THE HOMOLOGICAL QUADRATIC FORM OF A BIEXTENSION ALGEBRA

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Let A be a finite dimensional algebra over an algebraically closed field k. One of the fundamental problems in Representation Theory of Algebras is to determine the representation type of A and in case A is tame, to classify all the indecomposable A-modules.

We denote by mod_A the category of finite dimensional left A-modules. Roughly speaking, A is *tame* if for each dimension, the indecomposable A-modules occur (up to isomorphism) in a finite number of discrete and one-parameter families. If A is not tame, then the representation theory of A is as complicated as the study of vector spaces equipped with two non-commuting endomorphism, for which the classification of indecomposables is a famously unsolved problem.

Important classes of algebras are known to be tame: path algebras $k\Delta$ where Δ is a quiver with underlying graph of Dynkin or extended-Dynkin type; tubular algebras (see for example [7, 16]). For tame algebras of polynomial growth, the structure of the module category is well understood [15, 18], while the representation theory of tame non-polynomial algebras only in few cases is known [4, 6, 10]. Among the tools for the study of these algebras and their module categories we find integral quadratic forms, vector space categories, derived categories and others.

For the study of non-polynomial growth tame algebras, the following construction was introduced in [6]. Let $R_1, \dots, R_t$ and $R'_1, \dots, R'_s$ be two sequences of modules over an algebra B. We consider $R = \bigoplus_{i=1}^{t} R_i$ as a $B - k^t$-bimodule and $R' = \bigoplus_{j=1}^{s} R'_j$ as a $B - k^s$-bimodule. The *biextension* $[R']B[R]$ of B by the two sequences $R_1, \dots, R_t$ and $R'_1, \dots, R'_s$ is by definition the matrix algebra

$$\begin{pmatrix} k^t & 0 & 0 \\ R & B & 0 \\ DR' \otimes_B R & DR' & k^s \end{pmatrix}$$

equipped with the obvious addition and multiplication, where we denote by $D = \mathrm{Hom}_k(-, k)$ the usual duality. For any set of pairs of indices $L \subset \{1, \dots, t\} \times \{1, \dots, s\}$, consider the subspace $V = \bigoplus_{(i,j) \in L} DR_j \otimes_B R_i$ of the space $DR' \otimes_B R$ and the associated ideal

$$J(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V & 0 & 0 \end{pmatrix}$$

in $[R']B[R]$. The algebra $A = [R']B[R]/J(V)$ is called a *truncated biextension* of B.

For an algebra A with finite global dimension we consider the non-commutative bilinear form $\langle -,-\rangle_A\colon K_0(A) \times K_0(A) \to \mathbb{Z}^n$ defined on the Grothendieck group $K_0(A) \cong \mathbb{Z}^n$ as

$$\langle v, w\rangle_A = \sum_{s=0}^{\infty} \sum_{i,j=1}^{n} (-1)^s v(i) w(j) \dim_k \operatorname{Ext}_A^s(S_i, S_j),$$

where $S_1, \ldots, S_n$ is a complete set of the isoclasses of simple A-modules. The associated quadratic form $\chi_A(v) = \langle v, v\rangle_A$ is called the *homological form* of A (also called the *Euler form* of A). In several instances, arithmetical properties of χ_A are related with the representation type of A [5, 13, 14, 15]. The main result of this work is the following.

Theorem. *Let B be a strongly simply connected algebra with at least* 6 *vertices and $R_1, \ldots, R_t$; $R'_1, \ldots, R'_s$ be two sequences of indecomposable B-modules. Let $R = \bigoplus_{i=1}^{t} R_i$ and $R' = \bigoplus_{j=1}^{s} R'_j$ and consider a truncated biextension $A = [R']B[R]/J(V)$ of B.*

If χ_A is non-negative with $\operatorname{corank} \chi_A = 1 + s + t$ *and* $\operatorname{corank} \chi_B = 1$*, then A is tame.*

The proof of the Theorem uses heavily the results in [6], we present it in section 2. In section 1, besides some introductory remarks, we present many examples of algebras satisfying the hypothesis of the Theorem. In particular, we show that tree algebras A with global dimension 2 and non-negative Euler form χ_A accepting a positive sincere isotropic vector are in this class (hence they are tame, a result first shown in [5]).

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1. Some examples.

1.1. Let A be a basic, finite dimensional algebra over the fixed algebraically closed field k. It is well-known that A may be written as a quotient kQ/I of a path algebra kQ where Q is a finite quiver. We shall consider also A as a *spectroid*, that is, a k-category whose objects are the vertices $Q_0 = \{1, \ldots, n\}$ of Q and the space of maps from x to y is $A(x, y) = e_y A e_x$, where e_x denotes the primitive idempotent associated to the vertex x. See [7].

We recall that the *one-point extension* $A[M]$ of the algebra A by a module M is the spectroid with objects $Q_0 \cup \{s\}$ and morphism spaces $A[M](s, x) = M(x)$ for $x \neq s$, $A[M](s, s) = k$ and such that A is a full convex (= path closed) subcategory. We denote by $P_s = Ae_s$ the indecomposable projective cover of the simple module S_s. Observe that $M = \operatorname{rad} P_s$. See [16].

1.2. An algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there is a family $M_1, \dots, M_{s(d)}$ of $A - k[t]$-bimodules which are finitely generated free as right $k[t]$-modules and such that almost every indecomposable A-module X of dimension d is of the form $M_i \otimes_{k[t]} S_\lambda$ for some $1 \leq i \leq s(d)$ and $\lambda \in k$, where $S_\lambda = k[t]/(t-\lambda)$ is a simple $k[t]$-module. In this case, we denote by $\mu(d)$ the minimal possible $s(d)$ in the definition above. We say that A is of *polynomial growth* if $\mu(d) \leq d^m$ for some $m \in \mathbb{N}$ and all $d \in \mathbb{N}$.

