mu = \varphi^2 \cdot \nu$ are probability measures, it follows that $\varphi = \varphi_0$ ν–a.e. and assertion (ii) is proven. $\square$

Remark 4.6 If, as in Remark 1.8, we call μ satisfying (4.4) invariant measures, Theorem 4.5 (i), (ii) gives conditions when "invariant" implies "symmetrizing" resp. uniqueness of invariant measures. Theorem 4.5 (ii) immediately applies to the stochastic quantization of two– (space) dimensional Euclidean field theory in finite volume (cf. [20] for the terminology) with ν equal to the Euclidean free (Gaussian) field and φ_0 a renormalized polynomial integrated over the finite volume to give that $\varphi_0^2 \cdot \nu$ is the only invariant measure μ among all for which $|\nabla_H\varphi_0|_H/\varphi_0 \in L^2(\mu)$. We refer to [24, Sect. 7] for the precise details of the model. We only stress here that this statement is true for arbitrarily small regularization of the drift (i.e., with α in [24] resp.

$1-\varepsilon$ in [20] arbitrarily small). Finally, concerning results for the uniqueness of invariant measures in this model of other authors such as [17] we emphasize, that their notion of "invariant measure" is again different from ours. They define invariant measures in terms of the transition function of a particular (non–canonical) version of an associated diffusion process which in turn is constructed by a Girsanov transformation. That is, they consider the set Inv defined in Proposition A.3 below. But typically (because of $|\nabla_H \varphi_0|/\varphi_0 \in L^2(\nu)$) all such measures are equivalent, and since (e.g. by [19, Theorem 1.6]) Inv has extreme points, i.e., $\mathrm{Inv}_{ex} \neq \emptyset$, Proposition A.3 trivially implies that $\#\ \mathrm{Inv} = 1$ in this case.

Appendix

The following Poposition A.1 is a supplement to Corollary 2.3. It also renders simple proofs for the following Propositions A.3, A.5 (used in Remark 4.6) about the characterization of extremal invariant resp. symmetrizing measures of a semigroup of probability kernels. Both Proposition A.3, A.4 are, more or less, well–known to experts, but the elementary proofs given here appear to be new.

Proposition A.1 *In the situation of Corollary 2.3 assertions (i)–(iv) are in addition equivalent to:*
(v) If $f \in L^1(\mu)$ such that $T_t f = f$ for all $t > 0$, then $f =$ const.

For the proof we shall use the following

Lemma A.2 *Consider the situation of Corollary 2.3. Assume that S_t, $t > 0$, are sub–Markovian bounded linear operators on $L^1(\mu)$ and let ν be a probability measure on $(X, \mathcal{B})$, which is absolutely continuous w.r.t. μ and (S_t)–invariant. Suppose that assertion (v) holds and that for all $B \in \mathcal{B}$*

$$S_t 1_B = 0 \ \nu\text{–a.e. for all } t > 0 \text{ implies } \ T_t 1_B = 0 \ \nu\text{–a.e. for all } t > 0 \ . \tag{A.1}$$

Then ν is equivalent to μ.

Proof. Let $\nu = g\mu$ and set $Z := \{g > 0\}$. Let $t > 0$. Then $0 \le S_t 1_Z \le 1$ and hence

$$\nu(Z) = \int_X S_t 1_Z \, d\nu = \int_Z S_t 1_Z \, d\nu \le \nu(Z) \ .$$

Therefore, $S_t 1_Z = 1$ ν–a.e. and hence (A.1) implies that $T_t 1_Z = 1$ ν–a.e., i.e., $T_t 1_Z = 1$ μ–a.e. on Z. Consequently,

$$\mu(Z) = \int_X T_t 1_Z \, d\mu \geq \int_Z T_t 1_Z \, d\mu = \mu(Z),$$

and thus, $T_t 1_Z = 1_Z$ μ–a.e. Since $t > 0$ was arbitrary, (v) implies that $\mu(Z) = 1$. □

Proof of Proposition A.1. (v) $\Longrightarrow$ Corollary 2.3 (ii): This follows from the previous lemma applied to the case $T_t = S_t$, $t > 0$.

Corollary 2.3 (i) $\Longrightarrow$ (v): As in the proof of Proposition 2.2 for $t > 0$, let T_t^* denote the extension to $L^1(\mu)$ of the dual operator on $L^2(\mu)$ of the restriction of T_t to $L^2(\mu)$. We note that by Remark 2.4 μ is also (T_t^*)–invariant.

