Spectral Theory of Some Matrix Differential Operators of Mixed Order

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1. Introduction

Our purpose in this work is to study the spectral properties and the scattering theory of self-adjoint operator matrices of the form

$$H = \begin{pmatrix} -\Delta + q & -\nabla b + v \\ \bar{b}\mathrm{div} + v^* & h \end{pmatrix} \tag{1}$$

acting in the product of Hilbert spaces $\mathcal{H} = L_2(\mathbb{R}^n;\mathbb{C}^n) \times L_2(\mathbb{R}^n),\ n \geq 1$. Here $L_2(\mathbb{R}^n;\mathbb{C}^n)$ is L_2-space of vector functions $u : \mathbb{R}^n \to \mathbb{C}^n,\ q$ is an $n \times n$ Hermitian matrix function, b is a complex function, v is an $n \times 1$ matrix function, h is a real function. In the recent paper [AtLMS] (see also [S]) an abstract approach was proposed to study the essential spectrum of 2×2 block operator matrices acting in a Banach space. In particular, this approach was proved to be very useful for dealing with the spectral theory of some mixed order differential operators occuring in magnetohydrodynamics (see [AtLMS], [FMM], [LM]). On the other hand the structure of the continuous spectrum of such matrix differential operators has not been investigated. Here we give a sufficiently complete analysis of the self-adjoint operators of the form (1). In particular, we prove (under certain assumptions) the limiting absorption principle and the absence of singular continuous spectrum. Moreover, we describe the possible points of the eigenvalue accumulation. In Section 2 we study the essential spectrum of some general self-adjoint operator matrices of the form

$$L = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{2}$$

In particular, we prove (cf. [S]) that under some natural assumptions $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$, where

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & h_0 \end{pmatrix}, \tag{3}$$

$$h_0 = h - |b|^2 \tag{4}$$

In Section 3 we present some variant of Mourre's commutator method, which is suitable for dealing with the operators of the form (1). The limiting absorption principle for the operator H is proved in Section 4. In Section 5 we show that for the pair H, H_0 there exist complete wave operators. Note that the case $b(x) \equiv 0$ was investigated in [K].

2. The essential spectrum of some matrix operators.

First introduce some notations. Let T be a linear closed operator in a Hilbert space with the domain $D(T)$. Denote by $\rho(T)$ the resolvent set of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is the spectrum of T. The set $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not a Fredholm operator}\}$ is called the essential spectrum of T. If T is a self-adjoint operator, then $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \sigma(T) \mid \lambda \text{ is not an isolated eigenvalue of finite multiplicity}\}$. Denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ ($S_\infty(\mathcal{H}, \mathcal{K})$) the space of bounded (respectively compact) linear operators from $\mathcal{H}$ into $\mathcal{K}$. $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$, $S_\infty(\mathcal{H}) = S_\infty(\mathcal{H}, \mathcal{H})$.

Consider symmetric operator matrices of the form (2) acting in the product of Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Suppose that A and C are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let B be a densely defined closable operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ such that $D(B)$ is a core of C. We shall assume that

$$D(B^*) \supset D(|A|^{1/2}). \tag{5}$$

Under these conditions L is essentially self-adjoint on $D(A) \times D(B)$ (see [ALMS], [MS]). We retain the notation L for the closure of the operator (2). Define

$$S_0(\lambda) := B^*(A - \lambda)^{-1}B, \qquad \lambda \in \rho(A).$$

By (5), the closure $S := \overline{S_0(\lambda)}$ is a bounded operator in $\mathcal{H}_2$. Define on $D(C)$

$$T(\lambda) = C - S(\lambda), \qquad \lambda \in \rho(A). \tag{6}$$

Note that $\lambda \in \rho(T(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ [AL].

