On decomposing one relator products of cyclic groups into free products with amalgamation

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This paper deals with the problem of decomposing some finitely generated groups into non-trivial free products with amalgamation. We consider this problem for so called one-relator products of cyclic groups. Recall that a one-relator product of a family of groups $\{G_i\}$, $i \in I$ is the group $G = (*G_i)/N(S)$ where S is a cyclically reduced word in the free product $*G_i$ and $N(S)$ is its normal closure. S is called the relator. In general one-relator products share many properties with one-relator groups [6]. We consider the case where G_i's are cyclic and the relator is a proper power, i.e. $S = R^m$ where R is cyclically reduced in $(*G_i)$ and $m \geq 2$.

Definition 1 *A group G of the form*

$$G =< a_1, ..., a_n \mid a_1^{l_1} = ... = a_n^{l_n} = R^m(a_1, ..., a_n) = 1 >$$

where $n \geq 2$, $m \geq 2$, $l_i = 0$ or $l_i \geq 2$ for all i and $R(a_1, ..., a_n)$ is a cyclically reduced word in the free group on $a_1, ..., a_n$ is called one relator product of cyclics.

The first general result about decompositions of such groups into non-trivial free products with amalgamation was obtained by B. Fine, F. Levin and G. Rosenberger [3].

Theorem 1 **(**B. Fine, F. Levin, G. Rosenberger [3]) *If G is an one-relator product of n cyclics, $n \geq 3$, then G is a non-trivial free product with amalgamation.*

The proof of this theorem uses the techniques of representation and character varieties. We recall that if Γ is a finitely generated group then for any algebraic group H the set $R(\Gamma, H)$ of all representations $\rho : \Gamma \to H$ is known to have the natural structure of an algebraic variety, and endowed with this structure is called the variety of representations of Γ in H (cf. [7]). The group H acts on $R(\Gamma, H)$ by conjugation and if H is reductive then one can consider the categorical quotient $R(\Gamma, H)/H = X(\Gamma, H)$ which is usually called the character variety of Γ in H. M. Culler and P. Shalen [2] proved that if $\dim X(\Gamma, \mathrm{SL}_2(\mathbb{C})) > 0$ then Γ is either a non-trivial free product with amalgamation or a n0n-trivial HNN extension.

Let us come back to Theorem 1. Let $G =< a_1, ..., a_n \mid a_1^{l_1} = ... = a_n^{l_n} = R^m(a_1, ..., a_n) = 1 >$ be as in the theorem. We may assume without loss of generality that each $l_i \geq 2$. B. Fine, F. Levin and G. Rosenberger showed that $\dim X(G, \mathrm{SL}_2(\mathbb{C})) \geq 2$. Thus, G is either an HNN group or a non-trivial free product with amalgamation. Since the abelianization $G^{ab} = G/[G, G]$ is finite G is not HNN group and Theorem 1 is proved.

We consider now one-relator product of 2 cyclics, i.e. a group of the form

$$G =< a, b \mid a^m = b^n = R^l(a, b) = 1 >,$$

where $l \geq 2$, $R(a, b)$ is a cyclically reduced word in the free group on a and b. Such a group is usually called *the generalized triangle group.* In this case the situation is quite different. These groups sometimes admit decompositions into non-trivial free products with amalgamation and sometimes don't. For example, H. Zieschang [10] proved that *the ordinary triangle group*

$$T(m, n, p) =< a, b \mid a^m = b^n = (ab)^p = 1 >$$

with $m, n, p \geq 2$ is never a non-trivial free product with amalgamation. In fact, H. Zieshang [10] studied the problem of decompositions into non-trivial free products with amalgamation for discontinuous groups of transformations of the plane. He gave the full answer when such a group G has a such decomposition except the following two groups:

$$H_1 =< a, b \mid [a, b]^n = 1 >, \qquad n \geq 2,$$

and

$$H_2 =< c, d \mid c^2 = [c, d]^n = 1 >, \qquad n \geq 2.$$

G. Rosenberger [8] showed that H_1 and H_2 are non-trivial free products with amalgamation if n is not a power of 2. The following question is open:

Are H_1 and H_2 non-trivial free products with amalgamation if $n = 2^l$, $l \geq 1$?

Furthermore, there is the following conjecture.

Conjecture 1 (B. Fine, F. Levin, G. Rosenberger [3]) *A two generator one-relator group with torsion decomposes as a non-trivial free product with amalgamation.*

This conjecture was proved by B. Fine, F. Levin and G. Rosenberger in the special case when a group G is of the form

$$G =< a, b \mid R^m(a, b) = 1 >,$$

where $m \geq 2$ and $R(a, b)$ is a cyclically reduced word not contained in the commutator subgroup in the free product on a and b. Our goal is to prove the following theorem which answers these questions.

Theorem 2 *Let $G =< a, b \mid a^{2n} = R^m(a, b) = 1 >$, where $n = 0$ or $n \geq 2$, $m \geq 2$ and $R(a, b)$ is a cyclically reduced word in the free group on a and b which involves b. Then G is a non-trivial free product with amalgamation.*

The following corollaries follow immediately from Theorem 2.