Claim. Let $f \in L^1(\mu)$ such that $T_t^* f = f$ for all $t > 0$. Then $f = const.$

By Lemma 2.1 may assume that f is bounded, that $f \geq 0$ and that $\int f \, d\mu = 1$. But then $f \cdot \mu \in \mathcal{M}_\mu$, hence $f \equiv 1$ by assumption (i) in Corollary 2.3 and the claim is proven.

Let now $f \in L^1(\mu)$ such that $T_t f = f$ for all $t > 0$. By Lemma 2.1 we may assume that $f \geq 0$ and that $\int f \, d\mu = 1$. Furthermore, Lemma 2.1 implies that $T_t g = g$ for all $t > 0$ where $g := 1 - f \wedge 1$. If $\alpha := \int g \, d\mu = 0$, then $g = 0$, hence $f = 1$ μ–a.e. and we are done. If $\alpha > 0$, then $\nu := \alpha^{-1} g \mu$ is (T_t^*)–invariant. Applying Lemma A.2 with $T_t := S_t := T_t^*$, $t > 0$, we conclude that ν is equivalent to μ, hence $g > 0$ μ–a.e., i.e., $f < 1$ μ–a.e. But this contradicts the fact that $\int f \, d\mu = 1$, so indeed always $\alpha = 0$ and (v) is proven. □

Proposition A.3 *Let $(p_t)_{t>0}$ be a semigroup of probability kernels on a measurable space $(X, \mathcal{B})$. Let* Inv *be the (convex) set of all probability measures on $(X, \mathcal{B})$ which are invariant for $(p_t)_{t>0}$, and* $\mathrm{Inv}_{\mathrm{ex}}$ *its extreme points. Let $\mu \in$* Inv *and let $(T_t^\mu)_{t>0}$ denote the semigroup of Markovian operators on $L^1(\mu)$ naturally induced by $(p_t)_{t>0}$ (cf., e.g., [22, Proposition II.4.1]). Then the following assertions are equivalent:*

(i) $\mu \in \mathrm{Inv}_{\mathrm{ex}}$.

(ii) If $\rho \in L^1(\mu)$ is such that $\rho\mu \in \mathrm{Inv}$, then $\rho = 1$ μ-a.e.

(iii) If $\rho \in L^1(\mu)$ is such that $\rho\mu \in \mathrm{Inv}$, then $\rho > 0$ μ-a.e.

(iv) If $f \in L^1(\mu)$ is such that $T_t^\mu f = f$ for all $t > 0$, then f coincides with a constant μ-a.e.

Proof. (i) $\Longrightarrow$ (ii): Suppose $\mu \in \mathrm{Inv}_{\mathrm{ex}}$ and let $\rho\mu \in \mathrm{Inv}$. Let $\alpha := \int \rho \wedge 1 \, d\mu$ (> 0). If $\alpha < 1$, then by Proposition 2.2

$$\mu_1 := \alpha^{-1}(\rho \wedge 1)\, \mu \in \mathrm{Inv}, \quad \mu_2 := (1-\alpha)^{-1}(1 - \rho \wedge 1)\, \mu \in \mathrm{Inv}.$$

But $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, and thus, since μ is extreme, $\rho = 1$ μ-a.e. If $\alpha = 1$, then $\rho \leq 1$ μ–a.e., hence $\rho = 1$ μ–a.e., since $\rho\mu$ is a probability measure.

(ii) $\Longrightarrow$ (iii): This is trivial.

(iii) $\Longrightarrow$ (i): Let $\alpha \in]0,1[$; $\mu_1, \ \mu_2 \in \mathrm{Inv}$ such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$. Then $\mu_1 = \rho_1\mu$, $\mu_2 = \rho_2\mu$ for some $\rho_1, \ \rho_2 \in L^1(\mu)$. By assumption (iii) and Corollary 2.3, (ii) $\Longrightarrow$ (i) (with $T_t := T_t^\mu$, $t > 0$), it follows that $\rho_i = 1$ μ-a.e., $i = 1, \ 2$. Hence μ is extreme.

(ii) $\Longleftrightarrow$ (iv): This follows from Proposition A.1 $\square$

Remark A.4 (i) Proposition 2.2 immediately implies that in Proposition A.3 (iv) we may replace $L^1(\mu)$ by $L^\infty(\mu)$.

(ii) We note that for each $t > 0$ the restriction of T_t^μ to $L^p(\mu)$, $p > 1$, is a bounded linear operator on $L^p(\mu)$. We shall use this fact below without further notice and denote these restrictions again by T_t^μ, $t > 0$.

(iii) In Proposition A.3 as well as A.5 below $(p_t)_{t>0}$ can be replaced by any family $(p_t)_{t\in I}$ (with an arbitrary index set I) of probability kernels on $(X, \mathcal{B})$. The semigroup property has never been used.