Define

$$F(\lambda) := B^*(A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2), \qquad \lambda \in \rho(A).$$

Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$(L - \lambda)^{-1} = \begin{pmatrix} (A-\lambda)^{-1} + (F(\bar\lambda))^*(T(\lambda) - \lambda)^{-1}F(\lambda) & -(F(\bar\lambda))^*(T(\lambda) - \lambda)^{-1} \\ -(T(\lambda) - \lambda)^{-1}F(\lambda) & (T(\lambda) - \lambda)^{-1} \end{pmatrix} \tag{7}$$

(see [AL], [ALMS], [MS]). Denote

$$L(\mu) := \begin{pmatrix} A & 0 \\ 0 & T(\mu) \end{pmatrix}, \qquad \mu \in \rho(A).$$

Theorem 2.1. *Suppose that for some (and hence for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$*

$$G(\lambda) := (C - \lambda)^{-1}B^*(A - \lambda)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2). \tag{8}$$

Then for all $\lambda \in (\mathbb{C}\setminus\mathbb{R})\cap\rho(T(\mu))$ *the difference of resolvents* $D(\lambda) := (L-\lambda)^{-1} - (L(\mu)-\lambda)^{-1} \in S_\infty(\mathcal{H})$. *In particular,* $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L(\mu)) = \sigma_{\mathrm{ess}}(A) \cup \sigma_{ess}(T(\mu))$.

Remark 2.2. It is clear (see (6)) that for t large enough $it \in \rho(T(\mu))$.

Remark 2.3. Under the assumptions of Theorem 2.1 $\sigma_{\mathrm{ess}}(T(\mu))$ does not depend on $\mu \in \rho(A)$ (see the proof).

Remark 2.4. The essential spectrum of 2×2 block operator matrices acting in the product of Banach spaces has been studied by F.V. Atkinson, H. Langer, R. Menicken and A.A. Shkalikov [AtLMS], [S]. Their compactness conditions are more restrictive then (8). It was assumed that A has a compact resolvent or $B^*(A-\mu)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$. On the other hand they did not suppose (5). To weaken (5) and to extend Theorem 2.1 to the case of Banach spaces is beyond the scope of this paper.

Proof. By (7)

$$D(\lambda) = \begin{pmatrix} F(\bar\lambda)^*(T(\lambda)-\lambda)^{-1}F(\lambda) & F(\bar\lambda)^*(T(\lambda)-\lambda)^{-1} \\ -(T(\lambda)-\lambda)^{-1}F(\lambda) & (T(\lambda)-\lambda)^{-1} - (T(\mu)-\lambda)^{-1} \end{pmatrix}.$$

Consider

$$\begin{gathered}(T(\lambda)-\lambda)^{-1}F(\lambda) = (C-\lambda)^{-1}F(\lambda) + ((T(\lambda)-\lambda)^{-1} - (C-\lambda)^{-1})F(\lambda) = \\ G(\lambda) + (T(\lambda)-\lambda)^{-1}S(\lambda)G(\lambda) \in S_\infty(\mathcal{H}_1, \mathcal{H}_2).\end{gathered} \tag{9}$$

Analogously, $(F(\bar\lambda))^*(T(\lambda)-\lambda)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$. Furthermore, by (9)

$$(T(\lambda)-\lambda)^{-1} - (T(\mu)-\lambda)^{-1} = (T(\lambda)-\lambda)^{-1}(T(\mu)-T(\lambda))(T(\mu)-\lambda)^{-1} =$$

$$(\lambda-\mu)(T(\lambda)-\lambda)^{-1}F(\lambda)(F(\bar\mu))^*(T(\mu)-\lambda)^{-1} \in S_\infty(\mathcal{H}_2, H_2).$$

Hence $D(\lambda) \in S_\infty(\mathcal{H})$.

Let us apply Theorem 2.1 to the operator (1). Denote by Q_j the operator of multiplication by the variable x_j acting in $L_2(\mathbb{R}^n)$. Let $Q = (Q_1, \dots, Q_n)$ be the position operator, $a(Q)$ be an operator of multiplication by a $s \times t$ matrix function a acting from $L_2(\mathbb{R}^n; \mathbb{C}^t)$ to $L_2(\mathbb{R}^n; \mathbb{C}^s)$ $(s, t \in \mathbb{N})$. For simplicity we shall assume throughout the paper that (matrix) functions q, b, v are bounded over $\mathbb{R}^n$. Moreover, we assume that b is a Lipschitz function and h is continuous on $\mathbb{R}^n$. Denote $\mathcal{H}_1 = L_2(\mathbb{R}^n; \mathbb{C}^n)$, $\mathcal{H}_2 = L_2(\mathbb{R}^n)$. Now A is a vector Schrödinger operator in $\mathcal{H}_1$ defined by

$$A = -\Delta + q(Q), \qquad D(A) = D(-\Delta).$$

$B = -\nabla b(Q) + v(Q)$ with $D(B) = C_0^\infty(\mathbb{R}^n)$ and $C = h(Q)$. Clearly, (5) is satisfied and H is essentially self-adjoint on $(C_0^\infty(\mathbb{R}^n))^n \times C_0^\infty(\mathbb{R}^n)$. In the sequel we shall denote by the same symbol $|\cdot|$ the norm in $\mathbb{R}^n$ $(|x| = (x_1^2 + \cdots + x_n^2)^{1/2}, x = (x_1, \dots, x_n))$ and the norm of $s \times t$ (complex) matrices. Here s, t are arbitrary positive integers.