Important *examples* of tame algebras that we shall consider are the following:

(a) Tame concealed algebras. These algebras are of the form $A = \mathrm{End}_H(T)$ for a hereditary algebra $H = k\Delta$ for Δ an extended Dynkin diagram and T a tilting H-module whose indecomposable direct summands lie in the preprojective component of the Auslander-Reiten quiver Γ_H of H. For any such algebra A, we have $\mathrm{g}\ell \dim A \leq 2$ and the Euler form χ_A is non-negative of $\mathrm{corank}\, \chi_A = 1$ such that $\mathrm{rad}\, \chi_A = \{v \in K_0(A) \colon \langle v, - \rangle_A = 0\} = \mathbb{Z} v_0$ for a positive sincere vector v_0 (called a *critical vector*).

(b) Tubular algebras as defined in [16]. These algebras are of the form $A = B[M]$ for some representation infinite tilted algebra B of extended Dynkin type and M an indecomposable B-module. A tubular algebra A has $\mathrm{g}\ell \dim A \leq 2$ and the Euler form χ_A is non-negative of $\mathrm{corank}\, \chi_A = 2$ such that there are two positive vectors v_0 and v_∞ generating $(\mathrm{rad}\, \chi_A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition. [11, 16]. *Let M be an indecomposable B-module and consider the one-point extension $A = B[M]$. Then*

(i) *Assume that B is tame concealed. Then χ_A is non-negative if and only if A is tame and M is regular.*

(ii) *Assume B is tubular. Then χ_A is indefinite.* □

1.3. We recall that an algebra $A = kQ/I$ is said to be *strongly simply connected* if the first Hochschild cohomology group $H^1(B)$ vanishes for every convex full subcategory B of A, see [17]. Following [10], we say that a strongly simply connected algebra A is a pg-*critical* if it is tame minimal not of polynomial growth.

We recall also that the *Tits form* $q_A \colon \mathbb{Z}^n \to \mathbb{Z}$ is obtained as a 'truncation' of the Euler form χ_A in the following way:

$$q_A(v) = \sum_{s=0}^{2} \sum_{i,j=1}^{n} (-1)^s v(i) v(j) \dim_k \mathrm{Ext}_A^s(S_i, S_j).$$

The following facts are relevant:

(a) If $\mathrm{g}\ell \dim A \leq 2$, then $q_A = \chi_A$.

(b) [11] If A is tame, then q_A is weakly non-negative (that is, $q_A(v) \geq 0$ for every $v \in \mathbb{N}^n$).

(c) [14] If A is strongly simply connected and χ_A is weakly non-negative, then q_A is weakly non-negative.

(d) [14] Let $\hat{A}$ be the repetitive spectroid associated to A, with set of objects $Q_0 \times \mathbb{Z}$ (denoted as $s[i]$, for $s \in Q_0$ and $i \in \mathbb{Z}$) and the only possible non-zero morphism spaces $\hat{A}(r[i], s[i]) = A(r, s)$ and $\hat{A}(s[i], r[i+1]) = DA(r, s) \times \{i\}$. If every finite full subcategory of $\hat{A}$ is tame, then χ_A is non-negative.

(e) The converses of (b), (c) and (d) are not true. As an example, the algebra $A = kQ/I$ below is tame strongly simply connected but χ_A is not weakly non-negative (I is generated by the paths indicated with dotted lines).

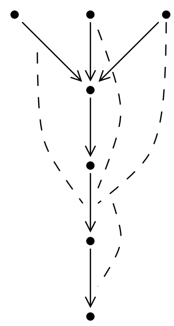


Figure 1:

(f) If A is a pg-critical algebra, then $\mathrm{g}\ell \dim A = 2$ and χ_A is non-negative with $\operatorname{corank} \chi_A = 2$.

1.4. We recall that a $A = kQ/I$ is a *tree algebra* if the underlying graph of Q has no cycles. We study certain trees as examples of truncated biextensions.

Proposition. *Let $A = kQ/I$ be a tree algebra such that q_A is non-negative with a sincere positive isotropic vector (that is, $v \in \mathbb{Z}^n$ with $0 < v(i)$ for all $1 \leq i \leq n$ and $q_A(v) = 0$). Then:*

(a) $\mathrm{g}\ell \dim A \leq 2$ *and* $q_A = \chi_A$*;*

(b) *there is a set of sources $\{i_1, \ldots, i_t\}$ of Q and a set of sinks $\{j_1, \ldots, j_s\}$ of Q such that $R_i = \operatorname{rad} P_{i_\ell}$ $(1 \leq \ell \leq t)$ and $R'_j = I_{j_\ell}/\operatorname{soc} I_{j_\ell}$ $(1 \leq \ell \leq s)$ are indecomposable modules;*

(c) *the full subcategory B of A with vertices $Q_0 \setminus (\{i_1, \dots, i_t\} \cup \{j_1, \dots, j_s\})$ is connected and* $\operatorname{corank} \chi_B = 1$*;*

(d) *A is a truncated biextension* $\left[\bigoplus_{j=1}^{s} R'_j\right] B \left[\bigoplus_{i=1}^{t} R_i\right] / J(V)$*;*

(e) $\operatorname{corank} \chi_A = 1 + s + t$.

Proof: In [5] we have shown that (a) holds. We divide the proof in several steps.