Proposition A.5 *Let $(p_t)_{t>0}$ be a semigroup of probability kernels on a measurable space $(X, \mathcal{B})$. Let Sym be the (convex) set of all probability measures μ on $(X, \mathcal{B})$ which are symmetric for $(p_t)_{t>0}$, (i.e., $\int f\, p_t\, g \, d\mu = \int g\, p_t\, f \, d\mu$ for all bounded, $\mathcal{B}$–measurable $f, g : X \to \mathbb{R}$ and all $t > 0$) and let $\mathrm{Sym}_{\mathrm{ex}}$ denote its extreme points. Then the following assertions are equivalent for $\mu \in \mathrm{Sym}$:*

(i) $\mu \in \mathrm{Sym}_{\mathrm{ex}}$.

(ii) If $\rho \in L^1(\mu)$ is such that $\rho\mu \in \mathrm{Sym}$, then $\rho = 1$ μ-a.e.

(iii) If $\rho \in L^1(\mu)$ is such that $\rho\mu \in$ Sym, then $\rho > 0$ μ-a.e.

(iv) If $f \in L^1(\mu)$ is such that $T_t^\mu f = f$ for all $t > 0$, then f coincides with a constant μ-a.e.

In particular, Sym is a face *in Inv, i.e., Sym $\subset$ Inv and Sym$_{ex} \subset$ Inv$_{ex}$.*

Proof. First note that Sym $\subset$ Inv and that hence T_t^μ, $t > 0$, are defined as in Proposition A.3. Then the proof is word by word the same as that of Proposition A.3 except that we have to replace Inv by Sym. $\square$

Remark A.6 Properties (i)–(iv) in Proposition A.5 are, in addition, equivalent to the following property:
(v) If $f \in L^\infty(\mu)$ such that $T_t^\mu(fg) = fT_t^\mu g$ for all $g \in L^1(\mu)$, $t > 0$, then f coincides with a constant μ–a.e.

This is an immediate consequence of a theorem due to Douglas (see [32, Chap. V, Theorem (4.4)]; take $\mathcal{L} := \{f\, p_t\, g - g\, p_t\, f \mid t > 0;\ f, g : X \to \mathbb{R}$, bounded, $\mathcal{B}$–measurable$\}$). This fact was pointed out to us by Marc Yor, whom we would like to thank at this point.

For completeness we add the following two corollaries extending the characterizations of extremality in Proposition A.5 in terms of properties of the corresponding Dirichlet form resp. the ergodicity of the semigroup.

Corollary A.7 *Consider the situation of Proposition A.5 and assume, in addition, that $(T_t^\mu)_{t>0}$ is strongly continuous on $L^2(\mu)$. Let $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$, $(L_\mu, D(L_\mu))$ denote its corresponding Dirichlet form and generator on $L^2(\mu)$ (cf. [22, Chapter I]). Then (i) – (iv) in Proposition A.5 are also equivalent to*

(v) If $u \in D(L_\mu)$ with $\mathcal{E}_\mu(u, u) = 0$, then $u = const$ μ-a.e.;

(vi) If $u \in D(L_\mu)$ with $L_\mu u = 0$, then $u = const$ μ-a.e.;

(vii) $\lim\limits_{t\to\infty} \int (T_t^\mu u - \int u\, d\mu)^2\, d\mu = 0$ for all $u \in L^2(\mu)$ (i.e., $(T_t^\mu)_{t>0}$ is ergodic).

Proof. See the appendix in [4]. $\square$

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References

[1] Albeverio, S., and Høegh-Krohn, R., *Dirichlet forms and diffusion processes on rigged Hilbert spaces.* Z. Wahr. theor. verw. Geb. 40, 1977, pp. 1–57.

[2] Albeverio, S., Kondratiev, Yu.G., and Röckner, M., *Dirichlet operators via stochastic analysis.* J. Funct. Anal. 128, 1995, pp. 102–138.

[3] Aida, S., and Shigekawa, I., *Logarithmic Sobolev inequalities and spectral gaps: perturbation theory.* J. Funct. Anal. 126, 1994, pp. 448–475.

[4] Albeverio, S., Kondratiev, Yu. G., and Röckner, M., *Ergodicity of L^2–semigroups and extremality of Gibbs states.* J. Funct. Anal. 144, 1997, pp. 394–423.

[5] Albeverio, S., Kusuoka, S., and Röckner, M., *On partial integration in infinite dimensional space and applications to Dirichlet forms.* J. London Math. Soc. 42, 1990, pp. 122–136.

[6] Albeverio, S., and Röckner, M., *Dirichlet forms on topological vector spaces – closability and a Cameron–Martin formula.* J. Funct. Anal. 88, 1990, pp. 395–436.

