Topological Invariance of Chow Varieties

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Abstract

By definition Chow varieties are invariants of embeddings of algebraic varieties in projective spaces. Hoyt showed that the algebraic homeomorphism type is determined by the algebraic isomorphism class of the variety. In this paper a rectified version of his proof is given. The use of resultants reduces to the problem, whether the (ring theoretic) greatest common devisor of a set of polynomials can be constructed via an algebraic map. The argument works for arbitrary characteristic. As an appendix we give a short description of the construction of Chow varieties in characteristic $p \neq 0$.

Introduction

In [6] Chow varieties of complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$ are defined using resultants. The resultant of a variety in $\mathbb{P}^n_{\mathbb{C}}$ is a principal ideal in a polynomial ring over $\mathbb{C}$ which is generated by the Chowform of the variety. This form is unique up to a factor in $\mathbb{C}^*$. Hence its coefficients define a point in $\mathbb{P}^N_{\mathbb{C}}$, where N only depends on the dimension, codimension, and degree of V. By multiplication of Chow forms one gets a definition for Chow forms of formal sums of algebraic varieties of fixed dimension. Chow forms of arbitrary algebraic cycles are defined by linear extension. Fixing dimension r, codimension $n-r$, and degree d of cycles the coefficients of the Chow forms of the cycles build an algebraic set, which is called Chow variety $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$.

The advantage of viewing Chow forms as generators of resultants is, that intersection of a variety with a hyperplane is given on Chow forms by evaluating the Chowform on the coefficients of the defining equation of the hyperplane.

Chow varieties of algebraic subsets $X \subset \mathbb{P}^n_{\mathbb{C}}$ are defined by restriction to cycles with support on X. Defining Chow varieties of arbitrary projective algebraic sets X now means choosing an embedding $i : X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ and taking the Chow variety $\mathcal{C}_{r,d}(X, i)$ of $i(X)$.

For the definition of Lawson homology one needs the homotopy type independent of the embedding i. Hoyt [4] proved the homeomorphism type of $\coprod_{d\geq 0} \mathcal{C}_{r,d}(X,i)$ to be independent of embeddings. The purpose of this note is to clarify and straighten out Hoyt's approach for arbitrary fields. In fact by Barlet [1] one knows that $\mathcal{C}_{r,d}(X,i)$ is independent of i up to isomorphism of algebraic sets in case of $K=\mathbb{C}$. On the other hand Nagata [5] shows that the Chow variety of 0-dimensional cycles of degree 2 in $\mathbb{P}^3_{\mathbb{Z}_2}$ is not normal, but there is an embedding into some $\mathbb{P}^n_{\mathbb{Z}_2}$ such that the Chow variety of the embedded $\mathbb{P}^3_{\mathbb{Z}_2}$ becomes normal. Hence those two Chow varieties can not be isomorphic algebraic sets.

As an appendix we describe the construction of Chow forms over arbitrary fields in section 3. We will explain how the Chow variety can be defined in case of inseparability. For the same reasons the Chowform of a variety is the generator of the resultant as in the case $K=\mathbb{C}$.

In the first section we will reduce the proof of Hoyt's theorem to the problem of determining the greatest common divisor of a set of multihomogeneous polynomials. The algebraicity of the greatest common divisor in the coefficients of the multihomogeneous polynomials is shown in section 2.

We will consider projective algebraic sets whose vanishing ideals are defined over an arbitrary field K, morphisms of algebraic sets are K-morphisms. The zero sets of such an homogeneous ideal are viewed as subsets of $\mathbb{P}^n_\Omega$, where Ω is an algebraically closed field extension of K of countably infinite transcendence degree over K. Irreducible algebraic sets are called varieties. In this sense the name Chow variety may be a bit confusing, because in general Chow varieties are reducible.

Points in $\mathbb{P}^n$ are denoted $\underline{p} := (p_0,\ldots,p_n)$. For a field K we denote by $K[\underline{p}]$ the ring $K[p_0,\ldots,p_n]$, where the representative $(p_0,\ldots,p_n)$ is chosen such that all non vanishing p_i are transcendental over K. $K(\underline{p})$ is the quotient field of $K[\underline{p}]$.

1 Independence of Chow varieties of embeddings

Definition 1.1. Let X be a projective variety (over K) with a (K-)embedding $j : X \hookrightarrow \mathbb{P}^n_\Omega$. Denote by $\mathcal{C}_{r,d}(X,j)$ the Chow variety of cycles V, such that $\deg V = d$ and $supp(V) \subset j(X)$. Further let $\mathcal{C}_r(X,j) := \coprod_{d\geq 0} \mathcal{C}_{r,d}(X,j)$.

Generally we know that Chow varieties of projective varieties depend on embeddings [5]. But in the case of ground field $\mathbb{C}$ the independence of the chosen embedding arises from analytic arguments as we will see below. In the case of characteristic $p \neq 0$ the algebraic sets can be topologized using the methods of Friedlander [3]. Then algebraic maps become continuous. This leads to the following definition.

Definition 1.2. A **continuous algebraic map** is a map $\varphi : V \to W$ — in the category of sets — between projective algebraic sets whose graph is an algebraic subset of the product $V \times W$.

In fact every algebraic map of projective subsets is continuous algebraic, but a continuous algebraic map need not be algebraic. We want to prove the following

Theorem 1.3. *Let X be a projective variety over an arbitrary field. The definition of $\mathcal{C}_r(X) := \mathcal{C}_r(X,j)$ is independent of a chosen embedding $j : X \hookrightarrow \mathbb{P}^n_\Omega$ up to algebraic homeomorphism.*

By Barlet [1, Thm. 6,p. 109] we know the space of analytic cycles to be independent of embeddings up to analytic isomorphism. Since the space of analytic cycles and the space of algebraic cycles are analytically isomorphic in case $K = \mathbb{C}$ ([1, Thm. 7]), we know by GAGA that $\mathcal{C}_r(X)$ is independent of embeddings up to isomorphism in case $K = \mathbb{C}$. This already proves Theorem 1.3 in the case of complex numbers.

In the following we want to prove Theorem 1.3 for K an arbitrary field. Suppose there are two embeddings $i : X \hookrightarrow \mathbb{P}^n_\Omega$ and $j : X \hookrightarrow \mathbb{P}^m_\Omega$. Then we get a third embedding by

$$i * j : X \xrightarrow{\Delta} X \times X \xrightarrow{i \times j} \mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega \xrightarrow{S} \mathbb{P}^{nm+n+m}_\Omega$$

where S denotes the Segre embedding. To prove the theorem it suffices to check that the projection

$$S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega) \cong \mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega \xrightarrow{pr_1} \mathbb{P}^n_\Omega$$

induces an algebraic map

$$\mathcal{C}_r(X, i * j) \xrightarrow{(pr_1)_*} \mathcal{C}_r(X, i)$$

for $((pr_1)_*, (pr_2)_*)$ then maps $\mathcal{C}_r(X, i * j)$ onto the graph of $(j \circ i^{-1})_*$: $\mathcal{C}_r(X,i) \to \mathcal{C}_r(X,j)$. Further images of projective algebraic sets under morphisms are closed, hence the graph of $(j \circ i^{-1})_*$ is closed by [7, I.5.2 Th. 2].