(1) Assume that A is of polynomial growth. As observed in [5, (1.4)], A is either tame concealed or tubular. If A is tame concealed, there is nothing to prove. Assume that A is tubular, then as recalled in (1.2), $A = B[M]$ for a representation-infinite tilted algebra B of extended Dynkin type and an indecomposable B-module M. Hence $\operatorname{corank} q_A = 1 + \operatorname{corank} q_B$ and the result holds in this case.

(2) Assume that A is not of polynomial growth. By [5, (1.4)], there are convex pg-critical subcategories $C_1, \dots, C_m$ of A such that $\bigcup_{i=1}^{m} \operatorname{supp} C_i = Q_0$. Moreover every tame concealed convex subcategory of A is of type $\tilde{\mathbb{D}}_t$.

Assume that C_1 has a zero relation of length 3. Then C_1 belongs to one of the following frames (or their duals):

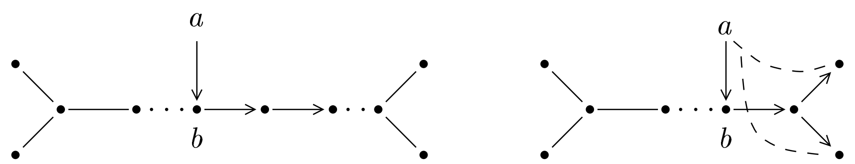


Figure 2:

In [5, Proposition (2.1)] it is shown that a has a unique neighbour b in Q. Then the subcategory A' of A with objects $Q_0 \setminus \{s\}$ is connected and $q_{A'}$ is non-negative with a sincere positive isotropic vector. Moreover $\operatorname{corank} q_{A'} = \operatorname{corank} q_A - 1$. By induction hypothesis we get that $A' = [R']B[R]/J(V')$ for some $R = R_1 \oplus \cdots \oplus R_t$ and $R' = R'_1 \oplus \cdots \oplus R'_s$ with R_i $(1 \leq i \leq t)$ and R'_j $(j \leq j \leq s)$ indecomposables B-modules, where B is connected and $\operatorname{corank} q_B = 1$; moreover, we have a subspace $V' = \bigoplus_{(i,j) \in L'} DR'_j \otimes_B R_i$ for some subset $L' \subset \{x_1, \dots, x_t\} \times \{y_1, \dots, y_s\}$, where $R_i = \operatorname{rad} P_{x_i}$ and $R'_j = I_{y_j}/\operatorname{soc} I_{y_j}$, and finally $\operatorname{corank} q_{A'} = 1 + s + t$. Set $x_0 = a$, $R_0 = B \otimes_A \operatorname{rad} P_a$ as B-module.

We have to check first that the vertices $x_1, \dots, x_t$ are still sources in Q. Assume x_1 is not a source in Q. Then there should be an arrow $a \to x_1$ in Q. Since a has a unique neighbour, we have $x_1 = b$. Since b has two neighbours, then the tree B is not connected, a contradiction. We consider the set $L = L' \cup \{(0,j)\colon 1 \leq j \leq s$, with $DR'_j \otimes_B R_0 \neq 0$ and $A(x_0, y_j) = 0\}$ and the subspace $V = \bigoplus_{(i,j)\in L} DR'_j \otimes_B R_i$ of $DR' \otimes_B (R_0 \otimes R)$. We shall show that $A = [R']B[R_0 \oplus R]/J(V)$. Indeed, if $A(x_0, y_j) \neq 0$, then $DR'_j \otimes R_0 \xrightarrow{\sim} \mathrm{Hom}_B(R_0, R'_j) = \mathrm{Hom}_A(\mathrm{rad}\, P_a, R'_j) = A(x_0, y_j)$; if $A(x_0, y_j) = 0$ and $DR'_j \otimes_B R_0 \neq 0$, then V contains $DR'_j \otimes_B R_0$ and $[R']B[R_0 \oplus R]/J(V)(x_0, y_j) = 0$. Finally, $\mathrm{corank}\, q_A = \mathrm{corank}\, q_{A'} + 1 = 1 + s + (t+1)$.

(3) We may assume that all the pg-critical subcategories of A contain only relations of length two. That is, each C_i has one of the following shapes (or their duals)

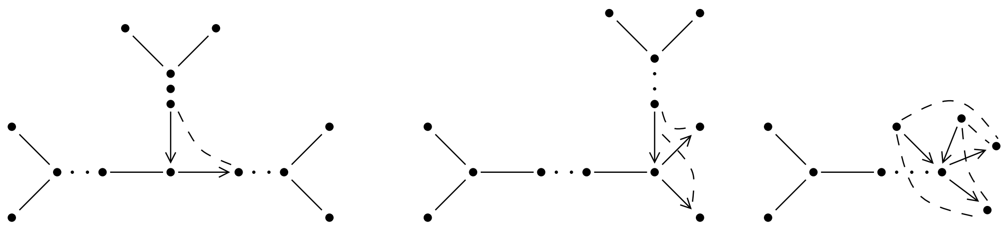


Figure 3:

As shown in [5, (2.1)(2)], there is a critical subcategory C of A containing a subcategory D of the following form (up to duality)

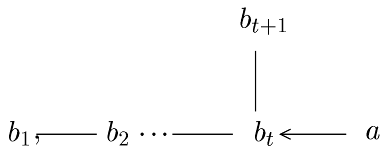


Figure 4:

such that C is the unique critical subcategory of A intersecting D. Then the full subcategory A' of A with objects $Q_0 \setminus \{b_1, \ldots, b_t, a\}$ satisfies the conditions of the Proposition and $\mathrm{corank}\, q_{A'} = \mathrm{corank}\, q_A - 1$. By induction hypothesis $A' = [R']B'[R]/J(V')$ for some connected algebra B' with $\mathrm{corank}\, q_{B'} = 1$ and $R, R', V', q_{A'}$ satisfying (a) to (e). Suppose $R_i = \mathrm{rad}\, P_{x_i}$ $(1 \leq i \leq t)$ and $R'_j = I_{y_j}/\mathrm{soc}\, I_{y_j}$ $(1 \leq j \leq s)$.

