CYCLICALLY PRESENTED GROUPS AND TAKAHASHI MANIFOLDS AS CYCLIC COVERINGS

Ann Chi Kim
Department of Mathematics, Pusan National University, Pusan, 609–735, Korea
e-mail: ackim@arirang.math.pusan.ac.kr

Andrei Vesnin
Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
e-mail: vesnin@math.nsc.ru

Abstract

We consider groups with cyclic presentations which arise as fundamental groups of a family of closed 3-dimensional manifolds. These manifolds were firstly described by Takahashi using Dehn surgeries on links. We demonstrate that the cyclic automorphisms of groups induce cyclic coverings of the 3-sphere branched over 2-bridge knots. Moreover, polynomials associated with cyclic presentations are equal to Alexander polynomials of corresponding knots.
Keywords: fundamental group, 3-manifold, cyclic covering.

1. Cyclically Presented Groups

The cyclically presented groups comprise a rich source of groups which are interesting from a topological point of views. The connection between cyclically presented groups and cyclic branched coverings of knots and links was studied, in particular, in [2], [5], [6] and [7].

Let $F_n = \langle x_1, \ldots, x_n \quad | \quad \rangle$ be the free group of rank n and $\eta : F_n \to F_n$ be an automorphism of order n such that $\eta(x_i) = x_{i+1}$ for $i = 1, \ldots, n$, where all indices by mod n. We recall [9, §9] that for a reduced word $w \in F_n$ the *cyclically presented group* $G_n(w)$ is given by

$$G_n(w) = \langle x_1, \ldots, x_n \quad | \quad w, \eta(w), \ldots, \eta^{n-1}(w) \rangle.$$

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A group G is said to *have a cyclic presentation* if $G \cong G_n(w)$ for some n and w. Clearly, the automorphism η of F_n induces an automorphism of $G_n(w)$. Such automorphism defines an action of the group $\mathbb{Z}_n = \langle \eta \mid \eta^n = 1 \rangle$ on $G_n(w)$. Let us consider a split extension $H_n = G_n(w) \lambda \mathbb{Z}_n$. The group H_n is said to be a *natural extension of a cyclically presented group.* It was remarked in [5] that the group $H_n = H_n(v)$ always has a 2-generator, 2-relator presentation of the form

$$H_n(v) \cong \langle \eta, x \quad | \quad \eta^n = v(\eta, x) = 1 \rangle,$$

where $v = v(\eta, x) = w(x, \eta^{-1} x \eta, \ldots, \eta^{-(n-1)} x \eta^{n-1})$.

Following [9], we define the *polynomial* $f_w(t)$ *associated with* the cyclically presented group $G \cong G_n(w)$ as

$$f_w(t) = \sum \alpha_i \, t^i,$$

where α_i is the exponent sum of x_i in w, $1 \leq i \leq n$.

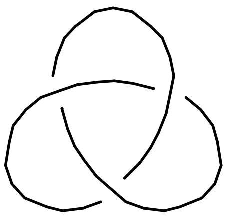
We recall that in [13] L. Neuwirth describes an algorithm for deciding is a group presentation with n generators and n relations corresponds to the spine of a closed compact 3-manifold. The nice approach to determine which cyclically presented groups arise as fundamental groups of 3-manifolds was devoloped by M. Dunwoody in [5], where he described an algorithm to enumerate Heegaard diagrams with the necessary cyclic symmetry. In particularly, he asked about covering properties of manifolds which have been arised in this way.

In the present paper we give another approach to describe a series of cyclically presented groups which arise as fundamental groups of 3-manifolds. We consider Dehn surgeries of a $2n$-component chain with cyclically symmetric surgery parameters. We demonstrate that this symmetry corresponds a cyclic coverings of two-bridge knots (Theorem 4.1 and 4.2) and descibe correspondong cyclically presented groups (Theorem 5.1). In particularly, Dunwoody's groups with $w = w(x_i, x_{i+1}, x_{i+2})$ belong to this series. We will start from two well-known examples.

Example 1.1. Let us consider *Sieradski groups* defined by the presentation

$$S(n) = \langle x_1, \ldots, x_n \quad | \quad x_i x_{i+2} = x_{i+1}, \quad i = 1, \ldots, n \rangle,$$

where all indices are taken mod n. As we see, this presentation is cyclic and $w(x_i, x_{i+1}, x_{i+2}) = x_i \, x_{i+2} \, x_{i+1}^{-1}$.