Corollary 1 *Groups $H_1 =< a, b \mid [a, b]^n = 1 >$ and $H_2 =< c, d \mid c^2 = [c, d]^n = 1 >$ are non-trivial free products with amalgamation for any $n \geq 2$.*

Corollary 2 *Let G be as in Conjecture 1 above. Then G is a non-trivial free product with amalgamation.*

Proof of corollary 2. It is well known that if G is a finitely generated one relator group with torsion then G can be presented in the form

$$G =< a_1, ..., a_n \mid R^m(a_1, ..., a_n) = 1 >,$$

where $m \geq 2$, $R(a_1, ..., a_n)$ is a cyclicaly reduced word in the free group on a_1, ..., a_n. Thus, in our case $G =< a, b \mid R^m(a, b) = 1 >$ and by Theorem 2 G is a non-trivial free product with amalgamation. $\square$

The following result of H. Bass [1] plays an important role in the proof of Theorem 2.

Theorem 3 (H. Bass[1]) *Let H be a finitely generated subgroup of $\mathrm{GL}_2(\mathbb{C})$. Then one of the following cases must occur:*

(1) There is an epimorphism $f : H \to \mathbb{Z}$ such that $f(u) = 0$ for all unipotent elements $u \in H$.

(2) H is conjugate to a subgroup of the group of triangular matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a and c roots of unity.

(3) H is conjugate to a subgroup of $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is a ring of algebraic integers in $\mathbb{C}$.

(4) H is a non-trivial free product with amalgamation.

Proof of Theorem 2. Let G be as in Theorem 2. If $n > 1$ we consider the group $G_1 =< a, b \mid a^2 = R^m(a, b) = 1 >$ which is an epimorphic image of G. It is well known that if $\varphi : \Gamma_1 \to \Gamma_2$ is an epimorphism of groups and Γ_2 is a non-trivial free product with amalgamation then so is Γ_1. Thus, we may assume without loss of generality that $n = 1$. We will try to find a representation $\varphi : G \to \mathrm{PSL}_2(\mathbb{C})$ for wich $\varphi(G)$ is a non-trivial free product with amalgamation. To do this it is sufficient to find some special matrices in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 1 *Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $A^2 = -E$, $R^{2m}(A, B) = E$. Suppose the group $H =< A, B >$ does not satisfy either of the conditions (1)–(3) of Bass's theorem. Then the group*

$$G =< a, b \mid a^2 = R^m(a, b) = 1 >$$

is a non-trivial free product with amalgamation.

Proof. Indeed, by the Bass theorem the group H is a non-trivial free product with amalgamation, i.e. $H = H_1 *_F H_2$ with $H_1 \neq F \neq H_2$. Since $-E \in Z(H)$ we must have $-E \in F$. Let $\overline{H}$, $\overline{H}_1$, $\overline{H}_2$, $\overline{F}$ be the natural epimorphic images of H, H_1, H_2, F in $\mathrm{PSL}_2(\mathbb{C})$ respectively. Then $\overline{H}_1 \neq \overline{F} \neq \overline{H}_2$ and hence $\overline{H} = \overline{H}_1 *_{\overline{F}} \overline{H}_2$ is a non-trivial free product with amalgamation. Then so is G as $\overline{H}$ is an epimorphic image of G.

To construct such matrices A, B we will use the following observation.

Lemma 2 *Let $m > 1$ and $X \in SL_2(\mathbb{C})$. If $\operatorname{tr} X = \pm 2\cos\frac{r\pi}{m}$, where $r \in \{1, ..., m-1\}$ then $X^{2m} = E$. In particular, if $\operatorname{tr} X = 0$ then $X^2 = -E$.*

Proof. Let, for example, $\operatorname{tr} X = 2\cos\frac{r\pi}{m}$. Then the characteristic polynomial of X has roots $\alpha = \cos\frac{r\pi}{m} + i\sin\frac{r\pi}{m}$ and $\alpha^{-1} = \cos\frac{r\pi}{m} - i\sin\frac{r\pi}{m}$, where α is a $2m$-th root of unity. Thus X is conjugate to a matrix $X_1 = \operatorname{diag}(\alpha, \alpha^{-1})$ hence $X^{2m} = E$. □

Let us come back to the proof of Theorem 2. Depending on the word $R(a, b)$ there are 3 cases which will be considered separately:

1) The word $R(a, b) = ab^{n_1}...ab^{n_s}$ is a cyclically reduced word lying in the subgroup generated by a^2 and the commutator subgroup of the free group on a and b and at least one of the numbers $|n_i| \geq 2$.

2) The word $R(a, b) = ab^{n_1}...ab^{n_s}$ is a cyclically reduced word lying in the subgroup generated by a^2 and the commutator subgroup of the free group on a and b and all $n_i = \pm 1, i = 1, ..., s$.

3) The word $R(a, b)$ is a cyclically reduced word which does not belong to the subgroup generated by a^2 and the commutator subgroup of the free group on a and b.

Case 1. Let $R(a, b) = ab^{n_1}...ab^{n_s}$. Since $R(a, b)$ belongs to the subgroup generated by a^2 and the commutator subgroup of the free group on a and b then s is even and $\sum_{i=1}^{s} n_i = 0$. We will try to find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ as in Lemma 1 such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = 2, \qquad \operatorname{tr} AB = z,$$

where z is a complex number which will be specified later. The condition $\operatorname{tr} A = 0$ implies that $A^2 = -E$, i.e. the matrix A has order 4, and the condition $\operatorname{tr} B = 2$ implies that B is unipotent. For any $z \in \mathbb{C}$ such matrices do really exist in view of the following well known

Lemma 3 *For any $\alpha, \beta, \gamma \in \mathbb{C}$ there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that*

$$\operatorname{tr} A = \alpha, \qquad trB = \beta, \qquad \operatorname{tr} AB = \gamma.$$

Proof. See, for example, [5].