We observe that the vertices $x_1, \ldots, x_t$ are sources of Q. Otherwise $x_1 \leftarrow b_1$ belong to a pg-critical subcategory of A with the shape

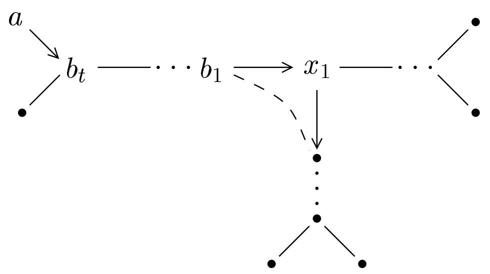


Figure 5:

Then B' is not connected, a contradiction. Let B be the full subcategory of A with objects $\operatorname{supp} B' \cup \{b_1, \dots, b_t, b_{t+1}\}$. Observe that B is connected since the neighbour $x \neq b_2$ of b_1 belongs to B'. Moreover, $\operatorname{corank} q_B = 1$ (since otherwise for $A'' = [R']B[R]/J(V')$ we have $\operatorname{corank} q_{A''} = \operatorname{corank} q_B + s + t \geq \operatorname{corank} q_A$. Then $\operatorname{corank} q_{A''} = \operatorname{corank} q_A$ which is impossible since C is not contained in A''). Then as in case (2), it is easy to see that $A = [R']B[R_0 \oplus R]J(V)$ for B, $R_0 = \operatorname{rad} P_a, R, R'$ and some V satisfying (a) to (e). $\square$

Remarks: (a) In case (3) in the Proposition above we may easily show that $V = 0$.

(b) The Proposition above and the main Theorem show that a tree algebra A with q_A non-negative accepting a positive sincere isotropic vector, is tame (the case in which B has 5 or less vertices is easy to handle separatly). This result was shown in [5].

2. Truncated biextensions and the Euler form.

2.1. Let A be an algebra with finite global dimension. We shall consider the derived category $D^b(A)$ of the module category mod_A. Two algebras A and B are said to be *derived equivalent* if there exists an equivalence $F\colon D^b(A) \to D^b(B)$ of triangulated categories. In this case, there is an induced isomorphism $f\colon K_0(A) \to K_0(B)$ such that $f(\mathbf{dim}\, X^\bullet) = \mathbf{dim}\, FX^\bullet$, for any object $X^\bullet \in D^b(A)$, where $\mathbf{dim}\, X^\bullet = \sum_{i\in\mathbb{Z}} (-1)^i (\mathbf{dim}\, X^i) \in K_0(A)$. Moreover f commutes with the corresponding homological bilinear forms (we say that f is an *isometry*). In particular, χ_A is non-negative if and only if so is χ_B and in this case, $\operatorname{corank} \chi_A = \operatorname{corank} \chi_B$.

Let $H = k\tilde{\Delta}$ be a tame hereditary algebra. The category of indecomposable objects in $D^b(H)$ is described in [8] as $\bigvee_{\nu\in\mathbb{Z}} C[\nu] \vee \bigvee_{\nu\in\mathbb{Z}} \mathcal{R}[\nu]$, where $C[\nu]$ is the mesh category of a translation quiver of type $\mathbb{Z}\Delta$ and $\mathcal{R}[\nu]$ is a tubular family of type Δ such that $\operatorname{Hom}_{D^b(H)}(\mathcal{R}[\nu], \mathcal{R}[\mu]) = 0$ if $\nu > \mu$. We identify mod_H as the full subcategory of $D^b(H)$ formed by the objects concentrated at level 0. If $F\colon D^b(B)\to D^b(H)$ is a derived equivalence, we say that a module $X\in \mathrm{mod}_B$ has *derived regular length m* if $F(X)\in\mathcal{R}[\nu]$ for some $\nu\in\mathbb{Z}$ and $F(X)$ has regular length m in the corresponding tube. Observe that this definition is independent of the derived equivalence F.

2.2. The following remark is useful.

Lemma. [2, 3] *Let $F\colon D^b(B)\to D^b(H)$ be a derived equivalence, where $H=k\tilde{\Delta}$ is a tame hereditary algebra. Let M be an indecomposable B-module such that $F(M)$ is concentrated at level 0 (that is, $F(M)$ is a H-module).*

Consider the isometry $f\colon K_0(B)\to K_0(H)$ induced by F. Then:

(a) *There is an isometry $\hat{f}\colon K_0(B[M])\to K_0(H[F(M)])$ extending f;*

(b) *If $\chi_{B[M]}$ is non-negative, then $F(M)$ is a regular H-module and $H[F(M)]$ is either domestic tubular algebra, a tubular algebra or $\tilde{\Delta}$ is of type $\tilde{\mathbb{D}}_n$ with $F(M)$ a module of regular length 2 lying on a tube of rank $n-2$.*

2.3. The initial step of our Theorem (i.e. case $A=B$) is given by the following result.

Theorem. [1, 3] *Let B be a strongly simply connected algebra which has a non-negative Euler form χ_B of corank one. Then B is derived equivalent to a hereditary algebra of type $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ for some $p=6,7,8$.*

The result is shown in [1] for the case when B is representation-finite and generalized in [3].

2.4. The main step of the proof of the Theorem will be the reduction to the case of truncated biextensions considered in [6]. The following is the main result we shall use.