First we want to determine the map $(pr_1)_*$ for a fixed cycle V. Without loss of generality let $V \subset X$ be a variety. Let $F_{i(V)}$ be the Chow form of $i(V)$ and $F_{i*j(V)}$ the Chow form of $i * j(V)$.

We need some notations: For $\underline{a} \in \mathbb{P}^n_\Omega$ and $\underline{b} \in \mathbb{P}^m_\Omega$ let $H(\underline{a})$ denote the hyperplane in $\mathbb{P}^n_\Omega$ given by $a_0x_0 + \cdots + a_nx_n = 0$ and $H(S(\underline{a}, \underline{b}))$ the hyperplane in $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ given by $a_0b_0z_{0,0} + a_0b_1z_{0,1} + \cdots + a_nb_mz_{n,m} = 0$.

1.4. Obviously $S(\underline{p},\underline{q}) \in H(S(\underline{a},\underline{b})) \Longleftrightarrow \underline{p} \in H(\underline{a})$ or $\underline{q} \in H(\underline{b})$ as we have

$$\begin{aligned} & a_0b_0 \cdot p_0q_0 + a_0b_1 \cdot p_0q_1 + \cdots + a_nb_m \cdot p_nq_m = 0 \\ \Longleftrightarrow \quad & (a_0p_0 + \cdots + a_np_n)(b_0q_0 + \cdots + b_mq_m) = 0 \end{aligned}$$

To determine $(pr_1)_*$ for a fixed cycle we will make some set theoretic rewritings using several times the following fundamental fact which arises from the property of Chow forms being the generator of resultants (Thm. 3.4):

1.5. For a variety $W \subset \mathbb{P}^n_\Omega$ with Chow form F_W and $\underline{a}^0,\ldots,\underline{a}^r \in \mathbb{P}^n_\Omega$ one has:

$$F_W(\underline{a}^0,\ldots,\underline{a}^r) = 0 \Longleftrightarrow W \cap H(\underline{a}^0) \cap \cdots \cap H(\underline{a}^r) \neq \emptyset\,.$$

For $i * j(V)$ we now obtain for fixed $\underline{a}^0,\ldots,\underline{a}^r \in \mathbb{P}^n_\Omega$:

$$\begin{aligned} & F_{i*j(V)}(S(\underline{a}^0,\underline{b}^0), S(\underline{a}^1,\underline{b}^1),\ldots,S(\underline{a}^r,\underline{b}^r)) = 0 \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \Longleftrightarrow \quad & \bigcap_{k=0}^{r} H(S(\underline{a}^k,\underline{b}^k)) \cap (i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \end{aligned}$$

Suppose $\underline{P} \in i*j(V)$, that means $\underline{P} = S(\underline{p},\underline{q})$ such that $j(i^{-1}(\underline{p})) = \underline{q}$. Using 1.4 we obtain

$$\begin{aligned} & \bigcap_{k=0}^{r} H(S(\underline{a}^k,\underline{b}^k)) \cap (i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \Longleftrightarrow \quad & \bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega\,. \end{aligned}$$

Lemma 1.6.

$$\begin{aligned} & \bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) \neq \emptyset \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega \\ \Longleftrightarrow \quad & \bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) \neq \emptyset\,. \end{aligned}$$

Proof: For $\underline{p} \in \bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V)$ we know that $(\underline{p}, j(i^{-1}(\underline{p})))$ is an element of $\bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V))$ for any $\underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega$. Hence it remains to show, if $\bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) = \emptyset$ there exist $\underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}^m_\Omega$ such that $\bigcap_{k=0}^{r} \left(H(\underline{a}^k) \times \mathbb{P}^m_\Omega \cup \mathbb{P}^n_\Omega \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) = \emptyset$.

Now $\bigcap_{k=0}^{r} H(\underline{a}^k) \cap i(V) = \emptyset$ means

$$\dim\left(H(\underline{a}^k) \cap i(V) \cap \bigcap_{l\in L_k} H(\underline{a}^l)\right) \neq \dim\left(i(V) \cap \bigcap_{l\in L_k} H(\underline{a}^l)\right)$$

for each $L_k \subset \{0,\ldots,r\}-\{k\}$. Now choose b_l^k, $l=0,\ldots,m$, $k=0,\ldots,r$ algebraically independent over $K(\underline{a}^0,\ldots,\underline{a}^r)$. Then $\mathbb{P}_\Omega^m \times H(\underline{b}^k)$ meets

$$S^{-1}(i*j(V)) \cap \left(\bigcap_{l\in L_k} H(\underline{a}^l)\times \mathbb{P}_\Omega^m\right) \cap \left(\bigcap_{l\in \overline{L}_k} \mathbb{P}_\Omega^n \times H(\underline{b}^l)\right)$$

transversally for each $L_k \subset \{0,\ldots,r\}-\{k\}$ and $\overline{L}_k = \{0,\ldots,r\}-(\{k\}\cup L_k)$. But this means $\bigcap_{k=0}^{r} \left(H(\underline{a}^k)\times \mathbb{P}_\Omega^m \cup \mathbb{P}_\Omega^n \times H(\underline{b}^k)\right) \cap S^{-1}(i*j(V)) = \emptyset$ for the chosen $\underline{b}^0,\ldots,\underline{b}^r$. $\square$

Using again 1.5 we have already proved

$$\begin{aligned} & F_{i*j(V)}(S(\underline{a}^0,\underline{b}^0),S(\underline{a}^1,\underline{b}^1),\ldots,S(\underline{a}^r,\underline{b}^r)) = 0 \quad \forall \underline{b}^0,\ldots,\underline{b}^r \in \mathbb{P}_\Omega^m \\ \iff\ & F_{i(V)}(\underline{a}^0,\ldots,\underline{a}^r) = 0 \end{aligned}$$

1.7. Now let $u_k^s, v_l^s \in \Omega$ be algebraically independent over K, $s=0,\ldots,r$, $k=0,\ldots,n$, $l=0,\ldots,m$. The above argument shows that

$$F_{i*j(V)}(S(\underline{u}^0,\underline{v}^0),\ldots,S(\underline{u}^r,\underline{v}^r)) = F_{i(V)}(\underline{u}^0,\ldots,\underline{u}^r)^t G(\underline{u}^0,\ldots,\underline{u}^r,\underline{v}^0,\ldots,\underline{v}^r)$$

for some $t\in\mathbb{N}^*$, and $G\in K[\underline{u}^0,\ldots,\underline{u}^r,\underline{v}^0,\ldots,\underline{v}^r]$ contains no factor in $K[\underline{u}^0,\ldots,\underline{u}^r]$.

If we could check $t=1$, the map $(pr_1)_*$ would be given pointwise by determining a unique factor of the evaluated Chow form. To prove $t=1$ it suffices to show that $H(S(\underline{u},\underline{v}))$ and $i*j(V)$ meet transversally.