Theorem 2.5. *Suppose that*

$$|q(x)| + \frac{|v(x)| + |b(x)|}{|h(x)| + 1} \to 0 \text{ as } |x| \to \infty. \tag{10}$$

Then

$$(H + i)^{-1} - (H_0 + i)^{-1} \in S_\infty(\mathcal{H}). \tag{11}$$

In particular, $\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H_0) = [0, \infty) \cup \overline{h_0(\mathbb{R}^n)}$.

Proof. Define the self-adjoint operator $\widetilde{H}$ in $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ by

$$\widetilde{H} = \begin{pmatrix} -\Delta & B \\ B^* & C \end{pmatrix}. \tag{12}$$

We first prove that

$$(H + i)^{-1} - (\widetilde{H} + i)^{-1} \in S_\infty(\mathcal{H})$$

As $D(H) = D(\widetilde{H})$ (q is bounded), it suffices to prove that

$$(H + i)^{-1}(H - \widetilde{H})(H + i)^{-1} \in S_\infty(\mathcal{H})$$

As $D(H) \subset D((-\Delta)^{1/2}) \times D(h(Q))$ (see [AtLMS]), this follows from the condition

$$(-\Delta + 1)^{-1/2} q(Q)(-\Delta + 1)^{1/2} \in S_\infty(\mathcal{H}_1), \tag{13}$$

which is a consequence of (10). Now we apply Theorem 2.1 to the operator $\widetilde{H}$. By (10) and (8)

$$(\widetilde{H} + i)^{-1} - (L(-1) + i)^{-1} \in S_\infty(\mathcal{H}),$$

where $L(-1) = \text{diag}\{-\Delta, T(-1)\}$, $T(-1) = C - B^*(-\Delta + 1)^{-1}B$ (see (6)). To prove (11), it suffices to show that

$$(T(-1) + i)^{-1} - (h_0(Q) + i)^{-1} \in S_\infty(\mathcal{H}_2). \tag{14}$$

We have

$$\begin{gathered} T(-1) = h(Q) - (\bar{b}(Q)\text{div} + v^*(Q))(-\Delta + 1)^{-1}(-\nabla b(Q) + v(Q)) = \\ h_0(Q) + b(Q)(-\Delta + 1)^{-1} b(Q) - v^*(Q)(-\Delta + 1)^{-1} v(Q) - \bar{b}(Q)\text{div}(-\Delta + 1)^{-1} v(Q) + \\ v^*(Q)(-\Delta + 1)^{-1} \nabla b(Q). \end{gathered} \tag{15}$$

As $D(T(-1)) = D(h(Q)) = D(h_0(Q))$, (14) is equivalent to the condition

$$(h(Q) + i)^{-1}(T(-1) - h_0(Q))(h(Q) + i)^{-1} \in S_\infty(\mathcal{H}_2),$$

which follows from (10) and (15).

Remark 2.6. Clearly, the condition (10) can be weakened. We need only (8), (13) and (14).

3. Conjugate operator method.

Here we give a variant of the abstract Mourre's method (see [M], [PSS], [ABG]), which is suitable for the study of the operators of the form (1).