Theorem. [6] *Let B be an algebra with a derived equivalence $F\colon D^b(B)\to D^b(H)$, where H is a hereditary tame algebra of type $\tilde{\mathbb{D}}_n$, $n>4$. Let $R_1,\dots,R_t$ and $R'_1,\dots,R'_s$ be two sequences of B-modules, $V=\bigoplus_{(i,j)\in L} DR'_j\otimes_B R_i$ and $A=\left[\bigoplus_{j=1}^{s} R'_j\right] B \left[\bigoplus_{i=1}^{t} R_i\right]/J(V)$ a truncated biextension. Assume that the following conditions are satisfied:*

(1) *each R_i (resp. R'_j) has derived regular length 2 and $F(R_i)$ (resp. $F(R'_j)$) lies on a tube of rank $n-2$;*

(2) *for each* $\nu \in \mathbb{Z}$, *the sets*

$$\mathcal{R}[\nu] \cap \{F(R_1), \dots F(R_t)\} \quad \textit{and} \quad \mathcal{R}[\nu] \cap \{F(R'_1), \dots, F(R'_s)\}$$

are formed by pairwise $\mathrm{Hom}_{D^b(H)}$*-orthogonal objects;*

(3) *if* $X = F(R_i)$ *and* $Y = \tau^- F(R'_j)[-1]$ *are in* $\mathcal{R}[\nu]$ *for some* $1 \le i \le t$ *and* $1 \le j \le s$, *then* $X \not\cong Y$ *and if* $\mathrm{Hom}_{D^b(H)}(Y, X) \neq 0$, *then* $(i, j) \in L$.

Then A *is tame.*

Notation: In the above statement, τ denotes the Auslander-Reiten translation in $D^b(H)$ and $\mathrm{Hom}_{D^b(H)}(X[-1], Y) = \mathrm{Ext}^1_H(X, Y) = D\mathrm{Hom}_H(Y, \tau X)$ for any two H-modules X and Y. Hence $\mathrm{Hom}_{D^b(H)}(\tau^- F(R'_j)[-1], F(R_i)) = \mathrm{Hom}_B(R_i, R_j) \simeq DR'_j \otimes_B R_i$.

2.5. *Proof of the Theorem:* We divide the proof in several steps.

(1) Since B is strongly simply connected and χ_B is non-negative with corank $\chi_B = 1$. Then by (2.3), we get a hereditary algebra $H = k\tilde{\Delta}$ of tame type $\Delta = \tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ ($p = 6, 7, 8$) and a derived equivalence $F\colon D^b(B) \to D^b(H)$.

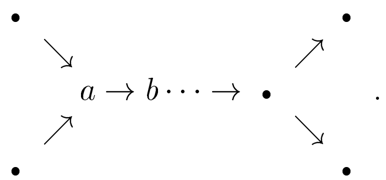
Consider the indecomposable B-module R_1. Without loss of generality, $F(R_1)$ is concentrated at level 0. Since $\chi_{B[R_1]}$ is non-negative, by (2.2), $F(R_1)$ is a regular H-module, that is, $F(R_1) \in \mathcal{R}[0]$. Moreover, $H[F(R_1)]$ is either a domestic tubular algebra, a tubular algebra or Δ is of type $\tilde{\mathbb{D}}_n$ and $F(R_1)$ has regular length 2 at a tube of rank $n-2$. The same holds for all extensions $H[F(R_i)]$, $1 \le i \le t$ and coextensions $[F(R'_j)]H$, $1 \le j \le s$.

(2) We show that $H[F(R_1)]$, is not domestic tubular. Indeed, observe that for any algebra $E = D[X]$ such that χ_E is non-negative, corank $\chi_E \le$ corank $\chi_D + 1$. Since A is a truncated extension of $B[R_1]$, then $1+t+s =$ corank $\chi_A \le$ corank $\chi_{B[R_1]} + t + s - 1$. Therefore $2 \le$ corank $\chi_{B[R_1]} =$ corank $\chi_{H[F(R_1)]}$ and $H[F(R_1)]$ is not domestic tubular. The same holds for all extensions $H[F(R_i)]$, $1 \le i \le t$ and coextensions $[F(R'_j)]H$, $1 \le j \le 5$.

(3) Assume that $H[F(R_1)]$ is a tubular algebra. Since $\tilde{\Delta}$ has $n \ge 6$ vertices, then the tubular type of H is one of $(2, 2, n-3)$ with $n > 5$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ and therefore the tubular type $\mathbf{t}$ of $H[F(R_1)]$ is one of $(2, 3, 6)$ and $n = 9$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$ respectively. In this case we claim that $t = 1$ and $s = 0$. Assume that $s \ge 1$ and consider the algebra $B' = [R'_1]B[R_1]/J(V_{11})$ where $V_{11} = DR'_1 \otimes R_1$ if $(1, 1) \in L$ and $V_{11} = 0$, else. Then $B' = [\bar{R}'_1](B[R_1])$ is one-point coextension of the algebra $B[R_1]$ by an indecomposable module $\bar{R}'_1$ such that the restriction to B_1 is R'_1. Observe that for B', corank $\chi_{B'} = 3$.