First we will check that $H(S(\underline{u},\underline{v}))$ meets $S(\mathbb{P}_\Omega^n\times\mathbb{P}_\Omega^m)$ transversally in all "sufficiently general" points of $H(S(\underline{u},\underline{v}))\cap S(\mathbb{P}_\Omega^n\times\mathbb{P}_\Omega^m)$. If $S(\underline{p},\underline{q})$ is an element of the intersection, then we know by 1.4 that $\underline{p}\in H(\underline{u})$ or $\underline{q}\in H(\underline{v})$. By a sufficiently general point we think of a point $S(\underline{p},\underline{q})$ of the intersection satisfying the following condition: In case $\underline{p}\in H(\underline{u})$ the elements $v_0,\ldots,v_m$ are still algebraically independent over $K(\underline{p})$. Or symmetrically in case $\underline{q}\in H(\underline{v})$ the $u_0,\ldots,u_n$ are algebraically independent over $K(\underline{q})$.

Lemma 1.8. *Let $u_0,\ldots,u_n,v_0,\ldots,v_m\in\Omega$ be algebraically independent over K. Suppose $S(\underline{p},\underline{q})\in H(S(\underline{u},\underline{v}))\cap S(\mathbb{P}_\Omega^n\times\mathbb{P}_\Omega^m)$ such that $v_0,\ldots,v_m$ are algebraically independent over $\overline{K(\underline{q})}$. Then $H(S(\underline{u},\underline{v}))$ meets $S(\mathbb{P}_\Omega^n\times\mathbb{P}_\Omega^m)$ transversally in $S(\underline{p},\underline{q})$.*

Proof: Without loss of generality let $p_0q_0\neq 0$.
Let $f_{i,j} := z_{0,0}z_{i,j} - z_{i,0}z_{0,j}$, $i=1,\ldots,n$, $j=1,\ldots,m$.

Step 1: The zero set of $A \cdot \underline{z}$, with

$$A := \left(\frac{\delta f_{i,j}}{\delta z_{k,l}} \left(S(\underline{p}, \underline{q}) \right) \right)_{(i,j),(k,l)} = \begin{pmatrix} \frac{\delta f_{1,1}}{\delta z_{0,0}}(S(\underline{p}, \underline{q})) & \cdots & \frac{\delta f_{1,1}}{\delta z_{n,m}}(S(\underline{p}, \underline{q})) \\ \vdots & & \vdots \\ \frac{\delta f_{n,m}}{\delta z_{0,0}}(S(\underline{p}, \underline{q})) & \cdots & \frac{\delta f_{n,m}}{\delta z_{n,m}}(S(\underline{p}, \underline{q})) \end{pmatrix}$$

is the tangent space of $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ in $S(\underline{p}, \underline{q})$.

To check this assertion we need A to have rank nm. We look at the nm columns $k = 1, \ldots, n$, $l = 1, \ldots, m$. Then we obtain the following submatrix of A:

$$\begin{pmatrix} p_0 q_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_0 q_0 \end{pmatrix}$$

which has rank nm.

Step 2: For the transversality of the intersection it remains to prove that adding the row $(u_0 v_0, \ldots, u_n v_m)$ to A enlarges the rank to $nm + 1$. We look at the columns (k, l) where $k = 1, \ldots, n$, $l = 1, \ldots, m$ and $(n, 0)$:

$$B := \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_{n-1} v_m & u_n v_1 & \cdots & \cdots & u_n v_m & u_n v_0 \\ p_0 q_0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & & & \vdots \\ \vdots & & \ddots & p_0 q_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & 0 & p_0 q_0 & 0 & \cdots & 0 & -p_0 q_1 \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots & -p_0 q_2 \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & p_0 q_0 & -p_0 q_m \end{pmatrix}$$

Now it suffices to show $\det B \neq 0$:

$$\det B \;=\; -(p_0 q_0)^{m(n-1)} \cdot \det \begin{pmatrix} u_n v_1 & \cdots & \cdots & u_n v_m & u_n v_0 \\ p_0 q_0 & 0 & \cdots & 0 & -p_0 q_1 \\ 0 & \ddots & \ddots & \vdots & -p_0 q_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & p_0 q_0 & -p_0 q_m \end{pmatrix}$$

$$
\begin{aligned}
&= -(p_0q_0)^{m(n-1)}u_np_0^m \cdot \det \begin{pmatrix} v_1 & \cdots & \cdots & v_m & v_0 \\ q_0 & 0 & \cdots & 0 & -q_1 \\ 0 & \ddots & \ddots & \vdots & -q_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & q_0 & -q_m \end{pmatrix} \\
&\neq 0
\end{aligned}
$$

because $v_0, \ldots v_m$ are algebraically independent over $K(\underline{q})$. $\square$

As a byproduct we obtain another proof of the well known result:

Remark 1.9. $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ has degree $\binom{n+m}{n}$.

Proof: Let $u_k^l, v_s^l \in \Omega$ be algebraically independent over K, $k = 0, \ldots, n$, $s = 0, \ldots, m$, $l = 1, \ldots, n+m$. Now we want to determine the set $H(S(\underline{u}^1, \underline{v}^1)) \cap \cdots \cap H(S(\underline{u}^{n+m}, \underline{v}^{n+m})) \cap S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$. Suppose $S(\underline{p}, \underline{q})$ is one point of this intersection. Then there is a subset $L \subset \{1, \ldots, n+m\}$ with n elements such that $\underline{p} \in \bigcap\limits_{l \in L} H(\underline{u}^l)$. Further a representative of $\underline{p}$ is contained in $\left(\overline{K(\underline{u}^l, l \in L)}\right)^{n+1}$.
Now $\underline{q}$ must be a point of $\bigcap\limits_{l \in \widetilde{L}} H(\underline{v}^l)$, $\widetilde{L} = \{1, \ldots, n+m\} - L$. And a representative of $\underline{q}$ is contained in $\left(\overline{K(\underline{v}^l, l \in \widetilde{L})}\right)^{m+1}$.
There are $\binom{n+m}{n}$ different sets L. Therefore the intersection consists of $\binom{n+m}{n}$ different points.
If each hyperplane $H(S(\underline{u}^l, \underline{v}^l))$ cuts $S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega)$ transversally in $S(\underline{p}, \underline{q})$ the proof is finished. But for $l \in L$ the $v_0^l, \ldots, v_m^l$ are algebraically independent over $\overline{K(\underline{q})}$, hence we know by Lemma 1.8 that $H(\underline{u}^l, \underline{v}^l)$ cuts transversally in $S(\underline{p}, \underline{q})$. For $l \in \widetilde{L}$ it cuts transversally for symmetrical reasons. $\square$

Now we can prove the multiplicity t in 1.7 to be actually one:

Theorem 1.10. *Let $u_k^s, v_l^s \in \Omega$ be algebraically independent over K, $s = 0, \ldots, r$, $k = 0, \ldots, n$, $l = 0, \ldots, m$. Then we have*

$$
\begin{gathered}
F_{i*j(V)}(S(\underline{u}^0, \underline{v}^0), S(\underline{u}^1, \underline{v}^1), \ldots, S(\underline{u}^r, \underline{v}^r)) = \\
F_{i(V)}(\underline{u}^0, \ldots, \underline{u}^r) \cdot G(\underline{u}^0, \ldots, \underline{u}^r, \underline{v}^0, \ldots, \underline{v}^r)
\end{gathered}
$$

and G contains no factor in $K[\underline{u}^0, \ldots, \underline{u}^r]$.