Therefore,

$$f_w(t) = t^i + t^{i+2} - t^{i+1} = t^i \, (t^2 - t + 1).$$

We recall that $\Delta(t) = t^2 - t + 1$ is the Alexander polynomial of the *trefoil knot* [1].

Consider the 2-generator group

$$H_n(v) = \langle \eta, x \quad | \quad \eta^n = v(\eta, x) = 1 \rangle,$$

where $v(\eta, x) = w(x, \eta^{-1} x \eta, \eta^{-2} x \eta^2) = x \eta^{-2} x \eta^2 (\eta^{-1} x \eta)^{-1} = x \eta^{-2} x \eta x^{-1} \eta$. Let λ be such that $x = \eta \lambda$. Then we get

$$H_n(v) \cong \langle \lambda, \eta \quad | \quad \eta^n = \lambda^n = 1, \quad \eta \lambda \eta^{-1} \lambda \eta \lambda^{-1} = 1 \rangle,$$

and we recall that the group

$$\langle \lambda, \eta \quad | \quad \lambda \eta^{-1} \lambda = \eta^{-1} \lambda \eta^{-1} \rangle$$

is isomorphic to the fundamental group of the trefoil knot [1], where generators λ and η are meridians. Thus the group $H_n(v)$ is isomorphic to the fundamental group of the orbifold whose underlying space is the 3-sphere and whose singular set is the trefoil knot as an order n cone axis.

Example 1.2. Let us consider *Fibonacci groups* defined by the presentation

$$F(2, 2n) = \langle x_1, \ldots, x_{2n} \quad | \quad x_i x_{i+1} = x_{i+2}, \quad i = 1, \ldots, 2n \rangle,$$

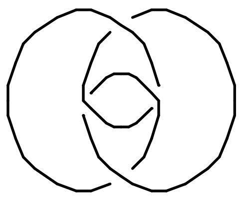
where all indices are taken mod $2n$. This presentation is cyclic and for this case $w(x_i, x_{i+1}, x_{i+2}) = x_i\, x_{i+1}\, x_{i+2}^{-1}$. Therefore

$$f_w(t) = t^i + t^{i+1} - t^{i+2} = t^i\,(-t^2 + t + 1).$$

Unfortunately, the polynomial $\Delta(t) = -t^2 + t + 1$ cannot be the Alexander polynomial of a knot [1]. So, we would like to consider another cyclic presentation of the group $F(2, 2n)$. Suppose $y_i = x_{2i}$ for $i = 1, \ldots, n$. Then $x_{2i+1} = x_{2i}^{-1}\, x_{2i+2} = y_i^{-1}\, y_{i+1}$. Hence

$$F(2, 2n) \cong \langle y_1, \ldots, y_n \quad | \quad (y_i^{-1} y_{i+1})\, y_{i+1} = (y_{i+1}^{-1} y_{i+2}) \quad i = 1, \ldots, n\rangle.$$

Thus we got the cyclic presentation with $w(y_i, y_{i+1}, y_{i+2}) = y_i^{-1} y_{i+1}^2\, y_{i+2}^{-1}\, y_{i+1}$.



Therefore

$$f_w(t) = -t^i + 2t^{i+1} - t^{i+2} + t^{i+1} = -t^i\,(t^2 - 3t + 1).$$

We recall that $\Delta(t) = t^2 - 3t + 1$ is the Alexander polynomial of the *figure–eight knot* [1].