By [2, 3], there is a derived equivalence $\hat{F}\colon D^b(B[R_1]) \to D^b(H[F(R_1)])$. By the description of $D^b(H[F(R_1)]$ in [9], the indecomposable object $Y = \hat{F}(\bar{R}'_1)$ is a module over a tubular algebra D of type $\mathbf{t}$ (which is a full convex subcategory of the repetitive category of $H[F(R_1)]$). As in (2.2), there is an isometry $K_0(B') \to K_0([Y]D)$ which implies that $\chi_{[Y]D}$ is non-negative. On the other hand, the arguments in [13] showing that the Tits form $q_{[Y]D}$ is indefinite also apply to $\chi_{[Y]D}$. Indeed, if v_0, v_∞ are positive geneators of $\operatorname{rad}\chi_D$ such that $\operatorname{supp} v_0$ and $\operatorname{supp} v_\infty$ induce critical subcategories of D, then either $\langle v_0, \mathbf{dim}\, Y\rangle_D \neq 0$ or $\langle v_\infty, \mathbf{dim}\, Y\rangle_D \neq 0$. If y is the coextension vertex of the quiver of $[Y]D$, then either $\langle v_0, e_y\rangle_{[Y]D} \neq 0$ or $\langle v_\infty, e_y\rangle_{[Y]D} \neq 0$ which implies that $\chi_{[Y]D}$ is indefinite. Therefore $s = 0$. Similarly, $t = 1$ and A is derived equivalent to $H[F(R_1)]$.

(4) We may therefore assume that condition (1) in Dräxler-Skowroński's Theorem is satisfied. We shall prove that in this case, condition (2) also holds. In particular we assume that $H = k\tilde{\Delta}$, where $\tilde{\Delta}$ is the following quiver with at least 6 vertices



Assume that $F(R_1)$ and $F(R_2)$ belong to $R[\nu]$ for some $\nu \in \mathbb{Z}$ and $\operatorname{Hom}_{D^b(H)}(F(R_1), F(R_2)) \neq 0$. Up to shifting we may assume that $\nu = 0$ and consider the algebra $E = H[F(R_1)][F(R_2)]$ which has χ_E non-negative of corank 3. By assumption $F(R_1)$ and $F(R_2)$ have both regular length 2 on the tube of rank $n-2$. Therefore we may assume that either $F(R_1) = F(R_2)$ or $F(R_1) = \tau_H F(R_2)$. Then E is a tilted algebra of one of the following algebras

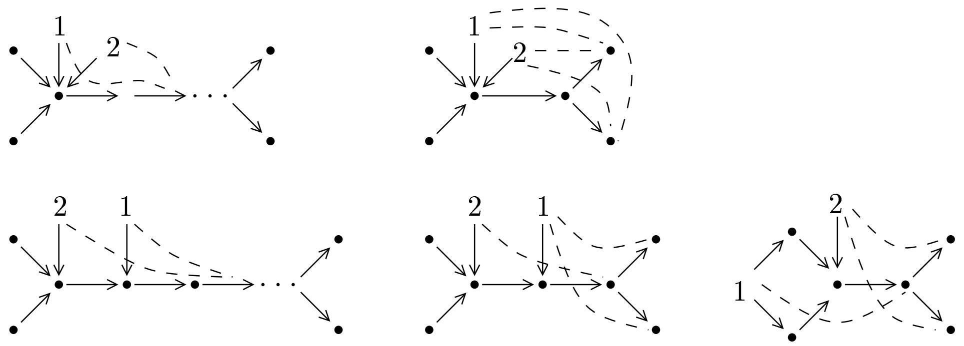


Figure 6:

In all cases, these algebras contain a convex hypercritical algebra and χ_E is indefinite. A contradiction showing the $\operatorname{Hom}_{D^b(H)}$-orthogonality of $F(R_1)$ and $F(R_2)$.

(5) We consider now indices $1 \leq \omega \leq t$ and $1 \leq \alpha \leq s$ and denote by $R = R_\omega$ and $R' = R'_\alpha$. Assume that $F(R)$ and $\tau^- F(R')[-1]$ belong to $\mathcal{R}[0]$. We shall check that condition (3) in Theorem (2.4) holds.

Observe first that for $i \geq 3$, $\mathrm{Ext}^i_A(S_\omega, S_\alpha) = \mathrm{Ext}^{i-2}_A(R, R') = \mathrm{Hom}_{D^b(H)}(F(R), F(R')[i-2])$ vanishes since $F(R') \in \mathcal{R}[1]$ and $F(R) \in \mathcal{R}[0]$.

Moreover $\mathrm{Ext}^2_A(S_\omega, S_\alpha) = \mathrm{Hom}_B(R, R') \xrightarrow{\sim} \mathrm{Hom}_{D^b(H)}(F(R), F(R')) = \mathrm{Hom}_{D^b(H)}(\tau^- F(R')[-1], F(R))$, which is at most one dimensional by the position of the modules on the tube and if it is not zero, then either $\tau^- F(R')[-1] = F(R)$ or $\tau^- F(R')[-1] = \tau F(R)$. Consider the truncated biextension $B' = [R']B[R]/J(V')$, where $V' = DR' \otimes_B R$ if $(\omega, \alpha) \in L$ and $V' = 0$, else.

(6) We shall analyze the case when $\tau^- F(R')[-1] = F(R)$ to get a contradiction. Let $f\colon K_0(B) \to K_0(H)$ be the isometry induced by the derived equivalence $F\colon D^b(B) \to D^b(H)$. We shall prove that f may be extended to an isometry $\hat{f}\colon D^b(B') \to D^b(E)$ for some algebra E with an indedinite Euler form χ_E. The definition of E depends on whether or not $(\omega, \alpha) \in L$. Observe that without loss of generality we may assume that the support of $F(R)$ is $\{a, b\}$, where a, b are as indicated in (4).