Proof: $i*j(V) \cap H(S(\underline{u}^1, \underline{v}^1)) \cap \cdots \cap H(S(\underline{u}^r, \underline{v}^r))$ consists of finitely many points. It suffices to check transversality of the intersection in one of these points which we denote $S(\underline{p}, \underline{q})$.

Step 1: Let $\{l_1, \ldots, l_r\}$ be an order of $\{1, \ldots, r\}$ such that

$$\underline{p} \in i(V) \cap \bigcap_{t=1}^{k} H(\underline{u}^{l_t}) \quad \text{and} \quad \underline{q} \in j(V) \cap \bigcap_{t=k+1}^{r} H(\underline{v}^{l_t})$$

then the coefficients of $\underline{u}^{k+1}, \ldots, \underline{u}^r, \underline{v}^1, \ldots, \underline{v}^k$ are algebraically independent over $K(\underline{u}^1, \ldots, \underline{u}^k, \underline{v}^{k+1}, \ldots, \underline{v}^r, \underline{p}, \underline{q})$.

To prove this assertion we use again 1.4. It says

$$\underline{p} \in i(V) \cap \bigcap_{t=1}^{k} H(\underline{u}^{l_t}) \text{ and } \underline{q} \in j(V) \cap \bigcap_{t=k+1}^{r} H(\underline{v}^{l_t})$$

for $S(\underline{p}, \underline{q}) \in i * j(V) \cap H(S(\underline{u}^t, \underline{v}^t))$ for all $t \in \{1, \ldots, r\}$.
Since $H(\underline{u}^{l_1}), \ldots, H(\underline{u}^{l_k})$ meet $i(V)$ transversally we know

$$\text{transdeg } \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})(\underline{p}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})\right] \leq r - k + 1 \ .$$

Analogously transdeg $\left[K(\underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{q}) : K(\underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})\right] \leq k + 1$.
The K-morphism $i \circ j^{-1}$ maps $\underline{q}$ to $\underline{p}$ for $S(\underline{p}, \underline{q}) \in i * j(V)$. That means $q_0, \ldots, q_m$ are expressible as fractions of homogeneous polynomials in $p_0, \ldots,$ p_n with coefficients in K and vice versa. Therefore also at $\underline{q}$

$$\text{transdeg } \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})(\underline{q}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})\right] \leq r - k + 1 \ .$$

This implies $\underline{q}$ is a general point of a subvariety of $j(V)$ of dimension at most $r - k$ over the ground field $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})$. Since the coefficients of $\underline{v}^{k+1}, \ldots, \underline{v}^r$ are algebraically independent over $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k})$ the intersection of $\bigcap_{t=k+1}^{r} H(\underline{v}^t)$ and this subvariety has dimension at most 0. Hence

$$\text{transdeg} \left[K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{q}) : K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})\right] = 1.$$

But now the coordinates of $\underline{u}^{k+1}, \ldots, \underline{u}^r, \underline{v}^1, \ldots, \underline{v}^k$ must be algebraically independent over $K(\underline{u}^{l_1}, \ldots, \underline{u}^{l_k}, \underline{v}^{l_{k+1}}, \ldots, \underline{v}^{l_r})(\underline{p}, \underline{q})$. This proves the first step.

Step 2: For the proof of transversality of the intersection of $i * j(V)$ and $\bigcap_{t=1}^{r} H(S(\underline{u}^t, \underline{v}^t))$ in $S(\underline{p}, \underline{q})$ it now suffices to check the transversality in case $r = 1$ because $i * j(V) \cap H(S(\underline{u}^1, \underline{v}^1))$ is again a variety but over a larger field $K(\underline{u}^1, \underline{v}^1)$, which allows us to repeat the argument for the intersection with $H(S(\underline{u}^r, \underline{v}^r))$ for $r > 1$.

For simplicity we may now assume $\underline{u} = \underline{u}^1$, $\underline{v} = \underline{v}^1$ and $\underline{p} \in H(\underline{u})$ which implies $v_0, \ldots, v_m$ to be algebraically independent over $K(\underline{q})$ by step 1.

We want to determine a line in the tangent space of $i * j(V)$ which meets $H(S(\underline{u}, \underline{v}))$ transversally.
Since $H(\underline{u})$ meets $i(V)$ transversally there is a line $g \subset T_{i(V),\underline{p}}$ in the tangent space of $i(V)$ in $\underline{p}$ which meets the tangent space $T_{H(\underline{u}),\underline{p}} = H(\underline{u})$ transversally. But this implies

$$T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} = T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})} \oplus T_{S(H(\underline{u}\times\mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} .$$

By 1.4 we know $S(H(\underline{u}) \times \mathbb{P}^m_\Omega) \subset H(S(\underline{u}, \underline{v}))$. On tangent space level this says

$$T_{S(H(\underline{u})\times\mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \subset T_{H(S(\underline{u},\underline{v})),S(\underline{p},\underline{q})} = H(S(\underline{u}, \underline{v})) .$$

On the other hand we can apply Lemma 1.8 which says

$$T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} \cap H(S(\underline{u}, \underline{v})) \neq T_{S(\mathbb{P}^n_\Omega \times \mathbb{P}^m_\Omega),S(\underline{p},\underline{q})} .$$

So we conclude $T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})}$ and $H(S(\underline{u}, \underline{v}))$ intersect transversally. Since $T_{S(g\times\{\underline{q}\}),S(\underline{p},\underline{q})} \subset T_{i*j(V),S(\underline{p},\underline{q})}$ the transversality of $i * j(V)$ and $H(S(\underline{u}, \underline{v}))$ is shown and the theorem proved. □

By linear extension to algebraic cycles Theorem 1.10 is also true for arbitrary purely dimensional, algebraic cycles V. Reformulating the theorem we obtain

Corollary 1.11. *Let V be a r−dimensional cycle in X and $F_{i*j(V)}$ the Chow form of $i * j(V)$ of multidegree d.*

$$F_{i*j(V)}(S(\underline{u}^1, \underline{v}^1), \ldots, S(\underline{u}^r, \underline{v}^r)) = \sum_{k=0}^{M} A_k(\underline{u}^0, \ldots, \underline{u}^r) M_k(v) ,$$

*where $M = \binom{m+d}{d}^{r+1} - 1$ and $M_k(v)$ are the multihomogeneous monomials in $\underline{v}^0, \ldots, \underline{v}^r$ of multidegree d. In this situation $F_{i(V)}$ is the **g**reates **c**ommon **d**ivisor of $A_0, \ldots, A_M$.*

The coefficients of $(A_0, \ldots, A_M)$ are exactly the coefficients of the form $F_{i*j}(S(\underline{u}^1, \underline{v}^1), \ldots, S(\underline{u}^r, \underline{v}^r)) = \sum_{k=0}^{M} A_k(\underline{u}^0, \ldots, \underline{u}^r) M_k(v)$, which coincide by definition with the Chow coordinates of $i * j(V)$ respectively with the coefficients of $F_{i*j(V)}$. Hence by Theorem 2.2, which says determination of gcd is an algebraic map in this special case, we get a morphism

$$\begin{array}{rccc} (pr_1)_* : & \mathcal{C}_r(X, i * j) & \longrightarrow & \mathcal{C}_r(X, i) \\ & \sum_k n_k (i * j)(V_k) & \longmapsto & \sum_k n_k i(V_k) \end{array}$$

which is bijective. Since the image of a morphism of projective algebraic sets is closed, it is obvious that

$$\mathcal{C}_r(X,i) \longrightarrow \mathcal{C}_r(X,j)$$

is an algebraic homeomorphism and Theorem 1.3 is proved.