Our matrices A, B must satisfy the following equation

$$R^{2m}(A, B) = E$$

By Lemma 2 it is sufficient to find a z such that

$$\operatorname{tr} R(A, B) = c, \tag{1}$$

where $c = \pm 2 \cos \frac{r\pi}{m}$, $r \in \{1, ..., m-1\}$. To solve (1) we now need to introduce so-called "Fricke characters".

Let $F_2 =< g, h >$ be a free group. The representation variety $R_2(F_2, \mathrm{SL}_2(\mathbb{C}))$ can be identified with $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. For any element $w = w(g, h) \in F_2$ one can consider the function

$$\tau_w : \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}, \qquad \tau_w(A, B) = \operatorname{tr}(w(A, B))$$

It was proved by Horowitz [5] (cf. also [4]) that for any $w \in F_2$

$$\tau_w = Q_w(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_w \in \mathbb{Z}[x, y, z]$ is a uniquely defined polynomial with integral coefficients. The function τ_w and the polynomial Q_w are called the Fricke character and the Fricke polynomial of the element $w \in F_2$, respectively. For arbitrary elements $u, v, w \in F_2$, the following relations for Fricke characters follow from the corresponding relations for traces of arbitrary matrices in $\mathrm{SL}_2(\mathbb{C})$, and can be easily verified:

$$1)\tau_{u^{-1}} = \tau_u; \qquad 2)\tau_{uv} = \tau_{vu}; \qquad 3)\tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}.$$

Next we need to know the explicit formula for the Fricke polynomial found by Traina. To formulate this result we need polynomials $P_n(\lambda)$ which satisfy the reccurence relations:

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda)$$

and the initial conditions

$$P_0(\lambda) = 1, \qquad P_{-1}(\lambda) = 0.$$

It is easy to check by induction on n that

$$P_n(2\cos\varphi) = \sin(n+1)\varphi / \sin\varphi.$$

The degree of $P_n(\lambda)$ in λ is n for $n \geq 0$. We set

$$P_n(\lambda) = -P_{|n|-2}(\lambda)$$

for $n < 0$. The following property of zeros of the polynomials $P_n(\lambda)$, $n \geq 1$, follows from examining their trigonometric representation and will be used later in our argument.

Lemma 4 *The polynomial $P_n(\lambda)$, $n \geq 1$, has n zeros, given by*

$$\lambda_{n,k} = 2\cos(k\pi/n+1), \qquad k = 1, 2, \ldots, n.$$

Let $w = g^{u_1}h^{v_1}\ldots g^{u_s}h^{v_s} \in F_2$ be a cyclically reduced word and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Consider the Fricke polynomial $Q_w(x, y, z)$ as a polynomial in z. Let

$$Q_w(x, y, z) = M_n(x, y)z^n + M_{n-1}(x, y)z^{n-1} + \ldots + M_0(x, y).$$

Proposition 1 (C. Traina [9]) *The degree of the Fricke polynomial $Q_w(x, y, z)$ in z equals s := the number of blocks $g^{u_i}h^{v_i}$ in w. The leading coefficient $M_s(x, y)$ of $Q_w(x, y, z)$ is of the form*

$$M_s(x, y) = \prod_{i=1}^{s} P_{u_i-1}(x)P_{v_i-1}(y).$$

The remaining coefficients can be determined recursively.

Thus, the equation (1) is equivalent to

$$\tau_R(A, B) = c$$

or

$$Q_R(0, 2, z) = c. \tag{2}$$

By Proposition 1 we can write (2) as

$$M_s(0, 2)z^s + M_{s-1}(0, 2)z^{s-1} + ... + M_0(0, 2) - c = 0, \tag{3}$$

where

$$|M_s(0, 2)| = \prod_{i=1}^{s} |P_0(0)||P_{n_i-1}(2)| = \prod_{i=1}^{s} |P_{n_i-1}(2)|.$$

One can easily show by induction that for any n

$$|P_{n-1}(2)| = |n|,$$

whence $|M_s(0, 2)| = \prod_{i=1}^{s} |n_i|$. Since s is even, $\sum_{i=1}^{s} n_i = 0$ and by assumption there exists an i such that $|n_i| > 1$, when $\prod_{i=1}^{s} |n_i| \geq 3$ and hence

$$|M_s(0, 2)| \geq 3. \tag{4}$$

Furthermore, one proves the following lemma.

Lemma 5 *The coefficient $M_0(0,2)$ in (3) equals $(-1)^{\frac{s}{2}}2$.*

Proof. First, we note the following two facts.
1) $\tau_{h^i}(A,B) = \operatorname{tr} B^i = 2$ for all i since B is a unipotent matrix.
2) $\tau_{gh^i}(A,B) = iz$.