(a) Suposse $(\omega, \alpha) \notin L$. Consider E defined as follows:

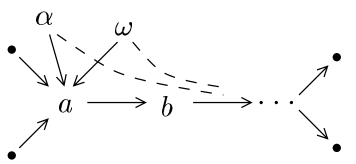


Figure 7:

Let $r = \mathbf{dim}\, F(R)$. We define $\hat{f}\colon K_0(B') \to K_0(E)$ by $x + x_\alpha e_\alpha + x_\omega e_\omega \mapsto f(x) + (\langle r, f(x)\rangle_H - x_\alpha)e_\alpha + x_\omega e_\omega$.

We shall show that $\hat{f}$ is an isometry. Indeed,

$$
\begin{aligned}
&\langle x + x_\alpha e_\alpha + x_\omega e_\omega, y + y_\alpha e_\alpha + y_\omega e_\omega \rangle_{B'} \\
&= \langle x, y\rangle_B - y_\omega \langle x, \mathbf{dim}\, R'\rangle_B + x_\alpha y_\alpha - \langle \mathbf{dim}\, R, y\rangle_B + x_\omega y_\alpha \langle e_\omega, e_\alpha\rangle_{B'} + x_\omega y_\omega \\
&= \langle f(x), f(y)\rangle_H - y_\omega \langle \mathbf{dim}\, \tau^- F(R')[-1], x\rangle_B + x_\alpha y_\alpha - \langle r, f(y)\rangle_H + x_\omega y_\omega \\
&\overset{!}{=} \langle \hat{f}(x + x_\alpha e_\alpha + x_\omega e_\omega), \hat{f}(y + y_\alpha e_\alpha + y_\omega e_\omega)\rangle_E
\end{aligned}
$$

In the step ! we have used that $\mathbf{dim}\, \tau^- F(R')[-1] = r$ and $e_\alpha + r$ (resp. $e_\omega + r$) is the class of the indecomposable projective cover of S_α (resp. S_ω) in $K_0(E)$.

Since clearly E contains a hypercritical quiver, then χ_E is indefinite and we get the desired contradiction.

(b) Suppose $(\omega, \alpha) \in L$. Consider E defined as follows:

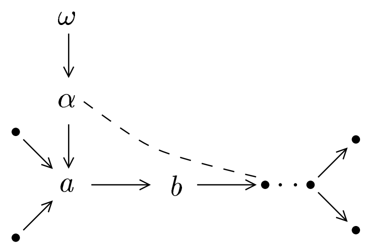


Figure 8:

As above we set $r = \mathbf{dim}\, F(R)$ and $\hat{f}\colon K_0(B') \to K_0(E)$, $x + x_\alpha e_\alpha + x_\omega e_\omega \mapsto f(x) + (\langle r, f(x)\rangle_H - x_\alpha)e_\alpha + x_\omega e_\omega$. We check that $\hat{f}$ is an isometry:

$$\begin{aligned}
&\langle x + x_\alpha e_\alpha + x_\omega e_\omega, y + y_\alpha e_\alpha + y_\omega e_\omega\rangle_{B'} \\
&\quad = \langle x, y\rangle_B - y_\omega\langle x, \mathbf{dim}\, R'\rangle_B + x_\alpha y_\alpha - \langle \mathbf{dim}\, R, y\rangle_B + x_\omega y_\alpha \langle e_\omega, e_\alpha\rangle_{B'} + x_\omega y_\omega \\
&\quad = \langle f(x), f(y)\rangle_H - y_\omega \langle r, f(y)\rangle_H + x_\alpha y_\alpha - \langle r, f(y)\rangle_H + x_\omega y_\alpha + x_\omega y_\omega = \\
&\quad \stackrel{!}{=} \langle \hat{f}(x + x_\alpha e_\alpha + x_\omega e_\omega), \hat{f}(y + y_\alpha e_\alpha + y_\omega e_\omega)\rangle_E
\end{aligned}$$

In step ! we have used that $e_\alpha + r$ (resp. $e_\omega + e_\alpha + r$) is the class of the indecomposable projective cover of S_α (resp. S_ω) in $K_0(E)$. It is clear that also χ_E is indefinite.

(7) We consider finally the case in which $\tau^- FR'[-1] \not\cong FR$ but they are not $\mathrm{Hom}_{D^b(H)}$-orthogonal. By (6) this happens only if $\tau^- FR'[-1] = \tau FR$. Moreover we assume that $(\omega, \alpha) \notin L$ to get a contradiction.

Indeed, we may suppose that $\operatorname{supp} F(R) = \{a, b\}$ and call $r = \mathbf{dim}\, \tau F(R)$. We consider the algebra E defined as follows:

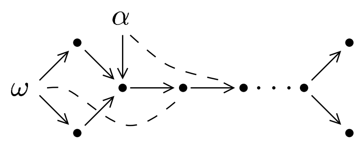


Figure 9:

where the relation at ω is a commutativity relation. We set $\hat{f}\colon K_0(B') \to K_0(E)$, $x + x_\alpha e_\alpha + x_\omega e_\omega \mapsto f(x) + (\langle r, f(x)\rangle_H - x_\alpha)e_\alpha + x_\omega e_\omega$ as before. We check that $\hat{f}$ is an isometry:

$$\begin{aligned}
&\langle x + x_\alpha e_\alpha + x_\omega e_\omega, y + y_\alpha e_\alpha + y_\omega e_\omega\rangle_{B'}\\
&= \langle x, y\rangle_B - y_\omega\langle x, \mathbf{dim}\, R'\rangle_B + x_\alpha y_\alpha - \langle \mathbf{dim}\, R, y\rangle_B + x_\omega y_\omega\\
&= \langle f(x), f(y)\rangle_H - y_\omega\langle f(\mathbf{dim}\,\tau^- F(R')[-1]), f(x)\rangle_H + x_\alpha y_\alpha - \langle r, f(y)\rangle_H + x_\omega y_\omega\\
&\overset{!}{=} \langle \hat{f}(x + x_\alpha e_\alpha + x_\omega e_\omega), \hat{f}(y + y_\alpha e_\alpha + y_\omega e_\omega)\rangle_E,
\end{aligned}$$

where in step ! we use that $e_\alpha + r$ (resp. $e_\omega + f(\mathbf{dim}\,\tau^- F(R')[-1]) = e_\omega + \mathbf{dim}\,\tau F(R))$ is the class of the projective cover of S_α (resp. S_ω) in $K_0(E)$. It is also clear that χ_E is indefinite.