In the next paragraph the algebraicity of forming the greatest common divisor in the case at hand is shown.

2 Determination of GCDs

Let $\underline{u}^j = (u_0^j, \ldots, u_n^j)$ for $j = 0, \ldots, r$. Further let $A_k \in K[\underline{u}^0, \ldots, \underline{u}^r]$ be a multihomogeneous polynomial of multidegree d in $\underline{u}^0, \ldots, \underline{u}^r$ or $A_k = 0$; $k = 0, \ldots, M$. At least one of $A_0, \ldots, A_M$ should not be equal to 0.

Notation 2.1. Let $N = \binom{n+d}{d}^{r+1} - 1$. Let $A_k = \sum\limits_{l=0}^{N} a_{k,l} M_l(u)$, where $M_l(u)$ are homogeneous monomials of multidegree d in $\underline{u}^0, \ldots, \underline{u}^r$, $a_{k,l} \in K$; $k = 0, \ldots, M$; $l = 0, \ldots, N$.
$(a_{k,l})$ defines a point in $\mathbb{P}^a_\Omega := \mathbb{P}^{NM+N+M}_\Omega$. Let $U \subset \mathbb{P}^a_\Omega$ denote the set of those points $(A_0, \ldots, A_M)$ with $gcd(A_0, \ldots, A_M)$ has degree $\widetilde{d}$.
Further the multihomogeneous polynomials in $K[\underline{u}^0, \ldots, \underline{u}^r]$ of multidegree $\widetilde{d}$ correspond to points in $\mathbb{P}^{\widetilde{N}}_\Omega$, $\widetilde{N} = \binom{n+\widetilde{d}}{\widetilde{d}}^{r+1} - 1$.
In the analogous way a set of $M+1$ multihomogeneous polynomials in $K[\underline{u}^0, \ldots, \underline{u}^r]$ of multidegree $d - \widetilde{d}$ corresponds to a point in $\mathbb{P}^b_\Omega$, where $b = (M+1)\binom{n+d-\widetilde{d}}{d-\widetilde{d}}^{r+1} - 1$.

Theorem 2.2. *The map $gcd : U \to \mathbb{P}^{\widetilde{N}}_\Omega$ which maps $A_0, \ldots, A_M$ to their greatest common divisor is algebraic.*

For the proof of this theorem we need some technical results. First observe that a locally closed subset of a (quasi) projective variety is the intersection of an open and a closed subset hence a quasi projective variety.

Lemma 2.3. *U is an irreducible, locally closed subset of $\mathbb{P}^a_\Omega$.*

Proof: The greatest common divisor of $A_0, \ldots, A_M$ has multidegree $\widetilde{d}$ if

1. There exist polynomials B_k such that $A_k B_l = A_l B_k$ for all $k, l = 0, \ldots, M$ and B_k has multidegree $d - \widetilde{d}$ if $A_k \neq 0$; $k = 0, \ldots, M$.

2. There exist no polynomials C_k such that $A_kC_l = A_lC_k$ for all $k, l = 0, \ldots, M$ and C_k has multidegree lower than $d - \widetilde{d}$ if $A_k \neq 0$; $k = 0, \ldots, M$.

In this situation the B_k are unique up to a common factor $c_k \in K$ and $A_k = c_kB_kD$, where $D = gcd(A_0, \ldots, A_M)$.

Let

$$\begin{array}{cccc} \varphi : & \mathbb{P}_\Omega^{\widetilde{N}} \times \mathbb{P}_\Omega^b & \longrightarrow & \mathbb{P}_\Omega^a \\ & (F, (G_0, \ldots, G_M)) & \longmapsto & (FG_0, \ldots, FG_M) \end{array}$$

where we think of $F, G_0, \ldots, G_M$ as multihomogeneous polynomials as in Notation 2.1.

Obviously φ is a regular map. Since $\mathbb{P}_\Omega^{\widetilde{N}} \times \mathbb{P}_\Omega^b$ is a projective variety, $Im(\varphi)$ is an algebraically closed [7, Thm 2, p. 57], irreducible set. Now the set U is the intersection of $Im(\varphi)$ and the open set defined by condition 2, which is the complement of a finite union of algebraically closed sets. Hence U is locally closed and irreducible. □

Condition 1 of the proof of Lemma 2.3 defines an algebraic subset of $U \times \mathbb{P}_\Omega^b$. We know for $(A_0, \ldots, A_M) \in U$ the polynomials $(B_0, \ldots, B_M) \in \mathbb{P}_\Omega^b$ are uniquely determined, i.e. we obtain a map $\psi : U \to U \times \mathbb{P}_\Omega^b$.

Lemma 2.4.

$$\begin{array}{cccc} \psi : & U & \longrightarrow & U \times \mathbb{P}_\Omega^b \\ & (A_0, \ldots, A_M) & \longmapsto & ((A_0, \ldots, A_M), (B_0, \ldots, B_M)) \end{array} \textit{ is algebraic.}$$

Proof: The B_k are uniquely determined by $B_kA_l = B_lA_k$ for all $k, l = 0, \ldots, M$. The polynomial B_kA_l is multihomogeneous of degree $2d - \widetilde{d}$ or equal to 0. If it is multihomogeneous we can check the condition $B_kA_l = B_lA_k$ in the coefficient of each monomial. Hence we get a finite set of polynomials $f_0, \ldots, f_Q$ (over $\mathbb{Z}$!) which are homogeneous in the coordinates of the point $(A_0, \ldots, A_M) \in U$ of degree 1 and homogeneous in the coordinates of the point $(B_0, \ldots, B_M)$ of degree 1. If we now fix the point $(A_0, \ldots, A_M) \in U$ we obtain a system of linear equations in the coordinates of $(B_0, \ldots, B_M)$. All coefficients of this linear equations are 0 or coordinates of $(A_0, \ldots, A_M)$. Since $(B_0, \ldots, B_M)$ are uniquely determined we can apply Cramer's rule and obtain that each coordinate of $(B_0, \ldots, B_M)$ can locally be written as a fraction of the determinant of matrices in the coefficients of $(A_0, \ldots, A_M)$. □

Analogously to the proof of Lemma 2.4 it is seen that

$$\begin{array}{cccc} \Psi : & \psi(U) & \longrightarrow & \psi(U) \times \mathbb{P}_\Omega^{\widetilde{N}} \\ & ((A_0, \ldots, A_M), (B_0, \ldots, B_M)) & \longmapsto & ((A_0, \ldots, A_M), (B_0, \ldots, B_M), D) \end{array}$$

such that $A_k = cB_kD$ for fixed $c \in K$ and $k = 0, \dots, M$ is algebraic. Now projection onto the second factor is again an algebraic map. Hence we obtain the morphism

$$gcd : U \xrightarrow{\psi} \psi(U) \xrightarrow{\Psi} \psi(U) \times \mathbb{P}_\Omega^{\tilde{N}} \xrightarrow{pr_2} \mathbb{P}_\Omega^{\tilde{N}}$$

such that $(A_0, \dots, A_M)$ is mapped to $gcd(A_0, \dots, A_M)$ and Theorem 2.2 is proved.