Consider the 2-generator group

$$H_n(v) = \langle \eta, y \quad | \quad \eta^n = v(\eta, y) = 1\rangle,$$

where $v(\eta, y) = w(y, \eta^{-1} y \eta, \eta^{-2} y \eta^2) = y^{-1}\eta^{-1}y^2\eta^{-1}y^{-1}\eta y \eta$. Let λ be such that $y = \eta\lambda$. Then we get

$$H_n(v) \cong \langle \lambda, \eta \quad | \quad \eta^n = \lambda^n = 1, \quad \eta^{-1}\,[\lambda, \eta] = [\lambda, \eta]\,\lambda\,\rangle,$$

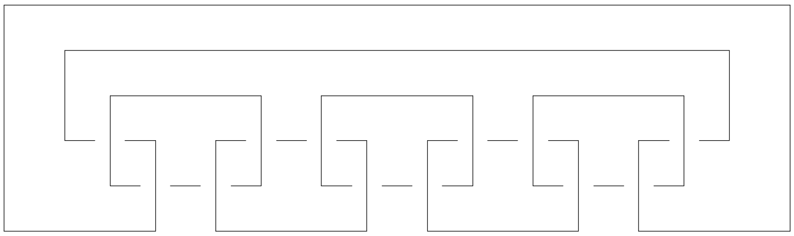
where $[\lambda, \eta] = \lambda^{-1}\eta^{-1}\lambda\eta$. We recall that the group

$$\langle \lambda, \eta \quad | \quad \eta^{-1}\,[\lambda, \eta] = [\lambda, \eta]\,\lambda\,\rangle$$

is the fundamental group of the figure-eight knot [1], where generators λ and η are meridians. Thus the group $H_n(v)$ is isomorphic to the fundamental group of the orbifold whose underlying space is the 3-sphere and whose singular set is the figure-eight knot as an order n cone axis.

2. Takahashi Manifolds

In this section we describe a nice series of closed orientable 3-manifolds whose fundamental groups were studied by M. Takahashi [15]. Our interest in studiing of these manifolds was inspired in part by the problem to classify the topological structure of these manifolds, formulated in [3]. For any integer $n \geq 2$ we consider a link L_{2n} with $2n$ components, each of which is unknotted and is linked with exactly two adjacent components, similar to the figure below, where the link L_6 is pictured.



The link L_6.

It was shown by W. Thurston [16, Section 6.8.7] that for $n \geq 3$ the link L_{2n} is hyperbolic and the hyperbolic volume of the complement $\mathcal{S}^3 \setminus L_{2n}$ is given by the formula

$$vol(\mathcal{S}^3 \setminus L_{2n}) \; = \; 8\,n \, \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2n} \right) \, + \, \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2n} \right) \right],$$

where $\Lambda(x)$ is the *Lobachevsky function* :

$$\Lambda(x) \; = \; - \int_0^x \ln |2 \sin \theta| \, d\theta.$$

Let us cyclically enumerate components of L_{2n}, and consider closed manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ obtained by Dehn surgeries on components of L_{2n}, where a surgery coefficient p_i/q_i, $i = 1, \ldots, 2n$, corresponds to the i-th component of L_{2n}. The manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ are refer as *Takahashi manifolds*. The presentations of the fundamental groups of manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ where studied in [15] where the following nice result was obtained.

THEOREM 2.1.[15] *The fundamental group of a manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ has the following presentation*

$$\langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1}^{q_{2k-1}} \, a_{2k}^{-p_{2k}} \; = \; a_{2k+1}^{q_{2k+1}}, \, a_{2k}^{q_{2k}} \, a_{2k+1}^{p_{2k+1}} \; = \; a_{2k+2}^{q_{2k+2}}, \qquad k = 1, \ldots, n \rangle$$

where all indices are taken mod $2n$.

Let us consider the following particular cases.

EXAMPLE 2.1. Assume that $p_i = q_i = 1$ for all $i = 1, \ldots, 2n$. Then the fundamental group of the manifold $M_{2n}(1, 1, \ldots, 1, 1)$ is given by

$$\begin{aligned} &\pi_1(M_{2n}(1, 1, \ldots, 1, 1)) \\ &\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1} \, a_{2k}^{-1} \; = \; a_{2k+1}, \, a_{2k} \, a_{2k+1} \; = \; a_{2k+2}, \qquad k = 1, \ldots, n \rangle \\ &\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1} \; = \; a_{2k+1} \, a_{2k}, \, a_{2k+1} \; = \; a_{2k}^{-1} \, a_{2k+2}, \qquad k = 1, \ldots, n \rangle \\ &\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k+1} \; = \; a_{2k+3} \, a_{2k+2}, \, a_{2k+1} \; = \; a_{2k}^{-1} \, a_{2k+2}, \qquad k = 1, \ldots, n \rangle. \end{aligned}$$