This completes the proof of our Theorem. □

2.6. As a simple *example* of a tree algebra satisfying the hypothesis of the main Theorem but not of the Proposition (1.4) we consider the following:

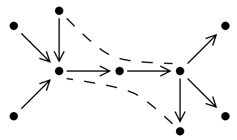


Figure 10:

2.7. The following version of the main Theorem seems easier to apply in some situations.

Theorem. *Let A be a strongly simply connected algebra which may be obtained as a truncated biextension $A = [R']B[R]/J(V)$ of a convex connected subcategory B.*

Assume that χ_A is non-negative and $\operatorname{corank} \chi_B = 1$. *Then A is tame.*

Proof: To apply the main Theorem we assume that $V = \bigoplus_{(i,j)\in L} DR'_j \otimes_B R_i$ and show that there exists (i) a strongly simply connected algebra B' which is a truncated biextension of B and a convex subcategory of A with $\operatorname{corank} \chi_{B'} = 1$; (ii) subsets $K \subset \{1, \ldots, t\}$ and $K' \subset \{1, \ldots, s\}$ and indecomposable B'-modules $\bar{R}_i$ (resp. $\bar{R}'_j$) with restriction R_i (resp. R'_j) to B for $i \in K$ (resp. $j \in K'$) such that A is the truncated biextension $\left[\bigoplus_{j\in K'} \bar{R}'_j\right] B' \left[\bigoplus_{i\in K} \bar{R}_i\right] / J(V')$ with $V' = \bigoplus_{(i,j)\in L'} D\bar{R}'_j \otimes \bar{R}_i$ for $L' = L \cap (K \times K')$ and (iii) $\operatorname{corank} \chi_A = 1 + t' + s'$, where t' (resp. s') is the cardinality of K (resp. K').

Indeed, we may proceed by induction on the number of extensions and coextensions $s+t$. Clearly, if $s+t=0$ there is nothing to show. Assume that $t>0$ and consider the truncated biextension $A_1 = \left[\bigoplus_{j=1}^{s} R'_j\right] B \left[\bigoplus_{i=1}^{t-1} R_i\right] / J(V_1)$, where $V_1 = \bigoplus_{(i,j)\in L_1} DR'_j \otimes_B R_i$ for $L_1 = L \cap [\{1,\dots,t-1\} \times \{1,\dots,s\}]$. By induction hypothesis we may suppose, without loss of generality, that $\operatorname{corank} \chi_{A_1} = 1 + (t-1) + s$. If $\operatorname{corank} \chi_A = 1 + t + s$, there is nothing to prove. Hence we assume that $\operatorname{corank} \chi_A = t + s = \operatorname{corank} \chi_{A_1}$.

Let $B' = B[R_t]$ and let $\bar{R}_i = R_i$. Further, for $1 \le j \le s$, if $(t,j) \notin L$, we set $\bar{R}'_j = (\operatorname{Hom}_B(R_t, R'_j), R'_j, \operatorname{id}_{\operatorname{Hom}_B(R_t,R'_j)})$ the canonical extension of R'_j to $\operatorname{mod}_{B'}$ and $\bar{R}'_j = R'_j$ else. We shall prove that (i) to (iii) hold.

(i): Since B' is a convex subcategory of the strongly simply connected algebra A, then B' is also strongly simply connected. Moreover, B' is a truncated biextension of B.

(ii): We shall prove that A is a truncated biextension of B'. Clearly, $\bar{R}_1, \dots, \bar{R}_{t-1}$ and $\bar{R}'_1, \dots, \bar{R}'_s$ are two sequences of indecomposable B'-modules. For $1 \le i \le t-1$ and $1 \le j \le s$, we have $D\bar{R}'_j \otimes_{B'} \bar{R}_1 = \operatorname{Hom}_{B'}(\bar{R}_i, \bar{R}'_j) = \operatorname{Hom}_B(R_i, R'_j)$. Let $E = \left[\bigoplus_{j=1}^{s} \bar{R}'_j\right] B' \left[\bigoplus_{i=1}^{t-1} \bar{R}_i\right]$ be the biextension of B' and let x (resp. y_j, $1 \le j \le s$) be the vertices of the quiver of E such that the restriction of the indecomposable projective E-module P_x to B is R_t (resp. the restriction of the indecomposable injective E-module I_{y_j} to B is R'_j). Then $e_{y_j} E e_x = \operatorname{Hom}_{B'}(P_x, \bar{R}'_j) = \operatorname{Hom}_B(R_t, R'_j)$ if $(t,j) \notin L$ and $e_{y_j} E e_x = 0$, else. This shows that $E/J(V_1) \xrightarrow{\sim} A$.

(iii): $\operatorname{corank} \chi_A = t + s = 1 + (t-1) + s$, as desired. □

2.8. We conclude by posing a restricted form of a problem raised in [14].

Problem: Let A be an algebra satisfying the hypothesis of the main Theorem. Is it true that the repetitive category $\hat{A}$ is tame?

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