3 Chow varieties in arbitrary characteristic

First we want to define the Chow form of a r-dimensional variety $V \subset \mathbb{P}_\Omega^n$ (cp. [2]). Let $f_1, \dots, f_k \in K[x_0, \dots, x_n]$ be homogeneous generators of the vanishing ideal of V. Further let $u_j^i \in \Omega$ be algebraically independent over K. Then the zero set $V \cap H_{n-r}$ of the polynomials

$$\begin{array}{l} f_1, \dots, f_k \\ u_0^1 x_0 + \dots + u_n^1 x_n =: \langle \underline{u}^1, \underline{x} \rangle \\ \qquad \vdots \\ u_0^r x_0 + \dots + u_n^r x_n =: \langle \underline{u}^r, \underline{x} \rangle \end{array}$$

is irreducible and has dimension 0 as variety over $K(\underline{u}^1, \dots, \underline{u}^r)$. Since $K(\underline{u}^1, \dots, \underline{u}^r)$ is not algebraically closed $V \cap H_{n-r}$ consists on a finite set of points $\underline{p}^1, \dots, \underline{p}^d \in \mathbb{P}_\Omega^n$. Further $\underline{p}^i$ is a general point of $V \cap H_{n-r}$ as a variety over $K(\underline{u}^1, \dots, \underline{u}^r)$ and a general point of V as a variety over K, for $i = 1, \dots, d$. This means a homogeneous polynomial f that satisfies $f(\underline{p}^i) = 0$ and has coefficients in $K(\underline{u}^1, \dots, \underline{u}^r)$ already belongs to the vanishing ideal of $V \cap H_{n-r}$. If its coefficients even belong to K the polynomial belongs to the vanishing ideal of V.

Remember that a K-automorphism of Ω maps general points of a K-variety to general points, when extended to $\mathbb{P}_\Omega^n$. On the other hand for two general points of a K-variety we can find a K-automorphism which maps one point onto the other, this is general points are conjugate to each other.

Therefore we may assume $p_j^i = 1$ for a fixed $j \in \{0, \dots, n\}$ and for all $i = 1, \dots, d$. Since $V \cap H_{n-r}$ has dimension 0 the p_j^i, $j = 0, \dots, n$, $i = 1, \dots, d$, are now elements of $\overline{K(\underline{u}^1, \dots, \underline{u}^r)}$.

In case of $char(K) = q \neq 0$ let $l \in \mathbb{N}$ be minimal such that $(p_j^i)^{(q^l)}$ is separable over $K(\underline{u}^1, \dots, \underline{u}^r)$ for all $i = 1, \dots, d$; $j = 0, \dots, n$. Remark l is independent of i because the $\underline{p}^i$ are conjugate to each other. Define

$$
\begin{aligned}
L_i(\underline{u}^0) &:= \begin{cases} \langle \underline{p}^i, \underline{u}^0 \rangle & , char(K) = 0 \\ \langle \underline{p}^i, \underline{u}^0 \rangle^{(q^l)} & , char(K) = q \neq 0 \end{cases} \\
G(\underline{u}^0) &= \prod_{i=1}^{d} L_i(\underline{u}^0) \,.
\end{aligned}
\tag{1}
$$

Let $\widetilde{Z}$ be the splitting field of the minimal polynomials of the p_j^i, $j = 0, \ldots, n$; $i = 1, \ldots, d$, and $Z \subset \widetilde{Z}$ the separable closure of $K(\underline{u}^1, \ldots, \underline{u}^r)$ in $\widetilde{Z}$.

By definition we have $L_i(\underline{u}^0) \in Z[\underline{u}^0]$ is irreducible, for $\langle \underline{p}^i, \underline{u}^0 \rangle^{q^k} \notin Z[\underline{u}^0]$ in case $q \neq 0$, $k < l$, for all $i = 1, \ldots, d$. Z is a Galois extension of $K(\underline{u}^1, \ldots, \underline{u}^r)$, for $i = 1, \ldots, d$. Now the Galois action of this extension still permutes the $L_i(\underline{u}^0)$, $i = 1, \ldots, d$. But this means the polynomial $G(\underline{u}^0)$ remains invariant under this Galois action, hence it has coefficients in $K(\underline{u}^1, \ldots, \underline{u}^r)$. Let $c \in K[\underline{u}^1, \ldots, \underline{u}^r]$ be the common denominator of the coefficients of $G(\underline{u}^0)$. Now we define

$$
F(\underline{u}^0, \ldots, \underline{u}^r) := cG(\underline{u}^0) \in K[\underline{u}^0, \ldots, \underline{u}^r] \tag{2}
$$

to be the Chow form of V.

By construction we know

Lemma 3.1. *Let $V \subset \mathbb{P}^n_\Omega$ be a variety of dimension r. The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r) \in K[\underline{u}^0, \ldots, \underline{u}^r]$ is uniquely determined up to a factor in K^* and irreducible.*

Since the $\underline{p}^i$ are general points of V, we also obtain different varieties having different Chow forms.

Definition 3.2. Let $V_1, \ldots, V_k \subset \mathbb{P}^n_\Omega$ be r-dimensional varieties with Chow forms $F_1, \ldots, F_k$. Then we define

$$
F(\underline{u}^0, \ldots, \underline{u}^r) := F_1^{n_1} \cdot \cdots \cdot F_k^{n_k}
$$

to be the Chow form of the **algebraic cycle** $V := \sum_{i=1}^{k} n_i V_i$, $n_i \in \mathbb{N}$.

The degree of $F(\underline{u}^0, \ldots, \underline{u}^r)$ as a polynomial in $K[\underline{u}^1, \ldots, \underline{u}^r][\underline{u}^0]$ is called the **degree** of V.

In characteristic 0 the degree of a r-dimensional variety coincides with the number of points where a generic linear subspace of dimension $n - r$ meets the variety.