Therefore $a_{2k+3} = a_{2k}^{-1}$ for all $k = 1, \ldots, n$. Hence using the notation $x_i = a_{2i}$ for $i = 1, \ldots, n$, we get

$$\begin{aligned} &\pi_1(M_{2n}(1, 1, \ldots, 1, 1)) \\ &\quad = \langle x_1, \ldots, x_n \quad | \quad x_{k-1}^{-1} \; = \; x_k^{-1} \, x_{k+1}, \qquad k = 1, \ldots, n \rangle \\ &\quad = \langle x_1, \ldots, x_n \quad | \quad x_k \; = \; x_{k-1} \, x_{k+1}, \qquad k = 1, \ldots, n \rangle, \end{aligned}$$

that is isomorphic to the Sieradski group $S(n)$. It was shown in [2] that groups $S(n)$ are isomorphic to fundamental groups of the n-fold cyclic coverings of the 3-sphere

$\mathcal{S}^3$ branched over the trefoil knot. Indeed, the Takahashi manifold $M_{2n}(1,1,\ldots,1,1)$ is homeomorphic to the n-fold cyclic covering of $\mathcal{S}^3$ branched over the trefoil knot (see discussion in [14] for small n).

Example 2.2. Assume that $q_i = 1$ for each i and $p_i = (-1)^{i+1}$, where $i = 1,\ldots,2n$. Then the fundamental group of the manifold $M_{2n}(1,-1,\ldots,1,-1)$ is given by

$$\begin{aligned}
&\pi_1(M_{2n}(1,-1,\ldots,1,-1)) \\
&\quad = \langle a_1,\ldots,a_{2n} \quad | \quad a_{2k-1}\, a_{2k} \;=\; a_{2k+1},\; a_{2k}\, a_{2k+1} \;=\; a_{2k+2}, \qquad k = 1,\ldots,n\rangle \\
&\quad = \langle a_1,\ldots,a_{2n} \quad | \quad a_i\, a_{i+1} \;=\; a_{i+2}, \qquad i = 1,\ldots,2n\rangle,
\end{aligned}$$

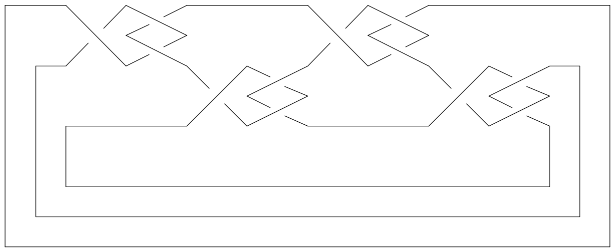
that is isomorphic to the Fibonacci group $F(2,2n)$. It follows from [6] and [7] that the group $F(2,2n)$ is isomorphic to the fundamental group of the n-fold cyclic covering of the 3-sphere $\mathcal{S}^3$ branched over the figure–eight knot. Indeed, it was shown in [4] that the Takahashi manifold $M_{2n}(1,-1,\ldots,1,-1)$ is homeomorphic to the n-fold cyclic covering of $\mathcal{S}^3$ branched over the figure-eight knot.

3. Two-fold branched coverings

Let us define a family of knots and links which are closely connected with the Takahashi manifolds. We recall [1] that any link can be obtained as a closed braid. For coprime integers p and q we denote by $\sigma_i^{p/q}$ a rational p/q-tangle [1] whose incoming arcs are i-th and $(i+1)$-th strings of the braid. For $n \geq 1$ and pairs of coprime integers p_i and q_i, $i = 1,\ldots,2n$, we denote by $K_{2n}(p_1/q_1,\ldots,p_{2n}/q_{2n})$ a *closed rational 3-strings braid*

$$\sigma_1^{p_1/q_1}\,\sigma_2^{p_2/q_2}\;\cdots\;\sigma_1^{p_{2n-1}/q_{2n-1}}\,\sigma_2^{p_{2n}/q_{2n}}.$$

As an example, the diagram of the link $K_4(3/2,-3/2,3/2,-3/2)$ is pictured below.



The link $K_4(3/2, -3/2, 3/2, -3/2)$.

There is the following connection between the Takahashi manifolds and the above links.