Indeed, if $i=1$ then $\tau_{gh}(A,B) = \operatorname{tr} AB = z$. For any i one holds

$$\tau_{gh^i} = \tau_h \tau_{gh^{i-1}} - \tau_{gh^{i-2}}$$

and by induction we have

$$\tau_{gh^i}(A,B) = 2(i-1)z - (i-2)z = iz.$$

Now let $R = gh^{n_1}\dots gh^{n_s}$. If $s=2$ then

$$\tau_R(A,B) = \tau_{gh^{n_1}gh^{n_2}}(A,B) = (\tau_{gh^{n_1}}\tau_{gh^{n_2}} - \tau_{h^{n_2-n_1}})(A,B) = n_1n_2z - 2.$$

For arbitrary s we have

$$\tau_R(A,B) = (\tau_{gh^{n_1}}\tau_{gh^{n_2}\dots gh^{n_s}} - \tau_{gh^{n_3}\dots gh^{n_s-n_1+n_2}})(A,B) = n_1zf(z) - g(z),$$

where

$$f(z) = \tau_{gh^{n_2}\dots gh^{n_s}}(A,B), \quad g(z) = \tau_{gh^{n_3}\dots gh^{n_s-n_1+n_2}}(A,B)$$

and the word $gh^{n_3}\dots gh^{n_s-n_1+n_2}$ has $s-2$ syllables. Thus we obtain by induction that

$$M_0(0,2) = -(-1)^{\frac{s-2}{2}}2 = (-1)^{\frac{s}{2}}2.$$

Lemma 5 is proved. □

Now we can write the equation (3) in the form

$$z^s + \tfrac{M_{s-1}(0,2)}{M_s(0,2)}z^{s-1} + \ldots + \tfrac{(-1)^{\frac{s}{2}}2-c}{M_s(0,2)} = 0. \tag{5}$$

The following lemma will be used in the sequel.

Lemma 6 *i) Given $s,m,M \in \mathbb{Z}$, where $m \geq 2$, $|M| \geq 3$, there exist numbers $\varepsilon \in \{-1,1\}$ and $r \in \{1,\dots,m-1\}$ such that the number $\alpha = \frac{(-1)^s2-c}{M}$, where $c = 2\varepsilon\cos\frac{r\pi}{m}$, is not an algebraic integer.*

ii) For $m \in \mathbb{Z}$, $m \geq 2$, there exists an $r \in \{1,\dots,m-1\}$ such that the number $\frac{1}{c+2}$, where $c = 2\cos\frac{r\pi}{m}$, is not an algebraic integer.

iii) If $m \in \mathbb{Z}$, $m \geq 3$, then the number $\cos\frac{\pi}{m}$ is not an algebraic integer.

Proof. i) Without loss of generality we may assume that s is even, so $(-1)^s = 1$. Suppose that for any $\varepsilon = \pm 1$ and $r \in \{1, \ldots, m-1\}$ the number $\frac{2-c}{M}$ is algebraic integer. Then the number

$$\frac{2-c}{M} + \frac{2+c}{M} = \frac{4}{M}$$

is algebraic integer. Since $|M| \geq 3$ this is possible only if $|M| = 4$. Let,for example, $M = 4$. If m is even we set $r = m/2$. Then $c = 0$ and the number $2/M = \frac{1}{2}$ is not an algebraic integer — a contradiction. Let m is odd and let $F_1 = \{2\cos\frac{2r\pi}{m} \mid r = 1, \ldots, m-1\}$. Then

$$\sum_{c \in F_1} \frac{2-c}{4} = \frac{2(m-1) - \sum_{c \in F_1} c}{4} = \frac{2(m-1)+1}{4},$$

because $1 + \sum_{c \in F_1} c$ is equal 0 as the sum of all roots of unity of degree m. The number $\frac{2(m-1)+1}{4}$ is not an algebraic integer — a contradiction.

ii) Since $\frac{1}{2+2\cos\frac{r\pi}{m}} = \frac{1}{4\cos^2\frac{r\pi}{2m}}$ it is sufficient to prove that for some $r \in \{1, \ldots, m-1\}$ the number $\frac{1}{2\cos\frac{r\pi}{2m}}$ is not algebraic integer. The polynomial $P_{2m-1}(\lambda)$ has the following roots:

$$0, \quad \pm 2\cos\tfrac{r\pi}{2m}, \qquad r = 1, \ldots, m-1.$$

It is easy to check that $P_{2m-1}(\lambda) = \lambda g(\lambda)$, where $g(\lambda)$ is a polynomial of the form

$$g(\lambda) = \lambda^{2m-2} + a_1\lambda^{2m-4} \ldots + (-1)^{m+1} m,$$

and so $g(\lambda)$ has the roots $\pm 2\cos\frac{r\pi}{2m}$, $r = 1, \ldots, m-1$. Then the numbers $\frac{1}{\pm 2\cos\frac{r\pi}{2m}}$ are the roots of the polynomial

$$g_1(\lambda) = (-1)^{m+1} m\lambda^{2m-2} + \ldots + 1,$$

and hence also of

$$\overline{g}_1(\lambda) = \lambda^{2m-2} + \ldots + (-1)^{m+1}\tfrac{1}{m}$$

Since $\frac{1}{m}$ is not an algebraic integer then at least one of the roots of $\overline{g}_1(\lambda)$ is not an algebraic integer.