Reminder 3.3. Let S be a subring of Ω, and let $f_1, \ldots, f_k \in S[x_0, \ldots, x_n]$ be homogeneous polynomials. The *resultant of the homogeneous equations*

$$f_1 = \cdots = f_k = 0$$

is an ideal $J(f) \subset S$, which is generated by determinants of matrices, where one fixed row of such a matrix consists of coefficients of one f_j [8, § 130]. Therefore there is a set of elements in the polynomial ring in the coefficients of $f_1, \ldots, f_k$ over $\mathbb{Z}$, whose image in S generates the ideal $J(f)$. Because the ideal $J(f)$ is generated by determinants, it is homogeneous in the coefficients of each f_j, $j = 1, \ldots, k$.
We also say $J(f)$ is the resultant of the homogeneous polynomials $f_1, \ldots, f_k$. The ideal $J(f)$ has the following property [8, §130]:

$$\begin{aligned} & f_1 = \cdots = f_k = 0 \text{ has a solution in } \mathbb{P}^n_\Omega \\ \iff \; & J(f) \text{ is the zero ideal in } S\,. \end{aligned}$$

Let $I(f) \subset S[x_0, \ldots, x_n]$ denote the ideal which is generated by $f_1, \ldots, f_k$. The zero set of $I(f)$ in $\mathbb{P}^n_\Omega$ is empty iff $J(f) \neq 0$, but the ideal $J(f)$ depends on the choice of generators of $I(f)$. The radical of $J(f)$ only depends on the radical of $I(f)$:

To see this let $g_1, \ldots, g_l \in S[x_0, \ldots, x_n]$ be homogeneous polynomials and $I(g)$ be the ideal which is generated by $g_1, \ldots, g_l$. Furthermore, let $J(g)$ be the resultant of $g_1 = \cdots = g_l = 0$. Suppose $\sqrt{I(g)} = \sqrt{I(f)}$. Then $g_1 = \cdots = g_l = 0$ has a solution in $\mathbb{P}^n_\Omega$ iff $f_1 = \cdots = f_k = 0$ has a solution in $\mathbb{P}^n_\Omega$. Hence $J(g) = 0$ if and only if $J(f) = 0$, which in turn implies $\sqrt{J(g)} = \sqrt{J(f)}$. Therefore we denote by $\sqrt{J(f)}$ the *resultant of the ideal* $I(f)$.

The Chow form of a variety V arises from the intersection with H_{n-r}. If one cuts this 0–dimensional set with another sufficiently general hyperplane, the intersection is empty. Therefore one gets a set of homogeneous polynomials, whose resultant is not the zero ideal. In fact the radical of this resultant is generated by the Chow form:

Theorem 3.4. *Let $V \subset \mathbb{P}^n_\Omega$ be a r-dimensional variety and $f_1, \ldots, f_k$ be homogeneous generators of the vanishing ideal of V. The Chow form F of V generates the resultant $\sqrt{R} \subset K[\underline{u}^0, \ldots, \underline{u}^r]$ of the following equations:*

$$\begin{aligned} f_1 = \cdots = f_k \\ \langle \underline{u}^0, \underline{x} \rangle = \cdots = \langle \underline{u}^r, \underline{x} \rangle \end{aligned} \tag{3}$$

Before we prove the theorem we need a technical lemma:

Lemma 3.5. *Let Z be a field of characteristic $q \neq 0$ and $u_0, \ldots, u_n \in \Omega$ algebraically independent over Z. Suppose $\widetilde{Z} \supset Z$ is a purely inseparable field extension. Let $L(\underline{u}) \in Z[\underline{u}]$ be irreducible and*

$$L(\underline{u}) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u} \rangle \in \widetilde{Z}[\underline{u}] \ , c \in \widetilde{Z}$$

Then $\underline{p}^1 = \underline{p}^i$ for all $i = 1, \ldots, d$ and $d = q^l$, where $l \in \mathbb{N}$ is minimal such that $(p_j^1)^{q^l} \in Z$ for all $j = 0, \ldots, n$.

Proof: The first step is to prove that if we have $p_j^i = 0$ for a fixed $j \in \{0, \ldots, n\}$ and for one $i \in \{1, \ldots, d\}$, then $p_j^i = 0$ for all $i = 1, \ldots, d$. Suppose $j = 0$ and $p_0^1 = \cdots = p_0^k = 0$, $p_0^{k+1} = \cdots = p_0^d = 1$. Now determine the coefficient $C \in Z$ of the monomial $(u_0)^{d-k} u_{i_1} \cdots u_{i_k}$ of $L(\underline{u})$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$:

$$C = c \sum_{J(d)} p_{i_1}^{j_1} p_{i_2}^{j_2} \cdots p_{i_k}^{j_k} p_0^1 \cdots \widehat{p_0^{j_1}} \cdots \widehat{p_0^{j_k}} \cdots p_0^d$$

$J(d) = \{(j_1, \ldots, j_k) \in \{1, \ldots, d\}^k \mid j_s \neq j_t \text{ for } s \neq t\}$, and $\widehat{p} := 1$. Since $p_0^1 = \cdots = p_0^k = 0$ we obtain

$$C = c \sum_{J(k)} p_{i_1}^{j_1} p_{i_2}^{j_2} \cdots p_{i_k}^{j_k} \underbrace{p_0^{k+1} \cdots p_0^d}_{=1}$$

But this implies

$$c \prod_{i=1}^{k} \langle \underline{p}^i, \underline{u}^0 \rangle$$

is a polynomial in $Z[\underline{u}]$ which implies $k = d$ because of the irreducibility of $L(\underline{u})$.

Now we can assume without loss of generality $p_0^i = 1$ for $i = 1, \ldots, d$ and therefore $c \in Z$. As a primitive polynomial $L(\underline{u}^0)$ is irreducible in $Z[\underline{u}] = Z[u_1, \ldots, u_n][u_0]$ if and only if it is irreducible in $Z(u_1, \ldots, u_n)[u_0]$.
The d zeros of $L(\underline{u}^0)$ in $Z(u_1, \ldots, u_n)$ are given by $z(i) = -p_1^i u_1 - \cdots - p_n^i u_n \in \widetilde{Z}(u_1, \ldots, u_n)$ for $i = 1, \ldots, d$. But since $\widetilde{Z} \supset Z$ is purely inseparable, we know $\widetilde{Z}(u_1, \ldots, u_n) \supset Z(u_1, \ldots, u_n)$ is purely inseparable. That means $z(i)$ are purely inseparable and zeros of an irreducible polynomial, therefore $z(1) = z(i)$ for all $i = 1, \ldots, d$ and $d = q^l$ for some $l \in \mathbb{N}$. $\square$

Proof of Theorem 3.4: The first step is to show $\sqrt{R}$ is a prime ideal in $K[\underline{u}^0,\dots,\underline{u}^r]$: Let

$$A := \{(\underline{p},\underline{y}^0,\dots,\underline{y}^r)\in \mathbb{P}^n_\Omega\times(\mathbb{P}^n_\Omega)^{r+1} \mid \underline{p}\in V\ ,\ \langle\underline{p},\underline{y}^0\rangle=\cdots=\langle\underline{p},\underline{y}^r\rangle=0\}.$$

The vanishing ideal of A in $K[\underline{u}^0,\dots,\underline{u}^r,\underline{x}]$ is generated by the polynomials in (3). The projection pr_1 onto the first factor is V and the projection pr_2 of A onto the second factor $(\mathbb{P}^n_\Omega)^{r+1}$ gives us the zero set of $R\subset K[\underline{u}^0,\dots,\underline{u}^r]$. Now we can apply [7, I.6.3 Th. 8] to pr_1 and obtain A being irreducible. Hence $pr_2(A)$ is irreducible, which proves $\sqrt{R}$ being prime.

Now we can prove the theorem:

The Chow form F of V is $F(\underline{u}^0,\dots,\underline{u}^r)=\prod_{i=1}^{d}L_i(\underline{u}^0)$ as defined in (2) and (1). We can assume the $\underline{p}^1,\dots,\underline{p}^d$ to be normed such that the factor c in (2) equals 1. Denote by $\widetilde{Z}\subset\overline{K(\underline{u}^1,\dots,\underline{u}^r)}$ the splitting field of the minimal polynomials of p^i_j, $i=1,\dots,d$, $j=0,\dots,n$. Further let Z be the separable closure of $K(\underline{u}^1,\dots,\underline{u}^r)$ in $\widetilde{Z}$.