THEOREM 3.1.[10] *Any Takahashi manifold* $M_{2n}(p_1/q_1, \dots, p_{2n}/q_{2n})$ *can be obtained as the two-fold branched covering of the link* $K_{2n}(p_1/q_1, \dots, p_{2n}/q_{2n})$.

The proof of the theorem is based on the Montesinos algorithm [12], which admits to describe a two-fold covering presentation for a manifold obtained by Dehn surgeries on a strongly invertible link.

Because for the case $q_i = 1$ for all $i = 1, \dots, 2n$, a rational 3-strings braid becomes an ordinary 3-strings braid, we get

COROLLARY 3.1. *For any closed 3-strings braid its two-fold branched covering is a Takahashi manifold.*

The following particular case of Theorem 3.1 was discussed in [14] for small n and in [2] for the general case.

COROLLARY 3.2. *Takahashi manifolds* $M_{2n}(1, 1, \dots, 1, 1)$ *are two-fold coverings of 3-strings torus knots* $T_{n,3} = (\sigma_1\sigma_2)^n$, *where in particular,* $T_{2,3}$ *is the trefoil knot* 3_1, *and* $T_{3,3} = 8_{19}$.

As a particular case of Theorem 3.1, in virtue of [4], we get the following result remarked in [11].

COROLLARY 3.3. *Takahashi manifolds* $M_{2n}(1, -1, \dots, 1, -1)$ *are two-fold coverings* *of Turks head links $Th_n = (\sigma_1 \sigma_2^{-1})^n$, where in particular, Th_2 is the figure-eight knot 4_1, Th_3 are Borromean rings 6_2^3, and Th_4 is the knot 8_{18}.*

Because the link $K_2(p_1/q_1, p_2/q_2)$ is a connected sum of 2-bridge (p_1/q_1)–link and (p_2/q_2)–link, the manifolds $M_2(p_1/q_1, p_2/q_2)$ can be easy described.

Corollary 3.4. *Takahashi manifolds $M_2(p_1/q_1, p_2/q_2)$ are connected sums of lens spaces L_{p_1,q_1} and $L_{p_2.q_2}$.*

4. Cyclic branched coverings

In this section we consider Takahashi manifolds with cyclic symmetries. We will be say that a manifold $M_{2n}(p_1/q_1, \dots, p_{2n}/q_{2n})$ is *n-periodic* if surgery parameters are such that $p_{2i-1}/q_{2i-1} = a/b$ and $p_{2i}/q_{2i} = c/d$ for $i = 1, \dots, n$, where a/b and c/d are some rational. In this case we will be use a notation

$$M_n(a/b, c/d) \ = \ M_{2n}(a/b, c/d, \dots, a/b, c/d).$$

According to this notations the Fibonacci manifolds can be written in the form $M_n(1, -1)$ where $\pi_1(M_n(1, -1)) = F(2, 2n)$.

Analogously, we consider n-periodic closed rational 3-strings braid $K_n(a/b, c/d)$, that is the closure of $(\sigma_1^{a/b} \sigma_2^{c/d})^n$. In particular cases we get following 2-bridge knots.

Lemma 4.1. *For any integers k and l the link $K_2(1/k, -1/l)$ is the 2-bridge $(2k+\frac{1}{2l})$–knot.*

Lemma 4.2. *For any integers k and l the link $K_2(1/k, 1/l)$ is the 2-bridge $(2k-\frac{1}{2l})$–knot.*

Similar to [8], we use the following notations for orbifolds whose singular set is a two-bridge knot or link. We denote by $(p/q)(n)$ an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the 2-bridge p/q–knot as an order

n cone axis. By $(p/q)(m,n)$ we denote an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the 2-bridge p/q–link with orders m and n corresponding to its components. By $K_n(a/b, c/d)(2)$ we denote an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the n-periodic closed rational 3-strings braid $K_n(a/b, c/d)$ each component of which is an order 2 cone axis.