iii) It is sufficient to prove that there exists an $r \in \{1, ..., m-1\}$ such that $\cos\frac{r\pi}{m}$ is not an algebraic integer because if $\cos\frac{\pi}{m}$ is an algebraic integer then so is $\cos\frac{r\pi}{m}$ for any r. The numbers $2\cos\frac{r\pi}{m}$, $r \in \{1, ..., m-1\}$, are the roots of the polynomial $P_{m-1}(\lambda)$. If $m = 2k$ is even then we have

$$P_{2k-1}(\lambda) = \lambda^{2k-1} + ... + (-1)^{k-1} k\lambda.$$

Then

$$(2\cos\tfrac{r\pi}{m})^{2k-1} + ... + (-1)^{k-1} k(2\cos\tfrac{r\pi}{m}) = 0,$$

and the numbers $\cos\frac{r\pi}{m}$, $r \in \{1, \ldots, m-1\}$, are the roots of the equation

$$\lambda^{2k-1} + ... + (-1)^{k-1}\tfrac{k}{2^{2k-2}}\lambda = 0,$$

Since $k \geq 2$ then $k < 2^{2k-2}$ and the number $\frac{k}{2^{2k-2}}$ is not an algebraic integer. Hence there exists an r such that $\cos\frac{r\pi}{m}$ is not an algebraic integer.

Let $m = 2k+1$ be odd. Then $P_{2k}(\lambda) = \lambda^{2k} + ... + (-1)^k$ and we obtain as above that the numbers $\cos\frac{r\pi}{m}$, $r \in \{1, \ldots, m-1\}$, are the roots of the equation

$$\lambda^{2k} + ... + \tfrac{(-1)^k}{2^{2k}} = 0.$$

Since $\frac{(-1)^k}{2^{2k}}$ is not an algebraic integer, there exists an r such that $\cos\frac{r\pi}{m}$ is not algebraic integer. Lemma 5 is proved. $\square$

We continue with the proof of Theorem 2. It follows from Lemma 6 i) that there exists a root z_0 of (5) which is not an algebraic integer. Now we can pick matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = 2, \qquad \operatorname{tr} AB = z_0.$$

By our construction, the following holds $A^2 = -E$ and $R^{2m}(A, B) = E$. To complete the proof in the first case it remains to show that the group $H =< A, B >$ does not satisfy either of the conditions (1)–(3) the Bass theorem. Suppose $f : H \to \mathbb{Z}$ is an epimorphism such that $f(u) = 0$ for all unipotent elements u in H. Then $f(A) = 0$ since A is a matrix of finite order, and $f(B) = 0$ since B is unipotent matrix. Thus $f(H) = \{0\}$ — a contradiction. Furthermore, by construction $z_0 = \operatorname{tr} AB$ is not algebraic integer so the group H does not satisfy the conditions (2) and (3) of the Bass theorem. Now Lemma 1 implies that the group G is a non-trivial free product with amalgamation.

Case ii). Let

$$R(a, b) = ab^{\varepsilon_1}ab^{\varepsilon_2}\ldots ab^{\varepsilon_s},$$

where s is even, $\varepsilon_i \in \{-1, 1\}$ for all $i = 1, \ldots, s$ and $\sum_{i=1}^{s}\varepsilon_i = 0$. Suppose for a moment that for some $j < s$ we have $\varepsilon_j = \varepsilon_{j+1}$. Then we can consider the new generators of the group G:

$$c = a, \qquad d = ab^{\varepsilon_j}$$

in terms of which G can be presented as follows

$$G =< ,d \mid c^2 = R_1^m(c, d) = 1 >,$$

where $R_1(c, d) = cd^{l_1}\ldots cd^{l_t}$ is a cyclically reduced word lying in the subgroup generated by c^2 and the commutator subgroup of the free group on c and d, and there exists an l_i such that $|l_i| \geq 2$. However this case has already been treated above.

So we may assume that

$$R(a,b) = abab^{-1}\dots abab^{-1} = (abab^{-1})^l$$

for some $l>0$, i.e. group G has following presentation:

$$G = < a, b \mid a^2 = (abab^{-1})^t = 1 > .$$

We will find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ as in Lemma 1 such that

1) $\operatorname{tr} A = 0$, i.e $A^2 = -E$;

2) $\operatorname{tr} AB^{-2}AB^3 = 2$, i.e. $AB^{-2}AB^3$ is an unipotent matrix.

3) $(ABAB^{-1})^{2m} = E$.

Let $F_2 = < g, h >$ be a free group. Using Fricke characters and Lemma 2 we may write conditions 2) and 3) as

$$\left\{\begin{array}{lcl} \tau_{ghgh^{-1}}(A,B) & = & c \\ \tau_{gh^{-2}gh^3}(A,B) & = & 2 \end{array}\right. \tag{6}$$

where $c = \pm 2\cos\frac{r\pi}{m}$, $r \in \{1,\dots,m-1\}$. It is easy to check that

$$\tau_{ghgh^{-1}} = -\tau_{gh}^2 + \tau_g\tau_h\tau_{gh} - \tau_h^2 + 2$$

and

$$\tau_{gh^{-2}gh^3} = (\tau_g\tau_{h^2} - \tau_g\tau_{gh} + \tau_g)(\tau_h^2\tau_{gh} - \tau_g\tau_h - \tau_{gh}) - \tau_h^5 + 5\tau_h^3 - 5\tau_h.$$