Now view the polynomials in (3) as polynomials in $Z[\underline{u}^0][x_0,\dots,x_n]$ and determine the resultant $\sqrt{R\cdot Z[\underline{u}^0]}$. The zero set of $R\cdot Z[\underline{u}^0]$ in $\mathbb{P}^n_\Omega$ equals the zero set of $\prod_{i=1}^{d}\langle\underline{p}^i,\underline{u}^0\rangle$. We know by Lemma 3.5 that if $B\in Z[\underline{u}^0]$ is divisible by $\prod_{i=1}^{d}\langle\underline{p}^i,\underline{u}^0\rangle$, then B is divisible by F. This means that the radical of the ideal which is generated by R in $Z[\underline{u}^0]$ is the ideal which is generated by F, i.e.

$$(F)_{Z[\underline{u}^0]}=\sqrt{R\cdot Z[\underline{u}^0]}\quad \text{in } Z[\underline{u}^0]\ .$$

Cutting down with $K[\underline{u}^0,\dots,\underline{u}^r]$ we first obtain

$$(F)_{K[\underline{u}^0,\dots,\underline{u}^r]}\subset(F)_{Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]\ .$$

Now let $h\in Z[\underline{u}^0]$ such that $F\cdot h\in(F)_{Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]$. And let σ be an element of the Galois group of Z over $K(\underline{u}^1,\dots,\underline{u}^r)$. Then $F\cdot h=\sigma(F\cdot h)=F\cdot\sigma(h)$, i.e. $h\in K(\underline{u}^1,\dots,\underline{u}^r)[\underline{u}^0]$, for Z is a Galois extension. Therefore $h=\frac{h_1}{h_2}$, $h_1\in K[\underline{u}^0,\dots,\underline{u}^r], h_2\in K[\underline{u}^1,\dots,\underline{u}^r]$, and the greatest common divisor of h_1 and h_2 is 1. Now h_2 must be a divisor of F, and since F is irreducible in $K[\underline{u}^0,\dots,\underline{u}^r]$ and $\underline{u}^0$ does not appear in h_2, then $h_2\in K^*$. Therefore

$$(F)_{Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]=(F)_{K[\underline{u}^0,\dots,\underline{u}^r]}\ .$$

Now it remains to show $\sqrt{R}$ equals $\sqrt{R\cdot Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]$. Obviously $\sqrt{R}\subset\sqrt{R\cdot Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]=(F)_{Z[\underline{u}^0]}\cap K[\underline{u}^0,\dots,\underline{u}^r]=(F)_{K[\underline{u}^0,\dots,\underline{u}^r]}$. For the inverse inclusion we have to show $F\in\sqrt{R}$.

Suppose $g_1,\dots,g_s \in K[\underline{u}^0,\dots,\underline{u}^r]$ generate the ideal $\sqrt{R}$. Since $\sqrt{R} \subset (F)_{K[\underline{u}^0,\dots,\underline{u}^r]}$ we know

$$g_i = F^{n_i} \cdot G_i \ \ , \ F \not| \, G_i \text{ in } K[\underline{u}^0,\dots,\underline{u}^r]$$

for some $n_i > 0$, $G_i \in K[\underline{u}^0,\dots,\underline{u}^r]$, $i = 1,\dots,s$. For the zero sets in $(\mathbb{P}^n_\Omega)^{r+1}$ we obtain

$$\begin{aligned} Z((F)) \subset Z(\sqrt{R}) &= Z((F^{n_i} \cdot G_i \mid i = 1,\dots,s)) \\ &= Z((F)) \cup Z((G_i \mid i = 1,\dots,s)) \ . \end{aligned}$$

But $\sqrt{R}$ is a prime ideal, which means $Z(\sqrt{R}) = Z((F))$ or $Z(\sqrt{R}) = Z((G_i \mid i = 1,\dots,s))$. The second case would imply $Z((F)) \subset Z((G_i \mid i = 1,\dots,s))$, hence the G_i would be divisible by F, which contradicts the properties of the G_i. That finishes the proof of the theorem. □

Remark 3.6. Since the equation in (3) remain invariant under permutation of $\underline{u}^0,\dots,\underline{u}^r$ we know $\sqrt{R}$ and hence $F(\underline{u}^0,\dots,\underline{u}^r)$ remains invariant under permutation of $\underline{u}^0,\dots,\underline{u}^r$.

Definition 3.7. A polynomial $F \in K[\underline{u}^0,\dots,u^r]$ is denoted multihomogeneous of (multi-)degree d, if it is homogeneous of degree d as a polynomial in $K[\underline{u}^0,\dots,\underline{u}^{i-1},\underline{u}^{i+1},\dots,\underline{u}^r][\underline{u}^i]$ for all $i = 0,\dots,r$.

Corollary 3.8. *The Chow form F of an algebraic cycle V of degree d is multihomogeneous of degree d.*

Lemma 3.5 now permits us to transfer the theorem of Chow and Van der Waerden [2, Satz 1], which characterizes those multihomogeneous form that actually are Chow forms, in a formal way to the case of arbitrary characteristic:

Theorem 3.9. *Let $F \in K[\underline{u}^0,\dots,\underline{u}^r]$ be multihomogeneous of multidegree d. F is a Chow form of an algebraic cycle in $\mathcal{C}_{r,d}(\mathbb{P}^n_K)$ iff the following conditions hold:*

There are d points $\underline{p}^i \in \overline{K(\underline{u}^1,\dots,\underline{u}^r)}^{n+1} - \{0\}$ not necessarily different and $c \in K(\underline{u}^1,\dots,\underline{u}^r)$ satisfying:

1. $F(\underline{u}^0,\dots,\underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, *in* $\overline{K(\underline{u}^1,\dots,\underline{u}^r)}[\underline{u}^0]$.

2. $L_i(\underline{u}^j) := \langle \underline{p}^i, \underline{u}^j \rangle = 0$ *for* $i = 1,\dots,d$, $j = 1,\dots,r$.

3. *For every $i \in \{1, \ldots, d\}$ one has:*
   Let S^j be skew symmetric $(n+1) \times (n+1)$–matrices, $j = 1, \ldots, r$, whose entries $s^j_{k,l} \in \Omega$ are algebraically independent over $K(\underline{p}^i)$, $k, l = 0, \ldots, n$. In this situation $L_i(\underline{u}^0)$ divides the form $F(\underline{u}^0, S^1\underline{p}^i, \ldots, S^r\underline{p}^i)$ in $K(\underline{p}^i)[s^1_{0,0}, \ldots, s^r_{n,n}][\underline{u}^0]$.

The three conditions of the theorem can be reformulated in homogeneous equations [2]. Hence we obtain for an algebraic set $X \subset \mathbb{P}^n_\Omega$ over K the Chow variety $\mathcal{C}_{r,d}(X) \subset \mathbb{P}^N_\Omega$ being an algebraic set over K.

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