Let M be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$. We shall say that $T \in C^1(M)$ if the bounded operator function

$$T(t) := e^{itM} T e^{-itM} \tag{16}$$

is differentiable in the strong operator topology. $T \in C^1(M)$ if and only if the commutator $[T, M] := TM - MT$ (defined in the form sense) extends to a bounded operator in $\mathcal{H}$. Analogously, one can define a class $C^k(M)$ ($k \in \mathbb{N}$). We shall say that a self-adjoint operator $H \in C^k(M)$ if for some (and hence for all) $z \in \rho(H)$ the resolvent $(H-z)^{-1} \in C^k(M)$. Throughout this section we shall assume that $H \in C^1(M)$. We shall say that M is conjugate to H on an open set J if there is a compact operator K and $\alpha > 0$ such that

$$E_H(J)[H, iM]E_H(J) \geq \alpha E_H(J) + K \tag{17}$$

Here E_H is the spectral measure of H. If we have the estimate (17) with $K = 0$, we shall say that M is strictly conjugate to H. We also need the following definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $C^{+0}(M)$ if

$$\int_0^1 \|T(t) - T\| t^{-1} dt < \infty. \tag{18}$$

Here $T(t)$ is defined by (16). Clearly, $C^{+0}(M) \subset C^1(M)$.

Theorem 3.1. *Assume that a self-adjoint operator H has a spectral gap (i.e. $\sigma(H) \neq \mathbb{R}$), $H \in C^1(M)$ and M is conjugate to H on an open set $J \subset \mathbb{R}$. Suppose also that for any $\varphi \in C_0^\infty(\mathbb{R})$ and some $z \in \rho(H)$*

$$\varphi(H)[(H-z)^{-1}, iM]\varphi(H) \in C^{+0}(M) \tag{19}$$

Then H has in J at most a finite number of eigenvalues (counting multiplicities) and no singular continuous spectrum. Moreover, for any compact $\Delta \subset J \setminus \sigma_p(H)$ and $\varepsilon > 0$

$$\sup_{\lambda\in\Delta, \mu\neq 0} \|(|M|+1)^{-1/2-\varepsilon}(H-\lambda-i\mu)^{-1}(|M|+1)^{-1/2-\varepsilon}\| < \infty. \tag{20}$$

If, in addition, M is strictly conjugate to H on J, then H has no eigenvalues in J.

Corollary 3.2. *The assertion of Theorem 3.1 remains true if (19) is replaced by the assumption*

$$(H-z)^{-1}[(H-z)^{-1}, iM](H-z)^{-1} \in C^{+0}(M), \qquad z \in \rho(H) \tag{21}$$

Proof. We have

$$\varphi(H)[(H-z)^{-1}, iM]\varphi(H) = \psi(H)T\psi(H),$$

where $\psi(x) = \varphi(x)(x - z) \in C_0^\infty(\mathbb{R})$ and T is the right side of (21). As $H \in C^1(M)$, we have $\psi(H) \in C^1(M)$ (see [ABG]). Therefore, (21) implies (19).

Remark 3.3. Apparently, the assumption that H has a spectral gap is not necessary, but it is very convenient for the proof (see [ABG]). The conditions (19), (21) weaken the condition

$$[(H-z)^{-1}, M] \in C^{+0}(M), \tag{22}$$

which was used in [BGM] and [BG]. Note that in our applications to the operators of the form (1) the condition (22) is not fulfilled. Moreover, we cannot apply the more general results of [BG]. On the other hand we can verify (21). Note at last that under the assumptions of Theorem 3.1 the limiting absorption principle (see (20)) can be proved in some Besov spaces (see [BG] and [ABG]).

The proof of Theorem 3.1 can be obtained following the line of [BGM] and [Y]. It will be given in the later paper of the author devoted to the abstract Mourre theory.

4. Limiting absorption principle.

In the following we shall assume that h and b are of class C^2. Moreover, we suppose that h_0'' (see (4)) is bounded. We shall say that a real number λ is a threshold of h_0 if for any open neighbourhood J of λ $\inf\{|h_0'(x)| \mid h_0(x) \in J\} = 0$. The set of all thresholds of h_0 will be denoted by $\tau(h_0)$. $\tau(h_0)$ is a closed subset of $\mathbb{R}$ such that $\tau(h_0) \supset c(h_0)$, where $c(h_0) = \{h_0(x) \mid h_0'(x) = 0\}$ is the set of critical values of h_0. Denote $P_j = -i\dfrac{\partial}{\partial x_j}$, $P = -i\nabla = (P_1, \ldots, P_n)$ is a momenum operator. As usual $f(P) = \mathcal{F}^{-1} f(Q)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform. Denote $\tau = \tau(h_0) \cup \{0\}$. Let J be a bounded open interval such that its closure $\bar{J} \subset \mathbb{R} \setminus \tau$. Let us construct an operator M conjugate to the unperturbed operator H_0 on J. Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(x) = 1$ if $x \in J$. Suppose that $\mathrm{supp}\theta \subset \mathbb{R}\setminus\tau$. Define