We set $y = \tau_h(A,B) = \operatorname{tr} B$, $z = \tau_{gh}(A,B) = \operatorname{tr} AB$. Since $\tau_g(A,B) = \operatorname{tr} A = 0$ we have $\tau_{ghgh^{-1}}(A,B) = -z^2 - y^2 + 2$ and $\tau_{gh^{-2}gh^3}(A,B) = -(y^3-y)z^2 - (y^5 - 5y^3 + 5y)$. Thus (6) is equivalent to

$$\left\{\begin{array}{rcl} z^2 + y^2 - 2 + c & = & 0 \\ (y^3-y)z^2 + y^5 - 5y^3 + 5y + 2 & = & 0. \end{array}\right. \tag{7}$$

It follows from (7) that

$$(c+2)y^3 - (c+3)y - 2 = 0$$

or

$$y^3 - (1 + \frac{1}{c+2})y - \frac{2}{c+2} = 0 \tag{8}$$

By Lemma 6ii) there exists $r \in \{1,\dots,m-1\}$ such that the number $\frac{1}{c+2}$ is not an algebraic integer hence there is a root y_0 of (8) which is not an algebraic integer either. Let (y_0, z_0) be a solution of the system (7). Now, we pick matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \qquad \operatorname{tr} AB = z_0.$$

and let $H =< A, B >$. Then the group H does not satisfy either of the conditions (1)–(3) of Bass's theorem. Indeed, if $f : H \to \mathbb{Z}$ is an epimorphism such that $f(u) = 0$ for all unipotent elements u in H then $f(A) = 0$ since A is an element of a finite order and $f(AB^{-2}AB^3) = f(B) = 0$ since by construction $AB^{-2}AB^3$ is an unipotent matrix. Thus $f(H) = \{0\}$ — a contradiction. Furthermore, $\operatorname{tr} B = y_0$ is not an algebraic integer, so the group H does not satisfy the conditions (2) and (3) of Bass's theorem. Now Lemma 1 completes the proof in the case ii).

Case iii). Let

$$R(a, b) = ab^{n_1} \ldots ab^{n_s},$$

and $R(a, b)$ does not belong to the subgroup generated by a^2 and the commutator subgroup of the free group on a and b. The case $\sum_{i=1}^{s} n_i \neq 0$ was considered in [3]. To be more precise the following theorem was proved in [3].

Theorem 4 (B. Fine, F. Levin, G. Rosenberger [3]) *Let $G =< a, b \mid a^n = R^m(a, b) = 1 >$ with $m \geq 2$, $n \geq 2$ and $R(a, b)$ a cyclically reduced word in the free group on a and b which involves both a and b. If $R(a, b) = a^p b^q c$ with $0 \leq p < n$, $q \neq 0$ and c in the commutator subgroup in the free group on a and b then G admits splitting as a non-trivial free product with amalgamation.*

Taking into account Theorem 4 it is sufficient to consider only a word $R(a, b)$ such that s is odd and $\sum_{i=1}^{s} n_i = 0$.

First we consider the case $m = 2$. Let $G_1 =< a, b \mid a^2 = b^2 = R^2(a, b) = 1 >$. We will show that $G_1 =< a, b \mid a^2 = b^2 = 1 >$, the free product of two cyclic groups of order two. It is sufficient to show that $R^2(a, b) = 1$ in G_1. We will prove a more general fact: if $w(a, b) = ab^{l_1} \ldots ab^{l_s}$, where $s = 2k + 1$ is odd and $\sum_{i=1}^{s} l_s$ is even, then $w^2(a, b) = 1$ in G_1. We will use the induction on s. For $s = 1$ the claim is obvious. For s arbitrary, at least one number among $l_1, \ldots, l_s$ must be even; let this be l_1. Then $b^{l_1} = 1$ in G_1 and

$$w(a, b) = b^{l_2} ab^{l_3} \ldots ab^{l_s},$$

i.e. $w(a, b)$ is conjugate to the word

$$w_1(a, b) = ab^{l_3} \ldots ab^{l_2 + l_s},$$

where $w_1(a, b)$ is an $(s - 2)$-syllable word and the sum $l_3 + \ldots + (l_2 + l_s)$ is even. By induction we obtain $w_1^2(a, b) = 1$ in G_1, therefore $w^2(a, b) = 1$ in G_1. Since the group G_1 is an epimorphic image of G we obtain that G is a non-trivial free product.

Now consider the case $m \geq 3$. Let the word $R(a, b)$ be as above. We will show how to find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ as in Lemma 1 such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \qquad \operatorname{tr} AB = 2,$$

where y_0 is a complex number which will be specified later. Let $F_2 =< g,h >$ be the free group, $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$ be correspondig Fricke characters, and $Q_R(x,y,z)$ be the Fricke polynomial of the word $R(g,h) \in F_2$.

Lemma 7 *1) $Q_R(0,y,z) = zf(y,z)$ for some polynomial $f(y,z) \in \mathbb{Z}[y,z]$.*

2) The polynomial $f_1(y) = f(y,2)$ is not constant.

Proof. 1) It is easy to check by induction on n that for any $n \in \mathbb{Z}$ the following holds

$$Q_{gh^n}(x,y,z) = P_{n-1}(y)z - P_{n-2}(y)x.$$

Thus, $Q_{gh^n}(0,y,z) = P_{n-1}(y)z$. If $s > 1$ then we have

$$\tau_{gh^{n_1}\dots gh^{n_s}} = \tau_{gh^{n_1}}\tau_{gh^{n_2}\dots gh^{n_s}} - \tau_{gh^{n_3}\dots gh^{n_s - n_1 + n_2}}$$

and by induction we obtain

$$Q_R(0,y,z) = P_{n_1-1}(y)f_1(y,z)z - zf_2(y,z) = zf(y,z)$$

for some polynomial $f(y,z) \in \mathbb{Z}[y,z]$.