THEOREM 4.1. *Let $M_n(1/b, -1/d)$, $n \geq 2$, $b > 0$, $d > 0$, be a n-periodic Takahashi manifold. Then the following covering diagram holds:*

$$\begin{tikzcd} & M_n(\frac{1}{b}, -\frac{1}{d}) \arrow[dl, "2"'] \arrow[dr, "n"] & \\ K_n(\frac{1}{b}, -\frac{1}{d})(2) \arrow[dr, "n"'] & & (2b + \frac{1}{2d})(n) \arrow[dl, "2"] \\ & (p/q)(2,n) & \end{tikzcd}$$

where $p = 8bd + 2$ and $0 < q < 4bd + 1$ such odd that $2dq = \pm 1 \pmod{4bd + 1}$.

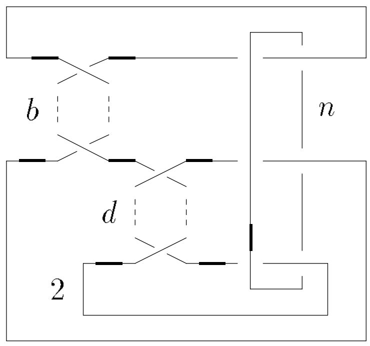
Proof. The 2-fold covering

$$M_n(1/b, -1/d) \xrightarrow{2} K_n(1/b, -1/d)$$

holds by Theorem 3.1. Obviously the orbifold $K_n(1/b, -1/d)(2)$ has the symmetry ρ of order n. Consider the quotient orbifold $\mathcal{O}^{b,d}(2,n) = K_n(1/b, -1/d)(2)/\rho$. Its singular set $\mathcal{L}^{b,d}$ is the 2-component link in the figure below with orders 2 and n corresponding to components. We remark that both components of $\mathcal{L}^{b,d}$ are unknotted and equivalent. Using the method from [11], we can define an epimorphism

$$\theta : \pi_1(\mathcal{O}^{b,d}(2,n)) \to \mathbb{Z}_2 \oplus \mathbb{Z}_n$$

such that $\theta^{-1}(\mathbb{Z}_n) = \pi_1((2b + 1/2d)(n))$, $\theta^{-1}(\mathbb{Z}_2) = \pi_1(K_n(1/b, -1/d)(2))$, and $Ker(\theta) = \pi_1(M_n(1/b, -1/d))$.



The link $\mathcal{L}^{b,d}$

The covering diagram holds from the diagram of subgroups and Lemma 4.1. 2

THEOREM 4.2. *Let $M_n(1/b, 1/d)$, $n \geq 2$, $b > 1$, $d > 0$, be a n-periodic Takahashi manifold. Then the following covering diagram holds:*

$$\begin{tikzcd}[column sep=small] & M_n(\frac{1}{b},\frac{1}{d}) \arrow[dl, "2"'] \arrow[dr, "n"] & \\ K_n(\frac{1}{b},\frac{1}{d})(2) \arrow[dr, "n"'] & & (2b-\frac{1}{2d})(n) \arrow[dl, "2"] \\ & (p/q)(2,n) & \end{tikzcd}$$

where $p = 8bd - 2$ and $0 < q < 4bd - 1$ such odd that $2dq = \pm 1 \pmod{4bd-1}$.

Proof. Analogously to the proof of Theorem 4.1. 2

5. GROUP PRESENTATIONS

THEOREM 5.1. *Denote by $\mathcal{M}_n(p/q)$ the n-fold cyclic branched covering of the 2-bridge (p/q)-knot. (i) If $p/q = 2k + \frac{1}{2l}$, then*

$$\pi_1(\mathcal{M}_n(p/q)) = \langle x_1, \ldots, x_n \quad | \quad (x_i^{-l}\, x_{i+1}^l)^k\, x_{i+1} = (x_{i+1}^{-l}\, x_{i+2}^l)^k, \quad i = 1, \ldots, n \rangle.$$

(ii) If $p/q = 2k - \frac{1}{2l}$, then

$$\pi_1(\mathcal{M}_n(p/q)) = \langle y_1, \ldots, y_n \quad | \quad (y_i^{-l}\, y_{i+1}^l)^k\, y_{i+1}^{-1} = (y_{i+1}^{-l}\, y_{i+2}^l)^k, \quad i = 1, \ldots, n \rangle.$$

Proof. (i) By Theorem 4.1 the manifold $\mathcal{M}_n(2k + \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, -1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\begin{aligned}\pi_1(\mathcal{M}_n(2k + 1/2l)) &= \pi_1(M_n(1/k, -1/l)) \\ &= \langle a_1, \ldots, a_{2n} \quad | \quad a_{2i+1}^k\, a_{2i+2} = a_{2i+3}^k, a_{2i}^l\, a_{2i+1} = a_{2i+2}^l, \quad i = 1, \ldots, n \rangle.\end{aligned}$$

The formula from the statement of the theorem will be obtained if we suppose $x_i = a_{2i}$, $i = 1, \ldots, n$.