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{23}$$

where

$$M_1 = \frac{1}{2}(F_1(P)Q + QF_1(P)), \tag{24}$$

$$M_2 = \frac{1}{2}(F_2(Q)P + PF_2(Q)), \tag{25}$$

$F_1(x) = \dfrac{1}{2}\theta(|x|^2)|x|^{-2}x$, $F_2(x) = -\theta(h_0(x)) \cdot |h_0'(x)|^{-2} h_0'(x)$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $x \in \mathbb{R}^n$. Since F_i are Lipschitz functions, M is essentially self-adjoint on $(S(\mathbb{R}^n))^n \times S(\mathbb{R}^n)$ [BG]. Here $S(\mathbb{R}^n)$ is the space of temperate test functions. It is easy to see that

$$\begin{aligned} [H_0, iM] &= \theta(H_0), \\ [[H_0, iM], iM] &= \theta(H_0)\theta'(H_0), \end{aligned} \tag{26}$$

so both commutators are bounded in $\mathcal{H}$ and $H_0 \in C^2(M)$. Moreover, (26) implies that M is conjugate to H_0 on J. Introduce the diagonal operator

$$\Lambda = \begin{pmatrix} (1+|Q|^2)^{1/2} & 0 \\ 0 & (1+|P|^2)^{1/2} \end{pmatrix} \tag{27}$$

acting in $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$.

Theorem 4.1. *Let H, H_0 be self-adjoint operators defined by (1) and (3) respectively. Assume that*

(i) For some $c > 0$, $\delta > 0$

$$|q(x)| + |b(x)| + |v(x)| \leq c(1+|x|)^{-1-\delta}, \qquad x \in \mathbb{R}^n.$$

(ii) b, v, h are bounded functions of class C^2 with bounded first and second derivatives. Then

(a) The only possible (finite) accumulation points for the eigenvalues of H are in τ. Any eigenvalue not in τ has finite multiplicity. H_0 has no eigenvalues in τ.

(b) The singular continuous spectrum of H lies in τ. In particular, if τ is countable, then H has no singular continuous spectrum.

(c) For any compact $\Delta \subset \mathbb{R} \setminus (\tau \cup \sigma_p(H))$ and $\varepsilon > 0$

$$\sup_{\lambda \in \Delta,\, \mu \neq 0} \|\Lambda^{-1/2-\varepsilon}(H - \lambda - i\mu)^{-1}\Lambda^{-1/2-\varepsilon}\| < \infty. \tag{28}$$

Remark 4.2. It is possible to generalize Theorem 4.1 on some unbounded functions h (see [K]). Moreover, in this situation we can weaken (i). It suffices to assume that for some $c > 0$ and $\delta > 0$

$$|q(x)| + \frac{|b(x)| + |v(x)|}{|h(x)| + 1} \leq c(1+|x|)^{-1-\delta}.$$

Proof. We first note that H is semibounded from below and hence it has a spectral gap. Let J be arbitrary bounded interval such that $\bar{J} \subset \mathbb{R} \setminus \tau$ and the operator M is defined by (23). The direct calculations (see (7)) show that the commutator $[(H-\lambda)^{-1}, iM] \in \mathcal{L}(\mathcal{H})$ $(\lambda \in \rho(H))$ and hence $H \in C^1(M)$. Furthermore, by Theorem 2.5

$$(H+i)^{-1} - (H_0+i)^{-1} \in S_\infty(\mathcal{H}).$$

Hence, M is conjugate to H on J [BG]. Now we shall verify the condition (21). Since $H_0 \in C^2(M)$, we have (for λ large enough)

$$(H+\lambda)^{-1}[(H_0+\lambda)^{-1}, iM](H+\lambda)^{-1} \in C^1(M) \subset C^{+0}(M).$$

Therefore, it suffices to show that

$$(H+\lambda)^{-1}[(H+\lambda)^{-1} - (H_0+\lambda)^{-1}, iM](H+\lambda)^{-1} \in C^{+0}(M) \tag{29}$$