2) Consider the ring T generated as $\mathbb{Z}$-algebra by all Fricke characters τ_w, $w \in F_2$. Any automorphism $\sigma \in \mathrm{Aut(F_2)}$ induces an automorphism

$$\sigma' : T \to T, \qquad \tau_w \mapsto \tau_{\sigma(w)}.$$

Consider now $\sigma \in \mathrm{Aut(F_2)}$ such that

$$\sigma(g) = g, \qquad \sigma(h) = gh.$$

Then $\sigma' \in \mathrm{Aut(T)}$ is an automorphism for which

$$\sigma'(x) = x_1 = x, \quad \sigma'(y) = y_1 = z, \quad \sigma'(z) = z_1 = xz - y.$$

Furthermore, we have

$$\sigma'(\tau_{R(g,h)}) = \tau_{\sigma(R(g,h))} = \tau_{R(\sigma(g),\sigma(h))}$$

hence

$$\sigma'(Q_R(x,y,z)) = Q_R(x_1,y_1,z_1) = Q_R(x,z,xz-y),$$

so $\sigma'(Q_R(0,y,2)) = Q_R(0,2,-y)$. By Proposition 1 the polynomial $Q_R(0,2,-y)$ has degree s since for its leading coefficient M_s we have $|M_s| = \prod_{i=1}^{s} |n_i| \neq 0$. Thus $Q_R(0,2,-y)$ is not constant. Since σ' is an automorphism hence $Q_R(0,y,2,) = 2f_1(y)$ is neither. Lemma is proved. $\square$

Now we can finish the proof of Theorem 2. The desired matrices A, B must satisfy the condition $R^{2m}(A,B) = E$. As we have seen above, it is sufficient to solve the equation

$$Q_R(0, y_0, 2) = 2\cos\frac{\pi}{m}.$$

By Lemma 7 it is equivalent to

$$f_1(y_0) = \cos\frac{\pi}{m}. \tag{9}$$

Since $\cos\pi/m$ is not an algebraic integer (Lemma 6iii)) there exists a root y_0 of (9) which is not an algebraic integer either. Pick now $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that:

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \qquad \operatorname{tr} AB = 2.$$

Let $H =< A, B >$. By construction, $A^2 = -E$ and $R^{2m}(A,B) = E$. We claim that the group H does not satisfy either of the conditions (1)–(3) of Bass's theorem. Suppose $f : H \to \mathbb{Z}$ is an epimorphism such that $f(u) = 0$ for all unipotent elements u in H. Then $f(A) = 0$ since A is an element of a finite order, and $f(AB) = f(B) = 0$ since AB is an unipotent matrix. Thus $f(H) = \{0\}$ — a contradiction. Furthermore, by construction $y_0 = \operatorname{tr} B$ is not an algebraic integer so the group H does not satisfy the conditions (2) and (3) of Bass's theorem. Now Lemma 1 implies that the group G is a non-trivial free product with amalgamation. Theorem 2 is proved.

A similar argument can be used to prove the following

Theorem 5 *Let $G =< a, b \mid a^n = R^m(a,b) = 1 >$, where $m \geq 3$, $n \geq 3$ and $R(a,b)$ a cyclically reduced word in the free group on a and b. If $R(a,b) = a^{u_1}b^{v_1}\dots a^{u_s}b^{v_s}$, where $0 < u_i < n$, $v_i > 0$ for $i = 1,\dots,s$ and $\prod_{i=1}^{s}|v_i| \geq 3$ then G is a non-trivial free product with amalgamation.*

Proof. Let $R(g,h) = g^{u_1}h^{v_1}\dots g^{u_s}h^{v_s}$ be an element of the free group $F_2 =< g, h >$ and Q_R be the Fricke polynimial of R. We assume for a moment that we find matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = \alpha, \qquad \operatorname{tr} B = 2, \qquad \operatorname{tr} AB = z,$$

where $\alpha = 2\cos\frac{\pi}{n}$, z is a complex number which is not an algebraic integer and satisfies the equation

$$Q_R(\alpha, 2, z) = c, \tag{10}$$

where $c = \pm 2\cos\frac{r\pi}{m}$ for some $r \in \{1,\dots,m-1\}$. We will show how this implies that G is a non-trivial free product with amalgamation. By Lemma 2 we have

$$A^{2n} = E, \qquad R^{2m}(A,B) = E.$$

Let $H = < A, B >$. Then the group H does not satisfy either of the conditions (1)–(3) of Bass's theorem. Indeed, let $f : H \to \mathbb{Z}$ be an epimorphism such that $f(u) = 0$ for all unipotent elements of H. Then $f(A) = 0$ since A is a matrix of finite order, and $f(B) = 0$ since B is unipotent matrix. Thus, $f(H) = 0$ — a contradiction. Furthermore, since $z = \operatorname{tr} AB$ is not an algebraic integer then H does not satisfy either of the conditions (2) and (3) of Bass's theorem. Now we can finish the proof of Theorem 5 in the same way as the proof of Lemma 2.