[7] Albeverio, S., Röckner, M., and Zhang, T.S., *Markov uniqueness for a class of infinite dimensional Dirichlet operators.* pp. 1–26 in: *Stochastic Processes and Optimal Control. Stochastic Monographs Vol. 7*, H.J. Engelbert et al. eds., Gordon & Breach. 1993.

[8] Albeverio, S., Röckner, M., and Zhang, T.S., *Girsanov transform for symmetric diffusions with infinite dimensional state space.* The Annals of Probability 21, 1993, pp. 961–978.

[9] Bogachev, V.I., *Differentiable measures and the Malliavin calculus.* Scuola Normale Superiore di Pisa, Preprint N 16 (1995), to appear in J. Math. Sci.

[10] Bogachev, V.I., Krylov, N.V., and Röckner, M., *Elliptic regularity and essential self-adjointness of Dirichlet operators.* Preprint 96–038 SFB 343, 1996, to appear in Ann. Scuola Normale Super. Pisa.

[11] Bogachev, V.I., and Röckner, M., *Regularity of invariant measures on finite and infinite dimensional spaces and applications.* J. Funct. Anal. 133, 1995, pp. 168–223.

[12] Bogachev, V.I., and Röckner, M., *A generalization of Hasminskii's theorem on existence of invariant measures for locally integrable drifts.* SFB–343–Preprint 1998, to appear in Theor. Probab. Appl.

[13] Bogachev, V.I., Röckner, M., and Zhang, T.S., *Existence and uniqueness of invariant measures: an approach via sectorial forms.* SFB–343–Preprint 1997, to appear in Appl. Math. Optim.

[14] Bogachev, V.I., and Smolyanov, O.G., *Analytic properties of infinite dimensional distributions.* Russian Math. Surveys 45, 1990, n 3, pp., 1–104.

[15] Cattiaux, P., and Fradon,M., *Entropy, reversible diffusion processes, and Markov uniqueness.* J. Funct. Anal. 138, 1996, n 1, pp. 243–272.

[16] Fukushima, M., Oshima, Y., and Takeda, M., *Dirichlet forms and symmetric Markov processes* Walter de Gruyter, Berlin – New York, 1994.

[17] Gatarek, D., and Goldys, B., *Existence, uniqueness and ergodicity for stochastic quantization equations.* Preprint 1995.

[18] Gross, L., *Logarithmic Sobolev inequalities and contractive properties of semigroups.* In: Lect. Notes Math. 1563, 1993, pp. 54–82.

[19] Hinssen, J., and Janssen, K., *Integral representation of invariant functions.* preprint 1996, to appear in Potential Analysis.

[20] Jona-Lasinio, G., and Mitter. P.K., *On the stochastic quantization of field theory.* Commun. Math. Phys. 101, 1985, pp. 406–436.

[21] Liskevich, V.A., and Semenov, Yu.A., *Some problems on Markov semigroups.* preprint 1995.

[22] Ma, Z.M., and Röckner, M., *An introduction to the theory of (non-symmetric) Dirichlet forms.* Springer, Berlin, 1992.

[23] Reed, M., and Simon, B., *Methods of modern mathematical physics II. Fourier Analysis, self–adjointness.* Academic Press, New York 1975.

[24] Röckner, M., and Zhang, T.S., *Uniqueness of generalized Schrödinger operators and applications.* J. Funct. Anal. 105, 1992, pp. 187–231.

[25] Röckner, M., and Zhang, T.S., *Uniqueness of generalized Schrödinger operators, Part II.* J. Funct. Anal. 119, 1994, pp. 455–467.

[26] Shigekawa, I., *Existence of invariant measures of diffusions on an abstract Wiener space.* Osaka J. Math. 24, 1987, n 1, pp. 37–59.

[27] Stannat, W., *First order perturbations of Dirichlet operators: existence and uniqueness.* J. Funct. Anal. 141, 1996, pp. 216–248.

[28] Stannat, W., *Generalized Dirichletforms and first–order perturbations of Dirichlet operators.* manuscript 1996, publication in preparation.

[29] Trudinger, N.S., *Linear elliptic operators with measurable coefficients.* Ann. Scuola Normale Super. Pisa (3) 27, 1973, pp. 265–308.

[30] Vakhania, N.N., Tarieladze, V.I., and Chobanyan, S.A., *Probability distributions in Banach spaces.* Kluwer Academic Publ., 1991.

[31] Varadhan, S.R.S., *Lectures on diffusion problems and partial differential equations.* Tata Institute of Fundamental Research, Bombay, 1980.

[32] Revuz, D., and Yor, M., *Continuous martingales and Brownian motion.* Springer, Berlin 1991.