To simplify calculations, we prove (29) in the case $q(x) \equiv 0$. The proof in the general case is of the same type (see also the proof of Theorem 2.5). Note that in order to verify that a symmetric bounded operator S is of class $C^{+0}(M)$ it suffices to show (see [BG]) that for some function $\xi \in C^\infty(\mathbb{R})$ with $\xi(t) = 0$ near zero and $\xi(t) = 1$ near infinity

$$\int_1^\infty \|\xi(\Lambda/r)S\| r^{-1} dr < \infty. \tag{30}$$

We have (for sufficiently large λ)

$$(H+\lambda)^{-1} - (H_0+\lambda)^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{pmatrix}, \tag{31}$$

where (see (7) and (15))

$$R_{11} = (-\Delta+\lambda)^{-1}(-\nabla b(Q) + v(Q))T(\bar{b}(Q)\mathrm{div} + v^*(Q))(-\Delta+\lambda)^{-1}, \tag{32}$$

$$R_{12} = -(-\Delta+\lambda)^{-1}(-\nabla b(Q) + v(Q))T, \tag{33}$$

$$\begin{gathered} R_{22} = (h_0(Q)+\lambda)^{-1}(v^*(Q)(-\Delta+\lambda)^{-1}v(Q) - \lambda\bar{b}(Q)(-\Delta+\lambda)^{-1}b(Q) + \\ \bar{b}(Q)\mathrm{div}(-\Delta+\lambda)^{-1}v(Q) - v^*(Q)(-\Delta+\lambda)^{-1}\nabla b(Q))T, \end{gathered} \tag{34}$$

$$T = (h(Q) - (\bar{b}(Q)\mathrm{div} + v^*(Q))(-\Delta+\lambda)^{-1}(-\nabla b(Q) + v(Q)) + \lambda)^{-1} \tag{35}$$

We shall write

$$(H+\lambda)^{-1} - (H_0+\lambda)^{-1} = R_1 + R_2,$$

where

$$R_1 = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & R_{12} \\ R_{12}^* & 0 \end{pmatrix}.$$

Simple calculations show that

$$\begin{pmatrix} (1+|Q|^2)^{\delta/2} & 0 \\ 0 & (-\Delta+1)^{1/2} \end{pmatrix} [R_1, iM] \in \mathcal{L}(\mathcal{H}).$$

Here $\delta > 0$ is defined by (i). In particular, $\Lambda^\delta [R_1, iM] \in \mathcal{L}(\mathcal{H})$ and (see (30))

$$[R_1, iM] \in C^{+0}(M). \tag{36}$$

On the other hand the condition (30) is not fulfilled for the operator $S = [R_2, iM]$. Therefore, we need to consider the operator

$$U = (H+\lambda)^{-1}[R_2, iM](H+\lambda)^{-1} = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{22} \end{pmatrix}$$

We have

$$[R_2, iM] = \begin{pmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{pmatrix},$$

where $T_{12} = i(R_{12}M_2 - M_1R_{12})$. Therefore,

$$U_{11} = R_{12}T_{12}^*((-\Delta+\lambda)^{-1} + R_{11}) + ((-\Delta+\lambda)^{-1} + R_{11})T_{12}R_{12}^*,$$

$$U_{12} = R_{12}T_{12}^*R_{12} + ((-\Delta+\lambda)^{-1} + R_{11})T_{12}((h_0(Q)+\lambda)^{-1} + R_{22}),$$

$$U_{22} = ((h_0(Q)+\lambda)^{-1} + R_{22})T_{12}^*R_{12} + R_{12}^*T_{12}((h_0(Q)+\lambda)^{-1} + R_{22}).$$

Using (ii), it is easy prove that $\Lambda^\delta U \in \mathcal{L}(\mathcal{H})$. Therefore (see also (36)), the condition (29) is verified. Now Theorem 4.1 directly follows from Corollary 3.2. Note only that, as $D(M) \supset D(\Lambda)$, the estimate (20) implies (28).

5. Wave operators.

We see that the unperturbed operator H_0 acts in one channel as the (vector) Laplace operator and in the other one as a multiplication operator. Therefore, we can expect existence and completeness of wave operators for the pair H, H_0 if the perturbation is of short-range in Q-variable in the first channel and in P-variable in the other one. In general, R_{22} (see (31),(34) and (35)) has only Coulomb decay in P-variable, and we can not prove the existence of wave operators. On the other hand, we shall see that if $\bar{b}(x)v(x) \in \mathbb{R}^n (x \in \mathbb{R}^n)$, R_{22} has a decay like $|P|^{-2}$. We shall prove that in this situation the wave operators for the pair H, H_0 exist and are complete.