So, it is sufficient to show that the equation (10) has a root which is not algebraic integer. By Proposition 1 we can rewrite (10) in the form

$$M_s(\alpha, 2)z^s + M_{s-1}(\alpha, 2)z^{s-1} + \ldots + M_0(\alpha, 2) - c = 0, \tag{11}$$

where $|M_s(\alpha, 2)| = \prod_{i=1}^s |P_{u_i-1}(\alpha)||P_{v_i-1}(2)| = \prod_{i=1}^s |P_{u_i-1}(\alpha)||v_i|$. By Lemma 4 $P_{u_i-1}(\alpha) \neq 0$ therefore $M_s(\alpha, 2) \neq 0$. Thus, (11) is equivalent to the eguation

$$z^s + \tfrac{M_{s-1}(\alpha,2)}{M_s(\alpha,2)} + \ldots + \tfrac{M_0(\alpha,2)-c}{M_s(\alpha,2)} = 0. \tag{12}$$

Suppose, there exist an $\varepsilon \in \{-1, 1\}$ and an $r \in \{1, \ldots, m-1\}$ such that the number $\frac{M_0(\alpha,2)-c}{M_s(\alpha,2)}$, where $c = 2\varepsilon \cos\frac{r\pi}{m}$, is not an algebraic integer. Then the equation (12) has a root which is not an algebraic integer either, and Theorem 5 is proved. Suppose, both of the numbers $\frac{M_0(\alpha,2)+2\cos\frac{r\pi}{m}}{M_s(\alpha,2)}$ and $\frac{M_0(\alpha,2)-2\cos\frac{r\pi}{m}}{M_s(\alpha,2)}$ are an algebraic integer for any $r \in \{1, \ldots, m-1\}$. Then the number $\frac{4\cos\frac{r\pi}{m}}{M_s(\alpha,2)}$ is an algebraic integer either. Since α is an algebraic integer hence numbers $P_{u_i-1}(\alpha)$, $i = 1, \ldots, s$, are algebraic integer either. Therefore the number $\frac{4\cos\frac{r\pi}{m}}{M}$, where $M = \prod_{i=1}^s |v_i| \geq 3$ by assumptions of the theorem, is an algebraic integer. Now, the following lemma completes the proof of Theorem 5.

Lemma 8 *Given $m, M \in \mathbb{Z}$, $m \geq 3$ and $|M| \geq 3$, there exists a number $r \in \{1, \ldots, m-1\}$ such that the number $\frac{4}{M}\cos\frac{r\pi}{m}$ is not an algebraic integer.*

Proof. By Lemma 4 numbers $2\cos\frac{r\pi}{m}$, $r = 1, \ldots, m-1$, are roots of the polynomial $P_{m-1}(\lambda)$. Therefore, numbers $\frac{4}{M}\cos\frac{r\pi}{m}$, $r = 1, \ldots, m-1$, are roots of the polynomial $P_{m-1}(\frac{M}{2}\lambda)$. Consider two cases.

Let $m = 2k+1$ is odd. Then $P_{2k}(\lambda) = \lambda^{2k} + \ldots + (-1)^k$ hence $P_{2k}(\frac{M}{2}\lambda) = (\frac{M}{2})^{2k}\lambda^{2k} + \ldots + (-1)^k$. Thus, numbers $\frac{4}{M}\cos\frac{r\pi}{m}$, $r = 1, \ldots, m-1$, are roots of the equation

$$\lambda^{2k} + \ldots + (-1)^k(\tfrac{2}{M})^{2k} = 0. \tag{13}$$

Since $|M| \geq 3$ we have $(-1)^k(\frac{2}{M})^{2k}$ is not an algebraic integer, whence the equation (13) has a root which is not an algebraic integer either.

Let now $m = 2k$ is even. Then $P_{2k-1}(\lambda) = \lambda^{2k-1} + \ldots + (-1)^{k+1}k\lambda$. Therefore, numbers $\frac{4}{M}\cos\frac{r\pi}{m}$, $r = 1, \ldots, m-1$, are roots of the equation

$$\lambda^{2k-1} + \ldots + (-1)^{k+1}k(\tfrac{2}{M})^{2k-2}\lambda = 0. \tag{14}$$

Clearly, the number $(-1)^{k+1}k(\frac{2}{M})^{2k-2}$ is not an algebraic integer. Therefore, the equation (14) has a root, which is not an algebraic integer either. Lemma 8 is proved. $\square$

We close this paper with a conjecture.

Conjecture 2 *The group $G =< a, b \mid a^n = R^m(a, b = 1) >$, where $m \geq 2$, $n = 0$ or $n \geq 2$ and $R(a, b)$ is a cyclically reduced word in the free group on a and b which involves b, decomposes as a non-trivial free product with amalgamation.*

In view of theorems 2, 4 and 5 this conjecture is not proved in the following case:

n is odd, $R(a, b) = a^l R_1(a, b)$ is a cyclically reduced word, where $0 \leq l < n$ and $R_1(a, b) = a^{u_1}b^{v_1} \ldots a^{u_s}b^{v_s}$ is a cyclically reduced word in the commutator subgroup of the free group on a and b such that $0 < u_i < n$, $v_i > 0$ for all $i = 1, \ldots, s$ and $\prod_{i=1}^{s} |v_i| \in \{1, 2\}$.

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