(ii) Analogously, by Theorem 4.2 the manifold $\mathcal{M}_n(2k - \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, 1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\begin{aligned}\pi_1(\mathcal{M}_n(2k - 1/2l)) &= \pi_1(M_n(1/k, 1/l)) \\ &= \langle a_1, \ldots, a_{2n} \quad | \quad a_{2i+1}^k\, a_{2i+2}^{-1} = a_{2i+3}^k, a_{2i}^l\, a_{2i+1} = a_{2i+2}^l, \quad i = 1, \ldots, n \rangle.\end{aligned}$$

The formula from the statement of the theorem will be obtained if we suppose $y_i = a_{2i}$, $i = 1, \ldots, n$. 2

Corollary 5.1. *The polynomial associated with the cyclic presentation of a group $\pi_1(\mathcal{M}_n)(p/q)$ from Theorem 5.1 is equivalent to the Alexander polynomial of the two-bridge p/q-knot: if $p/q = 2k + \frac{1}{2l}$, then*

$$\Delta(t) = k\,l\,t^2 - (2\,k\,l + 1)\,t + k\,l,$$

and if $p/q = 2k - \frac{1}{2l}$, then

$$\Delta(t) = k\,l\,t^2 - (2\,k\,l - 1)\,t + k\,l.$$

In conclusion we remark that the present paper was inspired in part by the nice paper of M. Dunwoody [5], where he constructed a family of 3-manifolds whose fundamental groups are cyclically presented, and asked are these manifolds cyclic branched coverings of knots or links. It is easy to check that all cyclically presented groups from [5] with $w = w(x_i, x_{i+1}, x_{i+2})$ are of the type (i) or of the type (ii) from Theorem 5.1 for some k and l.

COROLLARY 5.2. *Each cyclically presented group from [5] with $w = w(x_i, x_{i+1}, x_{i+2})$ is isomorphic to the fundamental group of the cyclic branched covering of the 2-bridge $(2k + \frac{1}{2l})$–knot or $(2k - \frac{1}{2l})$–knot for some k and l.*

In particularly, the follolwing table holds for groups with $w = w(x_i, x_{i+1}, x_{i+2})$ given in [5]:

$w(x_0, x_1, x_2)$	$f_w(t)$	p/q–knot
$x_1^{-1} x_2 x_0$	$t^2 - t + 1$	$(2 - 1/2) = 3_1$
$x_1^{-1} x_2 x_1^{-2} x_0$	$t^2 - 3t + 1$	$(2 + 1/2) = 4_1$
$x_1^{-2} x_2^2 x_1^{-1} x_0^2$	$2t^2 - 3t + 2$	$(2 - 1/4) = 5_2$
$x_1^{-1} x_2 x_1^{-1} x_2 x_0 x_1^{-1} x_0$	$2t^2 - 3t + 2$	$(4 - 1/2) = 5_2$
$x_1^{-2} x_2^2 x_1^{-3} x_0^2$	$2t^2 - 5t + 2$	$(2 + 1/4) = 6_1$
$x_1^{-1} x_2 x_1^{-1} x_2 x_1^{-2} x_0 x_1^{-1} x_0$	$2t^2 - 5t + 2$	$(4 + 1/2) = 6_1$

Because the goemetric structures on cyclic coverings of 2-bridge knots are well studied [8], we get that manifolds $\mathcal{M}_n(2k + \frac{1}{2l})$ and $\mathcal{M}_n(2k - \frac{1}{2l})$ are hyperbolic for all $n \geq 3$ if $k \geq 2$ or $l \geq 2$, and manifolds $\mathcal{M}_n(2 + \frac{1}{2})$ are hyperbolic for all $n \geq 4$. Therefore in these cases cyclically presented groups with presentations given in Theorem 5.1 are hyperbolic groups.

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