Denote by $P_{ac}(H)$ the projector on the (spectrally) absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of the operator H. Let $H_{ac} = HP_{ac}(H)$ be the absolutely continuous part of H, $\sigma_{ac}(H) = \sigma(H_{ac})$ be the absolutely continuous spectrum of H.

Theorem 5.1. *Suppose that*

(i) The assumptions of theorem 4.1 are fulfilled.

(ii) $\bar{b}v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

(iii) For some $c > 0$, $\delta > 0$

$$|b'(x)| + |v'(x)| \le c(1+|x|)^{-1/2-\delta}, \ x \in \mathbb{R}^n.$$

(iv) τ *is a countable set.*

Then

(a) $\sigma_{ac}(H) = \sigma_{ac}(H_0)$.

(b) H *has no singular continuous spectrum.*

(c) The wave operators

$$W^\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0)$$

exist and are complete: $\mathrm{Ran}(W^\pm) = \mathcal{H}_{ac}(H)$. *In particular, operators* H_{ac} *and* $H_{0,ac}$ *are unitary equivalent.*

Proof. The assertion *(b)* was proved in Theorem 4.1, *(a)* follows from *(c)*. By Theorem 4.1 and Theorem 2.5, it suffices to show that for some $\delta > 0$ and $\lambda > 0$ (large enough)

$$\Lambda^{1/2+\delta}((H+\lambda)^{-1} - (\widetilde{H}+\lambda)^{-1})\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}).$$

Let $\widetilde{H}$ be the operator defined by (12). By *(i)* of Theorem 4.1, for sufficiently small $\delta > 0$

$$\Lambda^{1/2+\delta}((H+\lambda)^{-1} - (\widetilde{H}+\lambda)^{-1})\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}).$$

Therefore, it is sufficient to show that for small $\delta > 0$

$$\Lambda^{1/2+\delta}((\widetilde{H}+\lambda)^{-1} - (H_0+\lambda)^{-1})\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}). \tag{37}$$

(37) is equivalent to the following conditions (see (31))

$$(1+|Q|^2)^{1/4+\delta/2} R_{11} (1+|Q|^2)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1), \tag{38}$$

$$(1+|Q|^2)^{1/4+\delta/2} R_{12} (-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1), \tag{39}$$

$$(-\Delta+1)^{1/4+\delta/2} R_{22} (-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2). \tag{40}$$

(38) easily follows from the assumptions of Theorem 4.1. Let us prove (39). Note that commutators $[(1+|Q|^2)^{1/4+\delta/2}, (-\Delta+1)^{-1}]$, $[(1+|Q|^2)^{1/4+\delta/2}, (-\Delta+1)^{-1}\nabla]$, $[(-\Delta+1)^{1/4+\delta/2}, T]$ (T defined by (35)) are bounded for $\delta \in (0, 1/2]$. Therefore, it suffices to show that for some $\delta > 0$

$$(-\Delta+1)^{-1}\nabla(1+|Q|^2)^{1/4+\delta/2} b(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2),$$

$$(-\Delta+1)^{-1}(1+|Q|^2)^{1/4+\delta/2} v(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$

These conditions directly follow from (iii). It remains to prove (40). Since the functions b, v, h are smooth with bounded first and second derivatives, it suffices to verify that for some $\delta > 0$

$$\begin{aligned}(-\Delta+1)^{1/4+\delta/2}(\bar{b}(Q)\mathrm{div}(-\Delta+\lambda)^{-1}v(Q)-\\ -v^*(Q)(-\Delta+\lambda)^{-1}\nabla b(Q))(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2).\end{aligned} \tag{41}$$

Clearly, for $\delta \in (0, 1/2]$

$$(-\Delta+1)^{1/4+\delta/2}[\bar{b}(Q), \mathrm{div}(-\Delta+\lambda)^{-1}]v(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2)$$

and

$$(-\Delta+1)^{1/4+\delta/2}[v^*(Q), (-\Delta+\lambda)^{-1}\nabla]b(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2).$$

Therefore, (41) follows from (ii).

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